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INVESTIGATION OF NEUTRON SCATTERING
FOR THE COMPLEX POTENTIAL MODEL

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I. Introduction

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Of the many models proposed to describe neutron reactions and scattering, the Feshbach, Porter, and Weisskopf(6)* complex well model has aroused the most interest in recent times. We shall discuss briefly some of the papers leading up to the FPW paper, emphasizing the more recent ones.

The use of a complex potential seems to have originated with Bethe(2). Bethe equated the formation of the compound nucleus with the absorption of the incoming beam, and wrote the continuity equation, assuming the number of particles absorbed per unit time at a given point in the nucleus is proportional to the density of particles at that point

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = \frac{-2 \sigma(\vec{r})}{\hbar} \rho \quad (1.1)$$

where $\vec{j}$ = current density

ρ = density of particles

σ = absorption potential.

Bethe showed that (1.1) is equivalent to the Schrödinger equation

$$\frac{-\hbar^2}{2m} \nabla^2 \psi + (V - i\sigma) \psi = i \hbar \frac{\partial \psi}{\partial t} \quad (1.2)$$

The potential is thus split into a real part, V , and an imaginary part, σ . Multiplying by ψ^* and subtracting the complex conjugate of the resultant expression, we have

*This paper will be denoted by FPW in succeeding references.

$$\frac{-\hbar^2}{2m} (\Psi^* \nabla^2 \Psi - \Psi \nabla^2 \Psi^*) - 2i\sigma \Psi^* \Psi = i\hbar \frac{\partial}{\partial t} \Psi^* \Psi. \quad (1.3)$$

But from vector analysis,

$$\phi \nabla \cdot \vec{a} = \nabla \cdot (\phi \vec{a}) - \vec{a} \cdot \nabla \phi. \quad (1.4)$$

Therefore,

$$\begin{aligned} & \frac{-\hbar^2}{2m} (\nabla \cdot \Psi^* \nabla \Psi - \nabla \Psi^* \cdot \nabla \Psi - \nabla \cdot \Psi \nabla \Psi^* \\ & + \nabla \Psi \cdot \nabla \Psi^*) + 2i\sigma \Psi \Psi^* = i\hbar \frac{\partial}{\partial t} \Psi \Psi^* \\ & \nabla \cdot \left[\frac{\hbar^2}{2m} (\Psi^* \nabla \Psi - \Psi \nabla \Psi^*) \right] + \frac{2\sigma \Psi \Psi^*}{\hbar} = -\frac{\partial}{\partial t} \Psi \Psi^*. \end{aligned} \quad (1.5)$$

Since the density and current are defined

$$\rho = \Psi \Psi^*, \quad \vec{j} = \frac{\hbar}{2im} (\Psi^* \nabla \Psi - \Psi \nabla \Psi^*), \quad (1.6)$$

we have the continuity equation,

$$\nabla \cdot \vec{j} + \frac{\partial \rho}{\partial t} = -\frac{2\sigma \rho}{\hbar}. \quad (1.7)$$

The concept of an imaginary potential was again used by Feshbach, Peaslee, and Weisskopf(4). They introduced a specific averaging procedure with respect to resonances, and were able to derive the Breit-Wigner formulae without having to contend with far away resonance levels. This had been a difficulty of earlier theories.

The formalism was set up in terms of the logarithmic derivative of the radial wave function, $f(E)$, defined as

$$f(E) \equiv a \left[\frac{d\phi/dr}{\phi} \right]_{r=a}, \quad (1.8)$$

where $\phi(r) = r \Psi$,

and a = nuclear radius.

They found it possible to express some characteristic quantities, such as reaction width $\Gamma_n^{(r)}$, absorption widths $\Gamma_a^{(r)}$, and resonance energies E_r in terms of general properties of the function $f(E)$.

To get numbers from their theory, they work with a real logarithm derivative and write the real part f_0 , in the form

$$f_0(W) = -K a \tan z(W)$$

where W is the energy, and $z(W)$ is a monotonically increasing function which is equal to multiples of π for the resonance values W_r :

$$z(W_r) = n\pi, \quad n \text{ integer}$$

The authors attempt to make plausible that K is a wave number corresponding to the kinetic energy of a particle in the nucleus. It is determined by a Fermi-Dirac model calculation with $a = 1.5 A^{1/3} \times 10^{-13} \text{ cm}$. The assumption is then made that $z(W)$ increases smoothly from $n\pi$ to $(n+1)\pi$, if the energy W goes from one resonance to the next, that is

$$\left(\frac{dz}{dW} \right)_{W_r} = \pi/D^* \quad (1.9)$$

where D^* is an energy of the order of the average level distance D . The various cross sections are then determined in terms of D^* .

A model in which the logarithmic derivative is pure imaginary was proposed by Feshbach and Weisskopf(5). They postulate that the wave function within the nucleus consists of an incoming beam only

$$u_l(r) = C e^{-iKr} \quad (1.10)$$

where K is the internal wave number and C a normalization constant. K is again determined by a Fermi-Dirac model, but the internal wave number used in contrast to that of Feshbach, Peaslee and Weisskopf, did not involve the

binding energy of the neutron since the authors felt it probable that a considerable portion of the binding energy is potential in virtue of the exchange force between nucleons. The choice of internal wave function leads to the imaginary logarithm derivative

$$f_l = i X, \quad (1.11)$$

where $X = KR$, $R = \text{nuclear radius}$

K and R being the two parameters of the theory. Of course, these extremely simple assumptions cannot yield any resonance phenomena and, when cross-sections calculated from the theory are compared to experimental data, require that the data in the resonance region be averaged.

Chronologically, the FPW model followed. A discussion of this model will be deferred to Sec. IV.

It should be mentioned that Lane, Thomas, and Wigner(11) have shown that the results of the FPW model can be obtained, as Wigner(22) put it, "as vestigial traces of the independent particle model." By modifying and reinterpreting the independent particle model, they have been able to obtain the maxima in the experimental total cross sections found by Barschall(1) and his collaborators, using real potentials.

Like all models based strongly upon the Bohr compound nucleus assumption, the first three models(2,4,5) suffer from the inadequacy of the compound nucleus model in the region of higher energy. It is a fundamental postulate of the Bohr compound nucleus theory that the decay of the compound nucleus is independent of the process which formed it. Thus one would expect that at a given energy a given compound nucleus, whether formed by an incident

proton or neutron, would have the same emission from the compound nucleus. This is not borne out by experiment. It has been found, for example, that in reactions induced by fast protons, protons are preferentially emitted, and, similarly, in neutron induced reactions, neutrons are preferentially emitted.

Of greater consequence to this investigation, however, is the behavior of these models in the region .050-3 Mev. It is here that the Feshbach, Peaslee, and Weisskopf models and Feshbach and Weisskopf models yield total cross-sections which decrease monotonically with energy, and, thus, do not reproduce the Barschall maxima.

II. Statement of the Problem

This work was motivated by a specific set of experimental data. Namely, it is an attempt to interpret theoretically the experimental data obtained by Langsdorf(12) and Lane at Argonne, data containing neutron differential scattering coefficients for a fairly complete mass spectrum in the range 50-1700 kev. The experiment will be described in Sec. X.

Our main attack upon the problem consisted in a modification and reinterpretation of the FPW model. We have also used a variational method to obtain the square-well phase shifts of all orders in closed form.

III. The Scattering Formalism

The FPW paper is a rather panoramic paper on neutron reactions. It sets up a formalism from which various differential and total cross-sections may be calculated utilizing few parameters. It interprets this simple model in terms of averages over resonances and illustrates how the so-called "compound elastic" scattering may be calculated. Further, it connects the parameters used in the theory with the experimentally measured resonance parameters Γ_0/D , the quotient of the average neutron half-width and the average level spacing.

Our interest is more confined. We will not attempt to calculate compound elastic scattering, though we will discuss it in Sec. VI-A, nor will we treat the connection of the parameters we use in our calculations with the resonance parameters.

Before proceeding with the FPW formalism let us set up the standard spin zero scattering formalism for completeness, following Blatt and Weisskopf(3), Chap. VIII, since the FPW notation is practically identical.

We assume we have an incident plane wave of neutrons e^{ikz} , where the wave is normalized to one neutron per unit volume, and the z-axis is taken in the direction of the incident neutron. The wave number of the incident neutron k is defined as

$$k = \frac{(2mE)^{1/2}}{\hbar} \quad (3.1)$$

where m is the reduced mass of the neutron and E is the center-of-mass kinetic energy of the neutron.

We then expand the plane wave in an infinite series of spherical harmonics.

$$e^{ikz} = e^{ikr \cos \theta} = \sum_{\ell=0}^{\infty} A_{\ell}(r) Y_{\ell,0}(\theta) \quad (3.2)$$

Multiplying both sides by $Y_{\ell',0}^*$ and integrating over the unit sphere, we have

$$\int Y_{\ell',0}^* e^{ikr \cos \theta} d\Omega = \sum_{\ell=0}^{\infty} A_{\ell}(r) \int Y_{\ell',0}^*(\theta) Y_{\ell,0}(\theta) d\Omega$$

and since the spherical harmonics are an orthonormal set on the unit sphere

$$\int Y_{\ell',0}^* e^{ikr \cos \theta} d\Omega = \sum_{\ell=0}^{\infty} A_{\ell} \delta_{\ell\ell'} \quad (3.3)$$

$$\therefore A_{\ell}(r) = \int Y_{\ell,0}^*(\theta) e^{ikr \cos \theta} d\Omega.$$

The asymptotic behavior of the expansion coefficients A_{ℓ} may be determined by doing a partial integration

$$A_{\ell} = 2\pi \int_0^{\pi} Y_{\ell,0}^* e^{ikr \cos \theta} \sin \theta d\theta$$

$$A_{\ell}(r) = 2\pi \left\{ -\frac{1}{ikr} e^{ikr \cos \theta} Y_{\ell,0}^* \Big|_0^{\pi} + \frac{1}{ikr} \int_0^{\pi} Y_{\ell,0}^*(\theta) e^{ikr \cos \theta} \sin \theta d\theta \right\} \quad (3.4)$$

Succeeding partial integrations give the expansion. Since we are interested in the asymptotic form we keep the leading term only

$$A_{\ell} \sim \frac{2\pi i}{kr} e^{ikr \cos \theta} Y_{\ell,0}^*(\theta) \Big|_0^{\pi} \quad (3.5)$$

$$\begin{aligned}
A_\ell &\sim \frac{2\pi i}{kr} \left[e^{-ikr} \left(\frac{2\ell+1}{4\pi} \right)^{1/2} (-1)^\ell - e^{ikr} \left(\frac{2\ell+1}{4\pi} \right)^{1/2} \right] \\
&\sim \left[\frac{4\pi(2\ell+1)}{kr} \right]^{1/2} i^\ell \frac{e^{ikr} - (-1)^\ell e^{-ikr}}{2i^{\ell+1}}
\end{aligned} \tag{3.6}$$

The term on the right is equal to $\sin(kr - \frac{1}{2}\ell\pi)$ as may readily be checked by application of the trigonometric relation for the sine of the sum of two angles. Thus,

$$A_\ell \cong i^\ell \left[4\pi(2\ell+1) \right]^{1/2} \frac{\sin(kr - \frac{1}{2}\ell\pi)}{kr}, \quad kr \gg \ell \tag{3.7}$$

and

$$\begin{aligned}
e^{ikz} &\cong \frac{\pi^{1/2}}{kr} \sum_{\ell=0}^{\infty} i^{\ell+1} (2\ell+1)^{1/2} \left[e^{-i(kr - \frac{1}{2}\ell\pi)} \right. \\
&\quad \left. - e^{i(kr - \frac{1}{2}\ell\pi)} \right] Y_{\ell,0}, \quad kr \gg \ell
\end{aligned} \tag{3.8}$$

The scattering will affect only the outgoing wave asymptotically.

Therefore, the wave function in the incident channel may be written

$$\begin{aligned}
\psi(\vec{r}) &= \frac{\pi^{1/2}}{kr} \sum_{\ell=0}^{\infty} (2\ell+1)^{1/2} i^{\ell+1} \left[e^{-i(kr - \frac{1}{2}\ell\pi)} \right. \\
&\quad \left. - \eta_\ell e^{i(kr - \frac{1}{2}\ell\pi)} \right] Y_{\ell,0}, \quad kr \gg \ell
\end{aligned} \tag{3.9}$$

where the complex reflection factor η_ℓ is the coefficient of the outgoing wave of angular momentum " ℓ ."

The elastically scattered wave is just the difference between the actual wave and the original plane wave

$$\psi_{el}(\vec{r}) = \psi(\vec{r}) - e^{ikz} \quad (3.10)$$

$$= \frac{\pi^{1/2}}{kr} \sum_{\ell=0}^{\infty} (2\ell+1)^{1/2} i^{\ell+1} (1-\eta_{\ell}) e^{i(kr - \frac{1}{2}\ell\pi)} Y_{\ell,0},$$

$$kr \gg \ell \quad (3.11)$$

In order that we may use the asymptotic form of the wave functions in computing cross-sections, we surround the scattering center by a sphere of very large radius R_0 . The number of particles elastically scattered per unit time n_{sc} is then the integrated flux through the sphere

$$n_{sc} = \int_A \vec{S}(\vec{r}) \cdot \hat{n} dA \quad (3.12)$$

$$\text{where } \vec{S}(\vec{r}) = \frac{\hbar}{2im} [\psi_{el}^* \nabla \psi_{el} - \psi_{el} \nabla \psi_{el}^*] \quad (3.13)$$

and $\hat{n}$ is a unit outward normal on the surface, and the integration extends over the enclosing surface A .

Now

$$\vec{S} \cdot \hat{n} = \frac{\hbar}{2im} \left[\psi_{el}^* \frac{\partial}{\partial r} \psi_{el} - \psi_{el} \frac{\partial}{\partial r} \psi_{el}^* \right] \quad (3.14)$$

and

$$\frac{\partial \psi_{el}}{\partial r} = ik \psi_{el} - \frac{1}{r} \psi_{el} \quad (3.15)$$

$$\begin{aligned} \therefore \vec{S} \cdot \hat{n} &= \frac{\hbar}{2im} \left[\psi_{el}^* \left(ik \psi_{el} - \frac{1}{r} \psi_{el} \right) \right. \\ &\quad \left. - \psi_{el} \left(-ik \psi_{el}^* - \frac{1}{r} \psi_{el}^* \right) \right] = \frac{\hbar k}{m} |\psi_{el}|^2 \end{aligned} \quad (3.16)$$

and

$$\begin{aligned}
 n_{sc} &= \frac{\hbar k}{m} R_o^2 \int |\gamma_{el}|^2 d\Omega \\
 &= \frac{\hbar k}{m} R_o^2 \left| \frac{\pi^{1/2}}{k R_o} \sum_{\ell=0}^{\infty} (2\ell+1)^{1/2} i^{\ell+1} (1 - \eta_{\ell}) e^{i(k R_o - \frac{1}{2}\ell\pi)} \right|^2 \\
 &\quad \times \int Y_{\ell,0}^* Y_{\ell,0} d\Omega
 \end{aligned} \tag{3.17}$$

The integration of the spherical harmonics again yields $\delta_{\ell\ell'}$, allowing us to break the double sum over ℓ 's into a single sum, the cross terms dropping out.

$$\begin{aligned}
 n_{sc} &= \frac{\hbar k}{m} R_o^2 \frac{\pi}{k^2 R_o^2} \sum_{\ell} (2\ell+1) |1 - \eta_{\ell}|^2 \\
 &= \frac{\hbar k}{m} \pi \lambda^2 \sum_{\ell} (2\ell+1) |1 - \eta_{\ell}|^2
 \end{aligned} \tag{3.18}$$

where $\lambda \equiv \frac{1}{k}$, and since

$$\begin{aligned}
 k &= \frac{(2mE)^{1/2}}{\hbar} = \frac{(m^2 v^2)^{1/2}}{\hbar}, \text{ where } v = \text{velocity of neutrons} \\
 v &= \frac{\hbar k}{m}.
 \end{aligned} \tag{3.19}$$

Having chosen a normalization of one particle per unit volume, the incident flux is then equal to v and we have

$$\sigma_{el} = \pi \lambda^2 \sum_{\ell} (2\ell+1) |1 - \eta_{\ell}|^2 \tag{3.20}$$

where $\sigma_{el} = \frac{\text{no. of neutrons elastically scattered per unit time per nucleus}}{\text{no. of incident neutrons per unit area per unit time.}}$

To determine the reaction cross section σ_r , we undertake a similar calculation: Since any neutron interacting with the nucleus and not being elastically scattered is removed from the incident channel, the reaction can be considered an absorption of particles from the incident channel. Therefore, we determine the net flux into the large sphere using the actual wave function $\psi(r)$.

$$n_r = - \int_A \vec{S} \cdot \hat{n} dA$$

where $S = \frac{\hbar}{2im} (\psi^* \nabla \psi - \psi \nabla \psi^*)$. This leads in a straightforward manner to

$$\sigma_r = \pi \lambda^2 \sum_{\ell} (2\ell + 1) (1 - |\eta_{\ell}|^2) \quad (3.21)$$

To find the differential elastic scattering cross-section $\sigma_{el}(\theta)$, we equate $\sigma_{el}(\theta) d\Omega$ to the flux of particles through a surface element of a large sphere $R_0^2 d\Omega$. Thus

$$\begin{aligned} n_{el}(\theta) d\Omega &= \vec{S}_{el} \cdot \hat{n} d\Omega \\ &= \frac{\hbar}{2im} \left(\psi_{el}^* \frac{\partial \psi_{el}}{\partial r} - \psi_{el} \frac{\partial \psi_{el}^*}{\partial r} \right) R_0^2 d\Omega \end{aligned} \quad (3.22)$$

After division by the incident flux v , we have

$$\sigma_{el}(\theta) d\Omega = \pi \lambda^2 \left| \sum_{\ell=0}^{\infty} (2\ell + 1)^{1/2} (1 - \eta_{\ell}) Y_{\ell,0} \right|^2 d\Omega \quad (3.23)$$

Let us now set up the apparatus for calculating the η_{ℓ} from the Schrödinger equation. $\psi(r)$ satisfies

$$\nabla^2 \psi + \left[k^2 - \frac{2m}{\hbar^2} V(r) \right] \psi = 0 \quad (3.24)$$

If we set $u_\ell = r \psi_\ell$, and assume our potential to have spherical symmetry we get the separation

$$\frac{d^2 u_\ell}{dr^2} + \left[k^2 - \frac{\ell(\ell+1)}{r^2} - \frac{2m}{\hbar^2} V(r) \right] u_\ell = 0 \quad (3.25)$$

If we now assume the nuclear potential to be negligible outside the nuclear radius R , we have two linearly independent solutions for the region $r > R$: the "regular" solution $F_\ell(r)$, and the "irregular" solution $G_\ell(r)$. The regular solution is the real solution which vanishes for $r = 0$. This determines it within a multiplicative constant and the constant is fixed so that the asymptotic behavior will be

$$F_\ell(r) \simeq \sin(kr - \frac{1}{2}\ell\pi), \quad kr \gg \ell \quad (3.26)$$

The multiplicative constant on the "irregular" solution is determined so that the asymptotic behavior will be

$$G_\ell(r) \simeq \cos(kr - \frac{1}{2}\ell\pi), \quad kr \gg \ell \quad (3.27)$$

We will need combinations of these solutions whose asymptotic behavior will be that of incoming and outgoing spherical waves. Let us define

$$u_\ell^{(+)}(r) \equiv G_\ell(r) + i F_\ell(r) \quad (3.28)$$

$$u_\ell^{(-)} \equiv u_\ell^{(+)*} = G_\ell(r) - i F_\ell(r). \quad (3.29)$$

Thus

$$u_\ell^{(+)} \simeq e^{i(kr - \frac{1}{2}\ell\pi)} \quad (3.30)$$

$$u_{\ell}^{(-)} = e^{-i(kr - \frac{1}{2}\ell\pi)} \quad (3.31)$$

The general solution may then be written as a linear combination

$$u_{\ell}(r) = c_1 u_{\ell}^{(+)}(r) + c_2 u_{\ell}^{(-)}(r) \quad (3.32)$$

and the constants determined from the asymptotic behavior of the wave function.

From (3.9)

$$u_{\ell}(r) = \frac{\pi^{1/2}}{k} (2\ell+1)^{1/2} i^{\ell+1} \left[e^{-i(kr - \frac{1}{2}\ell\pi)} - \eta_{\ell} e^{i(kr - \frac{1}{2}\ell\pi)} \right] \quad (3.33)$$

$$\therefore c_1 = \frac{\pi^{1/2}}{k} (2\ell+1)^{1/2} i^{\ell+1} \eta_{\ell}$$

$$c_2 = \frac{\pi^{1/2}}{k} (2\ell+1)^{1/2} i^{\ell+1}$$

and

$$u_{\ell} = \frac{\pi^{1/2}}{k} (2\ell+1)^{1/2} i^{\ell+1} (u_{\ell}^{(-)} - \eta_{\ell} u_{\ell}^{(+)}) \quad (3.34)$$

The F_{ℓ} and G_{ℓ} may be expressed in terms of Bessel functions

$$F_{\ell}(r) = \left(\frac{\pi kr}{2} \right)^{1/2} J_{\ell + \frac{1}{2}}(kr) \quad (3.35)$$

$$G_{\ell}(r) = - \left(\frac{\pi kr}{2} \right)^{1/2} N_{\ell + \frac{1}{2}}(kr) \quad (3.36)$$

where $J_n(z)$ and $N_n(z)$ are the Bessel and Neumann functions defined in Jahnke and Emde(10).

We shall use the more compact notation of Morse, Lowan, Feshbach, and Lax(14),

$$j_{\ell}(z) = \left(\frac{\pi}{2z}\right)^{1/2} J_{\ell + \frac{1}{2}}(z) \quad (3.37)$$

$$n_{\ell}(z) = \left(\frac{\pi}{2z}\right)^{1/2} N_{\ell + \frac{1}{2}}(z) \quad (3.38)$$

Thus, the "regular" and "irregular" solutions are

$$F_{\ell}(r) = kr j_{\ell}(kr) \quad (3.35a)$$

$$G_{\ell}(r) = -kr n_{\ell}(kr) \quad (3.36a)$$

The spherical waves may be described in terms of the spherical Hankel function

$$h_{\ell} = j_{\ell} + i n_{\ell} \quad (3.39)$$

giving

$$\begin{aligned} u_{\ell}^{(+)}(r) &= -kr n_{\ell}(kr) + i kr j_{\ell}(kr) \\ &= i kr h_{\ell}(kr) \end{aligned} \quad (3.40)$$

$$u_{\ell}^{(-)}(r) = -i kr h_{\ell}^{*}(kr) \quad (3.41)$$

We now have u_{ℓ} in a form in which its asymptotic behavior is proper.

Only η_{ℓ} remains free. We must fix η_{ℓ} so that the external solution matches the internal wave function at the nuclear boundary. For this purpose we introduce the logarithmic derivative f_{ℓ}

$$f_{\ell} \equiv R \left[\frac{d u_{\ell}/dr}{u_{\ell}} \right]_{r=R} \quad (3.42)$$

In addition, we introduce two real quantities completely independent of the internal wave function. The numbers Δ_{ℓ} and s_{ℓ} are defined

$$R \left[\frac{d u_{\ell}^{(+)} / dr}{u_{\ell}^{(+)}} \right]_{r=R} \equiv \Delta_{\ell} + i s_{\ell} \quad (3.43)$$

In terms of the spherical Hankel function,

$$\Delta_\ell + i s_\ell = 1 + \frac{x h'_\ell(x)}{h_\ell(x)} \quad (3.44)$$

where $x = kR$ and $h'_\ell(x) = \frac{d}{dx} h_\ell(x)$.

We may now find η_ℓ as a function of f_ℓ . Using the defining equation of f_ℓ and (3.34):

$$f_\ell = R \left[\frac{\frac{d u_\ell^{(-)}(r)}{dr} - \eta_\ell \frac{d u_\ell^{(+)}(r)}{dr}}{u_\ell^{(-)}(r) - \eta_\ell u_\ell^{(+)}(r)} \right]_{r=R} \quad (3.45)$$

Dividing through by $u_\ell^{(+)}(x)$

$$\eta_\ell \left[R \frac{d u_\ell^{(+)}(x)}{u_\ell^{(+)}(x)} - f_\ell \right] = R \frac{d u_\ell^{(-)}(x)/dr}{u_\ell^{(+)}(x)} - f_\ell \frac{u_\ell^{(-)}(x)}{u_\ell^{(+)}(x)}$$

We define the phase difference between the incoming and outgoing spherical waves at the nuclear boundary R as

$$e^{2i\xi_\ell} \equiv \frac{u_\ell^{(-)}(x)}{u_\ell^{(+)}(x)} \quad (3.46)$$

It is also easily shown that

$$R \left[\frac{d u_\ell^{(-)}/dr}{u_\ell^{(-)}} \right]_{r=R} = \Delta_\ell - i s_\ell \quad (3.47)$$

and noting that

$$\frac{d u_\ell^{(-)}/dr}{u_\ell^{(+)}} = \frac{d u_\ell^{(-)}/dr}{u_\ell^{(-)}} \frac{u_\ell^{(-)}}{u_\ell^{(+)}}$$

we have

$$\eta_l (\Delta_l + i s_l - f_l) = (\Delta_l - i s_l) e^{2i \xi_l} - f_l e^{2i \xi_l}$$

$$\therefore \eta_l = \frac{f_l - \Delta_l + i s_l}{f_l - \Delta_l - i s_l} e^{2i \xi_l} \quad (3.48)$$

If we define

$$M_l \equiv s_l - \text{Im } f_l \quad N_l \equiv -\Delta_l + \text{Re } f_l \quad (3.49)$$

and let

$$\delta_l \equiv -\xi_l \quad (3.50)$$

we may recast (3.48) into Eq. (3.3) in FPW

$$\begin{aligned} \eta_l &= \frac{\text{Re } f_l - \Delta_l + i(s_l + \text{Im } f_l)}{\text{Re } f_l - \Delta_l + i(\text{Im } f_l - s_l)} e^{-2i \delta_l} \quad (3.51) \\ &= \frac{N_l + i(s_l + \text{Im } f_l)}{N_l - i M_l} e^{-2i \delta_l} \\ &= \frac{i N_l - s_l - \text{Im } f_l}{M_l + i N_l} e^{-2i \delta_l} \\ &= \frac{M_l + i N_l - s_l - \text{Im } f_l - M_l}{M_l + i N_l} e^{-2i \delta_l} \\ &= \frac{M_l + i N_l - 2 s_l}{M_l + i N_l} e^{-2i \delta_l} \\ \therefore \eta_l &= \left(1 - \frac{2 s_l}{M_l + i N_l} \right) e^{-2i \delta_l} \quad (3.52) \end{aligned}$$

From (3.46)

$$e^{-2i \delta_l} = \frac{j_l - i n_l}{j_l + i n_l} = \frac{j_l^2 - n_l^2 - 2i j_l n_l}{j_l^2 + n_l^2} \quad (3.53)$$

$$\tan(-2\delta_e) = -\tan 2\delta_e = -\frac{2 \tan \delta_e}{1 - \tan^2 \delta_e} = -\frac{2 j_e n_e}{j_e^2 - n_e^2}$$

$$\frac{\tan \delta_e}{1 - \tan^2 \delta_e} = \frac{(-j_e/n_e)}{1 - (-j_e/n_e)^2}$$

$$\therefore \delta_e = \tan^{-1}(-j_e/n_e) \quad (3.54)$$

The three trigonometric functions required in the calculation may be found by computing δ_e from tables of spherical Bessel and Neumann functions, from tables of δ_e (14), or directly from the spherical Bessel and Neumann functions. From the triangle it is apparent that

$$\sin 2\delta_e = -\frac{2 j_e / n_e}{1 + (j_e/n_e)^2} \quad (3.55)$$

$$\cos 2\delta_e = \frac{1 - (j_e/n_e)^2}{1 + (j_e/n_e)^2} \quad (3.56)$$

and since

$$\sin^2 x = \frac{\tan^2 x}{1 + \tan^2 x}$$

$$\sin^2 \delta_\ell = \frac{(j_\ell / n_\ell)^2}{1 + (j_\ell / n_\ell)^2} \quad (3.57)$$

A recurrence relation for the Δ_ℓ and s_ℓ may be found from the recurrence relations for the spherical Bessel functions. From (3.44)

$$\begin{aligned} \Delta_\ell + i s_\ell &= 1 + \frac{x h'_\ell(x)}{h_\ell(x)} \\ &= \frac{h_\ell(x) + x h'_\ell(x)}{h_\ell(x)} \end{aligned}$$

From Morse, Lowen, Feshbach, and Lax, we have

$$z g'_\ell(z) + (\ell + 1) g_\ell(z) = z g_{\ell-1}(z) \quad (3.58)$$

$$z g'_\ell(z) - \ell g_\ell(z) = -z g_{\ell+1}(z) \quad (3.59)$$

where g_ℓ is any linear combination of spherical Bessel, Neumann, or Hankel functions. Using (3.58) we have

$$\begin{aligned} \Delta_\ell + i s_\ell &= \frac{h_\ell + x h_{\ell-1} - (\ell + 1) h_\ell}{h_\ell} \\ &= \frac{x h_{\ell-1}}{h_\ell} - \ell \end{aligned} \quad (3.60)$$

From (3.59) we have

$$g_\ell = \frac{1}{z} (\ell - 1) g_{\ell-1} - g'_{\ell-1}$$

$$\therefore \Delta_\ell + i s_\ell = \frac{x h_{\ell-1}}{\frac{1(\ell-1)h_{\ell-1} - h'_{\ell-1}}{x}} - \ell$$

$$= \frac{x}{\frac{1}{x} \left[(\ell-1) - \frac{x h_{\ell-1}'}{h_{\ell-1}} \right]} - \ell$$

$$= \frac{x^2}{\ell - \left[1 + \frac{x h_{\ell-1}'}{h_{\ell-1}} \right]} - \ell$$

$$= \frac{x^2}{\ell - (\Delta_{\ell-1} + i s_{\ell-1})} - \ell$$

$$\therefore \Delta_{\ell} + i s_{\ell} = \frac{x^2 \left[(\ell - \Delta_{\ell-1}) + i s_{\ell-1} \right]}{(\ell - \Delta_{\ell-1})^2 + s_{\ell-1}^2} - \ell \quad (3.61)$$

$$\Delta_{\ell} = \frac{x^2 (\ell - \Delta_{\ell-1})}{(\ell - \Delta_{\ell-1})^2 + s_{\ell-1}^2} - \ell \quad (3.62)$$

$$s_{\ell} = \frac{x^2 s_{\ell-1}}{(\ell - \Delta_{\ell-1})^2 + s_{\ell-1}^2} \quad (3.63)$$

with starting values,

$$\Delta_0 = 0,$$

$$s_0 = x.$$

IV. The FPW Formalism

Having set up the scattering formalism, we will not introduce the essential averaging procedure of FPW.* The reflection factor η_e is a rapidly varying function of energy due to the many close-spaced resonances of the compound nucleus. We will assume that if an energy interval I contains many resonances, the average reflection factor

$$\bar{\eta}_e = \frac{1}{I} \int_{\epsilon - I/2}^{\epsilon + I/2} \eta_e(\epsilon') d\epsilon' = \frac{1}{I} \int_{\epsilon - I/2}^{\epsilon + I/2} [\text{Re } \eta_e + i \text{Im } \eta_e] d\epsilon' \quad (4.1)$$

is a smooth function of ϵ .

We will also define

$$\bar{\sigma}_{el}^{(\ell)} = \pi \lambda^2 (2\ell + 1) \overline{|1 - \eta_\ell|^2} \quad (4.2)$$

$$\bar{\sigma}_r^{(\ell)} = \pi \lambda^2 (2\ell + 1) (1 - \overline{|\eta_\ell|^2}) \quad (4.3)$$

as the average elastic scattering cross section and average reaction cross-section, respectively, where the averages are taken over the interval I .

The interval $I \ll \epsilon$ so that $\overline{\lambda^2}$ may be approximated by λ^2 , being slowly varying in the interval.

*This will essentially be an elaboration of part of Sec. II of FPW.

Carrying through the averaging process

$$\begin{aligned}
 \overline{\sigma_{el}}^{(\ell)} &= \pi \chi^2 (2\ell + 1) \overline{|1 - \eta_\ell|^2} = \pi \chi^2 (2\ell + 1) \overline{(1 - \eta_\ell - \eta_\ell^* + \eta_\ell^* \eta_\ell)} \\
 &= \pi \chi^2 (2\ell + 1) (1 - \overline{\eta_\ell} - \overline{\eta_\ell^*} + \overline{\eta_\ell^* \eta_\ell}) \\
 &= \pi \chi^2 (2\ell + 1) (1 - \overline{\eta_\ell} - \overline{\eta_\ell^*} + \overline{\eta_\ell^* \eta_\ell} - \overline{\eta_\ell^*} \overline{\eta_\ell} + \overline{\eta_\ell^*} \overline{\eta_\ell}) \\
 \overline{\sigma_{el}}^{(\ell)} &= \pi \chi^2 (2\ell + 1) \left[|1 - \overline{\eta_\ell}|^2 + |\overline{\eta_\ell}|^2 - |\overline{\eta_\ell}|^2 \right] \quad (4.4)
 \end{aligned}$$

Now

$$\begin{aligned}
 \overline{\sigma_t}^{(\ell)} &= \overline{\sigma_{el}}^{(\ell)} + \overline{\sigma_r}^{(\ell)} = \pi \chi^2 (2\ell + 1) \left[|1 - \overline{\eta_\ell}|^2 + |\overline{\eta_\ell}|^2 \right. \\
 &\quad \left. - |\overline{\eta_\ell}|^2 + 1 - |\overline{\eta_\ell}|^2 \right] \\
 &= \pi \chi^2 (2\ell + 1) \left[|1 - \overline{\eta_\ell}|^2 + 1 - |\overline{\eta_\ell}|^2 \right] \quad (4.5)
 \end{aligned}$$

We may then take the average elastic scattering cross-section and split off the fluctuation term, call it the "compound-elastic" cross-section, $\sigma_{ce}^{(\ell)}$, and the remainder the "shape-elastic" cross-section, $\sigma_{se}^{(\ell)}$, by writing

$$\sigma_{se}^{(\ell)} = \pi \chi^2 (2\ell + 1) |1 - \overline{\eta_\ell}|^2 \quad (4.6)$$

$$\sigma_{ce}^{(\ell)} = \pi \chi^2 (2\ell + 1) \left[|\overline{\eta_\ell}|^2 - |\overline{\eta_\ell}|^2 \right] \quad (4.7)$$

FPW then combine the latter term with $\overline{\sigma_r}^{(\ell)}$ to form a new term

$\sigma_c^{(\ell)}$, the cross section for the formation of the compound nucleus

$$\sigma_c^{(\ell)} = \sigma_{ce}^{(\ell)} + \overline{\sigma_r}^{(\ell)} = \pi \chi^2 (2\ell + 1) \left[|\overline{\eta_\ell}|^2 - |\overline{\eta_\ell}|^2 + 1 - |\overline{\eta_\ell}|^2 \right] \quad (4.8)$$

$$= \pi \chi^2 (2\ell + 1) (1 - |\bar{\eta}_\ell|^2) \quad (4.9)$$

Thus $\sigma_{se}^{(\ell)}$ and $\sigma_e^{(\ell)}$ form a scattering and a reaction cross-section for a new problem whose phase is the slowly varying function $\bar{\eta}_\ell$, a problem which FPW call "the gross-structure problem."

We shall discuss the averaging process and "compound-elastic" scattering at length in Sec. VI-A.

Having completed our apparatus we may now calculate cross-sections.

From (3.23)

$$\frac{d\sigma_{se}}{d\Omega} = \pi \chi^2 \left| \sum_{\ell=0}^{\infty} (2\ell + 1)^{1/2} (1 - \bar{\eta}_\ell) Y_{\ell,0}(\theta) \right|^2$$

but since

$$P_\ell(\cos \theta) \equiv \left(\frac{4\pi}{2\ell + 1} \right)^{1/2} Y_{\ell,0}(\theta) \quad (4.10)$$

$$\frac{d\sigma_{se}}{d\Omega} = \frac{\chi^2}{4} \left| \sum_{\ell=0}^{\infty} (2\ell + 1) (1 - \bar{\eta}_\ell) P_\ell(\cos \theta) \right|^2 \quad (4.11)$$

From (3.52),

$$\begin{aligned} 1 - \bar{\eta}_\ell &= 1 - e^{-2i\delta_\ell} \left(1 - \frac{2s_\ell}{M_\ell + iN_\ell} \right) \\ &= 1 - e^{-2i\delta_\ell} + e^{-2i\delta_\ell} \frac{(2s_\ell)(M_\ell - iN_\ell)}{M_\ell^2 + N_\ell^2} \\ &= 1 - e^{-2i\delta_\ell} + \frac{(\cos 2\delta_\ell - i \sin 2\delta_\ell)(2s_\ell)(M_\ell - iN_\ell)}{M_\ell^2 + N_\ell^2} \end{aligned}$$

$$\begin{aligned}
1 - \overline{\pi}_\ell &= 1 - \cos 2\delta_\ell + 2s_\ell \frac{M_\ell \sin 2\delta_\ell - N_\ell \cos 2\delta_\ell}{M_\ell^2 + N_\ell^2} \\
&+ i \left[\sin 2\delta_\ell - 2s_\ell \frac{M_\ell \sin 2\delta_\ell + N_\ell \cos 2\delta_\ell}{M_\ell^2 + N_\ell^2} \right] \\
&= 2 \left[\sin^2 \delta_\ell + s_\ell \frac{M_\ell \sin 2\delta_\ell - N_\ell \cos 2\delta_\ell}{M_\ell^2 + N_\ell^2} \right] \\
&+ i \left[\sin 2\delta_\ell - 2s_\ell \frac{M_\ell \sin 2\delta_\ell + N_\ell \cos 2\delta_\ell}{M_\ell^2 + N_\ell^2} \right] \quad (4.12)
\end{aligned}$$

The real part of the expression on the right is equal to $1 - R_\ell \overline{\pi}_\ell$ and since it can easily be shown that

$$\overline{\sigma}_t^{(\ell)} = 2\pi\chi^2 (2\ell+1)(1 - R_\ell \overline{\pi}_\ell) \quad (4.13)$$

$$\frac{\overline{\sigma}_t^{(\ell)}}{\pi R^2} = \frac{4}{\chi^2} (2\ell+1) \left[\sin^2 \delta_\ell + s_\ell \frac{M_\ell \cos 2\delta_\ell - N_\ell \sin 2\delta_\ell}{M_\ell^2 + N_\ell^2} \right] \quad (4.14)$$

and using the expression above, we get the FPW formulae

$$\frac{1}{R^2} \frac{d\sigma_{\text{sc}}}{d\Omega} = (R_\ell \Sigma)^2 + (\text{Im } \Sigma)^2 \quad (4.15)$$

where*

$$R_\ell \Sigma = \frac{\chi}{4} \sum_\ell \frac{\overline{\sigma}_t^{(\ell)}}{\pi R^2} P_\ell(\cos \theta) \quad (4.15a)$$

*The real and imaginary have been interchanged in FPW and the Legendre Polynomial omitted in one of them.

$$\text{Im } \Sigma = \frac{1}{2x} \sum_{\ell} (2\ell+1) \left[\sin 2\delta_{\ell} - 2 s_{\ell} \frac{M_{\ell} \sin 2\delta_{\ell} + N_{\ell} \cos 2\delta_{\ell}}{M_{\ell}^2 + N_{\ell}^2} \right] P_{\ell}(\cos \theta)$$

From (4.9)

$$\sigma_c^{(\ell)} = \pi \chi^2 (2\ell+1) (1 - |\bar{\eta}_{\ell}|^2)$$

Now

$$\begin{aligned} |\bar{\eta}_{\ell}|^2 &= \left[1 - \frac{2 s_{\ell}}{M_{\ell} + i N_{\ell}} \right] \left[1 - \frac{2 s_{\ell}}{M_{\ell} - i N_{\ell}} \right] \\ &= 1 + \frac{4 s_{\ell}^2}{M_{\ell}^2 + N_{\ell}^2} - \frac{2 s_{\ell}}{M_{\ell} + i N_{\ell}} - \frac{2 s_{\ell}}{M_{\ell} - i N_{\ell}} \\ &= 1 + \frac{4 s_{\ell}^2 - 4 s_{\ell} M_{\ell}}{M_{\ell}^2 + N_{\ell}^2} = 1 + \frac{4 s_{\ell} (s_{\ell} - M_{\ell})}{M_{\ell}^2 + N_{\ell}^2} \\ &= 1 + \frac{4 s_{\ell} \text{Im } f_{\ell}}{M_{\ell}^2 + N_{\ell}^2} \end{aligned} \quad (4.16)$$

$$\frac{\sigma_c^{(\ell)}}{\pi R^2} = \frac{4}{x^2} (2\ell+1) s_{\ell} \left[- \frac{\text{Im } f_{\ell}}{M_{\ell}^2 + N_{\ell}^2} \right] = \frac{(2\ell+1) T_{\ell}}{x^2} \quad (4.17)$$

which is the defining equation for the penetrabilities, T_{ℓ} .

V. The Square Well

FPW have chosen for their internal calculations the simplest type of potential that will fit the formalism, the complex-square-well.

Let

$$\begin{aligned} V &= -V_0 (1 + i \int), & r < R \\ &= 0 & r > R \end{aligned} \quad (5.1)$$

where V_0 is the depth of the well, and $\int$ is what FPW call the "absorption constant." The only solution of (3.25) for $r < R$ is in terms of the spherical Bessel function since the spherical Neumann function would yield a non-integrable singularity in the probability density in any neighborhood of the origin for $\ell \neq 0$. However, n_0 requires special consideration. It yields a probability density which is integrable, but which can be excluded on other grounds.

The logarithmic derivatives may now be written

$$f_\ell = 1 + x \frac{j_\ell'(x)}{j_\ell(x)} \quad (5.2)$$

where

$$X^2 = x^2 + X_0^2 (1 + i \int), \quad X_0^2 = \frac{2m}{\hbar^2} V_0 R^2 \quad (5.2a)$$

A recursion relation for the f_ℓ may be written directly from the relation established for $\Delta_\ell + i s_\ell$. If one notes that the formulae for $\Delta_\ell + i s_\ell$ and f_ℓ

$$\Delta_\ell + i s_\ell = 1 + x \frac{h_\ell'(x)}{h_\ell(x)} \quad f_\ell = 1 + x \frac{j_\ell'(x)}{j_\ell(x)}$$

are identical if $h_\ell \rightarrow j_\ell$, $x \rightarrow X$ and since the fundamental recursion relations for the h_ℓ are identical to those for j_ℓ , we have

$$f_\ell = \frac{x^2}{\ell - f_{\ell-1}} - \ell \quad (5.3)$$

and for $\ell = 0$,

$$f_0 = \frac{1 + X j_0'(X)}{j_0(X)} \quad (5.4)$$

$$f_0 = \frac{1 + X \left(\frac{\cos X}{X} - \frac{\sin X}{X^2} \right)}{\frac{\sin X}{X}} \quad (5.5)$$

$$\text{Let } X = X_1 + i X_2 \quad (5.6)$$

$$\begin{aligned} f_0 &= X \frac{\cos X}{\sin X} \\ &= (X_1 + i X_2) \frac{\cos (X_1 + i X_2)}{\sin (X_1 + i X_2)} \\ &= (X_1 + i X_2) \frac{\cos X_1 \cos (i X_2) - \sin X_1 \sin (i X_2)}{\sin X_1 \cos (i X_2) + \cos X_1 \sin (i X_2)} \\ &= (X_1 + i X_2) \frac{\cos X_1 \cosh X_2 - i \sin X_1 \sinh X_2}{\sin X_1 \cosh X_2 + i \cos X_1 \sinh X_2} \\ f_0 &= (X_1 + i X_2) \frac{\frac{1}{2} (\sin 2X_1 - i \sinh 2X_2)}{\frac{1}{2} (\cosh 2X_2 - \cos 2X_1)} \quad (5.7) \end{aligned}$$

$$\text{Re } f_0 = \frac{X_1 \sin 2X_1 + X_2 \sinh 2X_2}{\cosh 2X_2 - \cos 2X_1} \quad (5.8)$$

$$\operatorname{Im} f_0 = \frac{X_2 \sin 2X_1 - X_1 \sinh 2X_2}{\cosh 2X_2 - \cos 2X_1} \quad (5.9)$$

and from (5.3)

$$\begin{aligned} f_\ell &= \frac{(X_1 + i X_2)^2}{\ell - (\operatorname{Re} f_{\ell-1} + i \operatorname{Im} f_{\ell-1})} - \ell \\ &= \frac{(X_1^2 - X_2^2 + 2i X_1 X_2)(\ell - \operatorname{Re} f_{\ell-1} + i \operatorname{Im} f_{\ell-1})}{(\ell - \operatorname{Re} f_{\ell-1})^2 + (\operatorname{Im} f_{\ell-1})^2} - \ell \end{aligned} \quad (5.10)$$

$$\operatorname{Re} f_\ell = \frac{(X_1^2 - X_2^2)(\ell - \operatorname{Re} f_{\ell-1}) - 2X_1 X_2 \operatorname{Im} f_{\ell-1}}{(\ell - \operatorname{Re} f_{\ell-1})^2 + (\operatorname{Im} f_{\ell-1})^2} - \ell \quad (5.10a)$$

$$\operatorname{Im} f_\ell = \frac{(X_1^2 - X_2^2) \operatorname{Im} f_{\ell-1} + 2X_1 X_2 (\ell - \operatorname{Re} f_{\ell-1})}{(\ell - \operatorname{Re} f_{\ell-1})^2 + (\operatorname{Im} f_{\ell-1})^2} \quad (5.10b)$$

VI. Discussion of the FPW Formalism

A. Compound-elastic Scattering.

Let us look more closely at some of the fundamental assumptions in the averaging procedure employed in FPW. From (3.23)

$$\frac{d \overline{\sigma}_{el}}{d \Omega} = \pi \chi^2 \left| \sum_{\ell=0}^{\infty} (2\ell+1)^{1/2} (1 - \eta_{\ell}) Y_{\ell,0} \right|^2 \quad (6.1)$$

where we again do not average over slowly varying quantities (χ^2). Now

$$\begin{aligned} (1 - \eta_{\ell}^*)(1 - \eta_{\ell'}) &= 1 - \overline{\eta_{\ell}^*} - \overline{\eta_{\ell'}} + \overline{\eta_{\ell}^* \eta_{\ell'}} \\ &= (1 - \overline{\eta_{\ell}})^*(1 - \overline{\eta_{\ell'}}) + (\overline{\eta_{\ell}^* \eta_{\ell'}} - \overline{\eta_{\ell}^*} \overline{\eta_{\ell'}}) \end{aligned} \quad (6.2)$$

The second term is a cross-correlation term between η_{ℓ} and $\eta_{\ell'}$. It is neglected for $\ell \neq \ell'$ by Porter(16). He states that we are dealing with a zero spin formalism and, consequently, ℓ and ℓ' represent total angular momenta of different levels in the compound nucleus and these are not necessarily correlated. This is probably a valid assumption for complex nuclei, but may not be valid for simple nuclei where the different levels of the compound nucleus may indeed be correlated. In any event, it is an assumption which is not made explicit in FPW. Thus we have

$$\overline{\eta_{\ell}^* \eta_{\ell'}} - \overline{\eta_{\ell}^*} \overline{\eta_{\ell'}} = (\overline{|\eta_{\ell}|^2} - |\overline{\eta_{\ell}}|^2) \delta_{\ell\ell'} \quad (6.3)$$

This allows FPW to separate the average differential elastic scattering cross section into two parts

$$\frac{d \overline{\sigma}_{el}}{d \Omega} = \frac{d \sigma_{se}}{d \Omega} + \frac{d \sigma_{ce}}{d \Omega}, \quad (6.4)$$

the so-called shape-elastic and compound-elastic cross sections, where the portion of the scattering arising from the fluctuation term, $\overline{|\eta_{\ell}|^2} - |\overline{\eta_{\ell}}|^2$, is identified with compound-elastic scattering. The identification

of the fluctuation term with compound-elastic scattering is not at all obvious and FPW have only attempted to make the assumption plausible by indicating that the fluctuation term would behave as one would expect compound-elastic scattering to behave, namely: it would be important at low energies where few levels of the compound nucleus are excited and unimportant at higher energies where so many levels are excited that decay through the entrance channel is highly improbable.

At the time this section was written Weisskopf and Friedman(20) published a paper in which the importance of the assumption, $\sigma_{ce} = \pi \chi^2 (|\overline{\eta}_\ell|^2 - |\overline{\eta}_\ell|^2)$, is stressed. They again give a plausibility argument. In addition, they develop the argument from a resonance approach utilizing a Breit-Wigner single level phase shift.

However, a basic difficulty remains. If one looks at the elastic scattering cross section in the standard scattering theory, one has

$$\sigma_{el}^{(\ell)} = \pi \chi^2 (2\ell + 1) |1 - \eta_\ell|^2$$

which may be written formally, (3) Chapter VIII,

$$\sigma_{el}^{(\ell)} = \pi \chi^2 (2\ell + 1) |A_{res}^\ell + A_{pot}^\ell|^2 \quad (6.5)$$

where

$$A_{res}^\ell = \frac{-2i s_\ell}{R_e f_\ell - \Delta_\ell + i (\text{Im } f_\ell - s_\ell)} \quad (6.5a)$$

$$A_{pot}^\ell = e^{-2i \xi_\ell} - 1 \quad (6.5b)$$

Although the division into the two scattering amplitudes is purely formal, it gives considerable physical insight into the problem. A most

important effect, from the standpoint of this discussion, is that compound-elastic scattering would result from resonances in the resonance scattering amplitude. True, when one uses the averaged phase shifts of the FPW formalism, these resonances are greatly broadened and contribute to the so-called Barschall maxima in the cross-sections. However, these resonances still contain a contribution from those levels which decay by the entrance channel.

Here, then, is one of the conceptual difficulties with the FPW formalism. While FPW postulate that all compound nucleus processes are to be considered as an absorption of the incoming beam, a contribution from the average compound-elastic scattering appears to be contained in the shape-elastic scattering.

B. The $\int$ Parameter in the FPW Model

Having been quite puzzled by the strong-coupling limit in FPW, we undertook a study to determine the effect of what FPW call the absorption parameter, $\int$. The study does not attempt to justify the complex potential used by FPW, but, rather, was designed to find the effect of the parameter $\int$ on the various cross-sections in the FPW formalism.

On the basis of our calculations, the following seems clear:

- (a) The absorption parameter is that and much more. Under certain conditions it will have a marked effect on the angular distribution of elastically scattered neutrons, particularly near resonances.
- (b) An increase of $\int$ leads to an increase in the absorption (in the sense of FPW) of those neutrons which penetrate the scatterer, but may well lead to decreased compound nucleus formation. In fact, we shall show $\lim_{\int \rightarrow \infty} \sigma_c = 0$.

$$\int \rightarrow \infty$$

- (c) There does not appear to be a bound, per se, on $\int$. That is, there does not appear to be any physical ground for excluding large $\int$. Any $\int$ which gives a good theoretical fit to the experimental data must be considered.

The effect of $\int$ may most clearly be seen by studying a one-dimensional case. As mentioned previously, we do not attempt to justify the potential used on firm quantum-mechanical grounds.

Let us treat the case of a plane wave of neutrons incident on a complex potential step

$$\begin{aligned} V &= 0, & x < 0 \\ V &= -V_1 - iV_2, & x > 0 \end{aligned} \quad (6.6)$$

where V_1 and V_2 are constants.

We shall solve the Schrödinger equation in the two regions divided by the potential break. The one-dimensional Schrödinger equation becomes, for $x < 0$,

$$\left[\frac{d^2}{dx^2} + k^2 \right] \psi = 0, \quad (6.7)$$

$$\text{where } k = \frac{(2mE)^{1/2}}{\hbar}$$

and E = Kinetic energy of incident neutron,

the solutions being

$$\psi = A e^{ikx} + B e^{-ikx}, \quad (6.8)$$

where A and B are constants to be determined from normalization and boundary conditions. Similarly for $x > 0$, we have

$$\left[\frac{d^2}{dx^2} + \frac{2m}{\hbar^2} (E + V_1 + iV_2) \right] \psi = 0 \quad (6.9)$$

With solutions

$$\psi = C e^{-i(F_1 + iF_2)x} + D e^{-i(F_1 + iF_2)x} \quad (6.10)$$

$$\text{where } F^2 = \frac{2m}{\hbar^2} (E + V_1 + i V_2) \quad (6.10a)$$

$$F \equiv F_1 + i F_2. \quad (6.10b)$$

Let us normalize to unit incident amplitude, setting $A = 1$. C must be set equal to zero since the term will increase without limit for $x \rightarrow \infty$, for F_2 is always positive. B and D may then be found from the boundary conditions at $x = 0$, namely: the wave function be continuous and have continuous first derivative at the boundary. Thus

$$\begin{aligned} \psi &= e^{-ikx} - \frac{F_2 + i(k - F_1)}{F_2 - i(k + F_1)} e^{-ikx}, \quad x < 0 \\ &= -\frac{2ik}{F_2 - i(k + F_1)} e^{i(F_1 + iF_2)x}, \quad x > 0. \end{aligned} \quad (6.11)$$

If we examine the probability flux density

$$\vec{S} = \frac{\hbar}{2im} \left[\psi^* \nabla \psi - (\nabla \psi^*) \psi \right], \quad (6.12)$$

we find that the probability flux of neutrons to the right for $x = 0$ is

$$\frac{4\hbar}{m} \frac{k^2 F_1}{F_2^2 + (k + F_1)^2} \quad (6.13)$$

and vanishes as $V_2 \rightarrow \infty$. (It also vanishes as $V_1 \rightarrow \infty$.)

Let us examine the wave function ψ for $x < 0$. Taking the limit as $V_2 \rightarrow \infty$, we have

$$\lim_{V_2 \rightarrow \infty} \psi = e^{ikx} - e^{-ikx}. \quad (6.14)$$

Thus ψ becomes proportional to a sine function in the limit $V_2 \rightarrow \infty$,

which is to be expected since the potential break now acts as an impenetrable wall and the required continuity of the wave function forces the wave function to vanish at the boundary. (Again, the identical limit is obtained for $V_1 \rightarrow \infty$.)

Thus, the imaginary part of the potential, V_2 , has two effects. It increases the absorption of those neutrons which cross the potential break and increases the reflection of neutrons from the break. As V_2 becomes very large the former effect is little felt since so few of the neutrons penetrate the potential break and we find $\lim_{\int \rightarrow \infty} \sigma_c = 0$.

The effect of $\int$ on σ_t , σ_{se} , and σ_c may be seen in Fig. 8. The various cross-sections show indications of structure at higher energies, as is to be expected. The peak in σ_c for all cases we have considered is below the geometric, $\sigma_c = \pi (R + \lambda)^2$.

Fig. 16 illustrates the effect of $\int$ for the mass spectrum. For extremely small $\int$ the peaks in the total cross-sections are much too high. However, small increments in $\int$ bring the peaks down rapidly, and very quickly, the peaks disappear completely and we approach a smooth curve almost proportional to $A^{2/3}$ which is characteristic of hard sphere scattering.

VII. Calculation of Scattering from a Complex Square Well by a Variational Method

Variational methods for treating scattering problems have been developed by Hulthén(8) and Schwinger(19). Since the exact calculations require considerable numerical work, it is of interest to develop variational expressions for the phase shifts, provided these can be easily computed with simple trial functions.

We assume the potential to be central and to have the form

$$V = -V_0 f(r/r_0) \quad , \quad (7.1)$$

where V_0 and r_0 serve as measures of the depth and range of the interaction and f describes the shape. The Schrödinger equation for a particle of mass M and energy E , moving in this potential, can be separated in polar coordinates. If u_ℓ is $r \times \mathcal{U}_\ell$, where $\mathcal{U}_\ell$ is the radial function for angular momentum ℓ , we obtain for u_ℓ the equation

$$u_\ell'' + \left[K^2 - \frac{\ell(\ell+1)}{x^2} \right] u_\ell = -\lambda f(x) u_\ell \quad , \quad (7.2)$$

$$\text{where } x = r/r_0, \quad K^2 = \frac{2ME}{\hbar^2} r_0^2, \quad \lambda = \frac{2MV_0}{\hbar^2} r_0^2,$$

and the primes denote differentiation with respect to x .

We can construct a Green's function $G_\ell(x, x')$ for (7.2) which satisfies

$$G_\ell'' + \left[K^2 - \frac{\ell(\ell+1)}{x^2} \right] G_\ell = -\delta(x - x'); \quad G_\ell(0, x') = 0 \quad . \quad (7.3)$$

G_ℓ will be a product $A_\ell(x_<) B_\ell(x_>)$, where $x_<$ and $x_>$ are respectively the smaller and the larger of the two variables x, x' , and A_ℓ ,

B_ℓ are solutions of the homogeneous equation with $A_\ell(0) = 0$. The jump in the derivative of G_ℓ at $x = x'$ is $\left[A_\ell B'_\ell + A'_\ell B_\ell \right]_{x=x'}$ which is just the Wronskian of the original differential equation. For eq. (7.3), the solutions A_ℓ and B_ℓ are $\left(\frac{\pi Kx}{2} \right)^{1/2} J_{\pm(\ell+1/2)}(Kx)$.

We set the Wronskian equal to -1 and determine the proportionality factor in G_ℓ by using the asymptotic forms of the Bessel functions.

This gives:

$$G_\ell(x, x') = (-1)^\ell \frac{\pi}{2} \sqrt{x_< x_>} J_{\ell+1/2}(Kx_<) J_{-(\ell+1/2)}(Kx_>) \quad (7.4)$$

Using this Green's function we obtain from (7.2) an integral equation for u_ℓ :

$$u_\ell = \left(\frac{\pi Kx}{2} \right)^{1/2} J_{\ell+1/2}(Kx) + \int_0^\infty dx' G_\ell(x, x') \lambda f(x') u_\ell(x'). \quad (7.5)$$

Asymptotically,

$$u_\ell \sim \sin\left(Kx - \frac{\ell\pi}{2}\right) + 1/K \cos\left(Kx - \frac{\ell\pi}{2}\right) \times \int_0^\infty dx' \left(\frac{\pi Kx'}{2} \right)^{1/2} J_{\ell+1/2}(Kx') \lambda f(x') u_\ell(x'). \quad (7.6)$$

so that, comparing with the asymptotic form of the perturbed wave

$$u_\ell \sim \sin\left(Kx - \frac{\ell\pi}{2} + \delta_\ell\right) \quad (7.6a)$$

we find

$$\frac{\lambda}{K} \cot \delta_\ell = \left[\int_0^\infty dx' \left(\frac{\pi Kx'}{2} \right)^{1/2} J_{\ell+1/2}(Kx') f(x') u_\ell(x') \right]^{-1} \quad (7.7)$$

(7.7) expresses the phase shift δ_ℓ in terms of an integral involving the unknown wave function u_ℓ . We can construct a functional $I_\ell[u]$

which has the following properties:

1. It reduces to $\frac{\lambda}{K} \cot \delta_\ell$ if u is the correct solution u_ℓ .
2. It is an extremum relative to small variations of the function

u in the neighborhood of the true solution.

The required functional is

$$I_\ell[u] = \frac{\int f u \left[u - \frac{\int G_\ell \lambda f u}{\left[\int f u \left(\frac{\pi K x}{2} \right)^{1/2} J_{\ell+1/2}(Kx) \right]^2} \right]}{\left[\int f u \left(\frac{\pi K x}{2} \right)^{1/2} J_{\ell+1/2}(Kx) \right]^2} \quad (7.8)$$

where we have omitted obvious limits and variables. If we choose for

u the u_ℓ satisfying (7.5), then I_ℓ becomes $\frac{\lambda}{K} \cot \delta_\ell$ as given by

(7.7). If we take the first variation δI we find that condition 2 is satisfied.

The simplest trial function which can be used in (7.8) is the unperturbed wave $\left(\frac{\pi K x}{2} \right)^{1/2} J_{\ell+1/2}(Kx)$. Then if we define

$$g_\ell(Kx) = \left(\frac{\pi K x}{2} \right)^{1/2} J_{\ell+1/2}(Kx) \quad (7.9)$$

we find

$$\tan \delta_\ell = \frac{\lambda/K \int f g_\ell^2}{1 - \frac{\lambda \iint f g_\ell G_\ell f g_\ell}{\int f g_\ell^2}} \quad (7.10)$$

The Green's function of (7.4) can be expressed as

$$G_\ell(x, x') = (-1)^{\ell/K} g_\ell(Kx_<) g_{-\ell-1}(Kx_>) \quad (7.11)$$

so that

$$\begin{aligned} \iint f g_\ell G_\ell f g_\ell &= \frac{(-1)^\ell}{K} \iint f g_\ell g_\ell(Kx_<) g_{-\ell-1}(Kx_>) f g_\ell \\ &= \frac{2}{K} (-1)^\ell \int_0^\infty dx f(x) g_\ell(Kx) g_{-\ell-1}(Kx) \int_0^x dx' f(x') g_{-\ell-1}^2(Kx') \end{aligned} \quad (7.12)$$

and we obtain from (7.10) our final result

$$\tan \delta_\ell = \frac{\lambda/K \int f g_\ell^2}{1 - \frac{2(-1)^\ell \lambda/K \int_0^\infty dx f g_\ell g_{\ell-1} \int_0^x dx' f g_\ell^2}{\int f g_\ell^2}} \quad (7.13)$$

It is important to emphasize that this solution is far superior to the Born approximation, which would give merely

$$\tan \delta_\ell = \lambda/K \int f g_\ell^2 \quad (7.14)$$

The denominator in (7.13) gives us the possibility of resonance effects in the scattering. Previous work has shown that the present method gives a good fit to the neutron proton scattering even down to thermal energies.

We shall apply the approximate equation (7.13) to the case of the complex square well. Then $f(x) = 1$ for $0 \leq x \leq 1$, $f(x) = 0$ for $x > 1$.

For this case, (7.13) becomes

$$\tan \delta_\ell = \frac{\lambda}{K} \frac{\int_0^1 dx g_\ell^2(Kx)}{1 - \frac{2(-1)^\ell \lambda/K \int_0^1 dx g_\ell(Kx) g_{\ell-1}(Kx) \int_0^x dx' g_\ell^2(Kx')}{\int_0^1 dx g_\ell^2(Kx)}} \quad (7.15)$$

so that the change of variables $Kx = y$ gives

$$\tan \delta_\ell = \frac{\lambda/K^2 \int_0^K dy g_\ell^2(y)}{1 - \frac{2(-1)^\ell \lambda/K^2 \int_0^K dy g_\ell(y) g_{\ell-1}(y) \int_0^y dy' g_\ell^2(y')}{\int_0^K dy g_\ell^2(y)}} \quad (7.15a)$$

or

$$\tan \delta_\ell = \frac{\lambda/K^2 P_\ell(K)}{1 - 2(-1)^\ell \lambda/K^2 \frac{Q_\ell}{P_\ell}} \quad (7.15b)$$

where

$$P_{\ell}(y) = \int_0^y dy \, g_{\ell}^2(y) \quad (7.16)$$

$$Q_{\ell}(y) = \int_0^y dy \, g_{\ell}(y) \, g_{-\ell-1}(y) \, P_{\ell}(y) \quad (7.16a)$$

Our functions g_{ℓ} are simply related to the usual spherical Bessel functions:

$$g_{\ell}(y) = y j_{\ell}(y) \quad (7.17)$$

so that, first, from (7.16)

$$P_{\ell}(y) = \int_0^y dy \, y^2 j_{\ell}^2(y) \quad (7.18)$$

and from (7.16a)

$$Q_{\ell}(y) = \int_0^y dy \, y^2 j_{\ell}(y) j_{-\ell-1}(y) P_{\ell}(y) \quad (7.18a)$$

The interesting point is that the integrals (7.18) can be expressed in simple closed form for easy computation, (despite the fact that (7.18a) involves products of four Bessel functions!).

We start from the well known recursion formulas:

$$x j_{\ell-1} = x j'_{\ell} + (\ell+1) j_{\ell} \quad (7.19)$$

$$x j_{\ell} = -x j'_{\ell-1} + (\ell-1) j_{\ell-1} \quad (7.20)$$

Changing ℓ to $\ell+1$,

$$x j_{\ell} = x j'_{\ell+1} + (\ell+2) j_{\ell+1} \quad (7.19a)$$

$$x j_{\ell+1} = -x j'_{\ell} + \ell j_{\ell} ; \quad (7.20a)$$

or changing ℓ to $-\ell$,

$$x j_{-\ell-1} = x j'_{-\ell} + (1-\ell) j_{-\ell} \quad (7.19b)$$

$$x j_{-\ell} = -x j'_{-\ell-1} - (1+\ell) j_{-\ell-1} \quad (7.20b)$$

Multiply (7.19) by $x^2 j_{\ell}$, (7.20a) by $x^2 j_{\ell}$ and add:

$$\begin{aligned}
x^3 j_\ell j'_\ell &= x^3 j_\ell j_{\ell-1} - (\ell+1) x^2 j_\ell^2 \\
x^3 j_\ell j'_\ell &= -x^3 j_\ell j_{\ell+1} + \ell x^2 j_\ell^2 \\
\hline
2x^3 j_\ell j'_\ell &= x^3 j_\ell (j_{\ell-1} - j_{\ell+1}) - x^2 j_\ell^2
\end{aligned}$$

or

$$\frac{d}{dx} (x^3 j_\ell^2) = x^3 j_\ell (j_{\ell-1} - j_{\ell+1}) + 2x^2 j_\ell^2 \quad (7.21)$$

Similarly multiply (7.19a) by $x^2 j_{\ell-1}$

$$x^3 j_{\ell-1} j'_{\ell+1} = x^3 j_\ell j_{\ell-1} - (\ell+2) x^2 j_{\ell-1} j_{\ell+1}$$

and (7.20) by $x^2 j_{\ell+1}$,

$$x^3 j_{\ell+1} j'_{\ell-1} = -x^3 j_\ell j_{\ell+1} + (\ell-1) x^2 j_{\ell-1} j_{\ell+1}$$

and add:

$$x^3 (j_{\ell-1} j_{\ell+1})' + 3x^2 j_{\ell-1} j_{\ell+1} = x^3 j_\ell (j_{\ell-1} - j_{\ell+1})$$

or

$$\frac{d}{dx} (x^3 j_{\ell-1} j_{\ell+1}) = x^3 j_\ell (j_{\ell-1} - j_{\ell+1}) \quad (7.22)$$

Combining (7.21) and (7.22)

$$\frac{d}{dx} \left[x^3 (j_\ell^2 - j_{\ell-1} j_{\ell+1}) \right] = 2x^2 j_\ell^2 \quad (7.23)$$

so that the integral (7.18) for P_ℓ is evaluated to be

$$P_\ell(y) = \frac{1}{2} y^3 (j_\ell^2 - j_{\ell-1} j_{\ell+1}) \quad (7.24)$$

Substituting this expression for P_ℓ in (7.18a), we get

$$Q_\ell(K) = \frac{1}{2} \int_0^K dy y^5 j_\ell j_{-\ell-1} (j_\ell^2 - j_{\ell-1} j_{\ell+1}) \quad (7.25)$$

o

We now evaluate (7.25) by again starting from the recursion formulas.

Multiply (7.19) by $x j_{-\ell}$, and (7.19b) by $x j_\ell$, and add:

$$\begin{aligned}
x^2 j_{-\ell} j_{\ell-1} &= x^2 j_{-\ell} j'_\ell + (\ell+1) x j_\ell j_{-\ell} \\
x^2 j_\ell j_{-\ell-1} &= x^2 j_\ell j'_{-\ell} + (1-\ell) x j_\ell j_{-\ell} \\
\hline
x^2 (j_\ell j_{-\ell-1} + j_{-\ell} j_{\ell-1}) &= \frac{d}{dx} (x^2 j_\ell j_{-\ell})
\end{aligned} \quad (7.26)$$

Multiply (7.20) by $x j_{-\ell-1}$, and (7.20b) by $x j_{\ell-1}$, and add:

$$\begin{aligned} x^2 j_{\ell} j_{-\ell-1} &= -x^2 j_{-\ell-1} j'_{\ell-1} + (\ell-1) x j_{\ell-1} j_{-\ell-1} \\ x^2 j_{-\ell} j_{\ell-1} &= -x^2 j_{\ell-1} j'_{-\ell-1} - (1+\ell) x j_{\ell-1} j_{-\ell-1} \\ \hline x^2 (j_{\ell} j_{-\ell-1} + j_{-\ell} j_{\ell-1}) &= -\frac{d}{dx} (x^2 j_{\ell-1} j_{-\ell-1}) \end{aligned} \quad (7.27)$$

Combining (7.26) and (7.27):

$$\frac{d}{dx} \left[x^2 (j_{\ell} j_{-\ell} + j_{\ell-1} j_{-\ell-1}) \right] = 0$$

so that the quantity in the bracket is independent of x , and can be evaluated by substituting the asymptotic forms of the Bessel functions, giving

$$x^2 (j_{\ell} j_{-\ell} + j_{\ell-1} j_{-\ell-1}) = (-1)^{\ell} \quad (7.28)$$

If we multiply (7.19a) by $x^2 j_{-\ell}$, (7.20a) by $-x^2 j_{-\ell-1}$, (7.19b) by $x^2 j_{\ell+1}$, (7.20b) by $-x^2 j_{\ell}$ and add, we find the relation:

$$x^3 \frac{d}{dx} (j_{\ell} j_{-\ell-1} + j_{\ell+1} j_{-\ell}) + x^2 (3 j_{\ell+1} j_{-\ell} + j_{\ell} j_{-\ell-1}) = 0$$

and adding $2x^2 j_{\ell} j_{-\ell-1}$ to both sides of the equation we have:

$$\frac{d}{dx} \left[x^3 (j_{\ell} j_{-\ell-1} + j_{\ell+1} j_{-\ell}) \right] = 2x^2 j_{\ell} j_{-\ell-1} \quad (7.29)$$

The integral which appears in (7.25) is

$$I = \int x^5 \left[j_{\ell}^3 j_{-\ell-1} - j_{\ell+1} j_{\ell} j_{\ell-1} j_{-\ell-1} \right] dx.$$

Using (7.28),

$$\begin{aligned} I &= \int x^5 \left[j_{\ell}^3 j_{-\ell-1} + \frac{(-1)^{\ell+1}}{x^2} j_{\ell+1} j_{\ell} + j_{\ell}^2 j_{\ell+1} j_{-\ell} \right] dx \quad (7.30) \\ &= (-1)^{\ell+1} \int x^3 j_{\ell+1} j_{\ell} + \int x^2 j_{\ell}^2 \left[x^3 (j_{\ell} j_{-\ell-1} + j_{\ell+1} j_{-\ell}) \right] dx \end{aligned}$$

Using (7.23), this equation becomes

$$\begin{aligned}
 I &= (-1)^{\ell+1} \int x^3 j_{\ell+1} j_{\ell} + \frac{1}{2} \left\{ \frac{d}{dx} \left[x^3 (j_{\ell}^2 - j_{\ell-1} j_{\ell+1}) \right] \right\} \\
 &\times \left\{ x^3 (j_{\ell} j_{\ell-1} + j_{\ell+1} j_{-\ell}) \right\} \\
 &= (-1)^{\ell+1} \int x^3 j_{\ell+1} j_{\ell} + \frac{1}{2} x^6 [j_{\ell}^2 - j_{\ell-1} j_{\ell+1}] \\
 &\quad \times [j_{\ell} j_{\ell-1} + j_{\ell+1} j_{-\ell}] \\
 &\quad - \frac{1}{2} \int x^3 [j_{\ell}^2 - j_{\ell-1} j_{\ell+1}] \frac{d}{dx} \left[x^3 (j_{\ell} j_{\ell-1} + j_{\ell+1} j_{-\ell}) \right]
 \end{aligned}$$

Now using (7.29), we see that the last term is just $-I$, so that

$$\begin{aligned}
 I &= \frac{(-1)^{\ell+1}}{2} \int x^3 j_{\ell+1} j_{\ell} + \frac{1}{4} x^6 \\
 &\times [j_{\ell}^2 - j_{\ell-1} j_{\ell+1}] \cdot [j_{\ell} j_{\ell-1} + j_{\ell+1} j_{-\ell}] \quad (7.31)
 \end{aligned}$$

The first term is easily integrated. From (7.20a)

$$\begin{aligned}
 x^3 j_{\ell+1} j_{\ell} &= -x^3 j_{\ell} j'_{\ell} + \ell x^2 j_{\ell}^2 \\
 \int x^3 j_{\ell+1} j_{\ell} &= \ell \int x^2 j_{\ell}^2 - \int x^3 j_{\ell} j'_{\ell} \\
 &= \ell \int x^2 j_{\ell}^2 - \frac{x^3 j_{\ell}^2}{2} + \frac{3}{2} \int x^2 j_{\ell}^2
 \end{aligned}$$

Then using (7.23), we have

$$\int x^3 j_{\ell+1} j_{\ell} = \frac{2\ell+3}{4} x^3 (j_{\ell}^2 - j_{\ell-1} j_{\ell+1}) - \frac{x^3}{2} j_{\ell}^2 \quad (7.32)$$

Using (7.31) and (7.32) we find for $Q_{\ell}(K)$ in (7.25):

$$\begin{aligned}
 Q_{\ell}(K) &= \frac{1}{2} \left\{ (-1)^{\ell+1} \left[\frac{2\ell+3}{4} K^3 (j_{\ell}^2 - j_{\ell-1} j_{\ell+1}) - \frac{K^3}{2} j_{\ell}^2 \right] \right. \\
 &\quad \left. + \frac{1}{4} K^6 [j_{\ell}^2 - j_{\ell-1} j_{\ell+1}] [j_{\ell} j_{\ell-1} + j_{\ell+1} j_{-\ell}] \right\} \quad (7.33)
 \end{aligned}$$

With eqs. (7.24) and (7.25), we find from (7.15b),

$$\tan \delta_\ell = \frac{1/2 \lambda K (j_\ell^2 - j_{\ell-1} j_{\ell+1})}{1 + \frac{\lambda}{K^2} \left[\frac{2\ell+3}{4} - 1/2 \frac{j_\ell^2}{j_\ell^2 - j_{\ell-1} j_{\ell+1}} + \frac{(-1)^{\ell+1}}{2} K^3 (j_\ell j_{\ell-1} + j_{\ell+1} j_\ell) \right]} \quad (7.34)$$

where for compactness we have omitted the argument K from all the Bessel functions.

The complex potential V_0 which can be written as

$$V_0 = V (1 + i \int)$$

appears only in the parameter λ , so that (7.34) can be used for rapid computation.

A few angular distributions were computed for comparison with the exact solutions. The results are shown in Figs. 1 - 3. The agreement with the exact results appears to be excellent.

The formal statement of these approximations is obtained in a modification of (9.9)

$$\omega_\ell \equiv \frac{a_\ell}{a_0 + \frac{Dc}{4\pi}}, \quad \ell \neq 0$$

$$\omega_0 \equiv 1. \quad (9.10)$$

Fig. 1

Fig. 2

Fig. 3

VIII. The Harmonic Oscillator

Since the shell model is, in a sense, bracketed by the square well and harmonic oscillator potential (13), the possibility of using the harmonic oscillator potential in a scattering problem rather than a bound state problem was examined.

It develops that the harmonic oscillator potential problem can be solved in terms of the confluent hypergeometric function. The potential itself has the same number of parameters as the square well and the logarithmic derivative can be calculated about as easily as the square well derivatives.

The harmonic oscillator potential used has the form

$$V = \begin{cases} \left[\frac{V_0 r^2}{R^2} - V_0 \right] (1 + i \zeta), & r < R \\ = 0 & r > R \end{cases} \quad (8.1)$$

The reduced radial equation is then

$$\frac{d^2 u_\ell}{dr^2} + \left\{ k^2 - \frac{\ell(\ell+1)}{r^2} - \frac{2m}{\hbar^2} \left[\frac{V_0}{R^2} (1 + i \zeta) r^2 - V_0 (1 + i \zeta) \right] \right\} u_\ell = 0 \quad (8.2)$$

$$\frac{d^2 u_\ell}{dr^2} + \left[k^2 + \frac{2m}{\hbar^2} V_0 (1 + i \zeta) - \frac{\ell(\ell+1)}{r^2} - \frac{2m}{\hbar^2} \frac{V_0}{R^2} (1 + i \zeta) r^2 \right] u_\ell = 0 \quad (8.2a)$$

Now let

$$K^2 = k^2 + \frac{2m}{\hbar^2} V_0 (1 + i \zeta) \quad (8.3)$$

$$\beta^2 = \frac{2m}{\hbar^2} \frac{V_0}{R^2} (1 + i \zeta) \quad (8.4)$$

The radial equation becomes

$$\frac{d^2 u_\ell}{dr^2} + \left[K^2 - \frac{\ell(\ell+1)}{r^2} - \beta^2 r^2 \right] u_\ell = 0 \quad (8.5)$$

Now let

$$u_\ell(r) = e^{-z/2} r^{\ell+1} F(z), \quad \text{where } z = \beta r^2 \quad (8.6)$$

$$\frac{du_\ell}{dr} = e^{-z/2} \left\{ \left[(\ell+1) r^\ell - \beta r^{\ell+2} \right] F(z) + 2 \beta r^{\ell+2} \frac{dF}{dz} \right\} \quad (8.7)$$

$$\begin{aligned} \frac{d^2 u_\ell}{dr^2} = e^{-z/2} & \left\{ \left[\ell(\ell+1) r^{\ell-1} - \beta(\ell+2) r^{\ell+1} \right] F(z) \right. \\ & + \left[(\ell+1) r^\ell - \beta r^{\ell+2} \right] (2\beta r) \frac{dF}{dz} + 2(\ell+2) \beta r^{\ell+1} \frac{dF}{dz} \\ & + 4 \beta^2 r^{\ell+3} \frac{d^2 F}{dz^2} + \left[-\beta(\ell+1) r^{\ell+1} + \beta^2 r^{\ell+3} \right] F \\ & \left. - 2 \beta^2 r^{\ell+3} \frac{dF}{dz} \right\} \\ \frac{d^2 u_\ell}{dr^2} = e^{-z/2} & \left\{ 4\beta^2 r^{\ell+3} \frac{d^2 F}{dz^2} + \left[2\beta(2\ell+3) r^{\ell+1} - 4\beta^2 r^{\ell+3} \right] \frac{dF}{dz} \right. \\ & \left. + \left[\ell(\ell+1) r^{\ell-1} - \beta(2\ell+3) r^{\ell+1} + \beta^2 r^{\ell+3} \right] F \right\} \quad (8.8) \end{aligned}$$

Substituting in eq. (8.5), we have

$$\begin{aligned} e^{-z/2} & \left\{ 4\beta^2 r^{\ell+3} \frac{d^2 F(z)}{dz^2} + \left[2\beta(2\ell+3) r^{\ell+1} - 4\beta^2 r^{\ell+3} \right] \frac{dF(z)}{dz} \right. \\ & + \left[\ell(\ell+1) r^{\ell-1} - \beta(2\ell+3) r^{\ell+1} + \beta^2 r^{\ell+3} \right] F(z) \\ & \left. + \left[K^2 - \frac{\ell(\ell+1)}{r^2} - \beta^2 r^2 \right] r^{\ell+1} F(z) \right\} = 0 \quad (8.9) \end{aligned}$$

Dividing through by $4 e^{-z/2} \beta r^{\ell+1}$

$$\begin{aligned} \beta r^2 \frac{d^2 F}{dz^2} + \left[\frac{2\ell+3}{2} - \beta r^2 \right] \frac{dF}{dz} + \left[\frac{\ell(\ell+1)}{4\beta r^2} - \frac{2\ell+3}{4} \right. \\ \left. + \beta r^2 + \frac{\kappa^2}{4\beta} - \frac{\ell(\ell+1)}{4\beta r^2} - \beta r^2 \right] F = 0 \end{aligned}$$

$$z \frac{d^2 F}{dz^2} + \left[\ell + 3/2 - z \right] \frac{dF}{dz} - \left[\frac{\ell+3/2}{2} - \frac{\kappa^2}{4\beta} \right] F = 0 \quad (8.10)$$

Equation (8.10) is the confluent hypergeometric equation (15, 17*, 18**)

$$z \frac{d^2 F}{dz^2} + (c - z) \frac{dF}{dz} - pF = 0 \quad (8.11)$$

whose solutions $F(p | c | z)$ are defined by the confluent hypergeometric series

$$F(p | c | z) \equiv 1 + \frac{p}{c} z + \frac{p(p+1)}{2! c(c+1)} z^2 + \frac{p(p+1)(p+2)}{3! c(c+1)(c+2)} z^3 + \dots \quad (8.12)$$

$$\text{with } p = \frac{\ell + \frac{3}{2}}{2} - \frac{\kappa^2}{4\beta},$$

$$c = \ell + 3/2, \text{ and } z = \beta r^2$$

* This reference contains recursion formulae for the confluent hypergeometric function $F(p | c | z)$. The reader should be warned of an important typographical error in formula (36). For the factor $\frac{\chi^\alpha}{\chi^\alpha} e^x \chi^{\alpha-\chi}$ read $\frac{\Gamma(\chi)}{\Gamma(\alpha)} e^x \chi^{\alpha-\chi}$.

** This reference contains tables of $F(p | c | z)$ for some of the desired half-integer "c."

The series for $\frac{dF}{dz}$ may be easily obtained

$$\frac{dF}{dz} (p | c | z) = \frac{p}{c} + \frac{p}{c} \frac{(p+1)}{c+1} z + \frac{p}{c} \frac{(p+1)(p+2)}{2!(c+1)(c+2)} z^2 + \dots$$

$$\frac{dF}{dz} (p | c | z) = \frac{p}{c} F(p+1 | c+1 | z) \quad (8.13)$$

$$\frac{dF}{dr} = 2 \beta r F(p+1 | c+1 | z) \quad (8.14)$$

The logarithmic derivative may then be evaluated directly from the series for $F(p | c | z)$ and $\frac{d}{dr} F(p | c | z)$ or a very convenient continued fraction representation may be obtained. From (8.6) and (8.7)

$$\begin{aligned} \frac{\frac{du_\ell}{dr}}{u_\ell} &= \frac{e^{-\beta r^2/2} \left\{ \left[(\ell+1) r^\ell - \beta r^{\ell+2} \right] F(\beta r^2) + r^{\ell+1} \frac{dF}{dr}(\beta r^2) \right\}}{e^{-\beta r^2/2} r^{\ell+1} F(\beta r^2)} \\ &= \frac{\ell+1 - \beta r^2}{r} + \frac{\frac{dF}{dr}}{F} \end{aligned}$$

and

$$\begin{aligned} f_\ell &= R \left[\frac{\ell+1 - \beta r^2}{r} + \frac{\frac{dF}{dr}}{F} \right]_{r=R} \\ &= \ell+1 - z(R) + 2z(R) \frac{\frac{dF}{dz}}{F} \bigg|_{r=R} \\ &= \ell+1 - z(R) + 2z(R) \frac{d}{dz} \ln F \bigg|_{r=R} \quad (8.15) \end{aligned}$$

From Ince (9)

$$z \frac{d}{dz} \ln F(p | c | z) = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \dots + \frac{a_n}{b_n} \quad (8.16)$$

where $a_n = (p + n - 1) z$

$$b_n = c - z + n - 1$$

The electronic digital computer calculations were accomplished using this continued fraction expansion.

IX. Legendre Polynomial Expansion Coefficients

Soon after we attempted to improve the fit of the FPW model using cosine plots of the angular distributions, it became apparent that this was a rather intractable, although sensitive, method of attempting a fit. It was just not possible to view the gross features of the fit in this manner.

Earlier, Langsdorf and Monahan had suggested we use the Legendre Polynomial expansion coefficients of the angular distributions and fit these to the same coefficients calculated from the experimental data. We had previously set up the apparatus for obtaining the phase shifts from the FPW formalism. It remained to obtain the expansion coefficients from the phase shifts.

From (4.11)

$$\sigma_{se}(\theta) = \frac{\chi^2}{4} \left| \sum_{\ell=0}^{\infty} (2\ell+1)(1-\eta_{\ell}) P_{\ell}(\cos \theta) \right|^2$$

We may also expand $\sigma_{se}(\theta)$ in the complete set of Legendre Polynomials

$$\sigma_{se}(\theta) = \sum_{\ell=0}^{\infty} a_{\ell} P_{\ell}(\cos \theta) \quad (9.1)$$

Equating the two expressions we have

$$\sum_{\ell=0}^{\infty} a_{\ell} P_{\ell}(\cos \theta) = \frac{\chi^2}{4} \left| \sum_{\ell=0}^{\infty} (2\ell+1)(1-\eta_{\ell}) P_{\ell}(\cos \theta) \right|^2 \quad (9.2)$$

Multiplying both sides of (9.2) by $P_k(\cos \theta)$ and integrating gives us

$$\begin{aligned}
& \sum_{\ell=0}^{\infty} a_{\ell} \int_{-1}^{+1} P_{\ell}(\cos \theta) P_k(\cos \theta) d(\cos \theta) \\
&= \frac{\lambda^2}{4} \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} (2\ell+1)(2m+1)(1-\eta_{\ell})(1-\eta_m^*) \\
& \times \int_{-1}^{+1} P_{\ell}(\cos \theta) P_m(\cos \theta) P_k(\cos \theta) d(\cos \theta) \quad (9.3)
\end{aligned}$$

From Whittaker and Watson (21), p. 331,

$$P_{\ell}(\mu) P_m(\mu) = \sum_{r=0}^{\ell} \frac{A_{\ell-r} A_r A_{m-r}}{A_{\ell+m-r}} \left(\frac{2\ell+2m-4r+1}{2\ell+2m-2r+1} \right) P_{\ell+m-2r}(\mu) \quad (9.4)$$

$$\text{where } A_{\ell} = \frac{1 \cdot 3 \cdot 5 \cdots (2\ell-1)}{\ell!},$$

ℓ, m positive integers, $\ell \leq m$.

Substituting (9.4) into the integral on the right of (9.3)

$$\begin{aligned}
& \int_{-1}^1 d(\cos \theta) P_{\ell}(\cos \theta) P_m(\cos \theta) P_k(\cos \theta) \\
&= \frac{2}{2k+1} \sum_{r=0}^{\ell} \frac{A_{\ell-r} A_r A_{m-r}}{A_{\ell+m-r}} \left(\frac{2\ell+2m-4r+1}{2\ell+2m-2r+1} \right) \delta_{k, \ell+m-2r} \quad (9.5)
\end{aligned}$$

ℓ, m positive integers, $\ell \leq m$

$$= \frac{2}{2k+1} \frac{A_{\frac{1}{2}(\ell-m+k)} A_{\frac{1}{2}(\ell+m-k)} A_{\frac{1}{2}(m-\ell+k)}}{A_{\frac{1}{2}(\ell+m+k)}} \cdot \frac{2k+1}{\ell+m+k+1} \quad (9.6)$$

$$|\ell - m| \leq k \leq \ell + m,$$

ℓ, m positive integers, $\ell + m + k$ even.

The restriction $\ell \leq m$ has been dropped, since the same expression obtains for $m \leq \ell$. For $\ell \leq m$

$$r \leq \ell.$$

We find from the δ function

$$\frac{1}{2} (\ell + m - k) \leq \ell$$

$$\therefore k \geq m - \ell.$$

Similarly for $m \leq \ell$

$$k \geq \ell - m$$

$$\therefore k \geq |\ell - m|.$$

Also, since $r \geq 0$

$$\frac{1}{2} (\ell + m - k) \geq 0$$

$$\therefore k \leq \ell + m.$$

Thus

$$|\ell - m| \leq k \leq \ell + m. \quad (9.7)$$

Since we sum over integral values of r only, the δ function yields a condition, along with (9.7), which insures that all the subscripts of the A 's will be positive integers.

From the orthonormality of the Legendre Polynomials, we find on the left of (9.3)

$$\frac{2}{2k+1} a_k$$

yielding

$$a_k = \frac{\chi^2}{4} \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} (2\ell+1)(2m+1)(1-\bar{\eta}_{\ell})(1-\bar{\eta}_m^*)$$

$$\times \frac{A_{\frac{1}{2}(\ell-m+k)} A_{\frac{1}{2}(\ell+m-k)} A_{\frac{1}{2}(m-\ell+k)}}{A_{\frac{1}{2}(\ell+m+k)}} \cdot \frac{2k+1}{\ell+m+k+1}, \quad (9.8)$$

where ℓ, m positive integers, $|\ell - m| \leq k \leq \ell + m$,

$\ell + m + k$ even.

Langsdorf and Monahan did not actually use the expansion coefficients in this form, but, rather a function defined

$$u_{\ell}' = \frac{a_{\ell}'}{a_0'}. \quad (9.9)$$

However, since the counters used in the experiment were flat, counting elastically scattered and inelastically scattered neutrons alike, the expansion coefficients obtained from the experimental data were for the angular distribution of all scattered neutrons. The angular distributions obtained from the FPW formalism are for elastically scattered neutrons, of course, and it was necessary to modify the coefficients obtained from theory to bring them into accord with the experimental coefficients. This was accomplished by making two approximations:

1. All compound nucleus processes result in the emission of a single neutron.
2. These neutrons are isotropic in their distribution.

The formal statement of these approximations is obtained in a modification of (9.9)

$$u_l \equiv \frac{a_l}{a_0 + \frac{Dc}{4\pi}}, \quad l \neq 0$$

$$u_0 \equiv 1. \quad (9.10)$$

X. The Experiment

As mentioned previously, this research was motivated by the experimental work of Langsdorf and Lane at Argonne. They accomplished a comprehensive survey of neutron scattering by thirty-nine elements. Most of these materials were measured over the range 60-1500 Kev, a few at 30 Kev also, and some up to 1700 Kev, all with about 80 Kev resolution in neutron energy.

They had two main objects in the survey: (1) To provide information of value in reactor design, especially for fast reactors which have a high flux of neutrons of a few hundred Kev energy. (The information is also useful for reactor shielding calculations.) (2) To provide data useful in nuclear theory, such as the FPW model.

The design of the apparatus is illustrated by the accompanying figures. Fig. 4 shows a scatterer in place, held by small clamps and wires in the neutron beam emergent from the collimator hole in the source shield directly behind the scatterer. The scatterer is typically a metal sheet 9" wide by 18" high, as illustrated. (Powdered samples are held in thin walled metal cans.) In Fig. 4 two detector tanks for scattered neutrons are shown to the right of the source tank. There are five such tanks around a circle with the scatterer at the center. The two shown detect neutrons scattered at 144° and 112° . Three more not shown in Fig. 4 detect neutrons scattered at 22° , 54° , and 91° . The entire survey was made with the detectors remaining at the five angles.

Fig. 5 shows details of one detector tank. The large steel tank, which is a truncated "pie" shape, contains 700 gallons of borated water shielding. Inserted into a cylindrical cavity in this tank is an aluminum

Fig. 4

Fig. 5

tank 22" in diameter by 4' high, filled with a clear mineral oil as a neutron moderator. Fig. 5 shows this oil tank with its lid off, so that the array of the BF_3 proportional counters may be seen. Neutrons are obtained from the $\text{Li}^6(\text{p},\text{n})$ reaction from protons accelerated by the Argonne electrostatic generator. Neutrons from the scatterer enter the oil moderator about 2" to the left of the two counters at the left side of the array. To get to this point, they passed through a collimator hole (the entrance to which shows in Fig. 4) 6" wide by 1" high by 36" long through the water tank, and then through extension of this collimator hole formed by a 6" x 16" x 10" pocket in the side of the oil tank, so that neutrons enter the moderator close to the center of the oil tank.

The BF_3 counters located as shown in Fig. 5 make a neutron detection system about 15% efficient for neutrons entering the moderator through the collimator. The response is nearly independent of the energy of the neutrons from 100 Kev to 1500 Kev. Tests showed the array was within 10% as "flat" as a long counter over this interval of energy. The high efficiency permits taking data on scatterers which scatter only 5 to 10% of the incident neutrons. Most of the background is caused by air in the path of the neutron beam, the shielding being so good.

XI. Comparison of Theory to Experiment

A. Square Well Total Cross Sections vs. $\int$

It was our hope, originally, that these plots might be used to determine a best fit of $\int$ to the experimental total cross sections. As discussed in VI, B., they illustrate the hard sphere limit for large and the low value of $\sigma_{c_{\max}}$ attainable relative to geometric.

From some of the plots, for example Fig. 8, it might be concluded that a best fit would be obtained for very large values of $\int$. However, these values of $\int$ are ruled out by the fits to the total cross sections in Sec. C.

${}^6\text{C}$ has been included although, strictly speaking, the FFW formalism should not be applied to such light nuclei.

Fig. 6

Fig. 7

65

Fig. 8

Fig. 9

Fig. 10

Fig. 11

B. Harmonic Oscillator Total Cross Sections vs. ζ .

These plots are similar to those in Sec. A. for the square well. Only the series for 50 Mev, the best fit found, have been reproduced for the sake of brevity. As in the case of the square well, there are some nuclei for which no value of ζ will give a good fit at 50 Mev.

The behavior of σ_c seems to be strikingly different for the harmonic oscillator. σ_c builds up rapidly for extremely small values of ζ and the curves are not nearly as smooth as those of the square well.

The parameters used are all indicated on the curves except for the radius, which in all cases is $R = 1.45 A^{1/3} \times 10^{-13}$ cm.

Fig. 12

Fig. 13

Fig. 14

Fig. 15

C. Square Well and Harmonic Oscillator Total Cross Sections vs.
Atomic Weight.

The 1 Mev square well total cross sections in this section have been described in VI, B. The "best fit" curve is for $\int = 0.03$.

The harmonic oscillator curves are for the parameters which appear to be the best on the basis of a survey, part of which is shown in Sec. B. As in the case of the square well, the heavy end of the mass spectrum is not too well reproduced. Also, there does not appear to be enough indication of the rise in the total cross section in the region of $A \sim 90$ at 350 kev.

75

Fig. 16

Fig. 17

Fig. 18

Fig. 19

D. Square Well Angular Distributions.

The curves in this section are a fraction of the angular distributions used in an attempt to make a best fit to the square well parameters. The theoretical curves are for $\sigma_{se}(\theta)$ as calculated from the indicated parameters, while the experimental curves are drawn from the data of Langsdorf and Lane. These latter are for the angular distributions of all scattered neutrons, not just elastically scattered neutrons. An ideal fit, then, would have the experimental curve always above the theoretical curve.

As one can see, the 20 Mev and 42 Mev curves are distinctly superior to the 30 Mev curves. A general deficiency of the curves is the excessive forward peaking evident. In addition, the peak at 90° for the heavier elements is not too well reproduced. The overwhelming effect of $\int$ on the elastic scattering in the neighborhood of a resonance is illustrated in the curve for K on p. 140.

The experimental data had not been completely corrected at the time these curves were drawn. However, the corrections were at most 10% and in general were much smaller. Parameters used are as indicated on the drawings with the exception of the radius, which is $R = 1.45 A^{1/3} \times 10^{-13}$ cm.

Note added in proof: The experimental data has just been published. Langsdorf, Lane, and Monahan, "Neutron Scattering Angular Distributions," Argonne National Laboratory Report No. 5567.

^{19}K

60 kev

^{29}Cu

60 kev

^{48}Cd

60 kev

^{73}Ta

60 kev

^{83}Bi

60 kev

^{19}K

350 kev

^{29}Cu

350 kev

^{48}Cd

350 kev

^{73}Ta

350 kev

^{83}Bi

350 kev

90

^{19}K

650 kev

^{29}Cu

650 kev

^{48}Cd

650 kev

^{73}Ta

650 kev

^{19}K

950 kev

^{29}Cu

950 kev

^{48}Cd

950 kev

^{73}Ta

950 kev

^{83}Bi

950 kev

^{19}K

1250 kev

^{29}Cu

1250 kev

^{48}Cd

1250 kev

^{73}Ta

1250 kev

^{83}Bi

1250 kev

^{92}U

1250 kev

^{19}K

60 kev

^{29}Cu

60 kev

^{48}Cd

60 kev

^{73}Ta

60 kev

^{83}Bi

60 kev

^{92}U

60 kev

^{19}K

350 kev

^{29}Cu

350 kev

^{48}Cd

350 kev

^{83}Bi

350 kev

^{19}K

650 kev

^{29}Cu

650 kev

^{48}Cd

650 kev

^{73}Ta

650 kev

^{83}Bi

650 kev

^{92}U

650 kev

^{19}K

950 kev

29 Cu

950 kev

^{48}Cd

950 kev

^{73}Ta

950 kev

^{83}Bi

950 kev

^{92}U

950 kev

^{19}K

1250 kev

^{29}Cu

1250 kev

^{48}Cd

1250 kev

^{73}Ta

1250 keV

^{83}Bi

1250 kev

^{92}U

1250 kev

^{19}K

60 kev

^{29}Cu

60 kev

^{48}Cd

60 kev

^{73}Ta

60 kev

^{83}Bi

60 kev

^{92}U

60 kev

^{19}K

350 kev

^{29}Cu

350 kev

^{48}Cd

350 kev

^{73}Ta

350 kev

^{83}Bi

350 kev

^{92}U

350 kev

^{19}K

650 kev

^{29}Cu

650 kev

^{48}Cd

650 kev

^{73}Ta

650 kev

^{83}Bi

650 kev

^{92}U

650 kev

^{19}K

950 kev

^{29}Cu

950 kev

^{48}Cd

950 kev

^{73}Ta

950 kev

^{83}Bi

950 kev

^{92}U

950 kev

^{19}K

1250 kev

^{29}Cu

1250 kev

^{48}Cd

1250 kev

^{73}Ta

1250 keV

^{83}Bi

1250 kev

^{92}U

1250 kev

E. Angular Distributions at 1 Mev.

This section contains a comparison of theoretical angular distributions to the experimental angular distributions of elastically scattered neutrons at 1 Mev by Walt and Barschall (23). The Walt and Barschall data may more easily be compared to the cross sections obtainable from the FPW formalism than the data of Langsdorf and Lane. A number of square well and harmonic oscillator parameters have been used, with the primary effort being to obtain a better fit for the heavy nuclei. The failings commented upon in Sec. D are still quite evident.

Several curves were plotted using a modified radius parameter.* There seems to be no general improvement.

The parameters used are as shown on the drawings with the exception of the radius, R. Unless there is an indication to the contrary, $R = 1.45 A^{1/3} \times 10^{-13}$ cm.

* Private discussion with Dr. W. S. Emmerich, Westinghouse Research Laboratories.

Fig. 20

Fig. 21

Fig. 22

Fig. 23

$$R = (1.26A^{1/3} + 0.7) \times 10^{-13} \text{ cm.}$$

Fig. 24

— Experimental
 - - - Square Well $V_0 = 42 \text{ Mev}$ $\xi = 0.03$
 - - - Harmonic Oscillator $V_0 = 40 \text{ Mev}$ $\xi = 0.03$ - - - Harmonic Oscillator $V_0 = 60 \text{ Mev}$ $\xi = 0.03$

$$R = (1.26A^{1/3} + 0.7) \times 10^{-13} \text{ cm.}$$

Fig. 25

F. Modified Legendre Polynomial Expansion Coefficients.

These plots are comparisons of the Legendre Polynomial expansion coefficients, modified as in IX, to the same coefficients calculated under the approximations in IX from the "best fit" FPW parameters:

$$\begin{aligned} V_0 &= 42 \text{ Mev} \\ \int &= 0.03 \\ R &= 1.45 A^{1/3} \times 10^{-13} \text{ cm.} \end{aligned}$$

Since the experimental data was taken at five angles, only five expansion coefficients, a_L , could be calculated. Thus, only four ω_L could be calculated. (ω_0 is a constant.) For most nuclei, these are more than adequate in the energy range considered. However, for the higher energies and heavier nuclei, the theoretical ω_5 and ω_6 were significant, though small. However, no theoretical ω_L higher than ω_4 are shown.

Although a poor fit at low energies might have been anticipated since it is proton magic, the fit to Sn is truly remarkable. The general shape of the ω_L 's is reproduced and the quantitative agreement is quite good. For the heavy nuclei, ω_1 theoretical is much too high. Since this is the dominant term, in general, in the determination of the angular distribution, it is a major factor in the disparity between theory and experiment at this end of the mass spectrum. It accounts for the forward peaking which plagues the theoretical angular distributions.

Fig. 26

Fig. 27

Fig. 28

181

Fig. 29

Fig. 30

183

Fig. 31

XII. Conclusions

How valid is the FPW model? Of course, this is in part a matter of taste. Certainly no one could examine the curve for Sn, Fig. 28, and not be struck by the fit. Yet if one examines the best fit FPW curve in Fig. 16, it is somewhat dismaying. On balance, it seems to us the model is quite remarkable, and that, in general, it does what its originators claim for it: it does reproduce the gross features of the scattering cross sections. However, the FPW model has not reached the stage of development, as yet, that any experimentalist need doubt his results (as has happened) because his data did not correspond to those given by the FPW parameters.

Despite our efforts, we have not been able to improve upon the FPW parameters. As far as the angular distributions are concerned, nothing we attempted substantially improved the forward peaking nor the peak at $\sim 90^\circ$ in the heavier elements.

We had hoped that the harmonic oscillator potential might be a step forward. In contrast to the square well, the potential is at least continuous. On the basis of the work we have completed with the harmonic oscillator we certainly cannot say that it is superior to the square well. And it may be worse. Artificial as it seems, the square well may be a better approximation to the potential seen by an incident neutron than the harmonic oscillator.

We have come to feel that it is best to work initially with the total cross sections in attempting a fit to the experimental data. One

must work with a great many nuclei over a reasonable energy range in order to have any hope of making a fit to the gross behavior of the data. We believe that a good fit to the total cross sections made in this manner will lead to a correspondingly good fit to the angular distributions. These latter may be made most easily by using the expansion coefficients of Langsdorf and Monahan. Of course, these presuppose a good fit to the total cross sections since the expansion coefficients are divided through by a_0 , which is proportional to the total cross section. We had planned originally to obtain the Legendre Polynomial expansion coefficients of the angular distributions for the harmonic oscillator as well as the square well. However, the calculation of the coefficients for the square well proved to be marginal for the capacity of the Argonne electronic digital computer, the AVIDAC, and it was decided to defer the calculation for the harmonic oscillator.

What of the future? Mr. J. W. Butler of Argonne National Laboratory has suggested that the complex potential, if one exists, may be found directly from the experimental angular distributions. On the basis of preliminary work, the problem seems feasible for the Langsdorf and Lane data using a reasonably fast computer such as the AVIDAC.

Although the variational formula for the phase shifts may be used for the calculation of cross sections, we believe it may most fruitfully be employed to gain fundamental insight into the FPW formalism. The formula has a resonance denominator and the complex portion of the potential

enters into the formula in a rather simple manner. As discussed in VI, there are some conceptual difficulties in the FPW formalism and this may be a convenient tool for studying the basic postulates of the FPW formalism.

At one time we attempted to set up a formalism under which the neutron, when inside the nucleus, would be treated as a damped harmonic oscillator. The velocity dependent classical damping term would then yield a complex term in the Hamiltonian in the quantum-mechanical operator formalism. And the complex term in the Schrödinger equation would yield wave functions having complex logarithmic derivatives, which are essential to the FPW formalism. We did not succeed in obtaining the Hamiltonian for the damped harmonic oscillator. We have not pursued it sufficiently, but it is possible that the potential for this problem does not exist.

Dr. H. Ekstein of Argonne National Laboratory has suggested that the problem might be carried through in a formalism due to Yang and Feldman (24) in which the Hamiltonian does not appear. Yang and Feldman work with the equations of motion which, in the case of the viscously damped harmonic oscillator, are well known. At present, Dr. Ekstein and Mr. D. Kaplan of the Illinois Institute of Technology are pursuing the problem.

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Vita

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