C00-1764-212

Variational Method for Solenoidal Excitation of Yang-Mills field*

By

H. Suura

Department of Physics and Astronomy
University of Minnesota
Minneapolis, Minnesota 55455

NOTICE

This report was prepared as an account of work sponsored by the United States Government. Neither the United States nor the United States Atomic Energy Commission, nor any of their employees, nor any of their contractors, subcontractors, or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness or usefulness of any information, apparatus, product or process disclosed, or represents that its use would not infringe privately owned rights.

July 16, 1974

* Work supported in part by the U. S. Atomic Energy Commission Contract Number AT(11-1) 1764.

MASTER

DISTRIBUTION OF THIS DOCUMENT IS UNLIMITED

10

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency Thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

DISCLAIMER

Portions of this document may be illegible in electronic image products. Images are produced from the best available original document.

Abstract

A vortex-like excitation mode of the massless Yang-Mills field coupled to fixed classical I-spin sources is investigated by the variational method. The lattice point quantization is introduced as an intermediate step to evaluate the variational integral of the field Hamiltonian. The method leads to the Ginzburg-Landau equations without the mass term. Some aspects of the model, especially the shielding of the Coulomb interaction by induced I-spin density, are discussed from the point of possible applications to the model of hadrons.

I. Introduction.

There have been suggestions^{1,2,3} that the non-abelian vector gauge field may be the agent for inter-quark forces. The idea is partly motivated by the so-called asymptotic freedom⁴ which obtains in the theory and also by the possible infrared slavery^{1,3} which may prevent the escape of free quarks. Kogut and Susskind² stressed in this connection the importance of the Schwinger phenomenon⁵ where the sea of quark-anti-quark pairs kills the free propagation of quarks. We take the gauge field theory from a somewhat different point of view. From the success of the quark model in the hadron spectroscopy we may believe in validity of the concept of an adiabatic potential between slowly moving quarks without taking into account quark pairs. At this point we may not exclude as yet any re-normalizable field, abelian gauge field included, as a possible candidate for the underlying field. We take the non-abelian gauge field here because there is a possibility of a string-like solution, a string attached to the quarks at the ends. To see this, we may recall the work of Nielsen and Olesen⁶ who showed that the Maxwell field coupled to the Higgs meson has a static vortex solution. In fact, the Higgs Lagrangian of such a system reduces in the static limit to the free energy in the Ginzburg-Landau theory of type II superconductors. The difficulty of this theory as a model of hadrons might be that there are no obvious ways to relate the source of the magnetic field to quarks, unless possibly we introduce magnetic monopoles. In non-abelian gauge fields, on the other hand, the scalar potential couples to the vector potential because of the non-linearity of the Lagrangian. Thus, there is a possibility in such a theory that a static

source (charge) can generate a string-like solution.

In the following we consider the quantized Yang-Mills⁷ field coupled to fixed I-spin sources (quarks) lying on the z-axis. We will not consider the precession of the quark I-spins in this preliminary work, so that the quark I-spins are classical vectors in the iso-space. We will use the Coulomb gauge throughout. This will enable us to use the Ritz variational principle because the metric of the Hilbert space is positive definite in this gauge. In section II, the gauge field is quantized, and only the vortex-like mode, or solenoidal mode, (the vector potential tangential around z-axis) is considered. This corresponds, as shown in the Appendix, to expanding the field into cylindrical waves and retaining one particular polarization state of the lowest (cylindrically symmetric) angular modes. To neglect the other modes may not be justified even in the sense of the variational method, since the solenoidal mode may couple to the vacuum fluctuation of the other modes. Our excuse at this point is merely to say that this is a preliminary attempt to obtain some idea about the characteristic feature of the solutions of this enormously complicated system. Once we neglect the other modes the Hamiltonian reduces to a scalar field energy corresponding to this excitation mode plus the Coulomb interaction between I-spin densities, which are the sum of the external source and the induced field I-spin density.

In section III, the variational method is applied to the reduced Hamiltonian. It is noted that without the gradient term in the field energy, each spatial point is equivalent to a three dimensional (equal to the I-spin degrees of freedom) harmonic oscillator. If all the

external I-spin sources are aligned in one direction, then the system reduces to a set of two dimensional harmonic oscillators at each local point. Thus, we introduce an infinite set of lattice points, at each of which we have a two-dimensional oscillator.

We take as the trial wave function at each lattice point a linear combination of the three lowest states of the two-dimensional oscillator. The variational equation we obtain can be written in the form closely resembling the Ginzburg-Landau equations for type II super-conductors.⁸ The essential difference is, however, the lack of the mass term (the term quadratic in the order parameter in the free energy) in our case. The lack of this term prevents the existence of the normal exponentially decaying vortex solution, although we can still have a string-line solution. (If we had given an imaginary mass to the gauge field, we would have obtained the Ginzburg-Landau equation).

We have not tried to obtain a rigorous solution to our variational equations, partly because it is extremely difficult and partly because we do not believe that our model is completely right. Instead we will discuss in section IV some peculiar feature of our model. It is the screening of the Coulomb interaction by the induced field charge. In fact, one can choose, in the sense of the variational method, the induced field charge to cancel completely the external charges. As a result, the Coulomb self-energy is reduced from g^2/a ($a \rightarrow 0$), where g is a large coupling constant, to $O(1)/a$. This would be a nice feature as a possible mechanism to reduce the physical mass of the quarks inside the hadrons. However, the mechanism seems to work for colorless (no total charge) as well as colored systems. This is an undesirable feature to be cured.

II. Solenoidal mode of the Yang-Mills field.

We consider the Yang-Mills field $A_\mu^a(x)$ ($a = 1, 2, 3$) generated by the external I-spin source

$$t_E^a(x) = \sum_A T_A^a \delta^3(x - x_A), \quad (1)$$

where T_A^a is the I-spin of the quark fixed at point x_A on the z-axis. In this paper we will assume that T_A^a are classical fixed vectors in the iso-space. The Lagrangian is given by

$$L = \int \left[-\frac{1}{4} F_{\mu\nu}^a(x) F^{a\mu\nu}(x) - g t_E^a(x) A_0^a(x) \right] d^3x, \quad (2)$$

where

$$F_{\mu\nu}^a(x) = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g \epsilon_{abc} A_\mu^b(x) A_\nu^c(x).$$

The equation of motion is

$$\begin{aligned} \partial^\mu F_{\mu\nu}^a(x) &= \bar{j}_\nu^a(x) \\ &= g \epsilon_{abc} A^{b\mu}(x) F_{\mu\nu}^c(x) + g \delta_{\nu 0} t_E^a(x). \end{aligned} \quad (3)$$

Since the quark I-spins T_A^a are fixed vectors, we have $\dot{t}_E^a = 0$, which requires, in order to obtain the conservation of current $\partial^\nu \bar{j}_\nu^a = 0$,

a constraint

$$\varepsilon_{abc} A_o^b(x) t_E^c(x) = 0. \quad (4)$$

If the quark I-spins are aligned in one direction, say the third axis in the iso-space

$$T_A^1 = T_A^2 = 0, \quad (5)$$

then the above constraint suggests a solution such that

$$A_o^1(x) = A_o^2(x) = 0. \quad (5')$$

We will later make this specification.

We choose the Coulomb gauge

$$\vec{\nabla} \cdot \vec{A}^a(x) = 0, \quad (6)$$

where

$$\vec{A}^a(x) = (A^{a1}, A^{a2}, A^{a3}).$$

The canonical momentum $\vec{\Pi}^a(x)$ conjugate to $\vec{A}^a$ must also be transversal,

$$\begin{aligned}\pi_i^a(x) &= \frac{\delta L}{\delta \dot{A}^{ai}(x)} = - (F_{0i}^a(x))^{tr} \\ &= (\dot{A}^{ai} + \nabla_i A_0^a - g \varepsilon_{abc} A_0^b A^{ci})^{tr}\end{aligned}$$

Here the superscript tr (or ℓ) means taking the transversal (or longitudinal) part. Thus

$$\vec{\pi}^a(x) = \vec{\dot{A}}^a(x) - g \varepsilon_{abc} (A_0^b(x) \vec{A}^c(x))^{tr} \quad (7)$$

A_0^a is not an independent variable and from (3) and (7) satisfies

$$\begin{aligned}-\nabla^2 A_0^a(x) &= g t_E^a(x) + g \varepsilon_{abc} [\vec{A}^b(x) \cdot \vec{\pi}^c(x) \\ &+ 2 \vec{\nabla} \cdot (A_0^b(x) \vec{A}^c(x))^\ell + g \varepsilon_{cde} \vec{A}^b(x) \cdot (A_0^d \vec{A}^e(x))^\ell].\end{aligned} \quad (8)$$

As the Hamiltonian we obtain

$$\begin{aligned}H &= \int \left\{ \frac{1}{2} \vec{\pi}^a(x)^2 + \frac{1}{2} (\vec{\nabla} \times \vec{A}^a)^2 \right. \\ &+ \frac{1}{2} g A_0^a(x) [t_E^a(x) + \varepsilon_{abc} \vec{A}^b(x) \cdot \vec{\pi}^c(x)] \\ &+ g \varepsilon_{abc} [(\vec{A}^b \cdot \vec{\nabla}) \vec{A}^a] \cdot \vec{A}^c \\ &\left. + \frac{g^2}{4} \varepsilon_{abc} \varepsilon_{cde} \vec{A}^a \cdot \vec{A}^d \vec{A}^b \cdot \vec{A}^e \right\} d^3x, \quad (9)\end{aligned}$$

where $A_0^a(x)$ should be determined from eq. (8).

So far, it is a standard procedure.⁹ Now we restrict ourselves to the 'solenoidal' mode,

$$\begin{aligned}\vec{A}^a(x) &= (\vec{n}_z \times \vec{\rho}/\rho) \alpha^a(z, \rho) \\ \vec{\pi}^a(x) &= (\vec{n}_z \times \vec{\rho}/\rho) \pi^a(z, \rho)\end{aligned}\tag{10}$$

where we have introduced a cylindrical coordinate (z, ρ, φ) . $\vec{n}_z$ is the unit vector in z-direction. In the appendix we describe the expansion of the transverse field into cylindrical waves and we will find that (10) corresponds to one particular polarization state with vanishing azimuthal quantum number. As mentioned in section I we are neglecting the coupling of the solenoidal mode to the vacuum fluctuation of the other modes, which may not be valid. According to the appendix, we can quantize the solenoidal mode by

$$[\alpha^a(z, \rho), \pi^b(z', \rho')] = \frac{i}{2\pi\rho} \delta_{ab} \delta(\rho - \rho') \delta(z - z').\tag{11}$$

An enormous simplification results for the solenoidal mode. For instance, we have

$$(A_0^a(x) \quad \vec{A}^b(x))^\dagger = 0,$$

as long as $A_0^a(x) = A_0^a(z, \rho)$. Hence we have from (8)

$$-\nabla^2 A_0^a(x) = g [t_E^a(x) + t_F^a(x)], \quad (12)$$

where

$$t_F^a(x) = \varepsilon_{abc} \alpha^b(x) \pi^c(x). \quad (13)$$

From (11), t_F^a satisfies a commutation relation

$$[t_F^a(z, p), t_F^b(z', p')] = i \varepsilon_{abc} t_F^c(z, p) \times \\ \times (2\pi p)^{-1} \delta(p-p') \delta(z-z'). \quad (14)$$

Thus, we may call t_F^a as the field I-spin density due to this mode.

The total field I-spin

$$T_F^a = \int t_F^a(x) d^3x$$

satisfies

$$[T_F^a, T_F^b] = i \varepsilon_{abc} T_F^c$$

It can also be seen very easily that the last two terms in H, eq. (9) vanish for (10). From (9), (12), (13), we have

$$A_0^a(x) = \frac{g}{4\pi} \int \frac{t^a(x')}{|x - x'|} d^3x', \quad (15)$$

and

$$\begin{aligned} H = & \frac{1}{2} \int [\pi^a(x)^2 + (\vec{\nabla} \alpha^a(x))^2] d^3x \\ & + \frac{g^2}{8\pi} \iint \frac{t^a(x) t^a(x')}{|x - x'|} d^3x d^3x', \end{aligned} \quad (16)$$

where

$$t^a(x) = t_E^a(x) + t_F^a(x) \quad (17)$$

is the total I-spin density from the source and the field. The first two terms of the Hamiltonian are the kinetic and magnetic energy of the field, respectively. The last is the Coulomb energy due to the external and the induced charges. It includes the Coulomb self-energy of the source.

III. Variational solution of the Yang-Mills field.

For quark I-spins aligned in one direction, case (5), we may assume (5'), which indicates from (12) that $t_F^1 = t_F^2 = 0$. From (13), we can then take $\alpha^3 = \pi^3 = 0$. Thus, our Hamiltonian further reduces to

$$\begin{aligned}
 H = & \frac{1}{2} \int [\pi^1(x)^2 + \pi^2(x)^2] d^3x \\
 & + \frac{1}{2} \int [(\vec{\nabla} \alpha^1)^2 + (\vec{\nabla} \alpha^2)^2] d^3x \\
 & + \frac{g^2}{8\pi} \iint \frac{t^3(x) t^3(x')}{|x - x'|} d^3x d^3x'.
 \end{aligned}
 \tag{18}$$

Notice that if the second term were the mass term without the derivative of the fields the Hamiltonian could be diagonalized immediately. Then the first two terms of H would correspond to continuously distributed two-dimensional harmonic oscillators, and $t^3(x)$ is the angular momentum of the oscillator at x . The eigenstate would be given by a simple functional of $\alpha^1(x)$ and $\alpha^2(x)$ in the representation where these 'coordinates' are diagonal. To evaluate derivative term in this state, we temporarily divide the space into cells at densely populated lattice points and deal with an enumerable set of harmonic oscillators. Let a lattice point labelled by an index i . The coordinate and the velocity of the oscillator at x_i are

$$\begin{aligned}
q_i^a &= \alpha^a(x_i) \\
\dot{q}_i^a &= \dot{\alpha}^a(x_i) = \pi^a(x_i), \quad a=1,2. \quad (19)
\end{aligned}$$

The canonical momentum is to be defined by

$$p_i^a = \Delta^3 \dot{q}_i^a \quad (20)$$

where Δ is the distance between neighboring lattice points. The normal commutation relation between q_i^a and p_i^a as defined by (20) is consistent with the commutation relation (11). The derivative of the field is now replaced by a difference $(q_{\bar{i}+\Delta\bar{i}} - q_{\bar{i}})/\Delta$ where $\bar{i}+\Delta\bar{i}$ represents a site neighboring to i . We also have from (19) and (20)

$$\begin{aligned}
\Delta^3 t_F^3(\bar{i}) &= \Delta^3 [\alpha^1(x_i) \pi^2(x_i) - \alpha^2(x_i) \pi^1(x_i)] \\
&= q_i^1 p_i^2 - q_i^2 p_i^1 = L_3(i) \quad (21)
\end{aligned}$$

which is the angular momentum of the i -th oscillator. In this way, we convert H in (18) to that for an infinite number of interacting two-dimensional oscillators.

$$H = H_0 + H_1 + H_2,$$

with

$$H_0 = \sum_i H_0(i) = \sum_i \left\{ \frac{1}{2\Delta^3} [(p_i^1)^2 + (p_i^2)^2] + \lambda \Delta [(q_i^1)^2 + (q_i^2)^2] \right\}, \quad (22)$$

$$H_1 = - \sum_{i < j = i + \Delta i} \Delta [q_i^1 q_j^1 + q_i^2 q_j^2] \quad (23)$$

$$H_2 = \frac{g^2}{8\pi} \sum_{i,j} \frac{[\Delta^3 t_E^3(i) + L_3(i)][\Delta^3 t_E^3(j) + L_3(j)]}{\xi_{ij}} \quad (24)$$

where ξ_{ij} is the distance between point i and j . In H_0 , r represents the number of independent derivatives. Thus, if $\alpha^a(x)$ is cylindrically symmetric, as is in our case, $r = 2$. The eigenvalues and eigenstates of $H_0(i)$ are standard.

$$H_0(i) \mathcal{U}_{n,m} = \mathcal{E}_{n,m} \mathcal{U}_{n,m}$$

with

$$\mathcal{E}_{n,m} = \omega (|m| + 2n + 1)$$

$$n = 0, 1, 2, \dots$$

$$m = 0, \pm 1, \pm 2, \dots \quad (25)$$

and

$$\begin{aligned}
 \mathcal{U}_{n,m}(q, \phi) = & \sqrt{\frac{n! \beta^{|m|+1}}{\pi [(n+|m|)!]^3}} e^{im\phi} x \\
 & \times q^{|m|} L_{n+|m|}^{|m|}(\beta q^2) e^{-\frac{1}{2}\beta q^2}
 \end{aligned} \tag{26}$$

Here

$$\omega = \sqrt{2\kappa}/\Delta, \quad \beta = \sqrt{2\kappa} \Delta^2$$

and

$$q^1 = q \cos \phi, \quad q^2 = q \sin \phi. \tag{27}$$

$L_p^q(x)$ is the Laguerre polynomial. We have suppressed the index i in the above.

We construct our trial state out of the lowest three states $m = 0, \pm 1$ with $n = 0$ for each point i . Hence

$$\Psi = \prod_i \Psi_i$$

with

$$\Psi_i = \sum_{m=0,\pm 1} f_m(i) |m\rangle \tag{28}$$

$|m\rangle$ stands for $\mathcal{U}_{0,m}$ given by (26). The normalization condition is

$$f_1^2 + f_{-1}^2 + f_0^2 = 1. \tag{29}$$

We have taken f_m to be real as a variational choice. It can be shown that the solution given in this section can be reproduced exactly from a more general treatment of H in (16) in terms of three-dimensional oscillators. Instead of $f_0, f_{\pm 1}$, two of which are independent because of (29), we may introduce two variational parameters by

$$\langle L_3(i) \rangle \equiv l_i = f_+(i)^2 - f_-(i)^2, \quad (30)$$

and

$$\langle \sqrt{L_3(i)^2} \rangle \equiv M_i = f_+(i)^2 + f_-(i)^2. \quad (31)$$

From (25) and (28) we have

$$\langle H_0(i) \rangle = \omega (M_i + 1). \quad (32)$$

From (23),

$$\langle H_1 \rangle = - \sum_{i < j = i + \Delta i} \frac{\Delta}{2} \langle | q_i q_j e^{i(\phi_i - \phi_j)} + h.c. | \rangle. \quad (33)$$

The matrix element of $q_i e^{i\phi_i}$ for (28) is

$$\begin{aligned} \langle \Psi_i | q_i e^{i\phi_i} | \Psi_i \rangle &= \sum_m f_{m+1}(i) f_m(i) \cdot \\ &\cdot \langle m+1 | q_i | m \rangle \end{aligned} \quad (34)$$

From (26) we obtain

$$\langle m+1 | q | m \rangle = -\frac{1}{\sqrt{2\hbar} \Delta^2} \begin{cases} \sqrt{m+1} & m \geq 0 \\ \sqrt{|m|} & m < 0 \end{cases} \quad (35)$$

From (33), (34) and (35)

$$\begin{aligned} \langle H_1 \rangle = & -\frac{1}{\sqrt{2\hbar} \Delta} \sum_{\bar{i} < \bar{j} = \bar{i} + \Delta \bar{i}} f_0(\bar{i}) f_0(\bar{j}) [f_1(\bar{i}) + f_{-1}(\bar{i})] \times \\ & \times [f_1(\bar{j}) + f_{-1}(\bar{j})]. \end{aligned} \quad (36)$$

We can express f_m in terms of l_i , M_i by (29), (30), (31). We will also assume that

$$l_i / M_i \sim O(\Delta) \quad (37)$$

which we will find to be a consistent assumption. Expanding in l_i / M_i we find

$$\begin{aligned} \langle H_1 \rangle = & \frac{1}{\sqrt{2\hbar} \Delta} \left\{ -\sum_{\bar{i} : \bar{j} = \bar{i} + \Delta \bar{i}} 2 \sqrt{M_i} \sqrt{M_j} \right. \\ & \left. + \sum_{\bar{i}} \hbar \left[\frac{1}{2} \frac{l_i^2}{M_i} + 2 M_i^2 \right] \right\}. \end{aligned} \quad (38)$$

We have neglected, beside the higher order terms, the difference between neighboring l_i 's and M_i 's except in the first term. These are negligible under the following assumptions we make.

$$\begin{aligned}\sqrt{M_i} &= \Delta U(x_i) \\ l_i &= \Delta^3 \mu(x_i)\end{aligned}\quad (39)$$

where U and μ are supposed to be finite in the limit of $\Delta \rightarrow 0$.

(39) is compatible with (37). Adding (32), (38) and (24) and using (30) and (39), we obtain (for $\ell = 2$)

$$\begin{aligned}\langle H \rangle &= \int \left[\frac{1}{2} (\vec{\nabla} U)^2 + 2 U^4 + \frac{1}{2} \frac{\mu^2}{U^2} \right] d^3x \\ &+ \frac{g^2}{8\pi} \iint \frac{[t_E^3(x) + \mu(x)][t_E^3(x') + \mu(x')]}{|x - x'|} d^3x d^3x'\end{aligned}\quad (40)$$

Notice that H_0 and the first term of H , combine to give the derivative term.

Now we take the variation of $\langle H \rangle$ with respect to U and μ , subject to the condition that the total induced charge be zero,

$$\int \mu(x) d^3x = 0. \quad (41)$$

The normalization condition is automatically satisfied since we eliminated $f_0(\bar{r})$ using (29). The variational equations are

$$\nabla^2 U(x) - 8 U(x)^3 + \mu(x)^2 U(x)^{-3} = 0, \quad (42)$$

$$\mu(x) U(x)^{-2} + \frac{g^2}{4\pi} \int \frac{t_E^3(x') + \mu(x')}{|x - x'|} d^3x' = \lambda, \quad (43)$$

where λ is a Lagrange multiplier for the constraint (41). A form similar to Ginzburg-Landau equations can be obtained by introducing

$$V(x) = \mu(x) U(x)^{-2}. \quad (44)$$

We obtain a set of coupled non-linear equations

$$\nabla^2 V(x) - g^2 U(x)^2 V(x) = g^2 t_E^3(x), \quad (45)$$

and

$$\nabla^2 U(x) + V(x)^2 U(x) - 8 U(x)^3 = 0. \quad (46)$$

If we added a term $b^2 U(x)$ to the left-hand side of eq. (46), then the set would be identical with Ginzburg-Landau equations in the source-free region. Formally our equations would correspond to the case of vanishing critical field. But such correspondence is misleading, since there is no obvious way to relate physically our V and U to the corresponding quantities in the superconductor, which would be the vector potential and the order parameter, respectively. In fact a closer analysis shows that $U(x)$ is more likely our vector potential and that $V(x)$ is the static Coulomb potential. Therefore, the correspondence is quite formal.

IV. Discussion.

The equations (45) and (46) do not give the exponentially decaying vortex solution of Ginzburg-Landau equations. It is rather easy to see the asymptotic form of the corresponding solution. In the source-free region, we may take one-dimensional solution $V(\rho)$ and $U(\rho)$, which satisfy

$$\frac{1}{\rho} \frac{d}{d\rho} \rho \frac{d}{d\rho} V(\rho) - g^2 U(\rho) V(\rho) = 0, \quad (47)$$

$$\frac{1}{\rho} \frac{d}{d\rho} \rho \frac{d}{d\rho} U(\rho) + V(\rho)^2 U(\rho) - 8 U(\rho)^3 = 0. \quad (48)$$

Assume that $V(\rho) \rightarrow 0$ faster than $U(\rho)$ as $\rho \rightarrow \infty$. Neglecting $V^2 U$ compared with U^3 term, the solution of (48) is

$$U(\rho) = \pm \frac{1}{\sqrt{8}} \frac{1}{\rho}$$

Inserting this into (47) we find

$$V(\rho) = \text{const. } \rho^{-g/\sqrt{8}}$$

In order to calculate energy density we have to find a solution near $\rho \sim 0$ which is extremely difficult to obtain. Instead of solving the variational equations exactly, we may examine the energy integral (40) for a particular set of $\mu(x)$ and $U(x)$ and compare it with the case of vanishing $\mu(x)$ and $U(x)$. For the latter case we have from (1),

$$\langle H \rangle = \frac{g^2}{8\pi} \left[\sum_A \frac{(T_A^3)^2}{\xi_{AA}} + \sum_{A \neq A'} \frac{T_A^3 T_{A'}^3}{\xi_{AA'}} \right], \quad (47)$$

which are static Coulomb energies of the external sources. For a large coupling constant g , eq. (43) suggests a complete screening

$$\mu(x) = -t_E^3(x). \quad (48)$$

(48) satisfies the constraint (41) as long as the system is colorless,

$$\sum_A T_A^3 = 0. \quad (49)$$

From (42), assuming U changes slowly, we may take

$$U(x) = \frac{1}{2} |\mu(x)|^{1/3}. \quad (50)$$

For these choices of $\mu(x)$ and $U(x)$, $\langle H \rangle$ becomes

$$\langle H \rangle = \int \left[\frac{1}{8} (\vec{\nabla} |t_E^3(x)|^{1/3})^2 + \frac{17}{8} |t_E^3(x)|^{4/3} \right] d^3x. \quad (51)$$

To give the meaning to this integral, we may replace the point external source by a smooth distribution of radius a . From the dimensional ground, then,

$$\langle H \rangle \sim O(1)/a,$$

which should be compared with (47)

$$\langle H \rangle \sim g^2/a.$$

Thus, the self-energy of the colorless system is reduced by a factor g^2 . This would be an extremely nice feature if it did not work for colored systems. Unfortunately it does. As an example, we consider a single source

$$t_E^3(x) = T_A S_a(x)$$

where $S_a(x)$ is a smooth function of radius a to replace the delta function. For instance

$$S_a(x) = \frac{1}{(\sqrt{\pi} a)^3} e^{-(x-x_A)^2/a^2}$$

which satisfies $\int S_a(x) d^3x = 1$.

We choose as $\mu(x)$

$$\mu(x) = T_A [-S_a(x) + S_b(x)], \quad (a \rightarrow 0),$$

which satisfies (41). The Coulomb energy in (40) is now replaced by

$$\frac{g^2}{8\pi} T_A^2 \iint \frac{S_b(x) S_b(x')}{|x-x'|} d^3x d^3x' \sim g^2/b.$$

Hence, by choosing $b \sim g^2 a$ we again reduce the Coulomb energy by a factor g^2 . The first three terms of (40) are again of order $1/a$. In other words, the external charge distribution is diffused by the induced charge, and the extended configuration is energetically favorable

compared to the original one. We must be missing either a term in the Hamiltonian or a new constraint which prevents this to happen. Allowing the precession of the quark I-spin, we will have to treat T_A^a quantum mechanically. This means that we have a constraint on the magnitude of total I-spin T^2 , which could be a desired mechanism to prevent the spreading of charge.

Appendix. Expansion of transverse field in cylindrical waves.

The cylindrical waves

$$U_{nk_{\perp}k_z}(z, \rho, \varphi) = \frac{1}{2\pi} J_n(k_{\perp}\rho) e^{ik_z z} e^{in\varphi} \quad (\text{A.1})$$

form a complete orthonormal set, i.e.

$$\begin{aligned} & \int_0^{2\pi} d\varphi \int_0^{\infty} d\rho \rho \int_{-\infty}^{\infty} dz U_{nk_{\perp}k_z}^*(z, \rho, \varphi) U_{n'k'_{\perp}k'_z}(z, \rho, \varphi) \\ &= k_{\perp}^{-1} \delta(k_{\perp} - k'_{\perp}) \delta(k_z - k'_z) \delta_{n,n'} \end{aligned} \quad (\text{A.2})$$

Also

$$\begin{aligned} & \sum_{n=-\infty}^{\infty} \int_0^{\infty} dk_{\perp} k_{\perp} \int_{-\infty}^{\infty} dk_z U_{nk_{\perp}k_z}(z, \rho, \varphi) U_{nk_{\perp}k_z}(z', \rho', \varphi') \\ &= \rho^{-1} \delta(\rho - \rho') \delta(z - z') \delta(\varphi - \varphi'). \end{aligned} \quad (\text{A.3})$$

For each cylindrical wave $(n k_{\perp} k_z)$, we can define two transversally polarized waves

$$\begin{aligned} \vec{U}_{nk_{\perp}k_z}^{(1)}(x) &= \frac{i}{k_{\perp}} (\vec{n}_z \times \vec{\nabla}) U_{nk_{\perp}k_z}(x), \\ \vec{U}_{nk_{\perp}k_z}^{(2)}(x) &= \frac{k}{k_{\perp}} \left(\vec{n}_z + \frac{1}{k^2} (\vec{n}_z \cdot \vec{\nabla}) \vec{\nabla} \right) U_{nk_{\perp}k_z}(x), \end{aligned} \quad (\text{A.4})$$

with $k^2 = k_\perp^2 + k_z^2$.

We see easily that

$$(\vec{u}_{nk_\perp k_z}^{(i)}, \vec{u}_{n'k'_\perp k'_z}^{(j)}) = \delta_{ij} \delta_{n,n'} k_\perp^{-1} \delta(k_\perp - k'_\perp) \delta(k_z - k'_z).$$

Using the above base, we can expand $\vec{A}^a(x)$ and $\vec{\Pi}^a(x)$ in the usual way,

$$\begin{aligned} \vec{A}^a(x) &= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} dk_z \int_0^{\infty} dk_\perp k_\perp \frac{1}{\sqrt{2\omega_k}} \times \\ &\quad \times \sum_i \left[C_{(i)}^a(nk_\perp k_z) \vec{u}_{nk_\perp k_z}^{(i)}(z, \rho, \varphi) + h.c. \right], \\ \vec{\Pi}^a(x) &= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} dk_z \int_0^{\infty} dk_\perp k_\perp \left(-i \sqrt{\frac{\omega_k}{2}} \right) \times \\ &\quad \times \sum_i \left[C_{(i)}^a(nk_\perp k_z) \vec{u}_{nk_\perp k_z}^{(i)}(z, \rho, \varphi) - h.c. \right] \quad (A.5) \end{aligned}$$

The quantization is given by

$$\begin{aligned} [C_{(i)}^a(nk_\perp k_z), C_{(j)}^b(n'k'_\perp k'_z)] \\ = \delta_{ab} \delta_{nn'} \delta_{ij} k_\perp^{-1} \delta(k_\perp - k'_\perp) \delta(k_z - k'_z), \end{aligned} \quad (A.6)$$

all other commutators = 0.

These lead to the commutation relation

$$[A^{ai}(x), \pi^{bj}(x')] = -i \left[\delta_{ij} - \frac{\nabla_i \nabla_j}{\nabla^2} \right] \delta^3(x-x'). \quad (\text{A.7})$$

The solenoidal mode (10) of Section II corresponds to $i = 1$ and $n = 0$.
From (A.1) and (A.4) we have

$$\vec{u}_{0k_{\perp}k_z}^{(1)}(z, \rho, \varphi) = \frac{i}{2\pi} (\vec{n}_z \times \frac{\vec{p}}{\rho}) J_0'(k_{\perp}\rho) e^{ik_z z} \quad (\text{A.8})$$

Comparing (A.5) and (10) we find

$$\alpha^a(x) = \int_{-\infty}^{\infty} dk_z \int_0^{\infty} dk_{\perp} k_{\perp} \frac{1}{\sqrt{2\omega_k}} \cdot [C_{(1)}^a(0k_{\perp}k_z) \frac{(-i)}{2\pi} J_1(k_{\perp}\rho) e^{ik_z z} + h.c.]$$

and

$$\pi^a(x) = \int_{-\infty}^{\infty} dk_z \int_0^{\infty} dk_{\perp} k_{\perp} (-i \sqrt{\frac{\omega_k}{2}}) \cdot [C_{(1)}^a(0k_{\perp}k_z) \frac{(-i)}{2\pi} J_1(k_{\perp}\rho) e^{ik_z z} - h.c.] \quad (\text{A.9})$$

Using (A.6) we obtain

$$[\alpha^a(z, \rho), \pi^b(z', \rho')] = i \delta_{ab} (2\pi\rho)^{-1} \delta(\rho - \rho') \times \delta(z - z'). \quad (\text{A.10})$$

References

1. S. Weinberg, Phys. Rev. Lett. 31 (1973) 497.
2. J. Kogut, L. Susskind, Cornell preprint CLNS-213 (1974).
3. J. Gross and F. Wilczek, Phys. Rev. D8, 3633 (1973).
4. D. J. Gross and F. Wilczek, Phys. Rev. Lett. 30, 1343 (1973),
H. D. Politzer, Phys. Rev. Lett. 30, 1346 (1973).
5. Schwinger; Phys. Rev. 128, 2425 (1962), A. Casher, J. Kogut and
L. Susskind, Phys. Rev. Lett. 31, 792 (1973).
6. H. B. Nielsen and P. Olesen, Niels Bohr Institute Preprint,
NBI-HE-73-13 (1973).
7. C. N. Yang and R. L. Mills, Phys. Rev. 96, 191 (1954).
8. C. f. for example, P. G. de Gennes, Superconductivity of metals
and alloys. Benjamin, New York (1966).
9. See for example: E. S. Ahers and B. W. Lee, Phys. Reports (Section C
of Phys. Lett.) 9, 1-141 (1973), Northland Pub. Co.