COO - 2271 - 19

INCLUSIVE ANNIHILATION PROCESSES IN ϕ^4 FIELD THEORY

A. H. Mueller

Department of Physics
Columbia University, New York, N. Y. 10027

This research was supported in part by the U. S. Atomic Energy Commission.

DISCLAIMER

ABSTRACT

The process $\phi(q) \to \phi(p)$ + anything, the process in ϕ^4 theory analogous to $e^+ + e^- \to$ hadron + anything, is examined in ϕ^4 field theory for large values of q^2. Some heuristic arguments as to the strength of mass singularities in a particular two-particle irreducible amplitude make it possible to argue that a light-cone-like expansion exists when $q^2 \to \infty$. This light cone expansion has virtually all of the properties of the usual light cone expansion except that it is not an expansion in terms of invariant amplitudes associated with local operators. In case ϕ^4 theory has an eigenvalue, $\beta(g_\infty) = 0$, the moments of the annihilation cross section will have a power behavior in q^2, a power unrelated to the powers of q^2 appearing in any deeply inelastic scattering process. Also, at an eigenvalue the average multiplicity of particles produced, a quantity governed by the Callan-Symanzik equation in this theory, grows like a fractional power of q^2.

I. Introduction

Two types of reactions occupy a very special position in high energy physics. These two reactions are e^+e^- total annihilation into hadrons and deeply inelastic electron scattering off hadrons. In each of these reactions one is testing the short-distance behavior of the underlying theory of hadrons. This is the region where the physical masses of the hadrons, and indeed the physical hadrons themselves, do not play an essential role. The case of e^+e^- total annihilation is the simpler of the two cases. The total annihilation cross section, $\sigma(q^2)$, is proportional to

$$\int e^{iqx} d^4x < j(x)\, j(0) >_0 \quad ,$$

where $j(x)$ is the hadronic part of the electromagnetic current. (We neglect vector indices for this casual discussion.) When q^2 becomes large, $x^2 \sim 1/q^2$. The Wilson expansion[1] then tells us that the large q^2 behavior depends only on the properties of the underlying zero-mass theory, and that these properties are severely constrained by renormalization requirements[2, 3].

Deeply inelastic electron scattering is proportional to

$$-i \int d^4x\, e^{iqx} < p \mid j(x)\, j(0) \mid p > \;=\; f^p(\omega, q^2)$$

where p represents some hadron and $\omega = -\frac{2p \cdot q}{q^2}$. In this case, for large q^2, the moments

$$f_n^p(q^2) = \int_1^\infty d\omega\, \omega^{-n} f^p(\omega, q^2)$$

factorize, $f_n^p(q^2) = F_n^p f_n(q^2)$, and $f_n(q^2)$ depends only on the short-distance behavior of the theory(4). F_n^p, on the other hand, depends on the details of the particular hadron off which the electrons scatter. It is the light cone expansion(5,6), which extracts particular invariant amplitudes from a Wilson expansion, that guarantees that $f_n(q^2)$ depends only on the underlying zero-mass theory.

It is probably not obvious whether e^+e^- inclusive annihilation into hadrons has properties similar to those of the above mentioned processes(7-10). The cross section for $e^+e^- \to$ hadron (p) + anything is proportional to

$$T^h(p^2, \omega, q^2) = -i\int d^4x\, d^4y\, d^4z\, e^{iqx+ip(y-z)} < \bar{T}(j_h(y)\, j(x))\, T(j_h(z)\, j(0)) >_0 ,$$

where $j_h(x)$ is an hadronic source function, $\bar{T}$ denotes an anti-time-ordered product, and now $\omega = \frac{2p \cdot q}{q^2}$ with q^2 time-like. When ω is fixed and $q^2 \to \infty$, kinematically $x^2 \to 0$ just as in the case of deeply inelastic electron scattering. However, here the Wilson expansion does not apply. Even if $x_\mu \to 0$ there may still be operators between $j(x)$ and $j(0)$ and no analog of the Wilson expansion has been established for products like

$$\lim_{x \to 0} j(x)\, j(z)\, j(0) .$$

If the Wilson expansion is not valid one cannot expect that the light cone expansion will be valid. The following question presents itself. Is the $q^2 \to \infty$ limit, and simultaneously $x^2 \sim 1/q^2 \to 0$, in e^+e^- inclusive annihilation simply a kinematic region on the light cone, or is there an analog of the light cone expansion, related to an underlying zero-mass theory[11], which applies to this case? In this paper it will be argued that the latter is in fact what occurs. That is, there is an expansion which resembles an ordinary light cone expansion in all its details except that local operators, or particular invariant amplitudes coming from local operators, do not occur in the expansion. Thus, if one defines

$$t^h_\sigma\,(p^2, q^2) = \int_0^1 d\omega\, \omega^\sigma \; T^h(p^2, \omega, q^2) \quad ,$$

then

$$\lim_{q^2 \to \infty} t^h_\sigma\,(p^2, q^2) = \sum_{i = \pm i\sigma} \Gamma^h(p^2)\, t_{i\sigma}(q^2)$$

so long as $\mathrm{Re}\,\sigma > 0$ (ϕ^4 theory). $t_\sigma(q^2)$ has no dependence on the produced hadron, h, obeys a coupled Callan-Symanzik equation [12, 13], and is determined by the zero-mass theory. The theory which will be examined in detail in this paper is ϕ^4 theory for which the above equation and statements are true in each order of perturbation theory.

An outline of this paper may be helpful. In Section II a familiar problem, that of deeply inelastic electron scattering, or at least a ϕ^4 analog of that process, is worked out in detail. This will be referred to as the space-like case since q^2 is space-like. The light cone expansion and Callan-Symanzik equation are derived by means of a diagonalization of the Bethe-Salpeter equation along with some additional subtractions.

The spirit, but not the details, follows treatments given by Symanzik in deriving the Wilson expansion from the Bethe-Salpeter equation[11]. The essential ingredient needed in order to obtain the light cone expansion is the statement that the two-particle irreducible part of the forward four-point function, $V(p, q)$, behaves as if it were not evaluated at an exceptional momentum. (Here two-particle irreducibility is in the channel of zero four momentum.) That means

$$V(p^2, \omega, q^2) - V(0, \omega, q^2)$$

is the order of p^2/q^2 for $q^2 \to \infty$ and ω fixed. This property can be obtained in a number of ways. In particular, Zimmerman's proof of the Wilson expansion can be used[14].

In Section III the problem of an annihilation amplitude, the time-like case, is worked out in detail. The procedure is imitative of the space-like case. The diagonalization procedure is only slightly different from that of Section II. The subtraction procedure is much as in Section II given the important property of the two-particle irreducible part,

$$\lim_{q^2 \to \infty} \left[\int_0^1 d\omega\, \omega^\sigma\, V(p^2, \omega, q^2) - \int_0^1 d\omega\, \omega^\sigma\, V(0, \omega, q^2) \right] = 0$$

so long as $\mathrm{Re}\ \sigma > -1$. A light cone expansion, Eq. (30), and a Callan-Symanzik equation, Eq. (42), are obtained. It should be mentioned, however, that there is no analog of the Wilson expansion given for this process.

In Section IV an example is worked out in detail both for the space-like and time-like cases.

The major part of this paper is an attempt to establish the fact that the zero-mass theory can determine many aspects of the large q^2 behavior of an amplitude which has time-like q^2 and for which the Wilson and operator light cone expansions do not apply. There has often been reluctance about applying renormalization group techniques to time-like regions. The apparent cause for concern is the absence of Weinberg's theorem[15]. We feel that time-like regions are not essentially different from space-like regions as far as the possibility of using renormalization group and Callan-Symanzik techniques. Consider, for example, the simplest of all possible amplitudes in ϕ^4 theory, the propagator,

$$\Delta'_F(q^2) = \int d^4x\, e^{iqx} < T\,\phi(x)\,\phi(0) >_0 \quad .$$

For q^2 space-like the dominant q^2 behavior, for any given graph of Δ'_F, is governed by the renormalization group. What about time-like q^2 ? Suppose one is considering a graph of n^{th} order in g . Then, if the renormalized mass of the ϕ particle is m , the highest possible threshold in q^2 is at $q_t^2 = \left[(n+1)m\right]^2$. So in order to reach an asymptotic domain, even for q^2 space-like, one needs $|q^2| >> q_t^2$. If q^2 is time-like and $q^2 >> q_t^2$ the thresholds are far away and the mass, m , should be irrelevant so long as renormalizations are carried out at a mass, Λ , independent of m . The question of whether q^2 is space-like or time-like, then, should not be relevant so long as the invariants are large compared to the possible thresholds. Further, suppose one takes the imaginary part of Δ'_F for q^2 large and time-like. Taking the imaginary part is just a numerical operation and cannot change the asymptotic behavior. But taking the imaginary part puts many internal particles on their mass shells so one might expect that mass dependences could not disappear in this case. A little thought, however, convinces one that setting internal particles on their mass shells does not in general introduce a mass dependence. Mass dependence arises when internal groups of particles are near thresholds. In the case of the propagator, the theorems on mass singularities[(16-18)] guarantee that these threshold dependences will be weak.

For the amplitude which is the topic of this paper, inclusive annihilation, the rigorous mass singularity arguments do not apply, as different internal propagators may have different signs for their $i\epsilon$ terms. However, the strengths of the mass singularities in the two-particle irreducible part of the amplitude can be estimated, heuristically, and are found to be sufficiently weak that renormalization group arguments apply. The full amplitude can be handled exactly as in the case of deeply inelastic electron scattering, by making additional subtractions in the Bethe-Salpeter equation.

Let me now comment on a few topics which are not considered in this paper.

1) Other theories. The only theory dealt with in this paper is $g\phi^4$ theory. I suspect that similar results will hold for other field theories also. $g\bar{\Psi}\gamma_5\Psi\phi$ and $g\bar{\Psi}\Psi\sigma$ theories are probably not very difficult to handle, and probably all the results which have been obtained in this paper will carry over there including the calculation of average multiplicities from the Callan-Symanzik equation. Massive quantum electrodynamics is a much more difficult theory to handle because of the spin one meson and gauge invariance. Presumably the q^2 dependence for fixed ω will still be governed by the zero-

mass theory, but quite likely the <u>very small</u> ω region, at which relatively few particles are produced in ϕ^4 theory, is quite important in this theory and will not be governed by the zero-mass theory[19].

2) Average multiplicities of produced particles. The average number of produced particles of the type h is

$$\bar{n}_h(q^2) \;\propto\; \frac{1}{\sigma(q^2)} \int_0^1 d\omega\, \omega\, T^h(\omega, q^2) \;=\; \frac{t_1^h(q^2)}{\sigma(q^2)}$$

for large q^2 with T^h as used previously in this introduction and $\sigma(q^2)$ the total annihilation cross section. Thus the Callan-Symanzik equation governs the average multiplicity of produced hadrons. We can obtain more information by using the energy-momentum sum rule [20]

$$\int_0^1 d\omega\, \omega^2 T^h(\omega, q^2) \;\propto\; \sigma(q^2)$$

so that

$$\bar{n}_h(q^2) \;\propto\; \frac{\int_0^1 d\omega\, \omega\, T^h(\omega, q^2)}{\int_0^1 d\omega\, \omega^2\, {}^h(\omega, q^2)} \;.$$

If the theory has an eigenvalue, $\beta(g_\infty) = 0$, then the moments

$$\int_0^1 d\omega\, \omega^\sigma\, T^h(\omega, q^2)$$

behave as $(q^2)^{\alpha_\sigma}$ for large q^2. Now α_σ must be a monotonic non-increasing function of σ, for real σ, so $\bar{n}_h(q^2) \propto (q^2)^{\alpha_1 - \alpha_2}$ with $\alpha_1 \geq \alpha_2$, and the average multiplicity grows like a power of q^2.

This is reminiscent of the bootstrap scheme of Polyakov[21].

3) The $\omega \to 0$ limit. For large q^2 write[7]

$$T(p^2, \omega, q^2) = \frac{1}{2\pi i} \int_{|a| - i\infty}^{|a| + i\infty} d\sigma \, \omega^{-\sigma-1} \, t_\sigma(p^2, q^2)$$

with

$$t_\sigma(p^2, q^2) = \int_0^1 d\omega \, \omega^\sigma \, T(p^2, \omega, q^2) \quad .$$

Now

$$t_\sigma(p^2, q^2) \sim \sum_i \Gamma_{i\sigma}(p^2) \, t_{i\sigma}(q^2) \quad .$$

The large q^2 behavior of $t_\sigma(q^2)$ is constrained by renormalization and thus the large q^2 behavior of $T(p^2, \omega, q^2)$ is constrained for fixed ω. When ω becomes small (ω^2 must be greater than $\frac{4p^2}{q^2}$) the $\omega^{-\sigma-1}$ factor would indicate that the σ contour should be distorted to the left. However, at $\sigma = 0$ there are singularities in t_σ, in perturbation theory, and as the example of Section IV shows, these singularities may even sum to an essential singularity at $\sigma = 0$. In the region $(\frac{p^2}{q^2}, \frac{m^2}{q^2}) \ll \omega^2 \ll 1$ the ω behavior of $T(p^2, \omega, q^2)$ should be determined by the zero mass since the singularities near $\sigma = 0$ of $\int_0^1 d\omega \, \omega^\sigma \, T(p^2, \omega, q^2)$ are determined by the zero-mass theory.

4) Continuation from the space-like region[8, 22-25]. The continuation from the non-forward space-like region to the time-like region is tortuous, and the singularity structure is difficult to determine. However, it seems clear that one could, heuristically, use continuation arguments to show that the time-like $V(P, \omega, Q)$ (see Eq. (28)) becomes independent of P when ω is fixed and Q is large if the analogous non-forward space-like quantity has this property.

5) Other amplitudes. Finally, the question arises as to which other sorts of amplitudes have their light cone behavior constrained by renormalization requirements. We have not attempted to answer this question in any detail. It seems reasonable, though, that Wightman functions in general should have such constraints and possibly also such amplitudes as occur in μ-pair production in proton-proton collisions.

II. Space - Like Equations

Consider the amplitude $f(p \cdot q\,,\, q^2)$ given by

$$f(p \cdot q,\, q^2) \;=\; i \int d^4x\, e^{iqx}\, (p \,|\, \phi(x)\, \phi(0) \,|\, p)\, (q^2 - m^2)^2 \quad . \tag{1}$$

(In what follows only a theory of the type $\mathcal{L}_I(x) = -\frac{g}{4!}\,\phi^4(x)$ will be considered.) Equation (1) represents the scattering of an off-shell field, ϕ, on a particle of mass m. This amplitude is analogous to amplitudes which occur in studies of deeply inelastic electron scattering. Further, define the completely off-shell amplitude

$$T(p^2,\, p \cdot q\,,\, q^2) \;=\; i \int d^4x\, d^4y\, d^4z\, e^{iq \cdot x + ip \cdot (y-z)} < \overline{T}(\phi(x)\, \phi(y))\; T(\phi(0)\, \phi(z)) >_0$$
$$\left[\Delta'_F(p^2)\;\Delta'_F(q^2)\right]^{-2} \tag{2}$$

where Δ'_F is the full, renormalized propagator for the ϕ field, and $\overline{T}$ denotes the anti-time-ordered product. Now, when q^2 and p^2 are below their thresholds

$$\mathrm{disc}_{(p+q)^2}\; \widetilde{T}(p^2, p \cdot q,\, q^2) \;=\; 2i\; \mathrm{Im}\, \widetilde{T}(p^2,\, p \cdot q,\, q^2) \;=\; T(p^2,\, p \cdot q,\, q^2) \tag{3}$$

where

$$\widetilde{T}(p^2,\, p \cdot q,\, q^2) \;=\; i \int d^4x\, d^4y\, d^4z\, e^{iq \cdot x + ip\,(y-z)} < T\, \phi(x)\, \phi(y)\, \phi(z)\, \phi(0) >_0$$
$$\left[\Delta'_F(p^2)\;\Delta'_F(q^2)\right]^{-2}$$

is the ordinary, amputated, time-ordered product. When q^2 is above its threshold Eq. (3) ceases to be true.

In the region of large space-like q^2, fixed $\omega = -\frac{2p\cdot q}{q^2}$, and fixed p^2 the behavior of the time-ordered product and that of the discontinuity are the same since the discontinuity is trivially related to the time-ordered product. (The fact that the time-ordered product and the discontinuity have the same behavior in the Bjorken scaling region is also guaranteed by the light cone expansion.) When q^2 becomes time-like and Eq. (3) ceases to hold, there is no known relation between the time-ordered product, Eq. (4), and the discontinuity, Eq. (2), in the $q^2 \to \infty$, ω fixed limit[8]. Clearly the Wilson expansion does not directly apply to an amplitude like T in Eq. (2), since the operators $\phi(x)$ and $\phi(0)$ may have additional operators between them as $x \to 0$.

In this section q^2 space-like is assumed. The Callan-Symanzik[12],[13] equation will be derived for the amplitude, T, by a method which can be generalized to time-like q^2. The previous methods of obtaining Callan-Symanzik equations for the moments of T [4],[26] utilized the light cone expansion and thus cannot be easily generalized.

A. Integral Equation and Diagonalization

An integral equation for T can be given in terms of a two-particle irreducible kernel, the potential V

$$T(p^2, p\cdot q, q^2) = V(p^2, p\cdot q, q^2) + \int d^4k\, T(p^2, p\cdot k, k^2) \left| \Delta'_F(k^2) \right|^2 V(k^2, k\cdot q, q^2) \quad . \qquad (5)$$

Equation (5) is illustrated in Figure 1. $V(p^2, p \cdot q, q^2)$ is defined as in Eq. (2) except only those graphs are included which do not have two-particle states in the channel of iteration. Because absorptive amplitudes are used, the Bethe-Salpeter equation, Eq. (5), is finite once internal subtractions are performed in V and Δ'_F. Equation (5) is an exact equation in renormalized perturbation theory.

Equation (5) is a four dimensional equation with one trivial variable. This equation can be reduced to a one-dimensional equation as pointed out by Nussinov and Rosner[27]. To this end introduce the variables Q, K, P, $\mathrm{ch}\,\zeta$, $\mathrm{ch}\,\zeta_1$, $\mathrm{ch}\,\zeta_2$ defined by

$$q^2 = -Q^2, \quad k^2 = -K^2, \quad p^2 = -P^2$$

$$-q \cdot k = QK\,\mathrm{ch}\,\zeta_1, \quad k \cdot p = KP\,\mathrm{ch}\,\zeta_2, \quad p \cdot q = PQ\,\mathrm{ch}\,\zeta .$$

We have chosen, for convenience, p^2 space-like. When p^2 and q^2 are space-like, kinematics requires that k^2 also be space-like so that Q, P, K are real and positive. We may write the volume of integration in terms of these new variables

$$d^4k = \frac{2\pi K^3}{\mathrm{sh}\,\zeta}\, dK\, d\,\mathrm{ch}\,\zeta_1\, d\,\mathrm{ch}\,\zeta_2\; \Theta(\zeta - \zeta_1 - \zeta_2) \tag{6}$$

where $(q-k)^2 \geqslant 4m^2$, $(k+p)^2 \geqslant 4m^2$ has been used to put the step function in the form used above.

Now define

$$T_\sigma(P, Q) = \int_0^\infty \mathrm{sh}^2\zeta\, d\zeta\, \frac{e^{-\sigma\zeta}}{\mathrm{sh}\,\zeta}\; T(P, Q, \zeta) \tag{7}$$

with the inversion

$$T(P, Q, \zeta) = \frac{1}{2\pi i} \int_{L-i\infty}^{L+i\infty} d\sigma \; \frac{e^{\sigma\zeta}}{\mathrm{sh}\,\zeta} \; T_\sigma(P, Q) \qquad (8)$$

where it is assumed that

$$(\mathrm{ch}\,\zeta)^{L+1} \, T(P, Q, \zeta) \underset{\zeta \to \infty}{\longrightarrow} 0 \; .$$

Equations (6) and (7), along with the expression

$$\Theta(\zeta - \zeta_1 - \zeta_2) = \frac{1}{2\pi i} \lim_{\epsilon \to 0^+} \int_{-i\infty}^{i\infty} d\sigma \; \frac{e^{\sigma(\zeta - \zeta_1 - \zeta)}}{\sigma + \epsilon}$$

give, when substituted into (5)

$$T_\sigma(P, Q) = V_\sigma(P, Q) + \frac{2\pi}{\sigma} \int K^3 \, dK \, T_\sigma(P, K) \, |\, \Delta_F'(K) \,|^2 \, V_\sigma(K, Q) \; . \qquad (9)$$

Equation (9) is a one-dimensional equation for the absorptive part, T, in terms of the potential, V. We wish to study this equation as $Q \to \infty$. To simplify this task it is helpful to recast this equation into one where additional subtractions have been performed.

B. Subtraction Procedure and Behavior of the Two-Particle Irreducible Part

The equation

$$T_\sigma(P, Q) = \int_0^\infty sh^2 \zeta \, d\zeta \, \frac{e^{-\sigma\zeta}}{sh\,\zeta} \, T(P, Q, \zeta)$$

can be rewritten in a slightly different form in terms of the variable
$\omega = -\frac{2p \cdot q}{q^2} = \frac{2P}{Q} \, ch\,\zeta$. Thus,

$$ch\,\zeta = \frac{Q}{2P}\omega , \quad e^\zeta = \frac{Q}{2P}\omega + \sqrt{\left(\frac{Q\omega}{2P}\right)^2 - 1} ,$$

and

$$T_\sigma(P, Q) = \left(\frac{Q}{2P}\right)^{-\sigma+1} t_\sigma(P, Q)$$

where

$$t_\sigma(P, Q) = \int_{1+\frac{p^2+4m^2}{Q^2}}^\infty d\omega \, \omega^{-\sigma} \left[1 + \sqrt{1 - \frac{4P^2}{Q^2\omega^2}}\right]^{-\sigma} T(P, Q, \omega) \; . \tag{10}$$

Similarly, defining

$$v_\sigma(P, Q) = \int_{1+\frac{4m^2+p^2}{Q^2}}^\infty d\omega \, \omega^{-\sigma} \left[1 + \sqrt{1 - \frac{4P^2}{Q^2\omega^2}}\right]^{-\sigma} V(P, Q, \omega) \tag{11}$$

Eq. (9) becomes

$$t_\sigma(P, Q) = v_\sigma(P, Q) + \frac{2^\sigma \pi}{\sigma} \int_0^\infty K^3 \, dK \, t_\sigma(P, K) \, |\Delta_F'(K)|^2 \, v_\sigma(K, Q) \; . \tag{12}$$

Now for large Q and fixed P

$$t_\sigma(P, Q) = \int_1^\infty d\omega\, \omega^{-\sigma}\, T(P, Q, \omega) \quad (13a)$$

$$v_\sigma(P, Q) = \int_1^\infty d\omega\, \omega^{-\sigma}\, V(P, Q, \omega) \quad (13b)$$

where the corrections are of the order m^2/Q^2 and P^2/Q^2. When P is fixed and Q is large

$$v_\sigma(P, Q) - v_\sigma(0, Q) \to 0 \quad (14)$$

with corrections again of order P^2/Q^2. That is, $v_\sigma(P, Q)$ loses its P dependence when Q becomes large. In the case of space-like q^2, the situation under discussion here, Eq. (14) follows from the light cone expansion[28].

$$(p \mid \phi(x)\,\phi(0) \mid p) \xrightarrow[\substack{x^2 \to 0 \\ p\cdot x \text{ fixed}}]{} \sum_{n=0}^{\infty} E_n(x^2)\, X^{\alpha_1} X^{\alpha_2} \ldots X^{\alpha_n} (p \mid O(0)_{\alpha_1 \cdots \alpha_n} \mid p)$$

where

$$O(x)_{\alpha_1 \cdots \alpha_n} = N_{n+2}\left\{ \phi(x)\, \overleftrightarrow{\partial}_{\alpha_1} \overleftrightarrow{\partial}_{\alpha_2} \cdots \overleftrightarrow{\partial}_{\alpha_n}\, \phi(x) \right\}.$$

But,

$$E_n(x^2)(p \mid O(0)_{\alpha_1 \alpha_2 \cdots \alpha_n} \mid p)$$

is immediately related, by Fourier transform, to

$$p_{\alpha_1} p_{\alpha_2} \cdots p_{\alpha_n} t_n(P, Q)$$

while

$$p_{\alpha_1} p_{\alpha_2} \cdots p_{\alpha_n} v_n(P, Q)$$

is related to the two-particle irreducible part of

$$E_n(x^2)(p|O(0)_{\alpha_1\alpha_2 \cdots \alpha_n}|p)$$

which can have no p dependence other than $p_{\alpha_1} p_{\alpha_2} \cdots p_{\alpha_n}$.

To obtain the desired subtraction procedure,[14] begin inductively. Write

$$v_\sigma(P, Q) = \left[v_\sigma(P, Q) - v_\sigma(0, Q)\right] + v_\sigma(0, Q)$$

$$= v_\sigma^{Reg}(P, Q) + v_\sigma(0, Q)$$

where

$$v_\sigma^{Reg}(P, Q) \underset{\substack{Q \to \infty \\ P \text{ fixed}}}{\longrightarrow} 0 .$$

Also, write

$$v_\sigma^{(2)}(P, Q) = \frac{2^\sigma \pi}{\sigma} \int K^3 dK\, v_\sigma(P, K)\, v_\sigma(K, Q)\, |\Delta_F'(K)|^2$$

$$v_\sigma^{(2)}(P, Q) = v_\sigma^{(2)Reg}(P, Q) + \Gamma_\sigma^{(1)}(P)\, v_\sigma(0, Q) + v_\sigma^{(2)}(0, Q) \qquad (15)$$

$$v_\sigma^{(2)Reg}(P,Q) = \frac{2^\sigma \pi}{\sigma} \int K^3 \mid \Delta_F'(K) \mid^2 dK \left[v_\sigma(P,K) - v_\sigma(0,K)\right] \left[v_\sigma(K,Q) - v_\sigma(0,Q)\right] ,$$

and

$$\Gamma_\sigma^{(1)}(P) = \frac{2^\sigma \pi}{\sigma} \int K^3 \mid \Delta_F'(K) \mid^2 dK \left[v_\sigma(P,K) - v_\sigma(0,K)\right] .$$

Clearly

$$v_\sigma^{(2)Reg}(P,Q) \xrightarrow[\substack{Q\to\infty \\ P \text{ fixed}}]{} 0$$

so to second order in the potential

$$v_\sigma(P,Q) + v_\sigma^{(2)}(P,Q) \xrightarrow[\substack{Q\to\infty \\ P \text{ fixed}}]{} \left[1 + \Gamma_\sigma^{(1)}(P)\right] v_\sigma(0,Q) + v_\sigma^{(2)}(0,Q) .$$

This looks like the start of a Wilson expansion. The iteration procedure can be completed by defining the standard renormalization operator r by

$$r\, v_\sigma(K,Q) = v_\sigma(0,Q) .$$

Then Eq. (12) can be written as

$$t_\sigma(P,Q) = (1 - r + r)\, v_\sigma(P,Q) + \frac{2^\sigma \pi}{\sigma} \int K^3 \mid \Delta_F'(K) \mid^2 dK\, t_\sigma(P,K)\,(1 - r + r)\, v_\sigma(K,Q) .$$

Considering $(1 - r)\, v_\sigma(P,Q)$ and $r\, v_\sigma(P,Q)$ as separate potentials, $t_\sigma(P,Q)$ can be written as

$$t_\sigma(P,Q) = t_\sigma^{Reg}(P,Q) + r\, t_\sigma(P,Q) + \frac{2^\sigma \pi}{\sigma} \int K^3 \mid \Delta_F'(K) \mid^2 dK \; t_\sigma^{Reg}(P,K)\, r\, t_\sigma(K,Q)$$

or

$$t_\sigma(P,Q) = t_\sigma^{Reg}(P,Q) + \Gamma_\sigma(P)\, t_\sigma(0,Q) \qquad (16)$$

where t_σ^{Reg} is defined by

$$t_\sigma^{Reg}(P,Q) = (1-r)\, v_\sigma(P,Q) + \frac{2^\sigma \pi}{\sigma} \int K^3 \mid \Delta_F'(K) \mid^2 dK \; t_\sigma^{Reg}(P,K)\,(1-r)\, v_\sigma(K,Q) \qquad (17a)$$

and

$$\Gamma_\sigma(P) = 1 + \frac{2^\sigma \pi}{\sigma} \int K^3 \mid \Delta_F'(K) \mid^2 dK \; t_\sigma^{Reg}(P,K) \; . \qquad (17b)$$

Equations (16) and (17) constitute a light cone expansion with σ a continued index of the operators in that expansion. In order to obtain the Callan-Symanzik equatione we need an equation for the four-point function with a mass insertion.

C. Mass Insertion and Callan-Symanzik Equation

The operator

$$\left[m^2\, \partial/\partial m^2 + \beta(g)\, \partial/\partial g - 2n\, \gamma(g) \right] \qquad (18)$$

inserts the operator

$$m^2 \phi(g) \int d^4w \, N_2 \left\{ \phi^2(w) \right\}$$

into the n-point amputated vertex function in ϕ^4 theory[11]. ($\beta(g)$, $\gamma(g)$, and $\phi(g)$, not to be confused with the field $\phi(x)$, are defined as in Symanzik[11].) Thus,

$$\left\{ m^2 \frac{\partial}{\partial m^2} + \beta \frac{\partial}{\partial g} - 4\gamma \right\} T(P, Q, \omega) \equiv \hat{T}(P, Q, \omega)$$

$$= 2i \, \mathrm{Im} \, m^2 \phi(g) \int d^4x \, d^4y \, d^4z \, d^4w \, e^{iqx + ip(y-z)}$$

$$\langle T \, \phi(x) \, \phi(y) \, \phi(z) \, \phi(0) \, N_2 \left\{ \phi^2(w) \right\} \rangle_0 \left[\Delta_F'(p^2) \, \Delta_F'(q^2) \right]^{-2} .$$

Applying (18) to (12) one obtains

$$\hat{t}_\sigma(P, Q) = \hat{v}_\sigma(P, Q) + \frac{2^\sigma \pi}{\sigma} \int K^3 \, | \Delta_F'(K) |^2 \, dK \, t_\sigma(P, K) \, \hat{v}_\sigma(K, Q)$$

$$+ 2 \, \frac{2^\sigma \pi}{\sigma} \int K^3 \, | \Delta_F'(K) \, \hat{\Delta}_F'(K) | \, dK \, t_\sigma(P, K) \, v_\sigma(K, Q) \qquad (19)$$

$$+ \frac{2^\sigma \pi}{\sigma} \int K^3 \, | \Delta_F'(K) |^2 \, dK \, \hat{t}_\sigma(P, K) \, v_\sigma(K, Q)$$

where ^ means that the amplitude has a mass insertion included. Now

$$\hat{v}_\sigma(P, Q) \xrightarrow[\substack{Q \to \infty \\ P \text{ arbitrary}}]{} 0$$

up to corrections of the order m^2/Q^2, so for large Q Eq. (19) reads

$$\hat{t}_\sigma(P,Q) = 2\,\frac{2^\sigma\pi}{\sigma}\int K^3\,|\,\Delta_F'(K)\,\hat{\Delta}_F'(K)\,|\;\,dK\; t_\sigma(P,K)\,v_\sigma(K,Q)$$

$$+\,\frac{2^\sigma\pi}{\sigma}\int K^3\,|\,\Delta_F'(K)\,|^2\,dK\,\hat{t}_\sigma(P,K)\,v_\sigma(K,Q)\quad . \tag{20}$$

The kernel of Eq. (20) is the same as Eq. (12) so we can immediately conclude that

$$\hat{t}_\sigma(P,Q) \xrightarrow[\substack{Q\to\infty\\ P\ \text{fixed}}]{} \hat{\Gamma}_\sigma(P)\,t_\sigma(0,Q) \tag{21a}$$

where, for completeness,

$$\hat{\Gamma}_\sigma(P) = \frac{2^\sigma\pi}{\sigma}\int K^3 dK\,|\,\Delta_F'(K)\,|^2\,\hat{v}_\sigma(P,K)\,\Gamma_\sigma(K) + 2\,\frac{2^\sigma\pi}{\sigma}\int K^3\,|\,\Delta_F'(K)\,\hat{\Delta}_F'(K)\,|\,dK\; t_\sigma(P,K)\,\Gamma_\sigma(K)$$

$$+\left[\frac{2^\sigma\pi}{\sigma}\right]^2\int K^3\,|\,\Delta_F'(K)\,|^2\,(K')^3\,|\,\Delta_F'(K')\,|^2\,dK\,dK'\;t_\sigma(P,K)\,\hat{v}_\sigma(K,K')\,\Gamma_\sigma(K')\ . \tag{21b}$$

Combining (16), (18) and (21) one obtains

$$\left\{m^2\frac{\partial}{\partial m^2} + \beta\frac{\partial}{\partial g} - 4\gamma\right\}\Gamma_\sigma(P)\,t_\sigma(0,Q) = \hat{\Gamma}_\sigma(P)\,t_\sigma(0,Q)\quad .$$

Now $\Gamma_\sigma(0) = 1$ by our method of subtraction while $\hat{\Gamma}_\sigma(0) = a_\sigma$, some function of g calculable, order by order, in perturbation theory. Thus

$$\left\{ m^2 \frac{\partial}{\partial m^2} + \beta \frac{\partial}{\partial g} - 4\gamma - a_\sigma \right\} t_\sigma(0, Q) = 0 \qquad (22)$$

which is the desired result.

There are only a few key steps involved in obtaining Eq. (22):

i) the integral equation (12) which is simply a property of the topology of ϕ^4 theory, and is easily generalized to the case of time-like q^2,

ii) Equation (14) which follows from the light cone expansion when q^2 is space-like, but which must be rederived when q^2 is time-like, and

iii) Equations (16) and (21) which follow directly from (14).

Thus, the key to the Callan-Symanzik equation is the relation

$$v_\sigma(P, Q) - v_\sigma(0, Q) \xrightarrow[\substack{P \text{ fixed} \\ Q \to \infty}]{} 0$$

which will be generalized to the case of time-like q^2 in the next part of this paper.

III. Time - Like Equations

A. Integral Equation and Diagonalization

Consider, again, the amplitude $T(p^2, p\cdot q, q^2)$ as given by Eq. (2), but now suppose that p^2 and q^2 are time-like and $p_0 q_0 - |p|\,|q| = p\cdot q < 0$. It is convenient to let $p \to -p$ so that $p\cdot q > 0$ and $p_0 > 0$. The integral equation (5) follows immediately as it is simply a property of the topology of the Feynman graphs. Again, introduce variables Q, K, P, $\mathrm{ch}\,\zeta$, $\mathrm{ch}\,\zeta_1$, $\mathrm{ch}\,\zeta_2$ by

$$q^2 = Q^2, \quad k^2 = K^2, \quad p^2 = P^2$$

$$q\cdot p = QP\,\mathrm{ch}\,\zeta, \quad q\cdot k = QK\,\mathrm{ch}\,\zeta_1, \quad k\cdot p = KP\,\mathrm{ch}\,\zeta_2 .$$

The momentum flow of the integral equation is indicated in Figure 2. When q^2 and p^2 are time-like K is real and positive. Now

$$d^4k = \frac{2\pi K^3}{\mathrm{sh}\,\zeta}\, dK\, d\,\mathrm{ch}\,\zeta_1\, d\,\mathrm{ch}\,\zeta_2 \left\{ \Theta(\zeta - |\zeta_1 - \zeta_2|) - \Theta(\zeta - \zeta_1 - \zeta_2) \right\} \quad (23)$$

replaces (6). Equation (7) no longer diagonalizes the integral equation because the step-functions in (6) and (23) are quite different. Equation (7) represents a diagonalization by means of representations of $O(3,1)$ in a non-compact $O(2,1)$ basis, often called $O(3,1)$ functions of the second kind. Such representation functions are appropriate when k^2, p^2, and q^2 are space-like, since the little group of a space-like vector can be taken to be $0(2,1)$. Here k^2, p^2, and q^2 are time-like so the natural representation functions are those of $0(3,1)$ in the compact $0(3)$ basis. When the external particles are spinless these functions are[29]

$$\frac{\operatorname{sh} \sigma \zeta}{\operatorname{sh} \zeta} .$$

No group theory is needed if one simply observes that

$$\Theta(\zeta - |\zeta_1 - \zeta_2|) - \Theta(\zeta - \zeta_1 - \zeta_2) = \frac{2i}{\pi} \int_{-i\infty}^{i\infty} \frac{d\sigma}{\sigma} \operatorname{sh}\sigma\zeta \operatorname{sh}\sigma\zeta_1 \operatorname{sh}\sigma\zeta_2 .$$

Using this expression along with

$$T_\sigma(P,Q) = 2 \int_0^\infty \operatorname{sh}^2\zeta \, d\zeta \, \frac{\operatorname{sh}\sigma\zeta}{\operatorname{sh}\zeta} \, T(P,Q,\zeta) \tag{24a}$$

and

$$T(P,Q,\zeta) = \frac{i}{2\pi} \int_{-i\infty}^{i\infty} d\sigma \, \frac{\operatorname{sh}\sigma\zeta}{\operatorname{sh}\zeta} \, T_\sigma(P,Q) \tag{24b}$$

the equation

$$T_\sigma(P,Q) = V_\sigma(P,Q) + \frac{2\pi}{\sigma} \int K^3 dK \, |\Delta_F'(K)|^2 \, T_\sigma(P,K) \, V_\sigma(K,Q) \tag{25}$$

is achieved. Equation (25) looks the same as Eq. (9), but it should be remembered that the regions of integration are quite different as are the definitions of T_σ in the two cases.

B. Subtraction Procedure and Behavior of the Two-Particle Irreducible Part

With the definition $\omega = \frac{2p \cdot q}{q^2} = \frac{2P}{Q} \operatorname{ch} \xi$, Eq. (24a) can be written as

$$T_\sigma(P,Q) = \left(\frac{Q}{2P}\right)^{\sigma+1} t_\sigma(P,Q) - \left(\frac{Q}{2P}\right)^{-\sigma+1} t_{-\sigma}(P,Q) \qquad (26)$$

where

$$t_\sigma(P,Q) = \int_{2P/Q}^{1+\frac{P^2-4m^2}{Q^2}} d\omega\, \omega^\sigma \left[1 + \sqrt{1 - \frac{4P^2}{\omega^2 Q^2}}\right]^\sigma T(P,Q,\omega) \ . \qquad (27)$$

Similarly,

$$V_\sigma(P,Q) = \left(\frac{Q}{2P}\right)^{\sigma+1} v_\sigma(P,Q) - \left(\frac{Q}{2P}\right)^{-\sigma+1} v_{-\sigma}(P,Q) \qquad (28)$$

with

$$v_\sigma(P,Q) = \int_{2P/Q}^{1+\frac{P^2-4m^2}{Q^2}} d\omega\, \omega^\sigma \left[1 + \sqrt{1 - \frac{4P^2}{\omega^2 Q^2}}\right]^\sigma V(P,Q,\omega) \ . \qquad (29)$$

Now define the renormalization operator, r, by

$$r\, V_\sigma(P,Q) = \left(\frac{Q}{2P}\right)^{\sigma+1} v_\sigma(0,Q) - \left(\frac{Q}{2P}\right)^{-\sigma+1} v_{-\sigma}(0,Q) \ ,$$

and

$$r\, T_\sigma(P,Q) = \left(\frac{Q}{2P}\right)^{\sigma+1} t_\sigma(0,Q) - \left(\frac{Q}{2P}\right)^{-\sigma+1} t_{-\sigma}(0,Q) \ .$$

Then,

$$T_\sigma(P,Q) = (1 - r + r)\, V_\sigma(P,Q) + \frac{2\pi}{\sigma} \int K^3 \,|\Delta_F'(K)|^2\, dK\, T_\sigma(P,K)\,(1 - r + r)\, V_\sigma(K,Q)$$

as in the space-like case. Thus, we can write

$$T_\sigma(P,Q) = T_\sigma^{Reg}(P,Q) + \sum_{i=\pm} i\,\Gamma_{i\sigma}(P,Q)\, t_{i\sigma}(0,Q) \qquad (30)$$

where

$$\Gamma_{i\sigma}(P,Q) = \left(\frac{Q}{2P}\right)^{1+i\sigma} + \frac{2\pi}{\sigma}\int K^3 dK |\Delta_F'(K)|^2\, T_\sigma^{Reg}(P,K)\left(\frac{Q}{2K}\right)^{1+i\sigma}$$

(31)

and T_σ^{Reg} obeys the integral equation

$$T_\sigma^{Reg}(P,Q) = \left(\frac{Q}{2P}\right)^{\sigma+1}(v_\sigma(P,Q) - v_\sigma(0,Q)) - \left(\frac{Q}{2P}\right)^{-\sigma+1}(v_{-\sigma}(P,Q) - v_{-\sigma}(0,Q))$$

$$+ \frac{2\pi}{\sigma}\int K^3 dK |\Delta_F'(K)|^2\, T_\sigma^{Reg}(P,K)\left\{\left(\frac{Q}{2K}\right)^{\sigma+1}\left[v_\sigma(K,Q) - v_\sigma(0,Q)\right]\right.$$

$$\left. - \left(\frac{Q}{2K}\right)^{-\sigma+1}\left[v_{-\sigma}(K,Q) - v_{-\sigma}(0,Q)\right]\right\}. \qquad (32)$$

If

$$v_\sigma(P,Q) - v_\sigma(0,Q) \xrightarrow[\substack{P \text{ fixed} \\ Q\to\infty}]{} 0$$

as in the space-like case, then $T_\sigma^{Reg}(P,Q)$ will go to zero in the above limit, and Eq. (30) is the analog of the light cone expansion. It should be emphasized, however, that in the space-like case $\Gamma_\sigma(P)$ represented part of the matrix element of a local operator when $\sigma = 0, 1, 2, \cdots$, while in the time-like case there is no obvious connection between $\Gamma_{i\sigma}(P)$ and the matrix elements of any local operators. It should be noted that (30) and (31) give the equation

$$t_\sigma(P,Q) \underset{Q\to\infty}{\longrightarrow} \left(\frac{2P}{Q}\right)^{\sigma+1} \Gamma_\sigma(P,Q)\, t_\sigma(0,Q) - \left(\frac{2P}{Q}\right)^{\sigma+1} \Gamma_{-\sigma}(P,Q)\, t_{-\sigma}(0,Q)$$

for $\mathrm{Re}\,\sigma > 0$, and that both terms on the right-hand side of this equation are equally important in general. Also, $\Gamma_\sigma(P,Q)$ has poles in σ, for example at $\sigma = 1$, but these poles do not appear in the asymptotic form of $t_\sigma(P,Q)$ given above.

Before directly confronting the question of the P independence of $V(P,Q)$, for large Q, let us first discuss a problem, many of whose aspects may be more familiar to the reader. Consider, then, the self-energy amplitude, $\Sigma(q^2)$, defined by $[\Sigma_{\phi\phi}(q^2) - (q^2 - m^2)]^{-1} = i\int e^{iqx}\, d^4x \;\langle T\,\phi(x)\,\phi(0)\rangle_0$. We envision a "$\phi^4$ type" field theory where there are two massive scalar mesons, ϕ and χ, whose renormalized masses are m_1 and m_2, interacting by means of $g_1\phi^4$, $g_2\chi^4$, and $g_3\phi^2\chi^2$ terms in the Lagrangian. Begin by regulating the theory, in a Pauli-Villars manner, with a regulator mass Λ. Consider, first, any particular graph contributing to $\Sigma_{\phi\phi}(q^2)$ which does not have any internal self-energy parts $\Sigma_{\phi\phi}$, $\Sigma_{\phi\chi}$, or $\Sigma_{\chi\chi}$. Such a graph we denote by $\tilde{\Sigma}_{\phi\phi}(q^2)$. The full functional dependence of $\tilde{\Sigma}$ is given by

$$\tilde{\Sigma}_{\phi\phi}(q^2) = \tilde{\Sigma}_{\phi\phi}(q^2, \Lambda^2, m_1^2, m_2^2)$$

where only coupling dependences, irrelevant for a fixed graph, have been suppressed.

Now for Λ^2, $|q^2| \gg m_1^2, m_2^2$

$$\tilde{\Sigma}_{\phi\phi}(q^2) \longrightarrow \tilde{\Sigma}_{\phi\phi}(q^2, \Lambda^2) = \tilde{\Sigma}_{\phi\phi}(q^2, \Lambda^2, 0, 0) \quad . \qquad (33)$$

That is, all m_1 and m_2 dependence disappears from $\tilde{\Sigma}$ when $|q^2|$ and Λ^2 are much greater than m_1^2 and m_2^2 (2), independently of whether q^2 is space-like, time-like, or complex. If subtractions are perfomed at a point large compared to m_1 and m_2 then the renormalized amplitude will also be asymptotically independent of m_1 and m_2. These statements are at the heart of the renormalization group approach to large off-shell behavior.

In Figure 3 a typical discontinuity of $\tilde{\Sigma}_{\phi\phi}$, corresponding to an intermediate state i, is shown. Straight lines indicate ϕ particles while wiggly lines indicate χ particles. We claim that for any particular discontinuity, i, of $\tilde{\Sigma}_{\phi\phi}(q^2)$

$$\text{disc}_i \; \tilde{\Sigma}_{\phi\phi}(q^2) \xrightarrow[q^2,\, \Lambda^2 \gg m_1,\, m_2]{} \text{disc}_i \; \tilde{\Sigma}_{\phi\phi}(q^2, \Lambda^2, 0, 0) \quad . \qquad (34)$$

The total discontinuity obeys (34) trivially from (33), but (34) is very non-trivial when a particular discontinuity is involved. To show that (34) is correct it is enough to show that

$$\lim_{\substack{m_1^2 \to 0 \\ m_2^2 = \rho m_1^2 \\ \rho \text{ fixed}}} m_1^2 \frac{\partial}{\partial m_1^2} \text{disc}_i \; \tilde{\Sigma}_{\phi\phi}(q^2) = 0 \qquad (35)$$

along with a similar statement with the roles of m_1 and m_2 interchanged. To show that (35) is correct we need to show that the strength of the mass singularity of

$\frac{\partial}{\partial m^2} \text{disc}_i \tilde{\Sigma}_{\phi\phi}(q^2)$ is only logarithmic. This is the same as showing that the strength of the possible mass singularity of $\text{disc}_i \tilde{\Sigma}_{\phi\phi}(q^2)$ is only like $m^2(\ln m^2)^n$ in the approach to the zero-mass theory. (For the purposes of counting the overall strength of mass singularities it is not necessary to distinguish m_1 from m_2.)

The possible mass singularities in $\text{disc}_i \tilde{\Sigma}_{\phi\phi}(q^2)$ occur when the invariant mass of some subset of the particles in the state i approaches the zero mass thresholds of A and B (see Figure 3). Thus the important region of phase space is where some set of particles $(1, 2, \cdots j)$ in the state i have momenta such that $(k_1 + k_2 + \cdot\cdot + k_j)^2 \to 0$. What needs to be ascertained is the strength of the singularity in the amplitudes A and B versus the suppressing factor of the phase space when a zero-mass threshold is approached. Call $K_j = k_1 + k_2 + \cdot\cdot + k_j$. Then, as shown in Appendix B,

$$d^4K_j \frac{d^3k_1}{E_1} \frac{d^3k_2}{E_2} \cdots \frac{d^3k_j}{E_j} \delta^4(k_1 + k_2 + \cdot\cdot + k_j - K_j)\,\theta(a^2 - K_j^2)\,\Theta(q_0 - K_{j0}) \propto (a^2)^{j-1}$$

for small a^2 when the particles $1, 2, \cdot\cdot j$ are massless. Thus a mass singularity in $\text{disc}_i \tilde{\Sigma}_{\phi\phi}(q^2)$ like $\ln m^2$ will occur if the product of the strengths of the threshold singularities in A and B are like $(a^2)^{-(j-1)}$. If j is an even number the maximum strength of the threshold singularity of A occurs when the j particles go into two particles. Then A has a singularity like $(a^2)^{-(j-2)/2}$. Similarly for B. Thus, the total strength of the mass singularity is

$$(a^2)^{j-1-2\frac{j-2}{2}} = a^2$$

for even j so that no mass singularity occurs. If j is odd the maximum strength of the a^2 singularity of A is $(a^2)^{-(j-3)/2}$ if the j particles can go into a minimum of three particles, and $(a^2)^{-(j-1)/2}$ if the j particles can go into a single particle. Similarly for B. If the j particles could go into a single particle in both A and B the total singularity would be like

$$(a^2)^{j-1-2\frac{(j-1)}{2}} = (a^2)^0$$

and a mass singularity would occur. But, such a case is ruled out by our assumption that $\tilde{\Sigma}_{\phi\phi}$ has no internal self-energy parts. Thus, the maximum singularity occurs when the j particles go into a single particle in one of the amplitudes, A or B, and into three particles in the other amplitude. Then the singularity is like

$$(a^2)^{j-1-\frac{j-1}{2}-\frac{j-3}{2}} = a^2 ,$$

and $\text{disc}_j\, \tilde{\Sigma}_{\phi\phi}(q^2)$ has mass singularities only like $m^2\left[\ln m^2\right]^n$. This completes the argument that if $\tilde{\Sigma}_{\phi\phi}(q^2)$ is an amplitude without internal self-energy parts then

$$\text{disc}_j\, \tilde{\Sigma}_{\phi\phi}(q^2, \Lambda^2, m_1^2, m_2^2) \xrightarrow[q^2,\, \Lambda^2 \gg m_1^2,\, m_2^2]{} \text{disc}_j\, \tilde{\Sigma}_{\phi\phi}(q^2, \Lambda^2, 0, 0) .$$

Furthermore, it is clear that one can add internal self-energy parts so long as discontinuities are added in such a way that total discontinuities are taken for all internal propagators. As an example consider Figure 4. If no three of the lines k_1, k_2, k_3, k_4, k_5 can go into a single line both in both A and B, and if no one of the lines k_1, k_2, k_3, k_4, k_5 goes into a self-energy part in either the upper or lower amplitudes, A or B, then the sum of the three discontinuities shown in Figure 4 will have the property (34) even though A and B may contain numerous self-energy insertions.

We now proceed to $v_\sigma(P, Q)$. For any given diagram, contributing to $V(P, Q, \omega)$,

$$\int V(P,Q,\omega)\,\frac{d^3p}{E_p} = \pi Q^2 \int_{2P/Q}^{1+\frac{P^2-4m^2}{Q^2}} \omega\, d\omega \sqrt{1-\frac{4p^2}{\omega^2 Q^2}}\quad V(P,Q,\omega)$$

corresponds to a sum of discontinuities of the type shown in Figure 5 where one of the internal lines has been given a mass, P, and thus singled out from the other internal lines. Since V is two particle irreducible the line p cannot be part of any self-energy insertion. Thus,

$$\int_{2P/Q}^{1+\frac{P^2-4m^2}{Q^2}} d\omega\, \omega \sqrt{1-\frac{4P^2}{Q^2\omega^2}}\quad V(P,Q,\omega) = v_1(P,Q)$$

must be independent of P for Q large so long as internal renormalizations are done at some point Λ. (However, it is clear that the P dependence of the above integral cannot depend on the point of renormalization so we may in fact renormalize at a point proportional to m if we choose.) Thus,

$$\int_0^1 \omega V(P, Q, \omega)\, d\omega = v_1(P, Q)$$

must have this same property. Since we do not expect delicate cancellations in the integration over ω we can conclude that $v_\sigma(P, Q)$ becomes independent of P, for large Q, so long as $\mathrm{Re}\,\sigma \geq 1$. A somewhat extended argument is given in Appendix C which indicates that $v_\sigma(P, Q)$ becomes independent of P, for large Q, so long as $\mathrm{Re}\,\sigma > -1$.

C. Mass Insertions and the Callan-Symanzik Equation

The operator $\left\{ m^2 \frac{\partial}{\partial m^2} + \beta \frac{\partial}{\partial g} - 4\gamma \right\}$ inserts $m^2 \phi(g) \int d^4 w\, N_2 \left\{ \phi^2(w) \right\}$ into the amplitude $T_\sigma(P, Q)$ or $V_\sigma(P, Q)$ exactly as in the space-like case. The argument then proceeds as in Part C of Section II. Write (25), symbolically, as

$$T_\sigma = V_\sigma + \frac{2\pi}{\sigma} T_\sigma [\Delta]^2 V_\sigma . \tag{25'}$$

Applying the Callan-Symanzik operator to (25') one obtains

$$\hat{T}_\sigma = \hat{V}_\sigma + \frac{2\pi}{\sigma} T_\sigma [\Delta]^2 \hat{V}_\sigma + \frac{4\pi}{\sigma} T_\sigma \Delta \hat{\Delta} V_\sigma + \frac{2\pi}{\sigma} \hat{T}_\sigma [\Delta]^2 V_\sigma . \tag{36}$$

For Q large we may drop the $\hat{V}_\sigma$ terms. (Remember $\hat{V}_\sigma$ stands for $\hat{V}_\sigma(P, Q)$ for the first $\hat{V}$ in (36) and $\hat{V}_\sigma(K,Q)$ for the second $\hat{V}$ in (36).) Then

$$\hat{T}_\sigma = \frac{4\pi}{\sigma} T_\sigma \Delta \hat{\Delta} V_\sigma + \frac{2\pi}{\sigma} \hat{T} [\Delta]^2 V_\sigma \quad . \tag{37}$$

Since the kernel of (37) is the same as in (25') we can write, $-1 < \mathrm{Re}\,\sigma < 1$,

$$\hat{T}_\sigma(P,Q) \underset{\substack{Q \to \infty \\ P \text{ fixed}}}{\longrightarrow} \sum_{i=\pm} i\, \hat{\Gamma}_{i\sigma}(P,Q)\, t_{i\sigma}(0,Q) \quad , \tag{38}$$

and

$$\left\{ m^2 \frac{\partial}{\partial m^2} + \beta \frac{\partial}{\partial \delta} - 4\gamma \right\} \sum_{i=\pm} i\, \Gamma_{i\sigma}(P,Q)\, t_{i\sigma}(0,Q) = \sum_{i=\pm} i\, \hat{\Gamma}_{i\sigma}(P,Q)\, t_{i\sigma}(0,Q) \quad . \tag{39}$$

Here,

$$\hat{\Gamma}_{i\sigma}(P,Q) = \frac{2\pi}{\sigma} \int K^3 \left[\Delta_F'(K)\right]^2 dK\; \hat{V}_\sigma(P,K)\, \Gamma_{i\sigma}(K,Q)$$
$$+ \left(\frac{2\pi}{\sigma}\right)^2 \int K^3 \left[\Delta_F'(K)\right]^2 (K')^3 \left[\Delta_F'(K')\right]^2 dK\, dK'\; T_\sigma(P,K)\, \hat{V}_\sigma(K,K')\, \Gamma_{i\sigma}(K',Q)$$
$$+ 2\,\frac{2\pi}{\sigma} \int K^3 \Delta_F'(K)\, \hat{\Delta}_F'(K)\; T_\sigma(P,K)\, \Gamma_{i\sigma}(K,Q) \quad . \tag{40}$$

Now, for $-1 < \mathrm{Re}\,\sigma < 1$ we may write (39) in detail as

$$\left\{ m^2 \frac{\partial}{\partial m^2} + \beta \frac{\partial}{\partial \delta} - 4\gamma \right\} \left\{ \Gamma_\sigma(P,Q)\, t_\sigma(0,Q) - \Gamma_{-\sigma}(P,Q)\, t_{-\sigma}(0,Q) \right\}$$
$$= \hat{\Gamma}_\sigma(P,Q)\, t_\sigma(0,Q) - \hat{\Gamma}_{-\sigma}(P,Q)\, t_{-\sigma}(0,Q) \quad .$$

Now, let $P \to 0$ to get

$$\left\{ m^2 \frac{\partial}{\partial m^2} + \beta \frac{\partial}{\partial g} - 4\gamma \right\} t_\sigma(0,Q) = \mathcal{B}\, t_\sigma(0,Q)$$

$$= \lim_{P\to 0} \left\{ \left(\frac{2P}{Q}\right)^{\sigma+1} \hat{\Gamma}_\sigma(P,Q)\, t_\sigma(0,Q) - \left(\frac{2P}{Q}\right)^{\sigma+1} \hat{\Gamma}_{-\sigma}(P,Q)\, t_{-\sigma}(0,Q) \right\} .$$

Calling

$$\lim_{P\to 0} \left(\frac{2P}{Q}\right)^{\sigma+1} \hat{\Gamma}_\sigma(P,Q) = a_\sigma \ ; \quad \lim_{P\to 0} \left(\frac{2P}{Q}\right)^{\sigma+1} \hat{\Gamma}_{-\sigma}(P,Q) = \left(\frac{m^2}{Q^2}\right)^{\sigma} b_{-\sigma}$$

one obtains

$$\mathcal{B}\, t_\sigma(0,Q) = a_\sigma\, t_\sigma(0,Q) - \left(\frac{m^2}{Q^2}\right)^{\sigma} b_{-\sigma}\, t_{-\sigma}(0,Q)$$

and

$$\mathcal{B}\, t_{-\sigma}(0,Q) = a_{-\sigma}\, t_{-\sigma}(0,Q) - \left(\frac{m^2}{Q^2}\right)^{-\sigma} b_\sigma\, t_\sigma(0,Q) \ .$$

These equations can be written as

$$(\mathcal{B} - a_\sigma)\, t_\sigma(0,Q) = -\left(\frac{m^2}{Q^2}\right)^{\sigma} b_{-\sigma}\, t_{-\sigma}(0,Q) \qquad (41a)$$

$$(\mathcal{B} - a_{-\sigma})\, t_{-\sigma}(0,Q) = -\left(\frac{m^2}{Q^2}\right)^{-\sigma} b_\sigma\, t_\sigma(0,Q) \qquad (41b)$$

or

$$(\mathcal{B} - a_{-\sigma}) \left(\frac{m^2}{Q^2}\right)^{-\sigma} \frac{1}{b_{-\sigma}} (\mathcal{B} - a_\sigma)\, t_\sigma(0,Q) = \left(\frac{m^2}{Q^2}\right)^{-\sigma} b_\sigma\, t_\sigma(0,Q) \ . \qquad (42)$$

a_σ and b_σ depend on g in a manner similar to β and γ. Equations (41) and (42) should be valid in the whole σ plane although the identification of $t_\sigma(0,Q)$ with $\lim_{P\to 0} t_\sigma(P,Q)$ is only valid for $\mathrm{Re}\,\sigma > 0$. For the examples of the next section Eq. (41) reduces to

$$(\mathcal{D} - a_\sigma)\, t_\sigma(0,Q) = 0 ,$$

but we cannot expect an uncoupled equation to hold in the general circumstance.

IV. Two Examples

As examples we shall consider the set of graphs generated by the Bethe-Salpeter equation, Eq. (5), when the potential is given by the graph shown in Figure 6.

$$V(p,q) = \frac{g^2}{8\pi} \sqrt{\frac{s-4m^2}{s}}$$

where $s=(p+q)^2$ in the space-like case and $s=(p-q)^2$ in the time-like case.

A. Space-like Case[30]

The diagonalized equation is obtained from (12) and reads

$$t_\sigma(P,Q) = v_\sigma(P,Q) + \frac{2^\sigma \pi}{\sigma} \int K^3 \mid \Delta_F(K) \mid^2 dK \, t_\sigma(P,K) \, v_\sigma(K,0) \qquad (42)$$

where

$$\Delta_F(K) = \frac{-i}{(2\pi)^4} \frac{1}{K^2+m^2-i\epsilon} ,$$

and

$$v_\sigma(P,Q) = \frac{g^2}{8\pi} \int_{1+\frac{P^2+4m^2}{Q^2}}^{\infty} d\omega \, \omega^{-\sigma} \left[1 + \sqrt{1 - \frac{4P^2}{Q^2\omega^2}} \right]^{-\sigma} \sqrt{\frac{1 - \frac{P^2+4m^2}{Q^2(\omega-1)}}{1 - \frac{P^2}{Q^2(\omega-1)}}}$$

(43)

with Re $\sigma > -1$. We can go directly to the Callan-Symanzik Eq. (22) with $\beta = \gamma = 0$.

Thus,

$$\left[m^2 \frac{\partial}{\partial m^2} - a_\sigma \right] t_\sigma(0, Q) = 0 \qquad (44)$$

where

$$a_\sigma = \hat{\Gamma}_\sigma(0) \quad .$$

In order to evaluate $\hat{\Gamma}_\sigma(0)$ one needs to know the solution for $t_\sigma(P, Q)$, as is clear from (21b). However, to order g^2 we can calculate a_σ trivially as

$$a_\sigma = \lim_{\Lambda^2 \to \infty} m^2 \frac{\partial}{\partial m^2} \frac{2^\sigma \pi}{\sigma} \int_0^\Lambda \frac{K^3\, dK}{|K^2 + m^2|^2} \frac{v_\sigma(0, K)}{(2\pi)^8}$$

$$= \lim_{\Lambda \to \infty} m^2 \frac{\partial}{\partial m^2} \frac{\pi g^2}{8\pi\sigma} \int_0^\Lambda \frac{K^3\, dK}{|K^2 + m^2|^2} \int_{1+\frac{4m^2}{K^2}}^\infty d\omega\, \omega^{-\sigma} \frac{\sqrt{1 - \frac{4m^2}{K^2(\omega - 1)}}}{(2\pi)^8}$$

$$= \lim_{\Lambda \to \infty} m^2 \frac{\partial}{\partial m^2} \frac{g^2}{16\sigma (2\pi)^8} \int_0^{\frac{\Lambda^2}{m^2}} \frac{y\, dy}{[y+1]^2} \int_{1+\frac{4}{y}}^\infty d\omega\, \omega^{-\sigma} \sqrt{1 - \frac{4}{y(\omega - 1)}}$$

$$= \lim_{\Lambda \to \infty} \frac{-g^2}{16\sigma (2\pi)^8} \frac{\left(\frac{\Lambda^2}{m^2}\right)^2}{\left[\frac{\Lambda^2}{m^2} + 1\right]^2} \int_{1+\frac{4m^2}{\Lambda^2}}^\infty d\omega\, \omega^{-\sigma} \sqrt{1 - \frac{4m^2}{\Lambda^2(\omega - 1)}}$$

$$= - \frac{g^2}{16\sigma (2\pi)^8} \int_1^\infty d\omega\, \omega^{-\sigma}$$

$$a_\sigma = \frac{g^2}{(2\pi)^8\ 16\sigma\ (\sigma - 1)} \tag{45}$$

so that

$$\int_1^\infty d\omega\ \omega^{-\sigma}\ T(P, Q, \omega) \xrightarrow[Q \to \infty]{} \left(\frac{Q^2}{m^2}\right)^{\frac{g^2}{(2\pi)^8\ 16\sigma(\sigma-1)}} \times \frac{g^2}{8\pi}\ \frac{1}{\sigma - 1} \tag{46}$$

in the leading logarithmic approximation.

B. Time-like Case

When $q^2 = Q^2 > 0$ the potential, V, has the same form as in the space-like case.

$$V(p, q) = \frac{g^2}{8\pi} \sqrt{\frac{s - 4m^2}{s}} \tag{47}$$

where now $s = (p - q)^2$. Equation (47) is shown in Figure 7. Now, in the leading logarithmic approximation,

$$\left[m^2 \frac{\partial}{\partial m^2} - \tilde{a}_\sigma\right] t_\sigma(0, Q) = 0 ,$$

where

$$\tilde{a}_\sigma = \lim_{\substack{Q \to \infty \\ P \to 0}} \left(\frac{2P}{Q}\right)^{\sigma + 1} \hat{\Gamma}_\sigma(P, Q) .$$

To first order in g^2

$$\tilde{a}_\sigma = \lim_{\Lambda\to\infty} m^2 \frac{\partial}{\partial m^2} \frac{2^{-\sigma}\pi}{(2\pi)^8 \sigma} \int_0^\Lambda \frac{K^3\, dK}{|K^2 - m^2|^2} v_\sigma(0, K) \quad .$$

Thus

$$\tilde{a}_\sigma = \lim_{\Lambda\to\infty} m^2 \frac{\partial}{\partial m^2} \frac{g^2}{(2\pi)^8\, 8\sigma} \int_{4m^2}^\Lambda \frac{K^3\, dK}{|K^2 - m^2|^2} \int_0^{1-\frac{4m^2}{K^2}} d\omega\, \omega^\sigma \sqrt{1 - \frac{4m^2}{K^2(1-\omega)}}$$

$$= \lim_{\Lambda\to\infty} m^2 \frac{\partial}{\partial m^2} \frac{g^2}{(2\pi)^8\, 16\sigma} \int_4^{\Lambda^2/m^2} \frac{y\, dy}{|y-1|^2} \int_0^{1-4/y} d\omega\, \omega^\sigma \sqrt{1 - \frac{4}{y(1-\omega)}}$$

$$= \lim_{\Lambda\to\infty} \frac{g^2}{(2\pi)^8\, 16\sigma} \frac{\left(-\frac{\Lambda^2}{m^2}\right)\left(\frac{\Lambda^2}{m^2}\right)}{\left|\frac{\Lambda^2}{m^2} - 1\right|^2} \int_0^{1-\frac{4m^2}{2}} d\omega\, \omega^\sigma \sqrt{1 - \frac{4m^2}{\Lambda^2(1-\omega)}}$$

$$= - \frac{g^2}{(2\pi)^8\, 16\sigma} \int_0^1 d\omega\, \omega^\sigma$$

$$\tilde{a}_\sigma = - \frac{g^2}{(2\pi)^8\, 16\sigma(\sigma+1)} \quad . \tag{48}$$

Thus

$$\int_0^1 d\omega\, \omega^\sigma \; T(P, Q, \omega) \longrightarrow \left(\frac{Q^2}{m^2}\right)^{\frac{g^2}{(2\pi)^8\, 16\sigma(\sigma+1)}} \frac{g^2}{8\pi} \frac{1}{\sigma+1} \quad . \tag{49}$$

Appendix A

In this appendix two examples will be given which show how $T(P, Q, \omega)$ and $V(P, Q, \omega)$ behave for large values of Q. We begin with the contribution to $T(P, Q, \omega)$ shown in Figure 8.

$$T(p, q) \propto \int \frac{d^3k_1}{E_1} \frac{d^3k_2}{E_2} \frac{\Theta\left[(q - p - k_1 - k_2)^2\right] \Theta(q_o - p_o - E_1 - E_2)}{\left| (k_1 + k_2 + p)^2 \right|^2} \qquad \text{(A.1)}$$

where internal particles have been taken to have zero mass. Write

$$\frac{d^3k_1}{E_1} \frac{d^3k_2}{E_2} = \frac{d(k_1)_z}{E_1} \frac{d(k_2)_z}{E_2} d^2\underline{k}_1 \, d^2\underline{k}_2$$

where

$$k_1 = (E_1, \underline{k}_1, k_{1z}) ; \; E_1 = \sqrt{\underline{k}_1^2 + k_{1z}^2} , \; \underline{k}_1 = (k_{1x}, k_{1y})$$

$$k_2 = (E_2, \underline{k}_2, k_{2z}) ; \; E_2 = \sqrt{\underline{k}_2^2 + k_{2z}^2} , \; \underline{k}_2 = (k_{2x}, k_{2y}) .$$

Now, choose a coordinate frame where

$$q = (Q, 0, 0, 0)$$

$$p \approx \left(\frac{\omega Q}{2} + \frac{P^2}{\omega Q}, 0, 0, \frac{\omega Q}{2}\right) .$$

then

$$(k_1 + k_2 + p)^2 \approx P^2 \left[1 + 2\frac{E_1 + E_2}{\omega Q} \right] + \omega Q \left[(E_1 - k_{1z}) + (E_2 - k_{2z}) \right]$$

$$+ 2(E_1 E_2 - k_{1z} k_{2z} - \underline{k}_1 \cdot \underline{k}_2)$$

if we are interested in finding the strength of the singularity at $P^2 = 0$, for fixed ω, $(k_1 + k_2 + p)^2$ must be the order of P^2. To achieve this it is necessary that $E_i \sim k_{iz}$ so that

$$E_i - k_{iz} \approx \frac{\underline{k}_i^2}{2k_{iz}}$$

and so

$$(k_1 + k_2 + p)^2 \approx P^2 \left[1 + 2\frac{k_{1z} + k_{2z}}{\omega Q} \right] + \frac{\omega Q}{2} \left[\frac{\underline{k}_1^2}{k_{1z}} + \frac{\underline{k}_2^2}{k_{2z}} \right]$$

$$+ \left(\frac{k_{1z}}{k_{2z}} \underline{k}_2^2 + \frac{k_{2z}}{k_{1z}} \underline{k}_1^2 - 2\underline{k}_1 \cdot \underline{k}_2\right) \tag{A.2}$$

clearly the result is a logarithmic divergence in P^2, since (A 2) is a positive definite quantity in $\underline{k}_1$ and $\underline{k}_2$, and the volume $d^2\underline{k}_1 \cdot d^2\underline{k}_2$ is four dimensional. Also the logarithmic singularity occurs when k_{1z} and k_{2z} are proportional to $\frac{\omega Q}{2}$ while $|\underline{k}_1|$ and $|\underline{k}_2|$ are on the order of P. The singularity occurs because the single particle poles are reached when k_1, k_2, and p are essentially parallel and proportional momenta. As described in Section III the volume of phase space for $(p+k_1+k_2)^2 \approx P^2$ is proportional to $(P^2)^2$ which almost balances the $(P^2)^{-2}$ coming from the two propagators, resulting in a $\ln P^2$ term.

The second example to be discussed involves the contribution to $V(P, Q, \omega)$ shown in Figure 9.

$$V(P, Q, \omega) \alpha \int \frac{d^3k}{E} \ln^2 \left[\frac{-(p+k)^2}{\Lambda^2} \right]$$

$$\lim_{m^2 \to 0} \mathrm{Im} \left\{ \frac{r^2 + \Lambda^2}{r^2 - m^2 + i\epsilon} \int_{9m^2}^{\infty} \frac{da^2 \rho(a^2, m^2)}{(a^2 + \Lambda^2)(a^2 - k^2 - i\epsilon)} \right\}$$

where $r = q - p - k$ and the term in $\{ \}$ is the g^2 term of the propagator in φ^4 theory. Again the zero mass limit has been taken. In the coordinate system where

$$q = (Q, 0, 0, 0)$$

$$p = \left(\frac{\omega Q}{2} + \frac{P^2}{\omega Q}, 0, 0, \frac{\omega Q}{2}\right)$$

we write

$$\frac{d^3k}{E} = \frac{dk_3}{E} d^2k = 2\pi \, dx \, dy ,$$

with

$$x = E , \; y = E - k_z .$$

Thus, for fixed ω ,

$$V(P, Q, \omega) \alpha\ 2\pi \int_0^{\infty} dx \int_0^{x} dy\ \ln^2 \left[\frac{P^2(1 + \frac{2x}{\omega Q}) + \omega Q y}{-\Lambda^2} \right]$$

$$\lim_{m^2 \to 0} \mathrm{Im} \left\{ \frac{Q^2(1-\omega) + P^2(1 - \frac{2}{\omega} + \frac{2x}{\omega Q}) + \omega Q y - 2Qx + \Lambda^2}{Q^2(1-\omega) + P^2(1 - \frac{2}{\omega} + \frac{2x}{\omega Q}) + \omega Q y - 2Qx - m^2 + i\epsilon} \right.$$

$$\int_{9m^2}^{\infty} \frac{\rho(a^2, m^2)\, da^2}{(a^2 + \Lambda^2) \left[a^2 - Q^2(1-\omega) - P^2(1 - \frac{2}{\omega} + \frac{2x}{\omega Q}) - \omega Q y + 2Qx - i\epsilon \right]} .$$

The dominant contribution to V comes from $x \sim y \sim Q$ which gives $V \alpha \ln^2 Q$. In order to obtain the dominant P dependence $y \sim P^2/Q$ while $x \sim Q$ so that

$$V \sim C_1\ P^2/Q^2\ \ln^2\ P^2/-\Lambda^2 + C_2\ P^2/Q^2\ \ln\ P^2/-\Lambda^2 + C_3\ P^2/Q^2 .$$

If ω is small then the dominant P dependence comes from $x \sim Q$, $y \sim P^2/\omega^2 Q$. Then the P^2 dependence looks like

$$V \sim C_1\ P^2/\omega^2 Q^2\ \ln^2\ P^2/-\Lambda^2\omega + C_2\ P^2/\omega^2 Q^2\ \ln\ P^2/-\Lambda^2\omega + C_3\ P^2/\omega^2 Q^2 .$$

Note that when $x >> y$, as is the case above,

$$x \approx k_z\ ,\ y \approx \frac{\underline{k}^2}{2k_z}$$

so that $k_z \sim Q$ while $|\underline{k}| \sim P/\omega$. Only when ω becomes as small as P/Q are $|\underline{k}|$ and k_z comparable. The above example illustrates the general case that so long as $\omega >> P/Q$ the dominant P dependence comes when a low mass system of particles moves parallel to p and with the same velocity as p .

As a special case of the last example we consider the graph in Figure 10. Here

$$V(P, Q, \omega) \alpha \ 2\pi \int_0^\infty dy \int_y^\infty dx \ \ln^2 \left[\frac{P^2 (1 + \frac{2x}{\omega Q}) + \omega Q y}{-\Lambda^2} \right] \ \delta(Q^2(1-\omega) - 2Qx + \omega Q y)$$

for small P^2 . The x integration can be done explicitly to give

$$V(P, Q, \omega) \alpha \ \frac{\pi}{Q} \int_0^{\frac{Q(1-\omega)}{2-\omega}} dy \ \ln^2 \left[\frac{P^2 (\frac{1}{\omega}) + \omega Q y}{-\Lambda^2} \right]$$

Let $y = P^2/Q\omega^2 \ z$. Then

$$V \ \alpha \ \frac{\pi P^2}{\omega^2 Q^2} \int_0^{\frac{Q^2\omega^2}{P^2} \frac{(1-\omega)}{(2-\omega)}} dz \ \ln^2 \left[\frac{P^2(1+z)}{-\omega \Lambda^2} \right] .$$

The singularity in P^2 occurs, roughly, when z is finite so that V has the form

$$V \ \alpha \ C_1 P^2/Q^2\omega^2 \ \ln^2 \left[P^2/-\Lambda^2\omega \right] + C_2 P^2/Q^2\omega^2 \ \ln \ P^2/-\Lambda^2\omega + C_3 P^2/Q^2\omega^2$$

for the dominant P dependence.

Appendix B

In this appendix the n-particle phase space, near the threshold of a zero mass theory, will be given. Let

$$d\Phi_n = \frac{d^3k_1}{E_1}\frac{d^3k_2}{E_2}\cdots\frac{d^3k_n}{E_n}\,\delta^4(k_1+k_2+\ldots+k_n-p)$$

where $(k_i)_\mu\,(k_i)^\mu = k^2 = 0$. Although $d\Phi_n$ is an invariant function of $P^2 = p^2$ we shall find it convenient to choose a coordinate frame where $P_o >> P$. Thus we write

$$P_\mu \approx (p + \frac{P^2}{2p}\,,\ 0, 0, p)$$

and

$$d\Phi_n = \frac{dk_{1z}}{E_1}\frac{dk_{2z}}{E_2}\cdots\frac{dk_{nz}}{E_n}\,d^2\underline{k}_1\,d^2\underline{k}_2\cdots d^2\underline{k}_n$$

$$\delta(E_1+E_2+\ldots+E_n - p - \frac{P^2}{2p})\,\delta(k_{1z}+k_{2z}+\ldots+k_{nz}-p)$$

$$\delta(\underline{k}_1+\underline{k}_2+\ldots+\underline{k}_n)$$

The δ functions require that $k_{iz} >> |\underline{k}_i|$ unless $|\underline{k}_i|$ and k_{iz} are both of the order $P^2/2p$. For the moment suppose that $k_{iz} >> |\underline{k}_i|$.

Then

$$d\Phi_n = \frac{dk_{1z}}{k_{1z}} \frac{dk_{2z}}{k_{2z}} \cdots \frac{dk_{nz}}{k_{nz}} d^2\underline{k}_1 d^2\underline{k}_2 \cdots d^2\underline{k}_n \quad \delta(\underline{k}_1 + \underline{k}_2 + \cdots + \underline{k}_n)$$

$$\delta\left(\frac{\underline{k}_1^2}{2k_{1z}} + \frac{\underline{k}_2^2}{2k_{2z}} + \cdots + \frac{\underline{k}_n^2}{2k_{nz}} - \frac{P^2}{2p}\right) \delta(k_{1z} + k_{2z} + \cdots + k_{nz} - p) .$$

Set $k_{iz} = \alpha_i p$ then

$$d\Phi_n = 2\frac{d\alpha_1}{\alpha_1} \frac{d\alpha_2}{\alpha_2} \cdots \frac{d\alpha_n}{\alpha_n} \quad d^2\underline{k}_1\, d^2\underline{k}_2 \cdots d^2\underline{k}_n \quad \delta(\underline{k}_1 + \underline{k}_2 + \cdots + \underline{k}_n)$$

$$\delta\left(\frac{\underline{k}_1^2}{\alpha_1} + \frac{\underline{k}_2^2}{\alpha_2} + \cdots + \frac{\underline{k}_n^2}{\alpha_n} - P^2\right) \delta(1 - \alpha_1 - \alpha_2 - \cdots - \alpha_n)$$

where $0 \leqslant \alpha_i \leqslant 1$. But

$$\int d^2\underline{k}_1\, d^2\underline{k}_2 \cdots d^2\underline{k}_n \quad \delta(\underline{k}_1 + \underline{k}_2 + \cdots + \underline{k}_n) \quad \delta\left(\frac{\underline{k}_1^2}{\alpha_1} + \frac{\underline{k}_2^2}{\alpha_2} + \cdots + \frac{\underline{k}_n^2}{\alpha_n} - P^2\right)$$

$$= \frac{\Pi^{n-1}}{(n-2)!} \alpha_1 \alpha_2 \cdots \alpha_n \, (P^2)^{n-2}$$

and

$$\int_0^1 d\alpha_1\, d\alpha_2 \cdots d\alpha_n \quad \delta(1 - \alpha_1 - \alpha_2 - \cdots - \alpha_n) = \frac{1}{(n-1)!} \quad ,$$

so

$$d\,\Phi_n = \frac{2\Pi^{n-1}}{(n-1)!\,(n-2)!}\,(P^2)^{n-2}$$

If $|\underline{k}_i|$ and k_{iz}, for a particular i, are both of the order $P^2/2p$ then an additional factor of $P^2/2p$ occurs so that this region is not important and the above result for $d\,\Phi_n$ is exact.

Appendix C

In Part B of Section III it was argued that

$$\int_0^1 \omega\, d\omega\; V(P, Q, \omega)$$

becomes independent of P for large Q, and that the correction terms are like P^2/Q^2 up to logarithms. In order to show that

$$\int_{2P/Q}^{1+\frac{P^2-4m^2}{Q^2}} d\omega\; \omega^\sigma \left[1 + \sqrt{1 - 4P^2/\omega^2 Q^2}\right]^\sigma \; V(P, Q, \omega)$$

has this same property it is necessary to determine how $V(P, Q, \omega)$ behaves both when ω is finite and when $\omega \sim 2P/Q$. The independence on P when $\sigma = 1$

indicates that for the region of finite ω

$$V(P, Q, \omega) - V(0, Q, \omega) \sim P^2/Q^2$$

and examples previously worked out in Appendix A suggest that

$$V(P, Q, \omega) - V(0, Q, \omega) \sim P^2/Q^2\omega^2 \tag{C.1}$$

when ω is small. We shall now give an argument to indicate that Eq. (C.1) should hold in general.

The argument given in Section III involved the product of factors coming from the phase space of a zero mass theory times the threshold singularities present. Refer now to Figure 5. Choose a coordinate system

$$q = (Q, 0, 0, 0)$$

$$p = \left(\frac{\omega Q}{2} + \frac{P^2}{\omega Q}, 0, 0, \frac{\omega Q}{2}\right)$$

when $\omega >> P/Q$. Now, for fixed ω and large Q a threshold involving p is reached when

$$(p + k_1 + k_2 + \ldots + k_j)^2 \sim P^2 .$$

As we have seen in Appendix A this means that

$$k_{iz} \sim \frac{\omega Q}{2} , \quad |\underline{k}_i| \sim P \qquad i = 1, 2, \ldots j$$

so that the j particles align with p. Now as ω becomes small less alignment will occur until there is no alignment at all when $\omega \sim P/Q$. This is illustrated in the examples of Appendix A. More quantitatively call

$$K_i = k_1 + k_2 + \ldots + k_i$$

then

$$(p + K_i)^2 = P^2 + 2p \cdot K_i + K_i^2$$

$$= P^2 + \omega Q (K_{io} - K_{iz}) + \frac{2P^2}{\omega Q} K_{io} + K_i^2$$

and

$$d\psi = d^4K_i \frac{d^3k_1}{E_1} \frac{d^3k_2}{E_2} \cdots \frac{d^3k_i}{E_i} \delta^4(K_i - k_1 - k_2 - \ldots - k_i)$$

$$\Theta(a^2 - (p + K_i)^2) \; \Theta(Q - K_{io})$$

attains a maximum when a^2 is chosen to be P^2/ω. (a^2 cannot be chosen significantly greater than P^2/ω or $d\psi$ loses all P dependence.)

$$d\psi \sim (a^2)^{i-2} \int d4K_i \, \Theta(Q - K_{io}) \, \Theta \boxed{a^2 - \omega Q \, K_{io} - K_{iz}) - \frac{2P^2}{\omega Q} K_{io} - K_i^2}$$

$$\sim (a^2)^{i-2} \, (P^2/\omega) \, (P^2/\omega^2) \sim (a^2)^{i-1} \, (P^2/\omega^2) \, .$$

The threshold singularity gives a factor $(a^2)^{-i+1}$ so altogether

$$V(P, Q, \omega) - V(0, Q, \omega) \sim P^2/\omega^2 Q^2 \qquad \text{(C.2)}$$

for large Q and small ω. The factor of $1/Q^2$ comes into (C.2) as the only possible quantity to set the scale for the mass singularity. Thus

$$\int_{2P/Q}^{1+\frac{P^2-4m^2}{Q^2}} d\omega\, \omega^\sigma \left[1+\sqrt{1-4P^2/Q^2\omega^2}\right]^\sigma V(P, Q, \omega) = v_\sigma (P, Q)$$

will becomes independent of P, for large Q, so long as $\mathrm{Re}\,\sigma > -1$.

ACKNOWLEDGEMENT

Part of this work was carried out while the author was a visitor at École Polytechnique and at Brookhaven National Laboratory. I wish to thank Tran Truong and Ronald Peierls for their hospitality.

REFERENCES

1. K. Wilson, Cornell Report No. LNS -64-15, (1964); Phys. Rev. 179, 1499 (1969).
2. M. Gell-Mann and F. Low, Phys. Rev. 95, 1300 (1954).
3. N. Bogoliubov and D. Shirkov, Introduction to the Theory of Quantized Fields (Interscience, New York, 1959).
4. N. Christ, B. Hasslacher and A. Mueller, Phys. Rev. D6, 3543 (1972).
5. R. Brandt and G. Preparata, Nucl. Phys. B27, 541 (1971).
6. Y. Frishman, Phys. Rev. Letters 25, 966 (1970).
7. V. Gribov and L. Lipatov, Soviet Journal of Nucl. Physics 15, 675 (1972).
8. R. Gatto and G. Preparata, Nucl. Phys. B47, 313 (1972).
9. J. Ellis and Y. Frishman, Phys. Rev. Letters 31, 135 (1973).
10. R. Brandt and W.-C. Ng, to be published.
11. The relationship between asymptotic behaviors in off shell variables and the properties of an underlying zero mass field theory are discussed extensively by K. Symanzik, Commun. Math. Phys. 23, 49 (1971) and DESY Report 73/6 March 1973.
12. C. Callan, Phys. Rev. D2, 1541 (1970).
13. K. Symanzik, Commun. Math. Phys. 18, 227 (1970).
14. W. Zimmerman in Lectures on Elementary Particles and Quantum Field Theory, edited by S. Deser et al. (MIT Press, Cambridge, Mass., 1971), Vol. I, p. 397.
15. S. Weinberg, Phys. Rev. 118, 838 (1960).
16. T. Kinoshita and A. Sirlin, Phys. Rev. 113, 1652 (1959).
17. T. Kinoshita, Journal of Math. Phys. 3, 650 (1962).

18. T. D. Lee and M. Nauenberg, Phys. Rev. 133, B1549 (1964).

19. P. Fishbane and J. Sullivan, Phys. Rev. D6, 3568 (1972).

20. I wish to thank G. Preparata for suggesting that such sum rules could prove useful.

21. A. Polyakov, JETP 59, 296 (1970).

22. R. Gatto, P. Menotti, and I. Vendramin, Nuovo Cimento Letters 4, 79 (1972).

23. A. Suri, Phys. Rev. D4, 570 (1971).

24. P. Landshoff and J. Polkinghorne, Phys. Rev. D6, 3708 (1972).

25. H. Dahmen and F. Steiner, Phys. Letters 43B, 217 (1973).

26. P. Mitter, Phys. Rev. D7, 2927 (1973).

27. S. Nussinov and J. Rosner, Journal of Math. Phys. 7, 1670 (1966).

28. Weinberg's theorem could presumably be applied to obtain this result also.

29. M. Naimark, Linear Representations of the Lorentz Group (Macmillan, New York, 1964) p. 186.

30. Amplitudes related to the discussion of this section have been considered by M. Baker and I. Muzinich, Phys. Rev. 132, 2291 (1963) in a spirit similar to this paper and also in a related context by S.-J. Chang and P. Fishbane, Phys. Rev. D2, 1084 (1970); M. Kugler and S. Nussinov, Nucl. Phys. B28, 97 (1971); R. Gatto and P. Menotti, Nuovo Cimento 2A, 881 (1971).

FIGURE CAPTIONS

Figure 1. An illustration of Eq. (5).

Figure 2. An illustration of Eq. (25).

Figure 3. A particular discontinuity of a self-energy graph.

Figure 4. A particular discontinuity of a self-energy graph where discontinuities over internal propagators are added.

Figure 5. An integrated inclusive cross section as related to a discontinuity of a self-energy amplitude.

Figure 6. A potential used in the Bethe-Salpeter equation.

Figure 7. A potential used in the Bethe-Salpeter equation.

Figure 8. A two particle reducible amplitude.

Figure 9. A two particle irreducible amplitude.

Figure 10. A two particle irreducible amplitude.

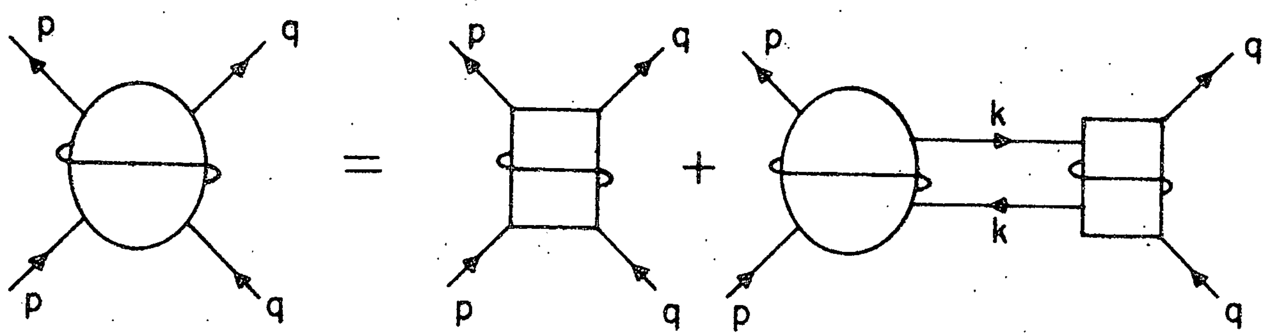

FIG. 1

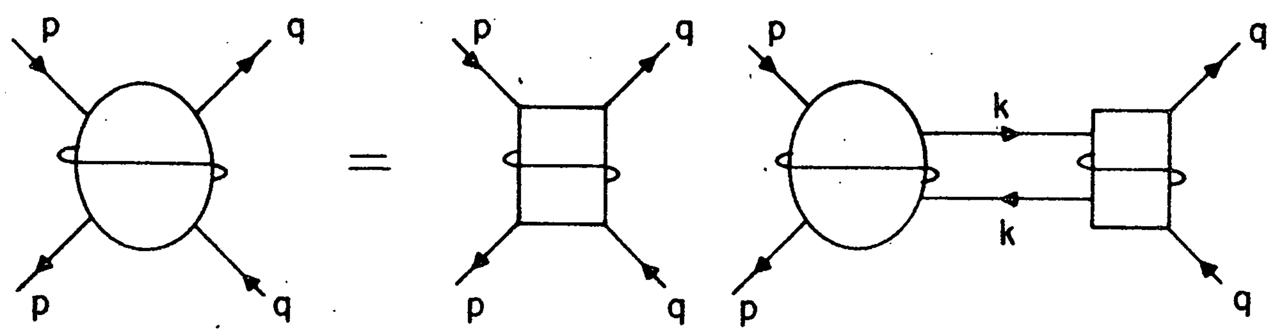

FIG. 2

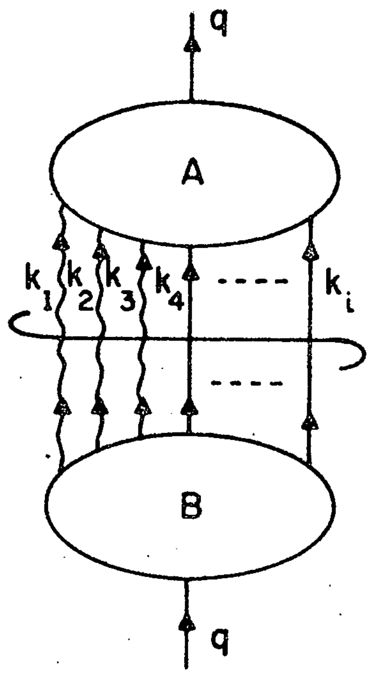

FIG. 3

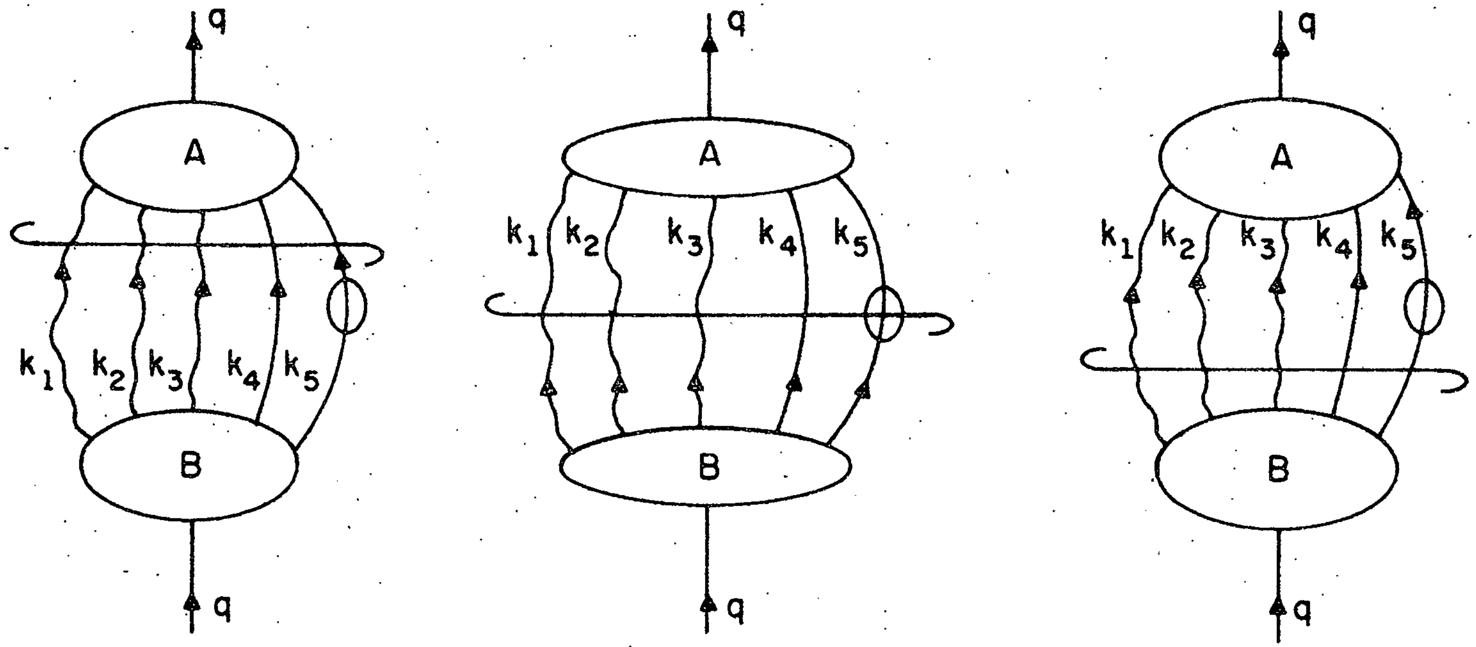

FIG. 4

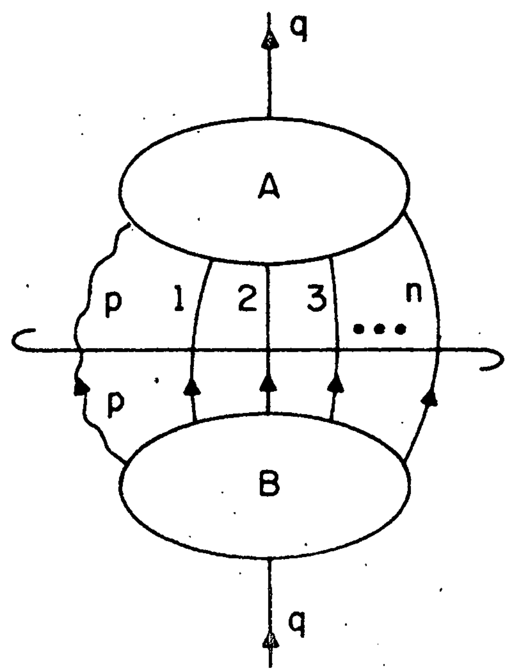

FIG. 5

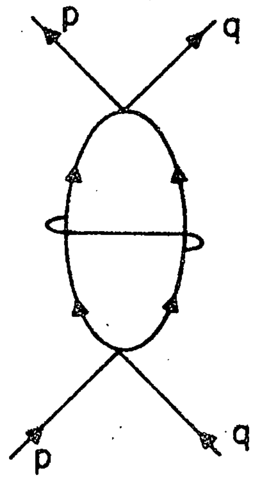

FIG. 6

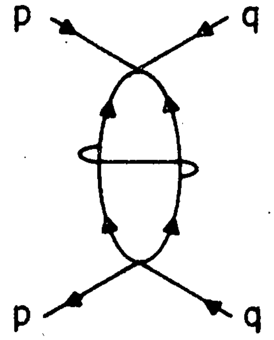

FIG. 7

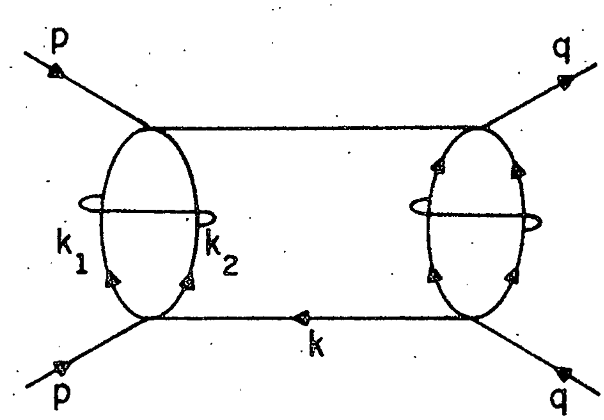

FIG. 8

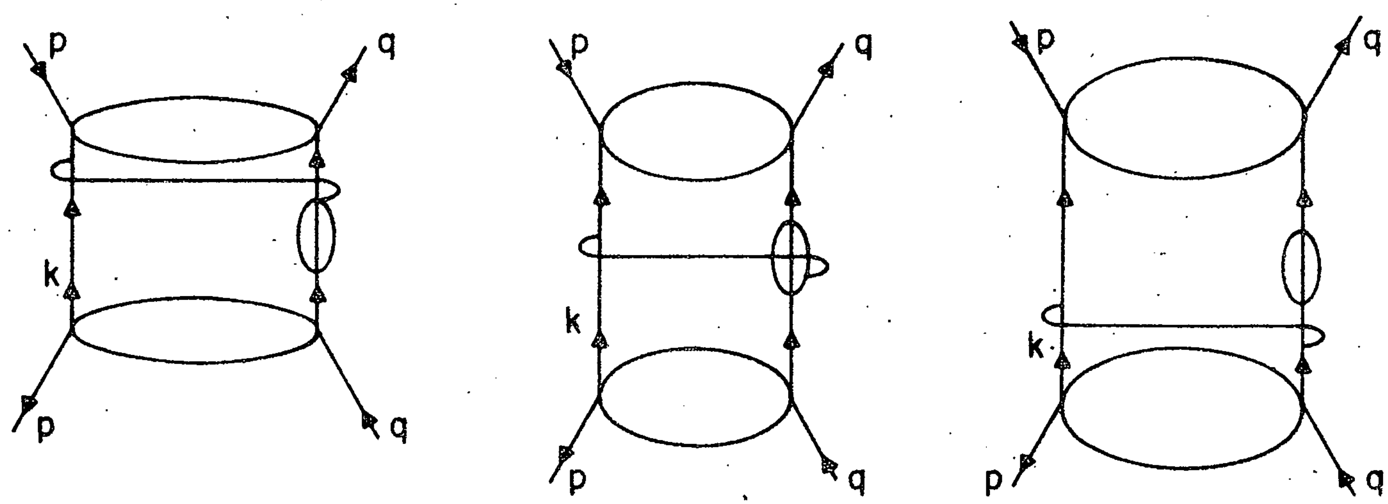

FIG. 9

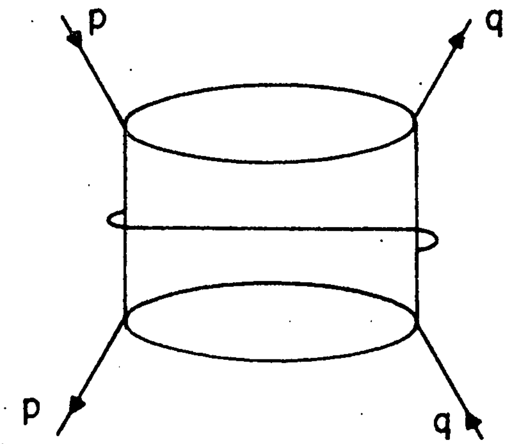

FIG. 10