Instabilities of Matter in

Strong External Fields

and at High Density.†*

by

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ABSTRACT

In the recent literature, there have been a number of theoretical studies of new phenomena involving the instability of the ground state (sometimes called the vacuum state) of various fundamental forms of matter under unusual perturbations. In this paper we consider in detail *two* examples of instability under the application of Coulomb fields of superheavy nuclei, against the emission of positrons and the creation of a condensate of charged pion pairs. The former may occur for atomic number $Z \sim 170$, whereas the latter requires $Z > 10^3$. The theory of possible forms of pion condensation inside nuclear matter and neutron star matter is also discussed. By careful attention to fundamentals, an effort is made to clarify some of the controversy now raging in the literature concerning these speculations.

I. Introduction

The purpose of this report is to review some recent theoretical developments which predict instabilities in the behavior of various forms of matter when subjected to sufficiently strong external electrostatic fields or when compressed to sufficiently high density. Except for the common feature of instability these phenomena, in detail, are quite distinct, as we shall see by a specific discussion of several examples. The best established theoretically⁽¹⁻¹¹⁾ is that an assembly of localized positive charges confined to a region of nuclear size ("superheavy nucleus") becomes unstable against the emission of a pair of positrons as we increase the nuclear charge number, Z , beyond $Z_c \sim 170$. Further discrete thresholds occur (the next one at $Z \sim 185$) until ultimately ($Z > 10^3$) we would see positive muon pairs. Since $Z_c > 170$ will soon be available, albeit transiently, in heavy ion collisions, the above phenomena may be the first of those discussed to be observed. In section II, the basic theoretical physics needed to understand this instability will be described. The relatively minor but interesting change in the structure of the nuclear charge distribution is seen to be a consequence of Fermi-Dirac statistics.

By contrast, we turn next to an academic example,^(12,9) sufficiently instructive in our view, however, that we devote

sections III and IV to its study. In this example, we imagine that charged pi mesons ($\pi_{\pm}$) are subject only to electrostatic interactions. We then show that the same (stripped) nuclear charge distribution described above would, for sufficiently large $Z > 10^3$, be unstable against the formation of an overall neutral cloud of $\pi_{\pm}$ pairs. The system can be stabilized by including the self-Coulomb interaction of the pion cloud. Remarkably enough, the condensation energy of these pions would ultimately be sufficient to neutralize the Coulomb repulsion of the protons. It is seen in these discussions how the contrasting phenomena are related to the character of the solutions of the single-particle Dirac and Klein-Gordon equations in an external potential.

In the final two sections we turn to the potentially most interesting, but also, at the moment, the most controversial of the phenomena reviewed in this paper. Various authors, principally Migdal in the USSR⁽¹¹⁻¹⁶⁾, Sawyer, Scalapino, and collaborators in the US⁽¹⁷⁻¹⁹⁾ and more recently Baym and Flowers^(20,21) and others^(22,23) have suggested and made model calculations illustrating the possible occurrence (or predicting the non-occurrence) of several kinds of pion condensation phenomena in nuclear matter (N , number of neutrons = Z , number of protons) or inside neutron stars ($N \gg Z$). These include neutral pion

and $\pi_{\pm}$ pair condensates in nuclear matter, $\pi_{\pm}$ condensates in neutron stars, and π_0 and $\pi_{\pm}$ pair condensates in neutron stars. The basic mechanism throughout is the attractive p-wave pion-nucleon interaction.

In our discussion, we have sought to extract some stable elements from the conflicts and the contrasting methodology. Thus, in section V, we have confined ourselves to purely formal results, which in our view establish a method for the study of these possible phase transitions. Within this context we are able to clarify some of the conflicts in methodology.

It seems to us that none of the calculations made so far is truly cogent, but several are highly suggestive. In section VI, we emphasize that perhaps the most accessible portion of the problem is the search for the transition point for a second order phase change. This problem is quite closely tied to the challenge of deriving from first principles the optical potential for pion-nucleus collisions. We conclude with some remarks concerning this relationship.

II. Solutions of the Dirac Equation in a Strong Coulomb Field and Instability of the Vacuum Against Positron Emission.

A. Physics of the Electron-Positron Field in a Strong Coulomb Field.

The electronic structure of superheavy atoms has been most thoroughly investigated in recent years.⁽¹⁻¹¹⁾ The spectrum found by Greiner and coworkers⁽³⁾ is shown in Fig. 1. First, it is seen that for $Z_c \sim 170$ the lowest bound state solution, the $1s_{\frac{1}{2}}$ state, reaches the top of the negative energy continuum and for further increase of Z ceases to be part of the point spectrum, as will be shown below. At this value of Z , the next higher state, the $2p_{\frac{1}{2}}$ state is several hundred KeV higher, though it, too, descends rapidly with Z and becomes "critical" at $Z \sim 185$. The many-body interpretation of this phenomenon is also well-understood. When the K-shell orbit reaches the energy $-m$ ($\hbar=c=1$), a state of a single such electron is degenerate in energy with a state of two K electrons and a positron at rest (neglecting the electron-electron interaction). If the nuclear charge is increased adiabatically through the critical value $Z_c \sim 170$, but the K shell remains filled throughout this process, then nothing is observed. If there is, however, a hole in the K shell, then behind $Z = Z_c$, the system will decay into a state of a filled K-shell and a free positron. If the K-shell is

completely empty, two positrons will materialize at infinity.

The only questionable aspect of these remarks concerns the meaning of "filled K-shell" when there is no bound K orbit. This is taken care of by a new definition of the vacuum state.

When bound orbits of negative (total) energy occur, it is sensible to redefine the vacuum state as the state in which all negative energy orbits, bound or unbound, are occupied, since this has a lower energy than the usual vacuum. With this definition, the vacuum is stable for all Z . Moreover, in the new description, a deeply bound hole in the K-orbit is interpreted as a loosely bound positron. When the hole reaches energy $-m$, the positron reaches energy m and beyond that decays into the continuum. The previous vacuum contains for $Z < Z_c$ two loosely bound positrons which are ultimately available for decay. It is a (crucial) detail of the Dirac equation that as the nuclear charge increases the effective potential binding the positron weakens, ultimately failing to bind. We then encounter standard resonance behavior. On Fig. 1 is marked the width of the resonance that would be observed ideally in the scattering of positrons of suitable energy by the superheavy nucleus.

For the mathematical description of the behavior of the system in the neighborhood of $Z = Z_c$, decisive simplification has been achieved by the introduction of a reduced Hamiltonian.

— In the remainder of this section we shall give a brief description of this model and its properties.

B. The Reduced Hamiltonian.

We are interested in studying the solutions of the Hamiltonian

$$H = \int \psi_{\alpha}^{\dagger}(\underline{r}) [\underline{\alpha} \cdot \underline{p} + \beta m + V(\underline{r})]_{\alpha\beta} \psi_{\beta}(\underline{r}) d^3r \equiv \int \psi^{\dagger} \mathcal{H} \psi d^3r, \quad (2.1)$$

$$\{\psi_{\alpha}(\underline{r}), \psi_{\beta}^{\dagger}(\underline{r}')\} = \delta_{\alpha\beta} \delta(\underline{r} - \underline{r}') \quad , \quad (2.2)$$

where $V(\underline{r})$ may be a superstrong (basically Coulomb) potential, also incorporating, possibly, some effects of vacuum polarization (see subsection F). We shall divide $V(\underline{r})$, the Coulomb field of the extrapolated charge distribution of the nucleus, into two parts,

$$V(\underline{r}) = V_0 + V_1 \quad , \quad (2.3)$$

Where the strength of V_0 is no more than critical and may be sub-critical; whether the additional V_1 carried us beyond critical is to be specified for each instance.

We divide the solutions of the one-particle Hamiltonian $\mathcal{H}_0 = \underline{\alpha} \cdot \underline{p} + \beta m + V_0(\underline{r})$ into three sets $u^{(q)}(\underline{r})$, $u^{(B)}(\underline{r})$, $v^{(q)}(\underline{r})$ designating positive energy continuum states (energies $(m + \epsilon(q))$), bound states (energies $(-m + \epsilon_B)$, measured from $-m'$), negative energy continuum states $(-m - \epsilon(q))$. Following the conventional quantization procedure, we expand in terms of fermion operators

$$a_q, a_B, b_q^\dagger,$$

$$\psi_\alpha(r) = \sum_q a_q u_\alpha^{(q)}(r) + \sum_B a_B u_\alpha^{(B)}(r) + \sum_q b_q^\dagger v_\alpha^{(q)}(r), \quad (2.4)$$

where the vacuum is defined by the conditions

$$a_q |\text{vac}\rangle = a_B |\text{vac}\rangle = b_q |\text{vac}\rangle = 0. \quad (2.5)$$

and of

With the help of (2.4) /its Hermitian conjugate, and of (2.5),

we find, discarding the usual infinite constant,

$$\begin{aligned} \hat{H} = & \sum_q (m + \epsilon(q)) (a_q^\dagger a_q + b_q^\dagger b_q) \\ & + \sum_B (-m + \epsilon_B) a_B^\dagger a_B + \left\{ \sum_{Bp} U_{Bp} a_B^\dagger b_p^\dagger \right. \\ & + \sum_{pp'} U_{pp'} a_p^\dagger b_{p'}^\dagger + \sum_{Bp} V_{Bp} a_B^\dagger a_p + \text{h.c.} \left. \right\} \\ & + \sum_{BB'} V_{BB'} a_B^\dagger a_{B'} + \sum_{pp'} V_{pp'} a_p^\dagger a_{p'} - \sum_{pp'} W_{pp'} b_{p'}^\dagger b_p, \quad (2.6) \end{aligned}$$

where the obvious definitions of the matrix elements U, V, W obtain, e.g.

$$U_{Bp} = (u^{(B)} | v_1 | v^{(q)}) \quad (2.7)$$

We are interested mainly in the fate of the pair of bound states for which $\epsilon_B \rightarrow 0$ as V_1 increases. If we attempt to apply perturbation theory to the K-orbit, it is natural first to include the diagonal contributions of the last three terms of (2.6) with the unperturbed Hamiltonian. Next, we see that the

only unperturbed states nearby in energy to the one-electron state are those with two 1s electrons and one slowly moving positron corresponding to the perturbation U_{Bp} . These latter states may be made as nearly degenerate with the bound state as we wish by taking V_0 sufficiently close to critical. The size of V_1 is also at our disposal and can be made as small as we please. Outside of the above class of perturbations, energy denominators are at least several hundred Kev (to the next higher bound state, $2p_{1/2}$, see Fig. 1).

We are thus able to conclude⁽²⁾ that the behavior of the field lowest bound state in the vicinity of critical/strength can be determined by studying a reduced Hamiltonian, H_r , of the form

$$H_r = \sum_{\sigma} \epsilon a_{\sigma}^{\dagger} a_{\sigma} + \sum_{q\sigma} \epsilon(q) b_{q\sigma}^{\dagger} b_{q\sigma} + \sum_{p\sigma} (U_p a_{\sigma}^{\dagger} b_{p\sigma}^{\dagger} + U_p^* b_{p\sigma} a_{\sigma}) , \quad (2.8)$$

where $\sigma = \pm$ now describes the two possible occupants of the lowest bound states by the projection of their total angular momentum ($\bar{\sigma} = -\sigma$). When necessary, we shall remember that

$$\epsilon = \epsilon_0 + \epsilon_1 , \quad (2.9)$$

and that both U_p and ϵ_1 are proportional to V_1 . We have, again, dropped an additive constant, taking into account that for H_r , the difference between the number of bound electrons and the number of positrons is a constant of the motion. Finally, the

apparent omission of the last term of (2.6) may be justified by the assumption that the part of H_r depending on continuum positron states has been pre-diagonalized. This involves only a change in the value of U_p .

C. The One-Electron Sector.

We now look for eigenstates of H_r of the form

$$|\psi_\sigma\rangle = A_0 a_\sigma^\dagger |\text{vac}\rangle + \sum_p A_p a_\sigma^\dagger a_{\bar{\sigma}}^\dagger b_{p\sigma}^\dagger |\text{vac}\rangle \quad (2.10)$$

The attendant Schrodinger equation

$$H_r |\psi_\sigma\rangle = W |\psi_\sigma\rangle \quad (2.11)$$

is equivalent to the coupled equations

$$(W - \epsilon) A_0 = \sum_p U_p^* A_p, \quad (2.12a)$$

$$[W - 2\epsilon - \epsilon(p)] A_p = U_p A_0. \quad (2.12b)$$

We look first for bound state solutions for which

$W_1 - 2\epsilon - \epsilon(p) \neq 0$. Solving for A_p from (2.12b) and substituting in (2.12a), we obtain the eigenvalue condition

$$x = F(x - \epsilon), \quad (2.13a)$$

$$F(y) = \sum_p \frac{|U_p|^2}{y - \epsilon(p)} \equiv \int_0^\infty dE \frac{|V_E|^2}{y - E}, \quad (2.13b)$$

$$x \equiv W_1 - \epsilon. \quad (2.13c)$$

It is known⁽⁴⁾ that for the case of interest $\epsilon_0 = 0$,

$$|V_E|^2 \sim \exp[-2\pi Z\alpha n/p], \quad (2.14)$$

(where Z is the atomic number and α the fine-structure constant), and thus vanishes very rapidly with ϵ . This is essentially the result of the strong repulsion by the nucleus of very slowly moving positrons. If we were to consider instead a model with short range forces, we can still expect $|V_E|^2$ to vanish like a (fractional) power of E , and therefore in any event the quantity $F(0)$ exists and is negative.

There are two cases to be considered, distinguished in Figs. (2a) and (2b). In (2a), $\epsilon > 0$. Since $F(-\epsilon) < 0$, and $F(y) \rightarrow 0$ as $y \rightarrow -\infty$, (2.13a) has a solution $x = W_1 - \epsilon < 0$, but as we are about to realize

$$0 < W_1 < \epsilon \quad (2.15)$$

Consider the case $\epsilon < 0$ (and $\epsilon_0 = 0$). In this case $F(x - \epsilon)$ exists only for $x \leq \epsilon$. As shown in the figure,

$$|F(0)| < |\epsilon| \quad (2.16)$$

and therefore no real root exists. This is obvious for V_1 small enough, since then $|\epsilon| \sim |V_1|$, whereas $|F(0)| \sim |V_1|^2$. These considerations represent the simplest proof that for $Z > Z_c$, the critical value, a bound ls state is no longer a member of the point spectrum.

Thus, if we have a bound state, it is represented by the vector

$$|\psi_\sigma\rangle = \begin{pmatrix} A_0 \\ A_p \end{pmatrix}, \quad (2.17a)$$

$$A_p = \frac{U_p A_0}{W - 2\epsilon - \epsilon(p)}, \quad (2.17b)$$

$$|A_0|^{-2} = 1 + \sum_p \frac{|U_p|^2}{(W - 2\epsilon - \epsilon(p))^2}, \quad (2.17c)$$

the last being the normalization condition.

There are also scattering solutions of the form (2.10) to be considered. We first remark that for any model considered above, in consequence of the Pauli principle the unperturbed state with two electrons $a_\sigma^\dagger a_{\bar{\sigma}}^\dagger |\text{vac}\rangle$ is a bound eigenstate with energy 2ϵ . The form (2.10) can then represent eigenstates with the boundary condition: positron incident on the state of two electrons, energy $W(q) = \epsilon(q) + 2\epsilon$. In this case we write

$$|\psi_{\sigma q}\rangle = \begin{pmatrix} A_{oq}^{(+)} \\ A_{pq}^{(+)} \end{pmatrix} \quad (2.18a)$$

and find for the outgoing wave solutions of the reduced Hamiltonian

$$A_{oq}^{(+)} = \frac{U_q^*}{\epsilon_+(q) + \epsilon - F(\epsilon_+(q))} \quad (2.18b)$$

$$A_{pq}^{(+)} = \delta(p-q) + \frac{U_q^* U_p}{[\epsilon_+(q) - \epsilon(p)] \sqrt{\epsilon_+(q) + \epsilon - F(\epsilon_+(q))}} \quad (2.18c)$$

where $\epsilon_+(q) = \epsilon(q) + i\eta$, η positive and small. The normalization is such that

$$\langle \psi_{\sigma q} | \psi_{\sigma q'} \rangle = \delta(q-q') \quad , \quad (2.19)$$

as one verifies by straightforward algebra. The scattering solutions are of course orthogonal to the bound state, if the latter exists.

In the event that we are supercritical, the denominator

$$\epsilon(q) + \epsilon - F_+(\epsilon(q)) \equiv \epsilon(q) + \epsilon - F_1(\epsilon(q)) + i\pi |V_{\epsilon(q)}|^2 \quad , \quad (2.20)$$

where F_1 represents the principal value sum, can have a vanishing real part at the resonance energy

$$\epsilon_R = |\epsilon| + F_1(\epsilon_R) \quad , \quad (2.21)$$

because $|F_1(0)| < |\epsilon|$ and $F_1(0) < 0$; the resonance is, of course, the analytic continuation of the bound ls state. When there is a true negative energy bound state, there is correspondingly no associated resonance scattering, as can be verified.

We remark finally on the completeness relation satisfied by our solutions. These read (bound state present)

$$|A_0|^2 + \sum_q |A_{0q}^{(+)}|^2 = 1 \quad , \quad (2.27a)$$

$$A_p A_0^* + \sum_q A_{pq}^{(+)} A_{0q}^{(+)*} = 0 \quad , \quad (2.27b)$$

$$A_p A_{p'}^* + \sum_q A_{pq}^{(+)} A_{p'q}^{(+)*} = \delta(p-p') \quad . \quad (2.27c)$$

If there is no bound state, the first term is missing. The proof of these relations is to be found in ref. 11.

D. Vacuum State Sector.

We turn next to the study of eigenstates of the form

$$\begin{aligned}
 |\psi_0\rangle = & C_0 |\text{vac}\rangle + \sum_{p\sigma} C_p a_{\bar{\sigma}}^\dagger b_{p\sigma}^\dagger |\text{vac}\rangle \\
 & + \sum_{pp'} C_{pp'} (a_{-p}^\dagger b_{p'}^\dagger) (a_{+p}^\dagger b_{p'}^\dagger) |\text{vac}\rangle, \quad (2.23)
 \end{aligned}$$

which includes the vacuum state. We shall quote only the results for this case. We can show that the condition for a stable vacuum state having the same charge as $|\text{vac}\rangle$ is the same as the conditions (2.13) for the existence of a stable one-electron state, consideration being given to an appropriate change in the definition of the energy. Thus these two states are stable or unstable together. In the reformulation given below this will be obvious, having the meaning described in subsection A.

E. Definition of a Stable Vacuum and Reformulation of Theory.

In the previous two subsections we have found that in supercritical fields, the state of two electrons bound in the K-shell is stable, but the vacuum and one-electron states are not. A solution to this dilemma is fairly immediate^(6,10). Within the present context the solution requires that we choose the state with a fully occupied K-orbit as the vacuum. Within the context of the original full Hamiltonian, this is a modification of Dirac's original quantization of the electron field, in which the vacuum was the state with the fully occupied sea of negative

energy continuum states. In general, we propose that a proper definition of the vacuum includes as well the filling of negative energy bound states. This redefinition simplifies the study of those bound states which encroach upon the top of the negative energy continuum.

The redefinition is effected in practice (as usual) by the interchange of creation and annihilation operators. We define

$$\begin{aligned} b_{\sigma}^{\dagger} &= a_{\bar{\sigma}} \quad , \\ b_{\sigma} &= a_{\bar{\sigma}}^{\dagger} \quad . \end{aligned} \quad (2.24)$$

We say that $b_{\sigma}^{\dagger}$ creates a bound positron, i.e., the absence of a deeply bound electron is reinterpreted as the presence of a weakly bound positron. This is seen by rewriting the reduced Hamiltonian (2.8), which to an additive constant becomes

$$\begin{aligned} H_r' &= - \epsilon \sum_{\sigma} b_{\sigma}^{\dagger} b_{\sigma} + \sum_p \epsilon_p b_{p\sigma}^{\dagger} b_{p\sigma} \\ &\quad - \sum_{p\sigma} U_p b_{p\sigma}^{\dagger} b_{\sigma} - \sum_{p\sigma} U_p^* b_{\sigma}^{\dagger} b_{p\sigma} \quad . \end{aligned} \quad (2.25)$$

We now have a theory exclusively of positrons. If $\epsilon > 0$, there can be bound positrons, which, as we have seen, remain bound under perturbation. If $\epsilon < 0$, the unperturbed system has a bound state in the continuum which becomes a resonance under the perturbation. Thus, if we use the new vacuum and the language of positrons, the situation concerning the behavior of

the K-shell in supercritical fields loses completely any aura of mystery and is reduced to the prototype problem of a weakly bound state transforming under further change in potential into a resonance, and the model is equivalent to one considered long ago by Fano. (23)

It is unnecessary to calculate the eigenstates of (2.25). They may be read off from the one-electron solutions of subsection C, changing only a few signs, in accordance with the change of sign in the interaction terms of (2.25). The solutions will be written in terms of the operators which diagonalize the Hamiltonian. Including the case of a bound system we introduce operators β_σ , $\beta_{q\sigma}$, given by the linear relations

$$\beta_\sigma = A_0 b_\sigma - \sum_p A_p b_{p\sigma} \quad , \quad (2.26a)$$

$$\beta_{q\sigma} = -A_{0q}^{(+)} b_\sigma + \sum_p A_{pq}^{(+)} b_{p\sigma} \quad , \quad (2.26b)$$

where A_0 and A_p are given by (2.17) and $A_{0q}^{(+)}$, $A_{pq}^{(+)}$ are given by (2.18b) and (2.18c). Instead of W_1 we shall use the eigenvalue

$$\omega = W_1 - 2\epsilon \quad . \quad (2.27)$$

The fermion operators (2.26) are the solutions of the equations of motion and render the Hamiltonian diagonal, e.g.,

$$H_r' = \sum_\sigma \omega \beta_\sigma^\dagger \beta_\sigma + \sum_{q\sigma} \epsilon(q) \beta_{q\sigma}^\dagger \beta_{q\sigma} \quad . \quad (2.28)$$

The orthogonality and completeness relations are unaffected

by the phase changes indicated in (2.26). The completeness relations allow us to invert the relations (2.26),

$$b_{\sigma} = A_0^* \beta_{\sigma} - \sum_q A_{0q}^{(+)*} \beta_{q\sigma} \quad , \quad (2.29a)$$

$$b_{p\sigma} = -A_p^* \beta_{\sigma} + \sum_q A_{pq}^{(+)*} \beta_{q\sigma} \quad . \quad (2.29b)$$

These transformations together with the orthogonality and completeness relations permit the discussion of any question of physical interest. For example we may follow a standard treatment⁽²⁴⁾ to derive the approximate exponential decay of the probability $|\langle \text{vac} | b_{\sigma} \exp[-iH'_r t] b_{\sigma}^{\dagger} | \text{vac} \rangle|^2$ and it follows as well from (2.29a) that $|A_{0q}^{(+)}|^2$ describes the energy spectrum of the decay positrons (when there is no bound state).

F. Effect of Vacuum Polarization.

We believe that the additional effects of vacuum polarization should have only a minor influence on the theory described up to this point. Consider the new definition of the vacuum. Divide the solutions of the Dirac equation into the positive energy ones $u_{\alpha}^{(v)}(r)$ and the negative energy ones $v_{\alpha}^{(\lambda)}(r)$, including bound states. The potential $V(r)$ in Eq. (2.1) and sequel should then be calculated self-consistently from the equation ($V = eA_0$, $\psi = u$ or v)

$$E_{\psi}^{(v)}(r) = [\alpha \cdot p + \beta m + eA_0(r)] \psi^{(v)}(r) \quad , \quad (2.30a)$$

$$\nabla^2 A_0(r) = -\rho_{\text{ext}}(r) - \rho_{\text{vac}}(r) \quad , \quad (2.30b)$$

where ρ_{ext} is the charge density of the "external field" (see below) and $\rho_{\text{vac}}(r)$ is the vacuum polarization charge.

$$\begin{aligned} \rho_{\text{vac}}(r) &= \sum_{\alpha} \langle \text{vac} | |e| \frac{1}{2} [\psi_{\alpha}(r), \psi_{\alpha}^{\dagger}(r)] | \text{vac} \rangle \\ &= \frac{1}{2} |e| \left\{ \sum_{\alpha\nu} |u_{\alpha}^{(\nu)}(r)|^2 - \sum_{\alpha,\lambda} |v_{\alpha}^{(\lambda)}(r)|^2 \right\}. \end{aligned} \quad (2.31)$$

By ρ_{ext} , we mean a quantity which includes the contribution of the negative energy bound electrons. This implies that in a linearized approximation below the critical field strength, the right hand side of (5.1b) is almost independent of the definition of the vacuum, since

$$\rho_{\text{ext}} = \rho_{\text{ext}}^{(0)} - \sum_{B,\alpha} |e| |v_{\alpha}^B(r)|^2, \quad (2.32)$$

where $\rho_{\text{ext}}^{(0)}$ is the conventional definition of the external charge.

Above the critical value of Z , this separation is no longer strictly possible for the lowest bound states, the corresponding terms disappearing from (2.31).

We note that with the new definition of the vacuum, ρ_{vac} has the same structure as in the weak coupling case (positive energies with one sign, negative energies with another). This suggests that the technique of Wichmann and Kroll⁽²⁶⁾ may continue to be applicable to the evaluation of vacuum polarization. In consequence, the nonlinear problem (5.1) may prove tractable, though

not trivial. There is no mechanism here to alter the essentials of the previous discussion, only the details. We believe that the problem posed by (2.30) is worthy of investigation.

III. Klein-Gordon Equation in a Strong Coulomb Field. Instability Against Charged Pair Condensation.

A. Forms of the K.G. Equation.

We study (with $\hbar = c = 1$) the equation

$$(E - V)^2 \psi(r) = (p^2 + m^2) \psi(r) , \quad (3.1)$$

where $\psi(r)$ is a complex scalar function and $V(r)$ is the potential energy of a negatively charged scalar particle of mass m (henceforth called pion) in the field of a fixed extended charge distribution. For example, we may take

$$V(r) = - \frac{Ze^2}{r} f(r) , \quad (3.2)$$

$$f(r) = 1 - e^{-\kappa r} \quad (3.3)$$

where $\kappa^{-1} \sim R$, the nuclear radius.

The formal properties of the equation are more easily studied if we introduce a formalism first order in the energy (time derivative) (27). In terms of the two-component vector

$$\underline{\phi} = \begin{pmatrix} \psi \\ (E-V)\psi \end{pmatrix} , \quad (3.4)$$

Eq. (3.1) becomes

$$E \tau_1 \underline{\phi} = \mathcal{K} \underline{\phi} , \quad (3.5)$$

$$\mathcal{K} = \frac{1}{2}(1 + \tau_3)(p^2 + \mu^2) + \frac{1}{2}(1 - \tau_3) + \tau_1 V . \quad (3.6)$$

Here τ_i , $i = 1, 2, 3$ are the usual Pauli spin matrices. Since $\mathcal{K}$ is Hermitian, we recognize that the fundamental scalar product is the integral of the density

$$d_{ab}(r) = \phi_a^\dagger(r) \tau_1 \phi_b(r) , \quad (3.7)$$

providing an orthogonality theorem for two solutions of (3.5) belonging to different energies.

For the norm of any given solution, we compute

$$\int \phi^\dagger \tau_1 \phi = 2 \int \psi^* (E-V) \psi . \quad (3.8)$$

Remembering the sign of V , Eq.(3.2), we see that (3.8) is certainly positive for $E \geq 0$. On the other hand, for negative energy continuum solutions, the norm must be negative, by analytic continuation from the limit $V = 0$. Thus if we choose the original K.G. scalar function to be normalized according to

$$\int |\psi|^2 = [2|E - \langle V \rangle|]^{-1} , \quad (3.9)$$

where $\langle V \rangle = \int \psi^* V \psi / \int \psi^* \psi$, we see that, in general, the solution set for given V divides into two subsets ϕ_p and ϕ_n characterized as follows:

$$\begin{aligned} \int \phi_p^\dagger \tau_1 \phi_p &= 1 , & E_p - \langle V \rangle_p &> 0 , \\ \int \phi_n^\dagger \tau_1 \phi_n &= -1 , & E_n - \langle V \rangle_n &< 0 . \end{aligned} \quad (3.10)$$

The considerations above obviously fail, should there occur an eigenvalue for which the inequalities (3.10) are replaced by an equality. The resolution of this difficulty, carried out below, represents the basic goal of this part of our work.

For the normally occurring situations for which (3.10)

applies, the completeness relation for the solutions of (3.5) takes the dyadic form

$$\sum_p \phi_p(r) [\phi_p^\dagger(r') \tau_1] - \sum_n \phi_n(r) [\phi_n^\dagger(r') \tau_1] = I \delta(r-r'), \quad (3.11)$$

and I is the unit two by two matrix.

A third form of the K.G. equation is sometimes useful because of its analogy with the non-relativistic Schrodinger equation. For this form, we write

$$\epsilon_{\text{eff}} \psi = (p^2/2m) \psi + V_{\text{eff}} \psi, \quad (3.12)$$

where

$$\epsilon_{\text{eff}} = E' [1 + (E'/2m)],$$

$$V_{\text{eff}} = V [1 + (E'/m) - (V/2m)],$$

$$E = E' + m. \quad (3.13)$$

B. Bound State Spectrum and Approach to the Critical Point. ⁽¹²⁾

For full understanding of the situation when Z becomes very large, we consider together the solutions for both negative and positive pions ($\pi_{\pm}$). The situation is illustrated schematically in Fig. 3. For a potential energy of the form (3.2), the curve marked E_{-} represents in its solid part the lowest bound state of a π_{-} . Two special values of Z are to be noted. For $Z > Z_0$, there emerges from the negative energy continuum a new bound state branch

for π_- which meets the branch E_- at a point of vertical tangency, $Z = Z_c$. For $Z > Z_c$ there is simply no bound state solution corresponding to this branch. The curve marked E_+ is a reflection of E_- with respect to the abscissa $E = 0$ and represents a solution branch for π_+ .

We shall now indicate the derivation of these results from the K.G. equation. To understand their physical implication we must have resort to the quantum field theory described in the next section.

It follows most easily from (3.1) that

$$\psi(r; E, e) = \psi(r; -E, -e) . \quad (3.14)$$

This shows that if branch E_- occurs for π_- , branch E_+ occurs for π_+ . Next consider the equation for π_+ near zero binding energy. From (3.13), remembering to change the sign of the charge, we have

$$V_{\text{eff}}(\pi_+, E' \sim 0) \cong |V(r)| - [V^2/2m] . \quad (3.15)$$

For Z large enough, the second term must dominate and give ($Z > Z_0$) a π_+ bound state - even in the field of a positively charged nucleus, - as shown in the solid part of the branch E_+ . The combination of (3.14) and (3.15) establishes the qualitative character of Fig. 3 .

It is furthermore easy to see from the considerations of

subsection A that points on the solid parts of curves $E_{\pm}$ correspond to solutions with positive norm whereas dashed portions represent solutions with negative norm. The meeting point must therefore be one at which $E = \langle V \rangle$. Now consider Eq. (3.1) for $V \rightarrow \lambda V$ and form the expectation value,

$$\langle (E - \lambda V)^2 \rangle = \langle p^2 \rangle + m^2 . \quad (3.16)$$

With the help of the K.G. equation, the first derivative of (3.16) at $\lambda = 1$ becomes

$$[E - \langle V \rangle] dE/d\lambda = \langle EV \rangle - \langle V^2 \rangle . \quad (3.17)$$

At the point $E = \langle V \rangle$, the right hand side cannot vanish. Therefore we must have $dE/d\lambda \rightarrow \infty$, a point of vertical tangency.

Let $E_- = -\mu$ represent this point occurring at $Z = Z_c$. Then we see that for the branch E_+ , we have $E_+ = \mu$. Therefore in a quantum field theory we could produce an indefinite number of $\pi_{\pm}$ pairs without energy cost (in the absence of other interactions). This apparent instability will be dealt with in the next section.

Let us estimate the value of Z_c predicted by our model, using the pion mass. This can be done with the help of a virial theorem. From the statement

$$\langle [r \cdot p, \{(E-V)^2 - p^2 - m^2\}] \rangle = 0 , \quad (3.18)$$

we derive

$$\langle p^2 \rangle = \langle (r \cdot \nabla V) (E - V) \rangle . \quad (3.19)$$

Combining this with Eqs. (3.16) ($\lambda=1$) and (3.2,3) we derive

$$(\alpha = e^2/\hbar c)$$

$$E[E - \langle V \rangle] = m^2 - Z\alpha E \langle f'(r) \rangle + Z\alpha \langle f'(r) V \rangle . \quad (3.20)$$

The correct order of magnitude for Z_c should be obtained if we examine (3.20) for $E = 0$. Thus we have

$$m^2 = (Z\alpha)^2 \langle e^{-\kappa r}/r \rangle \sim \frac{(Z\alpha)^2}{R^2} , \quad (3.21)$$

using the essential fact that the pion wave-function is largely confined to the nuclear interior. If we put $R^2 = (10^2/m^2)$ we find $Z\alpha \sim 10$, which is roughly consistent, but unfortunately renders our problem somewhat academic, at least under present conditions. (If we had a pion of electronic mass, Z_c would be reduced by an order of magnitude.) Thus, if there is to be an observable pion condensation phenomena, we shall have to look for its origin in the nuclear forces. This problem is considered in Secs. V and VI.

Returning now to the technical aspects of our problem, we notice that at $Z = Z_c$, there is only one solution remaining of the two we had for $Z < Z_c$. To understand the approach to the limit, we define in terms of the two solutions of interest $\phi_{\pm}$ ($\pm$ referring to norm), two new uncharged linear combinations

$$\begin{aligned}\varphi_e &= (1/\sqrt{2}) (\varphi_+ + \varphi_-) , \\ \varphi_o &= (1/\sqrt{2}) (\varphi_+ - \varphi_-) .\end{aligned}\quad (3.22)$$

With

$$d_e(r) = \varphi_e^\dagger(r) \tau_1 \varphi_e(r) , \quad (3.23)$$

etc., we have straightforwardly

$$\int d_e = \int d_o = 0 , \quad (3.24)$$

but

$$\int d_{eo} = \int d_{oe} = 1 . \quad (3.25)$$

It may be instructive to exhibit expressions for these densities. In the following we make use of the orthogonality between φ_+ and φ_- , which can be expressed in the form, assuming real wave functions,

$$E_+ + E_- = 2V_+ , \quad (3.26)$$

where

$$V_{ab} = (\psi_a, V\psi_b) / (\psi_a, \psi_b) , \quad (3.27)$$

and (as will be needed below) $V_{aa} = V_a$. We also record our expressions using unit normalization for ψ_a . We then have

$$\begin{aligned}d_e(r) &= \frac{1}{2} \left\{ \psi_+^2 - \psi_-^2 + \frac{(V_+ - V(r))\psi_+^2}{|E_+ - V_+|} + \frac{(V_- - V)\psi_-^2}{|E_- - V_-|} \right. \\ &\quad \left. + \frac{2(V_+ - V)\psi_+\psi_-}{[|E_+ - V_+||E_- - V_-|]^{\frac{1}{2}}} \right\} ,\end{aligned}\quad (3.28)$$

$$d_{oe}(r) = \frac{1}{2} \left\{ \psi_+^2 + \psi_-^2 + \frac{(V_- - V) \psi_-^2}{|E_- - V_-|} - \frac{(V_+ - V) \psi_+^2}{|E_+ - V_+|} \right\}, \quad (3.29)$$

$$d_o(r) = \frac{1}{2} \left\{ \psi_+^2 - \psi_-^2 + \frac{(V_+ - V) \psi_+^2}{|E_+ - V_+|} - \frac{(V_- - V) \psi_-^2}{|E_- - V_-|} \right\}. \quad (3.30)$$

From these expressions, we verify (3.24) and (3.25). From (3.28) we see that the limit $Z \rightarrow Z_c$ cannot be taken, since, if we define

$$\alpha = \frac{1}{2} \left\{ |E_+ - V_+| + |E_- - V_-| \right\}, \quad (3.30)$$

d_e behaves like $O(\alpha^{-1})$. By contrast, d_{oe} and d_o remain finite as $\alpha \rightarrow 0$.

IV. Quantization of the Klein-Gordon Equation in a Strong Coulomb Field. Stabilization of the Vacuum by Non-Linear Interactions.

A. Quantization Below the Critical Point. ^(28, 29)

This is achieved most elegantly in an external field by use of the first-order formalism. We adopt the Lagrangian

$$L(t) = \int d^3r [i\phi_{op}^\dagger(r,t)\tau_1 \dot{\phi}_{op}(r,t) - \phi_{op}^\dagger \kappa \phi_{op}] \quad (4.1)$$

This yields the canonical momentum

$$\pi_{op}(r,t) = i\phi_{op}^\dagger(r,t)\tau_1 \quad (4.2)$$

Thus the required commutation relations are

$$[\phi_{op}(r,t), \phi_{op}^\dagger(r',t)] = \tau_1 \delta(r-r'). \quad (4.3)$$

The Hamiltonian which follows from (4.1) and (4.2) is

$$H = \int \phi_{op}^\dagger \kappa \phi_{op} \quad (4.4)$$

To this we adjoin our candidate for total charge operator, namely

$$Q = -|e| \int \phi_{op}^\dagger \tau_1 \phi_{op} \quad (4.5)$$

It is understood that H and Q are to be taken in normal form with respect to the vacuum state to be defined below.

The completeness relation (3.11) will guarantee us a satisfactory quantum theory when used in conjunction with the expansion

$$\phi_{op}(r) = \sum_p a_p \phi_p(r) + \sum_n b_n^\dagger \phi_n(r) \quad , \quad (4.6)$$

if we assume that

$$\begin{aligned} [a_p, a_p^\dagger] &= \delta_{pp'} \quad , \\ [b_n, b_n^\dagger] &= \delta_{nn'} \quad , \end{aligned} \quad (4.7)$$

(and of course if the a 's and b 's commute, etc.). Thus (4.6) and (4.7) satisfy (4.3). Furthermore if the vacuum is annihilated by a_p and b_n , we find for (4.4) and (4.5),

$$H = \sum_p E_p a_p^\dagger a_p + \sum_n |E_n| b_n^\dagger b_n \quad , \quad (4.8)$$

$$Q = - |e| \left\{ \sum_p a_p^\dagger a_p - \sum_n b_n^\dagger b_n \right\} . \quad (4.9)$$

As long as $Z < Z_c$, this represents an unequivocally satisfactory theory of non-interacting bosons of either charge in the external field of a positively charged nucleus. As we approach the point $Z = Z_c$, we arrive, now on a proper quantum basis at the situation already described: the vacuum state is no longer unique, but approaches degeneracy with an infinite set of other states containing various numbers of $\pi_{\pm}$ of energy $\pm\mu$ each.

To remedy this situation, it is sufficient to include in the quantum theory the mutual Coulomb interaction of any produced pions. This will be demonstrated with sufficient rigor below, but the result can be anticipated on physical grounds. Thus as we approach $Z = Z_c$, including the Coulomb interaction of the pions, multi-pion states of small excitation energy can now mix

with the previously defined vacuum. But the expectation value of the self-Coulomb interaction of any assembly of charges is positive and since it is quartic in the amplitudes, it must ultimately dominate and prevent collapse. Moreover, if we consider the equation of motion for excitations of negative charge, which is the appropriate generalization of the unquantized K.G. equation, we must find that the charge distribution of the other pions screens the nuclear field on the average. The net result is an effective nuclear field which remains subcritical. A vacuum state is thus stabilized but in terms of the eigenstates of the Hilbert space defined by the expansion (4.6), its description will involve many components.

B. Quantization Above the Critical Point. The Reduced Hamiltonian.

Formally the quantization can be carried out precisely as above. We need only replace (4.4) by the Hamiltonian

$$H = \int \phi_{op}^\dagger \mathcal{K} \phi_{op} + \frac{1}{2} e^2 \int \frac{[\phi_{op}^\dagger(r) \tau_1 \phi_{op}(r)] [\phi_{op}^\dagger(r') \tau_1 \phi_{op}(r')]}{|r-r'|} . \quad (4.10)$$

Since V in $\mathcal{K}$ is now, by assumption, beyond critical, $|V| > |V_c|$ we choose V_0 such that $|V_0| < |V_c|$ and $[|V_c - V_0|/|V_c|] \ll 1$ and write

$$\begin{aligned} \mathcal{K} &= \mathcal{K}_0 - \tau_1 \Delta, \\ V - V_0 &= -\Delta, \Delta > 0 . \end{aligned} \quad (4.11)$$

We also define

$$\epsilon_{\pm} \tau_1 \phi_{\pm} = \chi_{0\pm} \phi_{\pm} , \quad (4.12)$$

$$\delta = \frac{1}{2} (\epsilon_+ - \epsilon_-) > 0 , \quad (4.13)$$

$$\mu_0 = -\frac{1}{2} (\epsilon_- + \epsilon_+) > 0 , \quad (4.14)$$

where these all refer to the near critical eigenvalues.

To establish the stability of the Hamiltonian (4.10), we write

$$\phi_{op}(r) = a\phi_+(r) + b^\dagger\phi_-(r) + \chi_{op}(r) . \quad (4.15)$$

For the remainder of the present discussion we drop the terms arising from χ_{op} , since these make reference to the non-dangerous levels only: their absence can effect quantitative details only. We also rewrite the first terms of (4.15) by means of the operators

$$q = (a + b^\dagger)/\sqrt{2} ,$$

$$p = i(a^\dagger - b)/\sqrt{2} , \quad (4.16)$$

yielding

$$\phi_{op}(r) = q\phi_e(r) + ip^\dagger\phi_o(r) , \quad (4.17)$$

where ϕ_e and ϕ_o are the combinations defined in (3.22). With the aid of definitions (4.11) - (4.14) and the further definitions

$$\Delta_{ab} = \int \phi_a^\dagger \tau_1 \Delta \phi_b , \quad (4.18)$$

$$U_{ab,cd} = e^2 \int (\phi_a^\dagger \tau_1 \phi_b) (\phi_c^\dagger \tau_1 \phi_d) / |r-r'| , \quad (4.19)$$

$$L = b^\dagger b - a^\dagger a = : i(q^\dagger p^\dagger - pq) : , \quad (4.20)$$

we obtain by straightforward transcription (where $U_{eeee} \equiv U_e$, etc.)

$$\begin{aligned} H_{\text{red}} = & (\delta - \Delta_{oo}) p^\dagger p + (\delta - \Delta_{ee}) q^\dagger q + (\mu_o + \Delta_{oe}) L \\ & + \frac{1}{2} U_e (q^\dagger q)^2 + \frac{1}{2} U_{e,o} \{ q^\dagger q, p^\dagger p \} + \frac{1}{2} U_o (p^\dagger p)^2 \\ & - \frac{1}{2} \{ q^\dagger q, L \} U_{e,oe} - \frac{1}{2} \{ p^\dagger p, L \} U_{o,oe} + \frac{1}{2} L^2 U_{oe,oe} . \end{aligned} \quad (4.21)$$

The subscript on H reminds us that we are describing only two degrees of freedom.

For $Z - Z_c \sim 1$, the basis defined by (4.11) and (4.12) should be very satisfactory for the diagonalization of the Hamiltonian (4.21). The unperturbed vacuum (we consider the subspace $L = 0$) will mix with states with at most a few pion pairs to yield a new stable vacuum. This is guaranteed by the positive definite character of the quartic terms of H .

For $Z - Z_c \equiv Z' \gg 1$, the mixing becomes large and the treatment by means of the basis defined above cumbersome. In this case the stability of the ground state is seen in a classical approximation for which

$$\langle (q^\dagger q)^2 \rangle \cong \langle q^\dagger q \rangle \langle q^\dagger q \rangle = (q^2)^2 \quad (4.22a)$$

$$\langle q^\dagger q p^\dagger p \rangle = \langle q^\dagger q \rangle \langle p^\dagger p \rangle = q^2 p^2 , \quad (4.22b)$$

etc.

With

$$\delta - \Delta_{co} = -B, \delta - \Delta_{ee} = -A, \quad (4.23)$$

we have ($L = 0$)

$$\begin{aligned} W(q^2, p^2) = \langle H_{red} \rangle = & -Aq^2 - Bp^2 \\ & + \frac{1}{2} U_e (q^2)^2 + U_{eo} q^2 p^2 + \frac{1}{2} U_o (p^2)^2. \end{aligned} \quad (4.24)$$

As will be seen below, we can, without loss of generality, choose $p^2 = 0$. The variation of W with respect to q^2 yields the minimum at

$$q^2 = A/U_e, \quad (4.25)$$

and

$$W = -\frac{1}{2} A^2/U_e, \quad (4.26)$$

which is proportional to Z'^2 .

Further progress and substantiation of the above simplification depends on recognizing that the variational expression (4.24) may be derived as the expectation value of H_{red} with respect to a coherent trial function

$$|a', b', \theta\rangle = \exp(a^\dagger a' + b^\dagger b' e^{-i\theta}) |vac\rangle. \quad (4.27)$$

(This is true insofar as $|a'|^2 \gg 1$, $|b'|^2 \gg 1$.) With the identifications

$$q^2 = |q_o|^2, \quad p^2 = |p_o|^2, \quad (4.28)$$

we have

$$\sqrt{2} \, q_0 = a' + b' e^{i\theta} , \quad (4.29)$$

$$\sqrt{2} \, p_0 = i(a' - b' e^{-i\theta}) . \quad (4.30)$$

Average charge neutrality is assured by choosing $a' = b'$.

The choice $\theta = 0$ for the arbitrary phase furthermore guarantees

$$p_0 = 0.$$

The observations just made suggest that the previous classical approximation can be improved by also varying the operators $a^\dagger$ and $b^\dagger$. The expectation value of H_{red} with respect to a general trial state (4.28) again has the form (4.24) with the altered definitions

$$A = - \int \underline{\phi}_e^\dagger \kappa \underline{\phi}_e , \quad (4.31a)$$

$$B = - \int \underline{\phi}_0^\dagger \kappa \underline{\phi}_0 . \quad (4.31b)$$

To obtain equations to define $\underline{\phi}_e$ and $\underline{\phi}_0$, we vary with respect to $q^2 \underline{\phi}_e^\dagger$ and $p^2 \underline{\phi}_e^\dagger$. These variations must be constrained by the condition of charge neutrality

$$-\mu \langle Q \rangle = - \mu [q^2 \int \underline{\phi}_e^\dagger \tau_1 \underline{\phi}_e + p^2 \int \underline{\phi}_0^\dagger \tau_1 \underline{\phi}_0] = 0 , \quad (4.32)$$

which, however, still doesn't define the scale of $\underline{\phi}_e$ and $\underline{\phi}_0$.

We therefore also subtract

$$-\delta [q^2 \int \underline{\phi}_e^\dagger \tau_1 \underline{\phi}_0 + p^2 \int \underline{\phi}_0^\dagger \tau_1 \underline{\phi}_e + \text{c.c.}] . \quad (4.33)$$

Thereby we have introduced the Lagrange multipliers μ and δ

(cf. (4.13) and (4.14) where μ corresponds to $-\mu_0$). In fact,

now carrying out the proposed variations, we obtain the equations

$$\mathcal{K}_{\text{eff}} \varphi_e = \mu \tau_1 \varphi_e + \delta \tau_1 \varphi_o \quad (4.34a)$$

$$\mathcal{K}_{\text{eff}} \varphi_o = \mu \tau_1 \varphi_o + \delta \tau_1 \varphi_e, \quad (4.34b)$$

of the form desired, with

$$\mathcal{K}_{\text{eff}} = \mathcal{K} + \mathcal{V}_{\text{eff}}, \quad (4.35)$$

$$\begin{aligned} \mathcal{V}_{\text{eff}}(r) &= \int |r-r'|^{-1} [q^2 \varphi_e^\dagger(r') \tau_1 \varphi_e(r') + p^2 \varphi_o^\dagger(r') \tau_1 \varphi_o(r')] \\ &\equiv \int |r-r'|^{-1} \rho_{\text{eff}}(r') \quad (4.36) \end{aligned}$$

Here the self-consistent Hamiltonian $\mathcal{K}_{\text{eff}}$ is to be subcritical, but to correspond to a situation in which the levels defined by φ_e and φ_o dominate the situation. We want to be in the neighborhood $\delta \sim 1$ (where unity means the energy change upon changing Z by one).

From Eq. (4.34) we can derive the following equations for q^2 and p^2 ,

$$q^2 U_e + p^2 U_{e,o} = A + \delta, \quad (4.37a)$$

$$q^2 U_{e,o} + p^2 U_o = B + \delta. \quad (4.37b)$$

These equations can all be simplified by the choice $p = 0$, but this simplification will not be carried out here.

The procedure just carried out provides the basis for a full discussion of H_{red} in the strong coupling regime $Z' \gg 1$. We shall allude only very briefly to this matter, since numerical studies are in progress and we intend to publish a full account elsewhere. One studies the Hamiltonian

$$H_{\text{red}}' = H_{\text{red}} - \mu Q, \quad (4.38)$$

writing the operator $\phi(r)$ as the sum of a classical and a quantum part

$$\phi(r) = \phi_{\text{eff}}(r) + q \chi(r), \quad (4.39)$$

where

$$\phi_{\text{eff}}(r) = q_0 \phi_e(r) + i p_0^\dagger \phi_0(r), \quad (4.40)$$

$$\chi(r) = q \phi_e(r) + i p^\dagger \phi_0(r). \quad (4.41)$$

There follows straightforward but lengthy algebra. To terms of relative order $(Z')^{-1}$, we find

$$H_{\text{red}}' \cong -\frac{1}{2} e^2 \int \phi_{\text{eff}}(r) |r-r'|^{-1} \phi_{\text{eff}}(r') + \frac{1}{2} e^2 \int \eta_{\text{op}}(r) |r-r'|^{-1} \eta_{\text{op}}(r'), \quad (4.42)$$

where

$$\eta_{\text{op}}(r) = \phi_{\text{eff}}^\dagger(r) \tau_1 \chi(r) + \text{h.c.} \quad (4.43)$$

Since η_{op} is Hermitian, the second term of (4.42) is positive definite. It is also quadratic in the variables $q, q^\dagger, p, p^\dagger$.

By a linear canonical transformation it can be brought to the form

$$\alpha P^\dagger P + \beta Q^\dagger Q , \tag{4.44}$$

where α and β are positive. All terms omitted from (4.42) can be treated by perturbation theory. This treatment then exhibits once more the existence of stable solutions.

V. Methods for the Study of Pion Condensation in Infinite Matter.

A. Neutral Pion Condensate.

Because it is formally simplest, we shall first consider the theory of a neutral pion condensate. It is convenient to use the quantum theory associated with the conventional second order wave-equation.

The action integral

$$I = \frac{1}{2} \int d^4x [-(\partial_\mu \phi)^2 - m_\pi^2 \phi^2] + \int d^4x \mathcal{L}^{(int)}(\phi, \partial_\mu \phi) + I_N, \quad (5.1)$$

where I_N is the action depending on variables other than ϕ , leads to the formal equations of motion

$$\partial_\mu^2 \phi - m_\pi^2 \phi = j, \quad (5.2)$$

$$j(x) = - \left(\frac{\partial \mathcal{L}^{int}}{\partial \phi(x)} - \partial_\mu \cdot \frac{\partial \mathcal{L}^{int}}{\partial \partial_\mu \phi(x)} \right). \quad (5.3)$$

with the definition

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(x)} = \dot{\phi}(x) + \frac{\partial \mathcal{L}^{int}}{\partial \dot{\phi}(x)}, \quad (5.4)$$

we compute the Hamiltonian

$$\begin{aligned} H = & \frac{1}{2} \int d^3x [\pi^2 + (\nabla \phi)^2 + m_\pi^2 \phi^2] \\ & - \int d^3x [\mathcal{L}^{(int)} + \frac{1}{2} \left(\frac{\partial \mathcal{L}^{int}}{\partial \dot{\phi}} \right)^2] + H_N. \end{aligned} \quad (5.5)$$

If, as we assume, the quantization is consistent, we derive the equations of motion

$$\begin{aligned}\dot{\pi} &= (\nabla^2 - m_\pi^2)\phi + \left(\frac{\partial \mathcal{L}^{(int)}}{\partial \phi} - \nabla \cdot \frac{\partial \mathcal{L}^{(int)}}{\partial \nabla \phi} \right), \\ \dot{\phi} &= \pi - \frac{\partial \mathcal{L}^{(int)}}{\partial \dot{\phi}},\end{aligned}\quad (5.6)$$

which upon elimination of π return Eq. (5.2). Formally we have

$$\begin{aligned}\dot{\pi} &= - \frac{\delta H}{\delta \phi}, \\ \dot{\phi} &= \frac{\delta H}{\delta \pi}.\end{aligned}\quad (5.7)$$

The occurrence of a condensate is signaled by a non-vanishing expectation value of ϕ and possibly of π . Because $\delta \langle H \rangle = \langle \delta H \rangle$, and considering a c-number variation we conclude that

$$\left\langle \frac{\delta H}{\delta \phi} \right\rangle = \frac{\delta \langle H \rangle}{\delta \langle \phi \rangle}, \text{ etc.} \quad (5.8)$$

Since there are no conservation laws constraining the variation of $\langle \phi \rangle$ and of $\langle \pi \rangle$, we now derive from (5.7)

$$\delta \langle H \rangle = 0 = \int [-\langle \dot{\pi} \rangle \delta \phi + \langle \dot{\phi} \rangle \delta \pi] \quad (5.9)$$

and we conclude that

$$\langle \dot{\pi} \rangle = \langle \dot{\phi} \rangle = 0. \quad (5.10)$$

Thus a neutral pion condensate wave function must be time independent.

Let $\langle \phi(x) \rangle = \varphi(r)$ and $\langle j(x) \rangle = J(r)$. Then the formally exact equation determining the condensate wave-function following either from (5.2) or (5.6) is

$$(\nabla^2 - m_\pi^2) \varphi(r) - J(r) = 0. \quad (5.11)$$

We now imagine that all other field equations, namely (5.6) as well as ^{these} for the "nucleon" variables have been solved in terms of $\varphi(r)$ and used to express J as a functional $J[\varphi]$. We also assume that we are interested in values of $\varphi(r)$ not too distant from the threshold of the condensation. If φ is a pseudoscalar, then J will be an odd functional of φ . Thus we write

$$J(r) = \int \mathcal{I}(r-r', 0) \varphi(r') + \frac{1}{3} \int \Lambda(r-r', r'-r'', r''-r''') \varphi(r') \varphi(r'') \varphi(r''') + \dots, \quad (5.12)$$

where, for instance

$$\begin{aligned} \mathcal{I}(r-r', 0) &= \int d(t-t') \delta \langle j(x) \rangle / \delta \langle \varphi(x) \rangle \Big|_{\langle \varphi \rangle = 0} \\ &= \int d(t-t') \mathcal{I}(r-r', t-t'), \end{aligned} \quad (5.13)$$

and $\mathcal{I}(r, t)$ is the pion self-energy. The quantity Λ is the corresponding pion-pion scattering kernel. The quantities $\pi, \Lambda, \dots$ characterize the normal fluid.

Since for sufficiently small condensate density (5.11) and (5.12) must constitute the exact equation determining the condensate function, we can *surmise* that it must be the variational condition following from $W[\varphi] = \langle H \rangle$ after all matrix elements other than φ have been eliminated by solution of the appropriate equations of motion. That is, Eqs. (5.11), (5.12) must be equivalent to the condition $\delta W[\varphi] / \delta \varphi = 0$. We conclude that the

condensation energy has the form

$$\begin{aligned}
 W[\varphi] = & \frac{1}{2} \int d^3x [(\nabla\varphi)^2 + m_\pi^2 \varphi^2] \\
 & + \frac{1}{2} \int \varphi(\underline{r}) \Pi(\underline{r}-\underline{r}', 0) \varphi(\underline{r}') + \frac{1}{4!} \int \varphi(\underline{r}) \varphi(\underline{r}') \varphi(\underline{r}'') \varphi(\underline{r}''') \\
 & \times \Lambda(\underline{r}-\underline{r}', \underline{r}'-\underline{r}'', \underline{r}''-\underline{r}''') \quad . \quad (5.14)
 \end{aligned}$$

In this method, which is essentially that of Migdal⁽¹⁴⁾, the problem "reduces" to obtaining convergent series for Π and for Λ . We shall comment in Sec. VI on the extent to which this has been done.

It is important, in attempting to resolve the various controversies that have arisen, to remark that Migdal's method differs in principle from the mean field method as it has been applied by a number of authors.⁽¹⁷⁻²¹⁾ To analyze this situation let us choose an example where

$$\mathcal{L}^{(int)} = -j\varphi \quad (5.15)$$

and j is independent of φ . With $\varphi = \varphi + \varphi'$, $\langle \varphi' \rangle = 0$, we compute

$$\begin{aligned}
 W[\varphi] = & \frac{1}{2} \int [(\nabla\varphi)^2 + m_\pi^2 \varphi^2] + \frac{1}{2} \int \langle [\pi'^2 + (\nabla\varphi')^2 + m_\pi^2 \varphi'^2] \rangle \\
 & + \int \langle j \rangle \varphi + \int \langle j \varphi' \rangle + \langle H_N \rangle \quad (5.16)
 \end{aligned}$$

as a formally exact expression. We observe first that the equations of motion for the nucleons are contained in the statement

$$\delta H_N + \int \delta j = \delta H_N + \int \delta(j\varphi) - \int j \delta\varphi = \int \sum_\alpha [-P_\alpha \delta Q_\alpha + \dot{Q}_\alpha \delta P_\alpha] \quad (5.17)$$

where Q_α , P_α are the canonical pairs for the nucleons. Consequently for the average in the ground state

$$\begin{aligned}
& \delta \langle H_N \rangle + \int [\langle j \rangle \varphi] + \delta \int \langle j \varphi' \rangle - \int \langle j \rangle \delta \varphi \\
& = \delta \langle H_N \rangle + \int \varphi \delta \langle j \rangle + \int \delta \langle j \varphi' \rangle = 0
\end{aligned} \tag{5.18}$$

Next we notice that in the mean field approximation we omit the second and fourth terms of (5.16), i.e.

$$W[\varphi] = \frac{1}{2} \int [(\nabla \varphi)^2 + m_\pi^2 \varphi^2] + \int \langle j \rangle \varphi + \langle H_N \rangle \tag{5.19}$$

Variation of this expression with respect to φ yields in consequence of (5.18), the equation

$$(-\nabla^2 + m_\pi^2) \varphi + J = \delta \int \langle j \varphi' \rangle / \delta \varphi, \tag{5.20}$$

which differs by the right hand side from the correct equation (5.11). This formal discrepancy can be rectified if we determine $\langle j \rangle$ from (5.18), omitting the last term. If this procedure is understood as part of the mean field approximation, we conclude that the latter is an approximate version of (5.14).

In practice then the method would consist in assuming a trial form for the condensate function $\varphi(r)$ for insertion into (5.14). With the further supposition that we have believable forms for π and Λ , we then seek a minimum for $W[\varphi]$ as a function of the parameters of $\varphi(r)$ and of the nuclear density. Further comment on this case is delayed to Section VI. Any tenable model must yield Λ as a positive definite operator in order to insure stability of the condensate phase. For the neutral pion case the

condensation energy can be determined without calculation of any observables relating to the nucleon components!

B. Charged Pion Condensate.

This case involves subtleties not present for the neutral case. We take for granted the steps analogous to (5.1) - (5.4) and proceed immediately to the Hamiltonian

$$H = \int [\pi^\dagger \pi + \nabla \phi^\dagger \cdot \nabla \phi + m_\pi^2 \phi^\dagger \phi] - \int [\mathcal{L}^{(int)} + \frac{\partial \mathcal{L}^I}{\partial \phi^\dagger} \frac{\partial \mathcal{L}^I}{\partial \phi}] + H_N . \quad (5.21)$$

where, e.g.

$$\pi(\underline{r}, t) = \partial \mathcal{L} / \partial \dot{\phi}^\dagger(\underline{r}, t) = \dot{\phi} + \partial \mathcal{L}^{(int)} / \partial \dot{\phi} . \quad (5.22)$$

The considerations leading to (5.10) must now be altered. The variation of $\langle H \rangle$ is now subjected to the constraint of the various conservation laws. For example, imagine a system consisting of charged pions, neutrons and protons. Let

$$\rho_\pi = i(\phi^\dagger \pi - \pi^\dagger \phi) \quad (5.23)$$

be the pion charge density ^{operator} in units of the electron's charge and let ρ_p and ρ_n be the proton and neutron contribution to the baryon number operators. The variation of the energy is to be carried out subject to conservation of charge and of nucleons, i.e.

$$\delta \langle H \rangle - \mu_\pi \int [\rho_\pi - \rho_p] - \mu_n \int (\rho_n + \rho_p) = 0 . \quad (5.24)$$

This equation has several important consequences. First, if we consider only the variation of the pion field, in place of (5.10) we obtain

$$\begin{aligned}\dot{\langle\phi\rangle} &= \dot{\phi}(\underline{r},t) = -i\mu_{\pi}\phi(\underline{r},t) \quad , \\ \dot{\langle\pi\rangle} &= \dot{\pi}(\underline{r},t) = -i\mu_{\pi}\pi(\underline{r},t) \quad ,\end{aligned}\tag{5.25}$$

or, e.g.

$$\phi(\underline{r},t) = \exp[-i\mu_{\pi}t]\phi(\underline{r}) \quad .\tag{5.26}$$

Second, we must be clear that μ_{π} is the chemical potential for adding unit negative ^{charge} at equilibrium, which is not ^{at} all the same as adding a negative pion. Thus the equilibrium condensate will in general contain some relative proportions of $\pi_{\pm}$ and protons and μ_{π} is determined by addition of charge maintaining these proportions. Thus even in the limit of vanishing condensate $\mu_{\pi} \neq \mu_{\pi^{-}}$ in general. Some confusion in the literature on this point has been settled by Sawyer.⁽¹⁹⁾ Finally we see that μ_n is the chemical potential for adding a baryon (or a neutron) and

$$\mu_n = \mu_{\pi} + \mu_p \quad .\tag{5.27}$$

We now turn to the equations required to study condensate possibility. The analogue of (5.14) is now

$$\begin{aligned}
W[\varphi] = & \int [|\nabla\varphi|^2 + (m_\pi^2 - \mu_\pi^2)|\varphi|^2] \\
& + \int [\varphi^*(\mathbf{r}) \mathcal{T}(\mathbf{r}-\mathbf{r}'; \mu_\pi) \varphi(\mathbf{r}')] \\
& + \left(\frac{1}{2!}\right)^2 \int \varphi^*(\mathbf{r}) \varphi^*(\mathbf{r}') \Lambda(\mathbf{r}-\mathbf{r}'', \mathbf{r}'-\mathbf{r}''', \mathbf{r}-\mathbf{r}') \varphi(\mathbf{r}'') \varphi(\mathbf{r}'''). \quad (5.28)
\end{aligned}$$

The main change (in addition to obvious factors and definitions) is that frequency μ_π must replace frequency zero in both $\mathcal{T}$ and Λ . The problem of the charged condensate is further complicated in that the dependence on μ_π and μ_N , implicit in (5.28) must be eliminated through the specification of charge density ρ_Q and baryon density ρ_B . This means a computation of

$$\langle [\rho_\pi - \rho_p] \rangle = \rho_Q ,$$

$$\langle [\rho_p + \rho_N] \rangle = \rho_B , \quad (5.29)$$

as part of the minimization procedure. As will be seen in the next section, these quantities can also profitably be treated as functional series in the condensate wave function.

VI. Summary and Critique of Calculations on Possible Pion Condensation in Infinite Matter.

A. Crude Model for Pion Condensation in Neutron Matter.

We shall first show, using the method of Migdal consistently that π^- condensation can occur in a highly simplified model. Our result will agree with that given by Sawyer⁽¹⁹⁾, using another method.

The transition point is determined by the solution of the equation

$$k^2 + m_\pi^2 - \mu_\pi^2 + \Pi(k, \mu_\pi) = 0 \quad (6.1)$$

We treat Π to lowest order perturbation theory in the fixed source limit. Below it is shown that this is equivalent to the mean field approximation in the same limit. The kernel Π is calculated in a familiar way from the limit of the simple closed loop diagram containing a neutron hole and a proton particle. The result is

$$\Pi(k, \omega) = - \frac{2f^2 k^2 \rho}{\omega m_\pi^2} \equiv - \frac{2M_k^2 \rho}{\omega} \quad (6.2)$$

Here $\rho = \rho_B$ is the baryon density and $f^2 = 1.1$.

To utilize (6.1) and (6.2) we need a relation between ρ and μ_π . This is obtained from the first of Eqs. (5.29) for $\rho_Q = 0$, namely

$$\langle \rho_\pi \rangle = \rho_p \quad (6.3)$$

Consider a plane wave condensate in the z direction

$$\begin{aligned}\langle \phi(r) \rangle &= \phi \exp(ikz) \\ \langle \pi \rangle &= \langle \dot{\phi} \rangle = i\mu_\pi \phi \exp(ikz),\end{aligned}\quad (6.4)$$

We thus have

$$\langle \rho_\pi \rangle = 2\mu_\pi |\phi|^2. \quad (6.5)$$

To calculate ρ_p , the correct Hamiltonian to use in the mean field approximation is the sum of the pion part and of an interaction of form

$$H^{(int)}(t) = -\sqrt{2}[iM_k \psi_n^\dagger(t)\sigma_3\psi_p(t)\phi + \text{h.c.}] \quad (6.6)$$

Here $\{\psi_p, \psi_p^\dagger\} = 1$, etc. and

$$\langle \psi_n^\dagger \psi_n \rangle = \rho_n, \quad \langle \psi_p^\dagger \psi_p \rangle = \rho_p \quad (6.7)$$

These are boundary values of the Green's function

$$G^{(ab)}(t) = i\langle T(\psi_a(t)\psi_b^\dagger(0)) \rangle, \quad (6.8)$$

Thus

$$\rho_p = iG^{(pp)}(t \rightarrow 0_-) = iG^{(pp)}(0) = \frac{i}{2\pi} \oint_{UHP} d\omega G^{(pp)}(\omega), \quad (6.9)$$

where

$$\tilde{G}^{(ab)}(\omega) = \int e^{i\omega t} G^{(ab)}(t) dt, \quad (6.10)$$

and the contour is closed in the upper half plane.

Including terms $[-\mu_n \psi_n^\dagger \psi_n - \mu_p \psi_p^\dagger \psi_p]$ in the Hamiltonian, standard methods yield the following equations

$$\begin{aligned}
(\partial_t - i\mu_p) G^{(pp)}(t) &= i \delta(t) + \sqrt{2} M_k \sigma_3 \varphi^* G^{(np)}(t) \\
(\partial_t - i\mu_n) G^{(np)}(t) &= -\sqrt{2} M_k \sigma_3 \varphi G^{(pp)}(t) . \quad (6.11)
\end{aligned}$$

Utilizing (6.10), we obtain

$$\begin{aligned}
\tilde{G}^{(np)}(\omega) &= \frac{-i}{\omega + i\mu_n} \sqrt{2} M_k \varphi \sigma_3 \tilde{G}^{(pp)}(\omega) \\
&= i \tilde{G}_0^{(nn)}(\omega) \sqrt{2} M_k \varphi \sigma_3 \tilde{G}^{(pp)}(\omega) , \quad (6.12)
\end{aligned}$$

where

$$[\tilde{G}_0^{(ab)}(\omega)]^{-1} = -(\omega + i\mu_a) \delta_{ab} , \quad (6.13)$$

and

$$\tilde{G}^{(pp)}(\omega) = \tilde{G}_0^{(pp)}(\omega) + \tilde{G}_0^{(pp)}(\omega) \tilde{G}_0^{(nn)}(\omega) 2M_k^2 |\varphi|^2 \tilde{G}^{(pp)}(\omega) . \quad (6.14)$$

Equation (6.14) is easily solved exactly, but comparing (6.3), (6.5), (6.9) and (6.14), it is seen to be sufficient to replace $\tilde{G}^{(pp)}$ by $\tilde{G}_0^{(pp)}$ in the second term of (6.14). We can then calculate ρ_p from (6.9). To do this we need the representations

$$\begin{aligned}
\tilde{G}_0^{(pp)}(\omega) &= \frac{-\rho_p}{\omega + i\mu_p - i\eta} - \frac{(1-\rho_p)}{\omega + i\mu_p + i\eta} \\
&\xrightarrow{\rho_p \rightarrow 0} \frac{-1}{\omega + i\mu_p + i\eta} , \quad (6.15)
\end{aligned}$$

$$\tilde{G}_0^{(nn)}(\omega) = \frac{-\rho}{\omega + i\mu_n - i\eta} - \frac{(1-\rho)}{\omega + i\mu_n + i\eta} . \quad (6.16)$$

We then obtain $(\mu_\pi = \mu_n - \mu_p)$ from (6.9)

$$\rho_p = \frac{2M_k^2 |\varphi|^2}{\mu_\pi^2} \rho, \quad (6.17)$$

or remembering (6.3) and (6.5), we find

$$\mu_\pi^3 = \rho M_k^2. \quad (6.18)$$

With (6.1) and (6.2), (6.18) yields

$$\omega = \sqrt{3} \mu_\pi, \quad (6.19)$$

$$\rho = \omega^3 / M_k^2 (3)^{3/2}. \quad (6.20)$$

The minimization of (6.20) now gives

$$k_c = \sqrt{2} m_\pi, \quad (6.21)$$

$$\rho_c = m_\pi^3 / 2f^2. \quad (6.22)$$

The specific numbers here are of little interest, because of the crudity of the model. The occurrence of a phase transition in this simplified model is of great interest as a stimulus for more careful and detailed investigation. It is important to emphasize that the transition should be characterized as a charged pion phase transition. The dynamics will determine the ratio (m_{π_+} / m_{π_-}) . For a plane wave mode, it is simple to show⁽²¹⁾ that

$$n_{\pi_+} / n_{\pi_-} = (\omega - \mu_\pi)^2 / (\omega + \mu_\pi)^2. \quad (6.23)$$

As one more formal result we shall show that in the approximation employed, Migdal's method and the mean field method are the same. In this connection (6.18) is already the mean field result

(as well as the perturbation result). It remains to show that the minimization of the mean field energy at threshold agrees with (6.1) and (6.2). We have first

$$\begin{aligned} \langle H_\pi \rangle &= \langle \int [\pi^\dagger \pi + \nabla \phi^* \nabla \phi + m_\pi^2 \phi^\dagger \phi] \rangle \\ &= [\mu_\pi^2 + k^2 + m_\pi^2] |\phi|^2. \end{aligned} \quad (6.24)$$

We must now calculate the expectation value of (6.6). Toward this end we need the solution (6.14). We find that $G^{(pp)}(\omega)$ has two poles⁽²¹⁾

$$E_\pm = -\frac{1}{2}(\mu_p + \mu_n) \pm \left[\mu_\pi^2 + 8M^2 |\phi|^2 \right]^{\frac{1}{2}}, \quad (6.25)$$

and that boundary conditions can be incorporated by writing

$$\tilde{G}^{(pp)}(\omega) = -\frac{(\omega + \mu_n)}{(E_+ - E_-)} \left[\frac{\rho_+}{\omega - E_+ - i\delta} + \frac{(1 - \rho_+)}{\omega - E_+ + i\delta} + (+ \leftrightarrow -) \right], \quad (6.26)$$

where $\rho_\pm$ will be determined imminently (with the equal help of the easily determined corresponding neutron Green's function

$$\tilde{G}^{(nn)}(\omega) = -\frac{(\omega + \mu_p)}{(E_+ - E_-)} [\quad], \quad (6.27)$$

where the bracket is the same as in (6.26)).

The application of (6.9) and the corresponding equation for ρ_n leads upon inversion to the following determination

$$\mu_\pi \rho_\pm = - (E_\mp + \mu_p) \rho_p + (E_\mp + \mu_n) \rho_n. \quad (6.28)$$

With the aid of the analogue of (6.9), (6.12), (6.26) and (6.28),

we now calculate

$$\begin{aligned} \langle \psi_p^\dagger \sigma_3 \psi_n \rangle &= -i \sqrt{2} M_k \varphi [\rho_+ - \rho_-] / [E_+ - E_-] \\ &= i \sqrt{2} M_k \varphi [\rho - 2\rho_p] / \mu_\pi, \end{aligned} \quad (6.29)$$

which we shall utilize in the limit $\rho_p \rightarrow 0$. This means that to terms quadratic in $|\varphi|^2$ the mean field energy becomes

$$W[\varphi] = \langle H \rangle = [\mu_\pi^2 + k^2 + m_\pi^2 - \frac{4M_k^2 \rho}{\mu_\pi}] |\varphi|^2. \quad (6.30)$$

The square bracket is, however, the same as (6.1) and (6.2) if we remember (6.18). Since the condition for transition is $\bar{W} = 0$, we have reached the result sought.

In contrast to the charged pion case, the neutral pion case does not have a sensible fixed nucleon limit.⁽¹⁹⁾ However, mean field theory, including the nucleon kinetic energy, would again appear to yield a possible second order phase transition.

B. Present Status and Possible Future Program.

It *requires* unusual optimism to accept uncritically the conclusions of any calculations of pion condensation carried out to date. For instance, there is no calculation which takes consistently into account at the same time all of the following effects present in the pion self-energy operator. (i) Resonant pion-nucleon scattering, (ii) Multiple pion-nucleon scattering, (iii) Short range nuclear correlations, (iv) Pion-pion interaction.

Regarding the problem from the point of view of Migdal's method, the main task is the calculation of the pion-self energy - quite analogous to the problem of computing the pion-optical potential in nuclear matter. It is our opinion that if the interesting range of densities does not much exceed nuclear densities, then the most promising approach to the problem is a nucleon hole line expansion, analogous to the Brueckner-Goldstone expansion for nuclear matter. A start has been made on this program⁽³⁰⁻³³⁾, but much work remains to be done. Similar methods⁽³⁴⁾ should suffice for the calculation of the nucleon densities needed for the study of the conservation laws of charge, baryon number, etc. We hope to return to these matters in a future publication.

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Figure Captions

- Fig. 1. Energy levels of an electron in the Coulomb field of the extended charge distribution of a nucleus (from ref. 3).
- Fig. 2a. Illustration that the equation $F(x-\epsilon) = x$ has a real solution for $\epsilon > 0$, indicating a stable one-electron bound state, $1s_{1/2}$.
- Fig. 2b. Illustration that for $\epsilon < 0$, the same equation has no real solution, indicating an unstable one-electron state $1s_{1/2}$.
- Fig. 3. Schematic representation of the lowest bound state branch for a negative pion in the Coulomb field of a superheavy nucleus (neglecting strong interactions). The branch for the positive pion related by charge conjugation is also shown.

(a)

(b)