On Standing Wave Solutions to the Inhomogeneous
Equation for Identical Particle Scattering

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Abstract

The standing wave solution to the inhomogeneous equation describing identical particle scattering is re-examined. Although the form of that solution is correct, its content is shown to be incorrect. A paradigm for constructing the correct solution, based on unitarity, is discussed, and the proper solution is then obtained. A comparison of different standing wave solutions with differing K operators is given. The results are shown to be an extension to the multi-channel case of those derived in the preceding paper for the potential well case.

1. Introduction

In an earlier article¹ [1] on the question of unitarity in approximate solutions to the set of coupled, inhomogeneous equations describing identical particle scattering, a K (reaction) operator solution was derived and discussed. As noted in the preceding paper²[2], an error was made in the derivation of the K operator solution used in I. The error resulted from use of an invalid expansion (Lippmann-Schwinger equation) for the full standing-wave Green's function. Thus, a standing wave solution was produced that was different in principle from the one actually sought, though this was not realized at the time. In A, we have compared the two standing wave solutions and their concomitant reaction matrices for the case of potential well scattering. That work is background for the development of the present article. Our purpose here is three-fold: first, we wish to prove that, with some reinterpretation, the standing wave solution finally used in I is correct; second, we extend the analysis of A to the case of identical particles; third, and closely related to our second aim, we also extend the result of A to the case of multi-channel scattering of a distinguishable particle. This latter result is actually contained in the extension to identical particles by proper interpretation of the symbols appearing in the solution, as we show below.

2. Form of the Correct Standing Wave Solution.

The proper standing wave solution to the inhomogeneous equation can be determined by using the unitarity condition on S matrix elements. Since one

1. We refer to ref. [1] as I and its equations as (I.1), (I.2) etc.

2. Reference [2] will be denoted A, and its equations as (A.1), (A.2), etc.

of the aims of the present paper is to elucidate this point, we consider it in some detail, including in our discussion some of the points noted in I.

When a particle is scattered by a system containing particles identical to it, the transition operator T becomes a linear combination of direct and exchange terms. That is, $T = \mathcal{T}$, with, for a system like the e^-H scattering system [3],

$$\mathcal{T} = T(d) - T(e)P_{12}, \quad (1)$$

where the two particle transposition operator P_{12} exchanges particle 2 initially incident with particle 1, initially bound; d refers to the direct scattering (no relabelling or rearrangement); and e refers to exchange or rearrangement scattering. Matrix elements of $\mathcal{T}$ for two particle scattering are to be taken between bound states of particle 1 and plane wave states of particle 2.

The inhomogeneous equation describing such scattering [1,4] is such that the solution to its homogeneous portion, ie, the complementary function, yields $T(d)$, while the particular integral yields $T(e)$. Numerical, approximate solutions to the inhomogeneous equation can be obtained by solving a finite set of coupled equations [4]. Let us consider the case where only the elastic channel is open. Then in a coupled representation leading to total angular momentum, the matrix elements of $\mathcal{T}$, $T(d)$, and $T(e)P_{12}$ are denoted [1] $\mathcal{T}$, $T(d)$ and $T(e)$ respectively (angular momentum labels suppressed as in I), and are obviously related via

$$\mathcal{T} = T(d) - T(e), \quad (2)$$

where

$$\mathcal{T} = e^{i\delta} \sin \delta \quad (3)$$

and

$$T(d) = e^{in} \sin \delta. \quad (4)$$

Here δ is the total scattering phase shift and n is the phase shift for "distinguishable particle" scattering. Thus, $T(e)$ must be such that, combined with $T(d)$ of (4), it leads to T of (3).

In applying the inhomogeneous equation to e^{-H} scattering, it was realized that an approximate solution as generated by the coupled equations may not yield values of $T(e)$ which lead to $T = \exp(i\delta) \sin \delta$, i.e., to a unitary S-matrix. The purpose of I was to show how to construct a reaction (K) matrix solution. That is, since matrix elements S of the scattering matrix, (angular momentum labels suppressed here also) are unitary, we know that S can always be written as

$$S = \frac{1+iK}{1-iK} = e^{2i\delta} \quad (5)$$

where K (in this instance $\tan \delta$) must be the matrix element of a properly symmetrized K operator χ composed of direct and exchange components. Since we also know that

$$S = 1 + 2iT, \quad (6)$$

with T given by Eq. (4), then it follows that

$$T = K/(1-iK), \quad (7)$$

which is a form of the damping equation that includes the effects of the Pauli principle.

The form of standing wave solution actually used in I was argued to be correct because it lead to Eq. (7). Unitarity is thus the key to choosing the proper standing wave solution out of the variety one can actually construct. Unfortunately, even though the form of the solution used in I is correct, its detailed structure is not correct, and for the same reasons that $|\chi_k^0\rangle$ of A is not equal to the correct standing wave solution $|\psi_k^0\rangle$. We shall discuss this in more detail below; it suffices

here to note that the result used in I is the analog of $|\chi_k^0\rangle$ and not of $|\psi_k^0\rangle$.

Although unitarity through Eq. (7) is the test of a solution, at this stage we do not know the relationship of K to the direct and exchange components $K(d)$ and $K(e)$, or similarly how χ is related to $K(d)$ and $K(e)P_{12}$. From the fact that the solution to the inhomogeneous equation is a sum of the complementary function and the particular integral, and that the complementary function referring as it does to distinguishable particle scattering, must yield $K(d)$ alone, we can infer that

$$\chi = K(d) - K(e)P_{12} \quad (8)$$

or that

$$K = K(d) - K(e). \quad (8')$$

That is, we require that the particular integral must yield the quantity $K(e)$.

Equations (8) and (8') are in fact inevitable and not simply inferential. From Eqs. (2), (3), (4) and (7) we can easily show that no form for $K(e)$ other than $K(e) = K(d) - K$ is allowed; $K(e) = f(K, K(d))$ with f an arbitrary function is not a solution of these equations. From this it follows that (8) is the desired relation³ between χ , $K(d)$ and $K(e)P_{12}$.

The paradigm for constructing the proper standing wave solution to the inhomogeneous equation is therefore as follows. The complementary function is to yield $K(d)$, while the particular integral is to yield $K(e)$ and, as we shall see, is to have the same asymptotic normalization as appears in the complementary function. The total matrix element is then obtained by using Eq. (8'). These rules will hold for both an exact solution and an approximate one, since even in an approximation, $K = \tan \delta'$ and $K(d) = \tan \eta'$, where δ' and η' are

3. We assume that there are no quantities in χ additional to $K(d)$ and $K(e)$ whose on-shell matrix elements are zero.

approximations to δ and η . We may conclude that unitarity plus the known form of $\tilde{J}$ and the relations $T = K/(1-iK)$ and $T(d) = K(d)/(1-iK(d))$ suffice to specify the proper standing wave solution.

With this paradigm available, we can now re-examine the standing wave solution to the inhomogeneous equation for e^-+H scattering. As noted in I, the e^-+H system is a prototype for the more general scattering problem involving one particle incident on a target containing particles identical to it. Hence these specific results are easily generalized.

We first redefine the problem and introduce the notation, most of which is the same as in I. We wish to solve

$$(E-H)|\psi_{\underline{k}}\rangle = -V_1|\phi_{\underline{k}}(1,2)\rangle \quad (9)$$

where, as in I,

$$\begin{aligned} H &= H_1 + H_2 + v_{12} \\ H_i &= -\frac{\hbar^2}{2M} \nabla_i^2 + v_i \\ v_{12} &= e^2/r_{12} \\ v_i &= -e^2/r_i \\ V_1 &\equiv v_1 + v_{12} \end{aligned}$$

and

$$|\phi_{\underline{k}}(i,j)\rangle = |\varphi_0(i)\phi_{\underline{k}}(j)\rangle,$$

with

$$H_1|\varphi_\alpha(i)\rangle = \epsilon_\alpha|\varphi_\alpha(i)\rangle$$

and

$$\langle \underline{r} | \phi_{\underline{k}}(i) \rangle = (2\pi)^{-3/2} e^{i\mathbf{k} \cdot \mathbf{r}_i}.$$

We use $\alpha=0$ to label the target ground state so that $|\psi_{\underline{k}}\rangle$ is always [1,4]

to be expanded via the set $\{\phi_a(1)\}$. Hence, in the direct term, particle 2 is taken to be incident and emergent. The right hand side of (11), containing $|\phi_k(1)\rangle$, has particle 1 incident and thus is the exchange term. We have departed from the notation $\Xi_k^\pm$ used in I, replacing it by $|\psi_k^\pm\rangle$. Also, no distinction is made here between singlet and triplet electronic spin states. They are both included by (mentally) inserting a $\pm$ sign in front of the V_1 factor in (9). We reserve the $\pm$ superscript to indicate outgoing and ingoing wave states.

These outgoing and ingoing wave solutions to (9) are denoted $|\psi_k^\pm\rangle$ and are given by [4]:

$$|\psi_k^\pm\rangle = |\phi_k(1,2)\rangle + (E^\pm - H)^{-1}V_2|\phi_k(1,2)\rangle - (E^\pm - H)^{-1}V_1|\phi_k(2,1)\rangle \quad (10)$$

in direct analogy with Eqs. (A.10). Expansion of $(E^\pm - H)^{-1}$ via $(E^\pm - H_2)^{-1}$ [as in Eq. (A.12)] then gives

$$\begin{aligned} |\psi_k^\pm\rangle = & |\phi_k(1,2)\rangle + (E^\pm - H_2)^{-1}T^\pm(d)|\phi_k(1,2)\rangle \\ & - (E^\pm - H_2)^{-1}T^\pm(e)|\phi_k(2,1)\rangle, \end{aligned} \quad (11)$$

where

$$T^\pm(d) \equiv T(d) = [T^-(d)]^\dagger = V_2 + V_2(E^\pm - H)^{-1}V_2 \quad (12)$$

and⁴

$$T^\pm(e) \equiv T(e) = [T^-(e)]^\dagger = V_1 + V_2(E^\pm - H)^{-1}V_1 \quad (13)$$

are the direct and exchange transition operators and satisfy

$$T(d) = V_2 + V_2(E^\pm - H_2)^{-1}T(d) \quad (12')$$

and

$$T(e) = V_1 + V_2(E^\pm - H_2)^{-1}T(e). \quad (13')$$

4. An alternate possibility for the exchange transition operator, which is equal to $T(e)$ on the energy shell, is $T'(e) = V_2 + V_2(E^\pm - H)^{-1}V_1$. We shall not need this operator here.

These solutions have been discussed elsewhere [4] and we do not comment on them here, except to note that the properly symmetrized transition operator $\mathcal{T}$, given by (1), is obviously seen to occur in $|\psi_k^+\rangle$:

$$|\psi_k^+\rangle = |\phi_k(1,2)\rangle + (E^+ - H_2)^{-1} \mathcal{T} |\phi_k(1,2)\rangle. \quad (14)$$

No separation into direct and exchange components is made in (14) as it is written. However, to obtain $\mathcal{T}$, we must combine the complementary function $|\psi_k^{+c}\rangle$, obeying

$$(E - H) |\psi_k^{+c}\rangle = 0$$

whose solution is

$$|\psi_k^{+c}\rangle = |\phi_k(1,2)\rangle + (E^+ - H_2)^{-1} T(d) |\phi_k(1,2)\rangle,$$

with the particular integral $|\psi_k^{+p}\rangle$ obeying

$$(E - H) |\psi_k^{+p}\rangle = -V_1 P_{12} |\phi_k(1,2)\rangle$$

and having the solution

$$|\psi_k^{+p}\rangle = - (E^+ - H_2)^{-1} T(e) P_{12} |\phi_k(1,2)\rangle.$$

Apart from the plane wave term $|\phi_k(1,2)\rangle$, both $|\psi_k^{+c}\rangle$ and $|\psi_k^{+p}\rangle$ are, in a space representation for particle 2, asymptotic ($r_2 \rightarrow \infty$) to an outgoing wave multiplied by the relevant direct or exchange T-matrix element.

Let us now consider the standing wave solution $|\psi_k^0\rangle$. From the preceding discussion, it must be of the form

$$|\psi_k^0\rangle = |\phi_k(1,2)\rangle + (E - H_2)^{-1} \chi |\phi_k(1,2)\rangle, \quad (15)$$

where, as in I and A, we suppress the principle value symbol $\mathcal{P}$ and simply write $(E - H_2)^{-1}$ for the principle value Green's function. Equation (15) is the obvious standing wave analog of the outgoing wave solution given by Eq. (14).

The paradigm tells us that $|\psi_k^0\rangle$ is the sum of a complementary function $|\psi_k^{0c}\rangle$ of the form

$$|\psi_{\underline{k}}^{0c}\rangle = |\phi_{\underline{k}}(1,2)\rangle + (E-H_2)^{-1}K(d)|\phi_{\underline{k}}(1,2)\rangle \quad (16)$$

plus a particular integral $|\psi_{\underline{k}}^{0p}\rangle$ given by

$$|\psi_{\underline{k}}^{0p}\rangle = - (E-H_2)^{-1}K(e)P_{12}|\phi_{\underline{k}}(1,2)\rangle. \quad (17)$$

These latter two equations guarantee that χ of (15) is given by

$$\chi = K(d) - K(e)P_{12}. \quad (18)$$

We may turn the paradigm around and ask if Eq. (15) as determined through Eqs. (16) - (18) is the one we are actually seeking. The answer of course is yes, since the analysis of I clearly establishes that a solution of the form of (15) does lead to Eq. (7). Furthermore, unlike the quantity $K(d)$ appearing in Eq. (I.15) and defined by Eq. (I.16), $K(d)$ of (16) above is the proper reaction operator for distinguishable particle scattering. This is proved by linking $K(d)$ with $T(d)$ and showing that $T(d) = \exp(in) \sin n$ leads directly to $K(d) = \tan n$, for the case that only the elastic channel is open. [A simple way to carry out this proof is by using the integral equation forms for $|\psi_{\underline{k}}^{+c}\rangle$ and $|\psi_{\underline{k}}^{0c}\rangle$ obtained from use of $T(d)|\phi_{\underline{k}}(1,2)\rangle = V_2|\psi_{\underline{k}}^{+c}\rangle$ and $K(d)|\phi_{\underline{k}}(1,2)\rangle = V_2|\psi_{\underline{k}}^{0c}\rangle$ in the equations for $|\psi_{\underline{k}}^{+c}\rangle$ and $|\psi_{\underline{k}}^{0c}\rangle$. The usual damping equation follows and this leads to $T(d) = K(d)/(1-iK(d))$. Finally, by the paradigm, only a particular integral of the form of Eq. (17) can lead to (18); the normalization as given by Eq. (17) is thus assured. Hence, Eq. (15) is the desired solution.

Since $|\psi_{\underline{k}}^0\rangle$ is the sum of $|\psi_{\underline{k}}^{0c}\rangle$, Eq. (16), and $|\psi_{\underline{k}}^{0p}\rangle$, Eq. (17), we may also inquire into the meaning of the solution (I.15) previously thought to be the valid one. First we note that $|\psi_{\underline{k}}^0\rangle$ is the precise analog for the identical particle case of $|\psi_{\underline{k}}^0\rangle$ of A [Eqs. (A.7)]. The solution of I we now denote $|\chi_{\underline{k}}^{(1)}\rangle$, analogous to $|\chi_{\underline{k}}^{(1)}\rangle$ of A; it has the form

$$|x_k^{(1)}\rangle = |\phi_k(1,2)\rangle + (E-H_2)^{-1}[\tilde{K}(d)|\phi_k(1,2)\rangle - \tilde{K}(e)P_{12}|\phi_k(1,2)\rangle], \quad (19)$$

where

$$\tilde{K}(d) = V_2 + V_2(E-H)^{-1}V_2 \quad (20)$$

and

$$\tilde{K}(e) = V_1 + V_2(E-H)^{-1}V_1 \quad (21)$$

It is evident that the second two terms in (19) can be combined so that $|x_k^{(1)}\rangle$ becomes

$$|x_k^{(1)}\rangle = |\phi_k(1,2)\rangle + (E-H_2)^{-1}\tilde{\chi}|\phi_k(1,2)\rangle. \quad (22)$$

The resemblance between $|x_k^{(1)}\rangle$ and the correct solution $|\psi_k^0\rangle$ is obvious. Hence we may expect that an angular momentum expansion of (22) would lead to an asymptotic form in which the K-operator-like quantity $\tilde{\chi}$ appears as the coefficient of the irregular solution (in coordinate r_2). In I, we incorrectly assumed that $|x_k^{(1)}\rangle$ was the desired solution, and linked what we have here denoted $\tilde{K}$ (the matrix element of $\tilde{\chi}$ in a coupled representation) to $\tilde{T}$ via a damping equation. But $|x_k^{(1)}\rangle$ is not the correct solution; $\tilde{K}(d)$ and $\tilde{K}(e)$ are not the proper K operators; and thus the damping equation of I is invalid since it has the right form but the wrong contents. On the other hand, simply by using $|\psi_k^0\rangle$ in the analysis of I, we are lead to the proper (valid) damping equation because both form and contents are correct, as the paradigm guarantees for us.

We may therefore ask as to meaning of $|x_k^{(1)}\rangle$. The answer is suggested by our notation. We define a solution $|x_k^0\rangle$ by the expression

$$|x_k^0\rangle = |\phi_k(1,2)\rangle + (E-H)^{-1}\{V_2\phi_k(1,2)\rangle - V_1P_{12}|\phi_k(1,2)\rangle\}. \quad (23)$$

Therefore, $|X_k^{(1)}\rangle$, as the analogue of $|X_k^{(1)}\rangle$, is an incomplete form of the solution $|X_k^0\rangle$. Furthermore, comparison of (23) with (I.14) shows these two states to be identical. Use of the invalid Lippmann-Schwinger equation $(E-H)^{-1} = (E-H_2)^{-1}[1+V_2(E-H)^{-1}]$ leads from (23) to (19) or (22), just as its use in (I.14) lead to (I.15). Thus, apart from the presence of the exchange term in these results, we have a repetition of the situation discussed in A: $|X_k^0\rangle$ and $|X_k^0\rangle$ are precise analogues. For this reason we use the notation $\tilde{K}(d)$ and $\tilde{K}(e)$ for the quantities defined by Eqs. (20) and (21), and $\tilde{X}$ for the difference $\tilde{K}(d) - \tilde{K}(e)P_{12}$. This completes the analogy.

Our aim is now to relate $|X_k^0\rangle$ of Eq. (23) with the proper standing wave solution $|\psi_k^0\rangle$ of Eq. (15). Formally, the relation is like that derived in A. It differs in important details, however, since the plane wave term $|\phi_k(1,2)\rangle$ is altered not merely by factors like $\cos^2\delta_1$ as in the one channel case but by additions from all the open channel states when inelastic processes can occur. For this reason, our analysis extends the results of A to the multi-channel case independent of whether Pauli principle exchange effects are present. Their presence or absence simply alters the content of the results, and not their form.

We carry out the analysis of $|X_k^0\rangle$ in two stages, just as in A. First we examine the formal relation of $|X_k^0\rangle$ to $|\psi_k^0\rangle$, and then we use an angular momentum expansion to relate partial wave elements.

As in A, an important ingredient is the damping equation in the identical particle case. We see a form of equation that involves $\mathcal{T}$, $\mathcal{X}$ and $\delta(E-H_2)$ and reduces to (7). The desired result is

$$\mathcal{T} = \mathcal{X} + i\pi\mathcal{X}\delta(E-H_2)\mathcal{T}. \quad (24)$$

By taking matrix elements of both sides of (24) between states $|\phi_{\underline{k}}(1,2)\rangle$ and $\langle\phi_{\underline{k}}(1,2)|$ and performing an angular momentum expansion to get states of good J as in I, it is easy to show that (24) leads, in the matrix notation of I employed so far, to

$$\mathbf{T} = \mathbf{K} + i\pi\mathbf{K}\mathbf{T} \quad (25)$$

whose solution for $\mathbf{T}$ is just Eq. (7). There are two important aspects of the ansatz (24). First, only properly symmetrized quantities enter it. Second, because we have chosen $\{\phi_{\alpha}(1)\}$ as our expansion set, the delta function $\delta(E-H_2)$ rather than, say, $\delta(E-H_1-H_2)$ or $\delta(E-H)$ appears in (24) [i.e., $H_1 + H_2$ and H are symmetrical under interchange of 1 and 2; H_2 is not]. Equation (24) is the identical particle analog of the usual Heitler damping equation [5] used in A. We shall comment on the derivation of (24) elsewhere since we only need its form here.

3. Comparison of $|\psi_{\underline{k}}^0\rangle$ and $|X_{\underline{k}}^0\rangle$.

Just as in A, we see almost by inspection that $|X_{\underline{k}}^0\rangle$ is given by

$$|X_{\underline{k}}^0\rangle = \frac{1}{2} [|\psi_{\underline{k}}^+\rangle + |\psi_{\underline{k}}^-\rangle]; \quad (26)$$

i.e., $|X_{\underline{k}}^0\rangle$ is one-half the sum of the outgoing and ingoing wave solutions to the inhomogeneous equation.

At this stage, the analysis is so similar to that of A that we can almost write out the results by simply replacing the quantities used in A with those defined above.

Comparison of (26) with (11) and (14) leads to

$$|X_{\underline{k}}^0\rangle = |\phi_{\underline{k}}(1,2)\rangle + (E-H_2)^{-1} \frac{1}{2} (\mathcal{J}^+ \mathcal{J}^+) |\phi_{\underline{k}}(1,2)\rangle - \frac{1}{2} i\pi\delta(E-H_2) (\mathcal{J}^- \mathcal{J}^+) |\phi_{\underline{k}}(1,2)\rangle, \quad (27)$$

or

$$|X_{\underline{k}}^0\rangle = (1 - \frac{1}{2} i\pi\delta(E-H_2) (\mathcal{J}^- \mathcal{J}^+)) |\phi_{\underline{k}}(1,2)\rangle + (E-H_2)^{-1} \frac{1}{2} (\mathcal{J}^+ \mathcal{J}^+) |\phi_{\underline{k}}(1,2)\rangle. \quad (28)$$

This latter equation is nearly identical to Eq. (A.22) for $|x_k^0\rangle$.

By solving Eq. (24) for $\mathcal{J}$, the combinations $\mathcal{J} \pm \mathcal{J}^\dagger$ can be found. Their use in (28) then gives

$$\begin{aligned} |x_k^0\rangle &= (1 + \pi^2 [\mathcal{X} \delta(E-H_2)]^2)^{-1} |\phi_k(1,2)\rangle \\ &\quad + (E-H_2)^{-1} (1 + \pi^2 [\mathcal{X} \delta(E-H_2)]^2)^{-1} \mathcal{X} |\phi_k(1,2)\rangle \\ &\equiv P |\phi_k(1,2)\rangle + (E-H_2)^{-1} \tilde{\mathcal{X}} |\phi_k(1,2)\rangle, \end{aligned} \quad (29)$$

with $P\mathcal{X} = \tilde{\mathcal{X}}$ in precise analogy to the distinguishable particle case. From this analogy, we can again conclude that (16) is the correct standing wave solution, and that the results of I, based on Eq. (I.15) are correct in form. As noted above, P here differs from the projection operator of A in that it contains contributions from all open channels, as we now show.

In order to evaluate P, we use the expansion [2] for $(E-H)^{-1}$

$$(E-H)^{-1} = (E-H_2)^{-1} [1 + V_2(E-H)^{-1}] - \pi^2 \delta(E-H_2) V_2 \delta(E-H) \quad (30)$$

In substituting Eq. (30) into Eq. (23) we find the alternative expression for $|x_k^0\rangle$ analogous to Eq. (A.20):

$$\begin{aligned} |x_k^0\rangle &= [1 - \pi^2 \delta(E-H_2) V_2 \delta(E-H) (V_2 - V_1 P_{12})] |\phi_k(1,2)\rangle \\ &\quad + (E-H_2)^{-1} \tilde{\mathcal{X}} |\phi_k(1,2)\rangle \end{aligned} \quad (31)$$

where $\tilde{\mathcal{X}} = \tilde{K}(d) - \tilde{K}(e) P_{12}$, with $\tilde{K}(d)$ and $\tilde{K}(e)$ given by Eqs. (20) and (21).

The term in square brackets in (31) is just P. We denote the second term by $|\mathcal{Q}\rangle$:

$$|\mathcal{Q}\rangle = - \pi^2 \delta(E-H_2) V_2 \delta(E-H) (V_2 - V_1 P_{12}) |\phi_k(1,2)\rangle. \quad (32)$$

Determination of $|\mathcal{Q}\rangle$ obviously yields P.

The evaluation of $|\mathcal{Q}\rangle$ proceeds along the same lines as in A, so we shall not give many details here. Complete sets of states will be used to

change H_2 and H to energies. The eigenstates of H_2 are ($|\phi_{\underline{k}\beta}(1,2)\rangle$):

$$H_2 |\phi_{\underline{k}\beta}(1,2)\rangle = (E_{\underline{k}\beta} + \epsilon_{\beta}) |\phi_{\underline{k}\beta}(1,2)\rangle \quad (33)$$

where $|\phi_{\underline{k}\beta}(1,2)\rangle = |\varphi_{\beta}(2)\phi_{\underline{k}\beta}(1)\rangle$ and $E_{\underline{k}\beta} = \hbar^2 k_{\beta}^2 / 2M$. We shall denote the continuum eigenstates of H by the subscripted symbol $|\psi_{\underline{k}\beta}^+(1,2)\rangle$; they are generated by the system initially in state $|\phi_{\underline{k}\beta}(1,2)\rangle$. Bound states of the full Hamiltonian H will be ignored since their energies will be assumed to be less than E and hence they cannot contribute to $|\mathcal{Q}\rangle$ because of the $\delta(E-H)$ factor.

Using these expansions in $|\mathcal{Q}\rangle$ and employing the methods of A, we get

$$\begin{aligned} |\mathcal{Q}\rangle &= -\pi^2 \sum_{\beta, \beta'} \rho_{\underline{k}\beta} \rho_{\underline{k}\beta'} \int d\Omega_{\underline{k}\beta} d\Omega_{\underline{k}\beta'} |\phi_{\underline{k}\beta}(1,2)\rangle T_{\underline{k}\beta \underline{k}\beta'}^{(d)} \\ &\quad \times [\langle \psi_{\underline{k}\beta}^+(1,2) | V_2 | \phi_{\underline{k}\beta}(1,2)\rangle - \langle \psi_{\underline{k}\beta}^+(1,2) | V_1 | \phi_{\underline{k}\beta}(2,1)\rangle] \\ &= -\pi^2 \sum_{\beta, \beta'} \rho_{\underline{k}\beta} \rho_{\underline{k}\beta'} \int d\Omega_{\underline{k}\beta} d\Omega_{\underline{k}\beta'} |\phi_{\underline{k}\beta}(1,2)\rangle T_{\underline{k}\beta \underline{k}\beta'}^{(d)} \mathcal{J}_{\underline{k}\beta \underline{k}\beta'}^{\dagger} \end{aligned} \quad (34)$$

In Eq. (34), $\mathcal{J}_{\underline{k}\beta \underline{k}\beta'}^{\dagger} = [T(d) - T(e)P_{12}]^{\dagger}$ and the prime on the β, β' summation symbol means that only those β, β' which are consistent with energy conservation are included in the sum; i.e., $E_{\underline{k}\beta} + \epsilon_{\beta} = E$ must be satisfied. The order of labels in the states occurring in the matrix elements is always (1,2). [We note parenthetically that $\underline{k}$ could have been written $\underline{k}_0$.]

The $T(d)\mathcal{J}$ product in (34) can be evaluated using Eqs. (12) and (1), plus the fact that $\langle \psi_{\underline{k}\beta}^+(1,2) | V_2 - V_1 P_{12} | \phi_{\underline{k}\beta}(1,2)\rangle = 0$. It is thus straightforward to show that

$$\mathcal{J}_{\underline{k}\beta \underline{k}\beta'} - \mathcal{J}_{\underline{k}\beta \underline{k}\beta'}^{\dagger} = -2\pi i \sum_{\beta'} \rho_{\underline{k}\beta'} \int d\Omega_{\underline{k}\beta'} T_{\underline{k}\beta \underline{k}\beta'} \mathcal{J}_{\underline{k}\beta \underline{k}\beta'}^{\dagger},$$

and therefore that $|\mathcal{Q}\rangle$ is equal to

$$|\mathcal{Q}\rangle = -\frac{\pi i}{2} \sum_{\beta} \rho_{\underline{k}\beta} \int d\Omega_{\underline{k}\beta} |\phi_{\underline{k}\beta}(1,2)\rangle [\mathcal{J}_{\underline{k}\beta \underline{k}\beta} - \mathcal{J}_{\underline{k}\beta \underline{k}\beta}^{\dagger}]. \quad (35)$$

In A, the term analogous to $\mathcal{J}_{\underline{k}\beta \underline{k}\beta} - \mathcal{J}_{\underline{k}\beta \underline{k}\beta}^{\dagger}$ was evaluated using phase shifts. In the present case, inelastic channels are open and it is more convenient to

evaluate this difference using the K matrix. The first step in doing this is to express all quantities in angular momentum eigenstates. We use the same representation as in I, viz.

$$|\phi_{\vec{k}}(1,2)\rangle = |\phi_{\vec{k}}(2)\rangle |\varphi(2)\rangle = |k_n, n'l'm'\rangle$$

and

$$|\phi_{\vec{k}}(1,2)\rangle = |k_n, nlm\rangle.$$

Then

$$\mathcal{T}_{\vec{k}\vec{k}} = \langle k_n, n'l'm' | \mathcal{T} | k_n, nlm \rangle$$

and from Eq. (I.24) we have

$$\begin{aligned} \mathcal{T}_{\vec{k}\vec{k}} - \mathcal{T}_{\vec{k}\vec{k}}^\dagger &= \sum_{JM} \sum_{L'M_L} C(L'l'J; M_L, m'M) C(LlJ; M_L m m') \\ &\quad \times Y_{L'}^{M_L}(\hat{k}_n) Y_L^{M_L*}(\hat{k}_n) [T_{n'L'l', nLl}^J - T_{n'L'l', nLl}^{J\dagger}]. \end{aligned} \quad (36)$$

In addition, we write

$$|\phi_{\vec{k}}(1,2)\rangle = \sum_{L'M_L} |k_n, L'M_L\rangle Y_L^{M_L*}(\hat{k}_n) |n'l'm'\rangle, \quad (37)$$

so that by substituting (36) and (37) into (35) and carrying out the $d\Omega_{\vec{k}_n}$ integration, we obtain

$$\begin{aligned} |\mathcal{Q}\rangle &= \frac{i}{2} \sum_{n'l'm'} \sum_{JM} \sum_{L'M_L} \left(\frac{\rho_{\vec{k}_n}}{\rho_{\vec{k}_n}'} \right)^{1/2} |k_n, L'M_L\rangle |n'l'm'\rangle Y_L^{M_L*}(\hat{k}_n) \\ &\quad \times C(L'l'J; M_L, m'M) C(LlJ; M_L m m') \\ &\quad \times \{ -\pi \rho_{\vec{k}_n}^{1/2} \rho_{\vec{k}_n}^{1/2} [T_{v',v}^J - T_{v',v}^{J\dagger}] \}, \end{aligned} \quad (38)$$

where $v' = n'l'l'$ and $v = nLl$.

The term in the curly brackets in (38) is just $T_{v',v}^J - T_{v',v}^{J\dagger}$ as defined in I. Using the damping equation result of I [ie, Eq. (7)] we have

$$\begin{aligned} T_{\nu,\nu}^J - T_{\nu,\nu}^{J\dagger} &= [K^J(1-iK^J)^{-1}]_{\nu,\nu} - [K^J(1-iK^J)^{-1}]_{\nu,\nu}^* \\ &= 2i[(K^J)^2(1+(K^J)^2)^{-1}]_{\nu,\nu}, \end{aligned}$$

the second line following from the fact that K is hermitian. We therefore obtain for $|\mathcal{Q}\rangle$ the expression

$$\begin{aligned} |\mathcal{Q}\rangle &= - \sum_{n',\ell',m'} \sum_{JM} \sum_{\substack{LM_L \\ L'M_L'}} \left(\frac{\rho_{k_{n'}}}{\rho_{k_n}} \right)^{1/2} |k_n, L'M_L'\rangle |n',\ell',m'\rangle Y_L^{M_L*}(\hat{k}_n) \\ &\quad \times C(L'\ell'J; M_L'm'M) C(L\ell J; M_L m M) [(K^J)^2(1+(K^J)^2)^{-1}]_{\nu,\nu}. \end{aligned} \quad (39)$$

We now wish to combine this result with the $|\phi_k(1,2)\rangle$ term, which can be written as

$$\begin{aligned} |\phi_k(1,2)\rangle &\equiv |k, n\ell m\rangle = \sum_{LM_L} |k_n, L'M_L'\rangle Y_L^{M_L*}(\hat{k}_n) |n\ell m\rangle \\ &= \sum_{JM} \sum_{n',\ell',m'} \sum_{\substack{LM_L \\ L'M_L'}} |k_n, L'M_L'\rangle |n',\ell',m'\rangle Y_L^{M_L*}(\hat{k}_n) \\ &\quad \times C(L'\ell'J; M_L'm'M) C(L\ell J; M_L m M) \\ &\quad \times \delta_{nn'} \delta_{LL'} \delta_{\ell\ell'} \left(\rho_{k_n} / \rho_{k_n'} \right)^{1/2} \end{aligned} \quad (40)$$

If we note that $\delta_{nn'} \delta_{LL'} \delta_{\ell\ell'} = \delta_{\nu,\nu}$, then (39) and (40) can be combined to give

$$\begin{aligned} |\phi_k\rangle + |\mathcal{Q}\rangle &= \sum_{n',\ell',m'} \sum_{JM} \sum_{\substack{LM_L \\ L'M_L'}} |k_n, L'M_L'\rangle |n',\ell',m'\rangle Y_L^{M_L*}(\hat{k}_n) \\ &\quad \times C(L'\ell'J; M_L'm'M) C(L\ell J; M_L m M) (\rho_{k_n} / \rho_{k_n'})^{1/2} \\ &\quad \times [(1 + (K^J)^2)^{-1}]_{\nu,\nu}. \end{aligned} \quad (41)$$

We notice that for a one channel case, the $n'\ell'm'$ summation in (41) contains one term ($L = L' = J$ only), so that the term in square brackets reduces to

$\cos^2 \delta_L$, thus verifying our earlier result [δ_L is the L^{th} order total phase shift, including direct and exchange effects].

By quantizing along $\hat{k}_n$ and going into a space representation in r_2 , $P|\phi_k(1,2)\rangle$ may be expressed as

$$\begin{aligned} \langle r_2 | P | \phi_k(1,2) \rangle &= \langle r_2 | \phi_k + q \rangle = (2\pi)^{-3/2} \sum_{n'l'} \sum_{LL'JM} \left(\frac{\rho_{k_n'}}{\rho_{k_n}} \right)^{1/2} |L'n'l', JM\rangle \\ &\times \sqrt{4\pi(2L+1)} C(LlJ; 0MM) F_L(k_n' r_2) \\ &\times \{ (1 + (K^J)^2)^{-1} \}_{v',v} \end{aligned} \quad (42)$$

where as in I we use $F_L(\rho)$ in place of $j_L(\rho)$ and $|L'n'l', JM\rangle$ is given by [1]

$$|L'n'l', JM\rangle = \sum_{M_L'm'} C(L'l'J; M_L'm'M) {}_1^L Y_{L'}^{M_L'}(r_2) |n'l'm'\rangle$$

The remaining term $(E-H_2)^{-1} \chi | \phi_k \rangle$ in Eq. (31) can be treated in the same way by taking its r_2 representative. We shall not give the details here.

However by combining this result with Eq. (42), we find that

$$\begin{aligned} \langle r_2 | \chi_k^0 \rangle &= (2\pi)^{-3/2} \sum_{n'l'} \sum_{LL'JM} \left(\frac{k_n'}{k_n} \right)^{1/2} |L'n'l', JM\rangle \sqrt{4\pi(2L+1)} \\ &\times C(LJ; 0MM) \tilde{\Theta}_{v',v}^J \end{aligned} \quad (43)$$

in direct analogy with Eq. (I.27). For r_2 greater than the range of forces we have (as in I)

$$\begin{aligned} \tilde{\Theta}_{v',v}^J &= F_{v',v}(k_n, r_2) \{ (1 + (K^J)^2)^{-1} \}_{v',v} \\ &+ \tilde{K}_{v',v}^J G_L(k_n, r_2). \end{aligned} \quad (44)$$

Comparison of (44) with (I.28) shows that all $F_{v',v}$ consistent with energy conservation enter into $\tilde{\Theta}_{v',v}^J$ and not just the term $v' = v$.

From the foregoing discussion we conclude that

$$\tilde{K}_{v',v}^J = \sum_{v''} [(1 + K^J)^2]^{-1} \}_{v',v''} K_{v'',v}^J.$$

Hence $\tilde{\Theta}_{v',v}^J$ of (44) and $\Theta_{v',v}^J$ of Eq. (I.28) are related by

$$\sum_{v''} [(1 + (K^J)^2)]_{vv''} \tilde{\Theta}_{v''v}^J = \Theta_{v',v}^J$$

or in matrix form

$$\Theta^J = [(1 + (K^J)^2)] \tilde{\Theta}^J$$

which is the exact analog of Eq. (A.37) in the distinguishable particle case.

We have thus proved that $|\psi_{\underline{k}}^0\rangle$ and $|\chi_{\underline{k}}^0\rangle$ are related in precisely the same way as are their distinguishable particle analogs $|\psi_{\underline{k}}^0\rangle$ and $|\chi_{\underline{k}}^0\rangle$. This relationship is of course not accidental; it can arise only if $|\psi_{\underline{k}}^0\rangle$ of (15) is the correct standing wave solution to (9).

These results do not refer in detail to the composition of the K matrix elements K and $\tilde{K}$. Hence the distinguishable particle and identical particle cases are both included in this development. The case of interest will be specified only by stating if K (or $\tilde{K}$) has an exchange component. Thus, we may regard our final results as the extension of the analysis of A to the multichannel case: the form is the same for both identical particles and distinguishable particles. In each instance, we see that, while $\Theta_{v',v}^J$ has only the initial state channel present in its regular portion (ie in F_v), $\tilde{\Theta}_{v',v}^J$ has all open channel states present such that each regular solution F_v corresponds to elastic scattering from an excited state.

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