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ABSTRACT

It is shown how a momentum space formulation of the p-d and n-d scattering problem makes possible a good calculation of the no polarization approximation. Calculations are carried out for a central Yukawa shaped Serber potential for values of the angular momentum $\ell = 0 - 6$ in the energy range 0 - 95 Mev. These show that Born's approximation is not a good approximation at any energy, even for $\ell = 3$. It is shown that a Fredholm approximation reproduces the results of the no polarization approximation for $\ell \gg 1$. The results are compared with those of Christian and Gammel and the experimental data.

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ELASTIC SCATTERING OF PROTONS AND NEUTRONS BY DEUTERONS

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I. Introduction.

It will be shown how a momentum space formulation of the p-d and n-d scattering problem makes possible a good calculation of the no polarization approximation. There need be no restriction on the energy or angular momentum. Yukawa potentials and a good approximation to the deuteron ground state wave function can be used. Coulomb forces can be included. Tensor forces and inelastic scattering are neglected, however.

The results show that Born's approximation and the no polarization approximation do not agree no matter what the energy even for $\ell = 3$. A Fredholm approximation reproduces the results of the no polarization approximation accurately for $\ell \geq 1$, but not for $\ell = 0$.

Angular distributions calculated from the no polarization approximation (and the potential used here) do not agree with experiment. This could be due to neglect of polarization¹ or we may have the wrong

¹It is not likely that it is due to neglect of inelastic scattering.

potential (perhaps we should have hard cores, or perhaps tensor forces

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are not negligible). We plan a calculation of the effect of polarization.

Without a calculation of the effect of polarization, further interpretation of our results amounts to speculation, which we prefer to avoid. We prefer the simplest possible view of our results; namely, a complete calculation of the no polarization approximation with the potential used here does not agree with experiment, and the no polarization approximation is not the same as Born's approximation, even for $\ell = 3$, no matter what the energy.

However, we feel obliged to state all we know about the problem, and so we offer the following observations.

If one uses the no polarization approximation for $\ell = 0$ and Born's approximation for $\ell \geq 1$ (this is essentially what Christian and Gammel² did), one gets good agreement with experiment. We have shown

²R. S. Christian and J. L. Gammel, Phys. Rev. 91, 100 (1953).

this by comparing our calculated 4S and 2S " $k \cot \delta$ " plots with those they found from their phase shift analysis (which was based on the assumption that Born's approximation is valid for $\ell \geq 1$) of the p-d experimental data. We do not understand why this apparently inconsistent combination of approximations works.

We have also inquired which of the phase shifts calculated in no polarization approximation seem most to cause the lack of agreement with experiment. We find that we cannot get agreement by changing the S phase shifts; we can get agreement by increasing the magnitude of the P phase shifts, especially the $4P$ phase shift, which is in the direction of Born's approximation.

Although to say so is to abandon our policy of avoiding speculation, it appears that it would be desirable if polarization were not important in S states, and had the effect of making Born's approximation more valid than the no polarization approximation in the 4P state. There are many other possibilities, however, and all that we really know firmly is contained in the simple statement of our results.

II. The No Polarization Approximation.

For n-d scattering, let particle 1 be the incoming neutron, particle 2 the proton in the deuteron, and particle 3 the neutron in the deuteron. Then let

$$\underline{w} = -\underline{r}_1 + \frac{1}{2} (\underline{r}_2 + \underline{r}_3) \quad (1)$$

$$\underline{r} = \underline{r}_2 - \underline{r}_3$$

The no polarization approximation results from the assumption that the wave function has the form

$$\psi = (1 - P_{13}) \phi(r) f(\underline{w}) \chi \quad (2)$$

where ϕ is the wave function for the ground state of the deuteron and f is a function to be determined. χ is a spin function formed from the spin functions of the incoming particle (α_1 and β_1) and the deuteron spin functions ($\alpha_2 \alpha_3$, $(\alpha_2 \beta_3 + \beta_2 \alpha_3)/\sqrt{2}$, $\beta_2 \beta_3$) in such a way to

give either $S = 3/2$ or $S = 1/2$ where S is the total spin. The factor $(1 - P_{13})$, where P_{13} exchanges the two neutrons' space and spin coordinates, anti-symmetrizes ψ .

The complete solution has the form

$$\psi = (1 - P_{13}) \sum_{\underline{K}} (\phi_{\underline{K}}^t(\underline{r}) f_{\underline{K}}^t(\underline{w}) \chi^t + \phi_{\underline{K}}^s(\underline{r}) f_{\underline{K}}^s(\underline{w}) \chi^s) \quad (3)$$

where the sum is over the bound state and the continuum states of the deuteron. The superscripts "t" and "s" refer to the triplet and singlet states of the deuteron (χ^s means χ is to formed from α_1 and β_1 and the singlet deuteron spin function $(\alpha_2 \beta_3 - \beta_2 \alpha_3)/\sqrt{2}$).

The no polarization approximation proceeds with the substitution of ψ of Eq. (2) into Schrodinger's equation. The result is multiplied through by $\phi(\underline{r})$ and integrated over $\underline{r}$. This results in an equation for $f(\underline{w})$. The name "no polarization" arises from the neglect of the continuum states which occur in Eq. (3) and the fact that after Eq. (2) is substituted Schrodinger's equation the result is projected on the $\phi(\underline{r})$ axis of a function space, the complete basis of which includes the continuum states of the deuteron. The physical idea is that the deuteron is not much distorted during the collision; that is, it remains in its ground state.

The complete solution results from the substitution of Eq. (3) into Schrodinger's equation. The result is multiplied through by $\phi_{\underline{K}}(\underline{r})$ and integrated over $\underline{r}$ for every value of $\underline{K}$. This gives an infinite set of coupled equations for the $f_{\underline{K}}(\underline{w})$. The no polarization approximation results from the neglect of the coupling of the $f(\underline{w})$ which multiplies the deuteron bound state wave function with all other $f_{\underline{K}}(\underline{w})$ except itself.

III. Momentum Space Formulation of the No Polarization Approximation.

The equation which results from the no polarization approximation may be put in the form of an integral equation. We find

$$\begin{aligned}
 f(\underline{w}) + \beta \int d\underline{r} \phi(r) \phi(|\underline{w} + \frac{1}{2} \underline{r}|) f(-\frac{1}{2} \underline{w} + \frac{3}{4} \underline{r}) & \quad (4) \\
 = \exp(i \underline{k}_0 \cdot \underline{w}) + \frac{1}{4\pi} \int d\underline{r} d\underline{w}' \frac{\exp(i \underline{k}_0 \cdot |\underline{w} - \underline{w}'|)}{|\underline{w} - \underline{w}'|} \phi(r) U(|\underline{w}' + \frac{1}{2} \underline{r}|) \\
 \left\{ \alpha \phi(r) f(\underline{w}') + \beta \phi(|\underline{w}' + \frac{1}{2} \underline{r}|) f(-\frac{1}{2} \underline{w}' + \frac{3}{4} \underline{r}) + \gamma \phi(|-\underline{w}' + \frac{1}{2} \underline{r}|) f(-\frac{1}{2} \underline{w}' - \frac{3}{4} \underline{r}) \right\}
 \end{aligned}$$

where $\underline{k}_0$ is the wave number vector of the incident particle in the center of mass system; α , β , and γ depend on the potentials³ and are given in

³In all of our calculations we use ${}^3V_{np}^+ = 1$, ${}^1V_{np}^+ = .69$, and assume that all odd parity forces are zero and that the forces are charge independent. this gives $\alpha = .5$, $\beta = -1$, $\gamma = .5$ for $S = 3/2$, and $\alpha = .90125$, $\beta = .5$, $\gamma = .90125$ for $S = 1/2$.

Ref. 2, Eqs. (9) and (10), and $U(r)$ has the meaning described in Ref. 2 at the beginning of Section III.

The Fourier integral representation of the integral over $\underline{r}$ on the right hand side of Eq. (4) is

$$\int d\underline{r} \phi(r) U(|\underline{w}' + \frac{1}{2} \underline{r}|) \{ \} = \frac{1}{\pi^2} \int d\underline{k}' b(\underline{k}') \exp(i \underline{k}' \cdot \underline{w}') \quad (5)$$

This defines $b(\underline{k}')$, and the Fourier integral theorem makes it possible to write $b(\underline{k}')$ explicitly:

$$b(\underline{k}') = \frac{1}{8\pi} \int d\underline{w}' \exp(-i \underline{k}' \cdot \underline{w}') \int d\underline{r} \phi(\underline{r}) u(|\underline{w}' + \frac{1}{2} \underline{r}|) \{ \} \quad (6)$$

We may write the Green's function in Eq. (4) as follows:

$$\frac{\exp(i \underline{k}_0 \cdot \underline{w} - \underline{w}')}{|\underline{w} - \underline{w}'|} = \frac{1}{2\pi^2} \int d\underline{k}'' \frac{\exp(i \underline{k}'' \cdot (\underline{w} - \underline{w}'))}{k''^2 - k_0^2} \quad (7)$$

where the principal part of the integral on the right is to be taken.

When Eqs. (5) and (7) are substituted into it, Eq. (4) takes the form:

$$\begin{aligned} f(\underline{w}) + \beta \int d\underline{r} \phi(\underline{r}) \phi(|\underline{w} + \frac{1}{2} \underline{r}|) f(-\frac{1}{2} \underline{w} + \frac{3}{4} \underline{r}) \\ = \exp(i \underline{k}_0 \cdot \underline{w}) + \frac{1}{8\pi^5} \int d\underline{w}' \int d\underline{k}'' \frac{\exp(i \underline{k}'' \cdot (\underline{w} - \underline{w}'))}{k''^2 - k_0^2} \int d\underline{k}' \exp(i \underline{k}' \cdot \underline{w}') b(\underline{k}') \end{aligned} \quad (8)$$

the integrals over $\underline{w}'$, $\underline{k}'$ can be carried out if it is remembered that according to Fourier's integral theorem,

$$b(\underline{k}') = \frac{1}{(2\pi)^3} \int d\underline{x} \exp(-i \underline{k}' \cdot \underline{x}) \int d\underline{k} b(\underline{k}) \exp(i \underline{k} \cdot \underline{x}) \quad (9)$$

The result is

$$\begin{aligned} f(\underline{w}) + \beta \int d\underline{r} \phi(\underline{r}) \phi(|\underline{w} + \frac{1}{2} \underline{r}|) f(-\frac{1}{2} \underline{w} + \frac{3}{4} \underline{r}) \\ = \exp(i \underline{k}_0 \cdot \underline{w}) \frac{1}{\pi^2} \int d\underline{k}'' \frac{\exp(i \underline{k}'' \cdot \underline{w})}{k''^2 - k_0^2} b(\underline{k}'') \end{aligned} \quad (10)$$

To proceed further, let

$$\beta \int d\underline{r} \phi(\underline{r}) \phi(|\underline{w} + \frac{1}{2} \underline{r}|) f(-\frac{1}{2} \underline{w} + \frac{3}{4} \underline{r}) = \int d\underline{k}' a(\underline{k}') \exp(i \underline{k}' \cdot \underline{w}') \quad (11)$$

the inverse of this is

$$a(\underline{k}') = \frac{\beta}{(2\pi)^3} \int d\underline{r} d\underline{w} \exp(-i \underline{k}' \cdot \underline{w}) \phi(\underline{r}) \phi(|\underline{w} + \frac{1}{2} \underline{r}|) f(-\frac{1}{2} \underline{w} + \frac{3}{4} \underline{r}) \quad (12)$$

Using Eq. (11), we may write Eq. (10) as

$$f(\underline{w}) = \exp(i \underline{k}_0 \cdot \underline{w}) + \frac{1}{\pi^2} \int d\underline{k}'' \frac{\exp(i \underline{k}'' \cdot \underline{w})}{k''^2 - k_0^2} b(\underline{k}'') - \int d\underline{k}'' \exp(i \underline{k}'' \cdot \underline{w}) a(\underline{k}'') \quad (13)$$

The final step is to substitute Eq. (13) back into Eqs. (6) and (12). The result is a pair of coupled integral equations for $b(\underline{k}'')$ and $a(\underline{k}'')$; namely,

$$\begin{aligned} b(\underline{k}') &= (\underline{k}' | V | \underline{k}_0) + \frac{1}{\pi^2} \int d\underline{k}'' \frac{(\underline{k}' | V | \underline{k}'')}{k''^2 - k_0^2} b(\underline{k}'') - \int d\underline{k}'' (\underline{k}' | V | \underline{k}'') a(\underline{k}'') \\ a(\underline{k}') &= (\underline{k}' | 0 | \underline{k}_0) + \frac{1}{\pi^2} \int d\underline{k}'' \frac{(\underline{k}' | 0 | \underline{k}'')}{k''^2 - k_0^2} b(\underline{k}'') - \int d\underline{k}'' (\underline{k}' | 0 | \underline{k}'') a(\underline{k}'') \end{aligned} \quad (14)$$

$(\underline{k}' | V | \underline{k}'')$ is just the matrix element of the potential between plane wave states:

$$(\underline{k}' | V | \underline{k}'') = \frac{1}{8\pi} (\alpha J_1 + \beta J_2 + \gamma J_3) \quad (15)$$

where

$$J_1 = \int d\mathbf{r} d\mathbf{w} \exp(-i \mathbf{k}' \cdot \mathbf{w}) \phi(\mathbf{r}) U(|\mathbf{w} + \frac{1}{2} \mathbf{r}|) \exp(i \mathbf{k}'' \cdot \mathbf{w}) \phi(\mathbf{r})$$

$$J_2 = \int d\mathbf{r} d\mathbf{w} \exp(-i \mathbf{k}' \cdot \mathbf{w}) \phi(\mathbf{r}) U(|\mathbf{w} + \frac{1}{2} \mathbf{r}|) \exp(i \mathbf{k}'' \cdot (\mathbf{w} + \frac{1}{2} \mathbf{r})) \phi(|\mathbf{w} + \frac{1}{2} \mathbf{r}|) \quad (16)$$

$$J_3 = \int d\mathbf{r} d\mathbf{w} \exp(-i \mathbf{k}' \cdot \mathbf{w}) \phi(\mathbf{r}) U(|\mathbf{w} + \frac{1}{2} \mathbf{r}|) \exp(i \mathbf{k}'' \cdot (\mathbf{w} - \frac{1}{2} \mathbf{r})) \phi(|\mathbf{w} - \frac{1}{2} \mathbf{r}|)$$

$(\mathbf{k}' | 0 | \mathbf{k}'')$ has a similar form, but does not involve potentials:

$$(\mathbf{k}' | 0 | \mathbf{k}'') = \frac{\beta}{(2\pi)^3} \int d\mathbf{w} d\mathbf{r} \exp(-i \mathbf{k}' \cdot \mathbf{w}) \phi(\mathbf{r}) \phi(|\mathbf{w} + \frac{1}{2} \mathbf{r}|) \exp(i \mathbf{k}'' \cdot (\mathbf{w} + \frac{1}{2} \mathbf{r})) \phi(|\mathbf{w} + \frac{1}{2} \mathbf{r}|) \quad (17)$$

When one transforms the usual two body scattering problem to momentum space, one finds a result like Eq. (14) except that one has a single integral equation involving $b(\mathbf{k}')$ only. The coupling seen in Eq. (14) is a result of Pauli's principle. Except for this coupling, Eq. (14) is not of much interest by itself. The techniques used in its derivation are, of course, well known.

The technique of eliminating the dependence of $b(\mathbf{k}')$ and $a(\mathbf{k}')$ on the direction of $\mathbf{k}'$ by the method of partial waves is also well known, but we repeat it here anyway (briefly, however).

Let

$$a(\mathbf{k}') = \sum_{\ell} \left(\frac{2\ell + 1}{2} \right) a_{\ell}(\mathbf{k}') P_{\ell}(\hat{\mathbf{k}}' \cdot \hat{\mathbf{k}}_0) \quad (18)$$

and define $b_\ell(k')$ similarly, where $P_\ell(\underline{k}', \underline{k}_0)$ means $P_\ell(\cos \theta_1)$, where θ_1 is the angle between $\underline{k}'$ and $\underline{k}_0$. Let

$$(\underline{k}' | V | \underline{k}'') = \sum_{\ell} \left(\frac{2\ell+1}{2} \right) (\underline{k}' | V | \underline{k}'')_{\ell} P_{\ell}(\underline{k}', \underline{k}'') \quad (19)$$

and define $(\underline{k}' | O | \underline{k}'')_{\ell}$ similarly. When Eq. (18) and (19) are substituted into Eq. (14), it is possible to integrate out the angles of $\underline{k}''$ by using the relation

$$P_{\ell}(\underline{k}', \underline{k}'') = P_{\ell}(\underline{k}', \underline{k}_0) P_{\ell}(\underline{k}'', \underline{k}_0) + (\quad) \cos \phi \quad (20)$$

where ϕ is the azimuth of $\underline{k}''$, so that the integral of the term with $\cos \phi$ is zero. The orthonormal properties of the Legendre polynomials lead immediately to

$$b_{\ell}(k') = (\underline{k}' | V | \underline{k}_0)_{\ell} + \frac{2}{\pi} \int \frac{k''^2 dk'' (\underline{k}' | V | \underline{k}'')_{\ell} b_{\ell}(k'')}{k''^2 - k_0^2} - 2\pi \int k''^2 dk'' (\underline{k}' | V | \underline{k}'')_{\ell} a_{\ell}(k'') \quad (21)$$

$a_{\ell}(k') =$ same with O's instead of V's.

The factors involving π may be eliminated by the transformations:

$$b = \frac{\pi}{2} b'$$

$$V = \frac{\pi}{2} V'$$

$$a = \frac{1}{2\pi} a' \quad (22)$$

$$O = \frac{1}{2\pi} O'$$

Dropping the primes we have

$$b_{\ell}(k') = (k'|V|k_0)_{\ell} + \int \frac{k''^2 dk'' (k'|V|k'')_{\ell} b_{\ell}(k'')}{k''^2 - k_0^2} - \int k''^2 dk'' (k'|V|k'')_{\ell} a_{\ell}(k'') \quad (23)$$

$a_{\ell}(k') =$ same thing with O 's instead of V 's.

Now the dependence on the direction of $\underline{k}'$ is eliminated, but we have to solve Eq. (23) for every value of ℓ (the angular momentum $\ell = 0, 1, 2, \dots$). Of course, in practice, we need only $\ell = 0$ through $\ell = 6$ even for 100 Mev. A program for a high speed computer which solves Eq. (23) for one value of ℓ solves it as well for another.

$b_{\ell}(k_0)$ is related to the phase shift for the ℓ -th partial wave as follows⁴

$$b_{\ell}(k_0) = \frac{2}{\pi} \frac{\tan \delta_{\ell}}{k_0} \quad (24)$$

⁴Walter Kohn, Phys. Rev. 84, 495 (1951). See the derivation of Eq. (2.11).

When Coulomb forces are present, the Green's function is no longer given by Eq. (7) but by

$$G(\underline{w}, \underline{w}') = \frac{1}{2\pi^2} \int d\underline{k}'' \frac{F(\underline{k}'', \underline{w}) F^*(\underline{k}'', \underline{w}')}{k''^2 - k_0^2} \quad (25)$$

where the F 's are the Coulomb functions which reduce to plane waves when the charge is zero. (Because the charge of the deuteron is not concentrated at its center but spread out, two integrals like J_1 and J_3 occur

in V .⁵ However, these are very small and the deuteron charge distribution

⁵These are precisely the two integrals given in Ref. 2, Eqs. (29) and (30).

cannot be studied by scattering protons off deuterons. It is readily seen that Eq. (23) is not changed; the only change required is that V 's and O 's must be calculated with Coulomb functions instead of plane wave functions. This may be done approximately by computing the V_ℓ 's and O_ℓ 's with plane wave functions and multiplying the result by $C_\ell(k')$ $C_\ell(k'')$, where the C_ℓ 's are the usual Coulomb penetration factors which reduce to one when the charge is zero. Our p-d calculations were done in this way.

IV. Calculation.

All calculations were carried out on the Los Alamos Maniac or an IBM 701 computer.

For $\ell \geq 1$, the accuracy of a Fredholm approximation made it unnecessary to carry out the exact (numerical) solutions of Eq. (23) described in this section except for a couple of sample energies (3 Mev and 12 Mev). For $\ell = 0$, they were carried out for several energies between 0 and 20 Mev.

Eq. (23) was solved by replacing it by a system of linear equations. This technique has been described in detail⁶. Fifty equally

⁶J. L. Gammel, Phys. Rev. 95, 209 (1954).

spaced mesh points (twenty-five for b and twenty-five for a) were used. The integrals in Eq. (23) were cut off at $k'' = 3 \times 10^{13} \text{ cm}^{-1}$. This is

sufficiently large for $\ell = 0$, and sufficiently large to establish the accuracy of a Fredholm approximation for $\ell \gg 1$.

It is necessary to calculate the 4 kernels in Eq. (23) for 25 x 25 values of k' and k'' . This was done from Eqs. (15), (16), and the inverse of Eq. (19):

$$(\underline{k}' | V | \underline{k}'')_{\ell} = \int_{-1}^1 d\mu P_{\ell}(\mu) \frac{2}{\pi} \frac{1}{8\pi} (\alpha J_1 + \beta J_2 + \gamma J_3) \quad (26)$$

where μ is the cosine of the angle between $\underline{k}'$ and $\underline{k}''$. (The factor $2/\pi$ comes from the transformation Eq. (22).) The integral in Eq. (26) was done numerically using $\Delta\mu = .05$ for each of the 4 kernels and 625 values of k' and k'' .

The J 's and O 's were calculated using a potential

$$U(r) = \frac{4M}{3\hbar^2} V_0 \frac{\exp(-\gamma r)}{\gamma r} \quad (27)$$

with $V_0 = 68$ Mev and $\gamma = 1/1.18 \times 10^{-13}$ cm, and a deuteron wave function

$$\phi(r) = \frac{\alpha\beta(\alpha + \beta)}{2\pi(\alpha - \beta)^2}^{1/2} \frac{\exp(-\alpha r) - \exp(-\beta r)}{r} \quad (28)$$

where $\alpha^2 = ME_b/\hbar^2$ (E_b is the binding energy of the deuteron and $\beta = 7.0 \alpha$ ($\alpha = .2316 \times 10^{13} \text{ cm}^{-1}$ and $\beta = 1.6212 \times 10^{13} \text{ cm}^{-1}$). This is a well known accurate approximation to the deuteron wave function⁷.

⁷See, for example, G. F. Chew, Phys. Rev. 74, 809 (1948), or R. L. Gluckstern and H. A. Bethe, Phys. Rev. 81, 761 (1951).

The evaluation of J_1 , J_2 and O is straight forward. J_3 has not been evaluated before for a Yukawa potential and a good deuteron wave function like Eq. (28). It is evaluated in the first appendix.

We find

$$\frac{2}{\pi} \frac{1}{8\pi} J_1 = 2g \frac{1}{|\underline{k}' - \underline{k}''|^2 + \nu^2} \frac{1}{|\underline{k}' - \underline{k}''|} \quad (29a)$$

$$\left\{ \arctan \frac{|\underline{k}' - \underline{k}''|}{4\alpha} - 2 \arctan \frac{|\underline{k}' - \underline{k}''|}{2(\alpha + \beta)} + \arctan \frac{|\underline{k}' - \underline{k}''|}{4\beta} \right\}$$

$$\frac{2}{\pi} \frac{1}{8\pi} J_2 = g \frac{1}{|\underline{k}' + \frac{1}{2} \underline{k}''|} \left\{ \frac{1}{|\underline{k}'' + \frac{1}{2} \underline{k}'|^2 + \alpha^2} - \frac{1}{|\underline{k}'' + \frac{1}{2} \underline{k}'|^2 + \beta^2} \right\} \quad (29b)$$

$$\left\{ \arctan \frac{|\underline{k}' + \frac{1}{2} \underline{k}''|}{\nu + \alpha} - \arctan \frac{|\underline{k}' + \frac{1}{2} \underline{k}''|}{\nu + \beta} \right\}$$

where

$$g = \frac{2}{\pi} \frac{4M}{3\hbar^2} \frac{U_0}{\nu} \frac{\alpha \beta (\alpha + \beta)}{(\alpha - \beta)^2} = .59474 \times 10^{26} \text{ cm}^{-2} \quad (30)$$

and

$$(\underline{k}' | O | \underline{k}'') = \frac{2}{\pi} \frac{\alpha \beta (\alpha + \beta)}{(\alpha - \beta)^2} \left\{ \frac{1}{|\underline{k}' + \frac{1}{2} \underline{k}''|^2 + \alpha^2} - \frac{1}{|\underline{k}' + \frac{1}{2} \underline{k}''|^2 + \beta^2} \right\} \quad (31)$$

$$\left\{ \frac{1}{|\frac{1}{2} \underline{k}' + \underline{k}''|^2 + \alpha^2} - \frac{1}{|\frac{1}{2} \underline{k}' + \underline{k}''|^2 + \beta^2} \right\}$$

J_3 is given in the appendix.

The results of these calculations⁸ are summarized in Fig. 2

⁸These calculations were done on the Los Alamos Maniac. It took two hours to generate the kernels for Eq. (23). There were 625 magnitudes of $\underline{k}'$ and $\underline{k}''$, and for each of these 40 values of the cosine of the angle between them. V involves 3 J 's, and O consists of a single term, so that there were $4 \times 625 \times 40 = 100,000$ values of J 's and O 's to be computed from complicated formulae like (29a,b). It took only eleven minutes to solve the 50×50 linear system which approximates Eq. (23).

for $\ell = 0$ and in Table I for $\ell = 0, 1$, and 2.

V. A Fredholm Approximation

Born's approximation to the solution of Eq. (25) is

$$b_\ell(\underline{k}') = (\underline{k}' | V | \underline{k}_0)_\ell \quad (32)$$

An approximate solution suggested by the Fredholm theory of integral equations like it is⁹

$$b_\ell(\underline{k}') = (\underline{k}' | V | \underline{k}_0)_\ell + \frac{\int \frac{k''^2 dk''}{k''^2 - k_0^2} (\underline{k}' | V | \underline{k}'')_\ell (\underline{k}'' | V | \underline{k}_0)_\ell - \int k''^2 dk'' (\underline{k}' | V | \underline{k}'')_\ell (\underline{k}'' | O | \underline{k}_0)_\ell}{1 - \int \frac{k''^2 dk''}{k''^2 - k_0^2} (\underline{k}'' | V | \underline{k}'')_\ell + \int k''^2 dk'' (\underline{k}'' | O | \underline{k}'')_\ell} \quad (33)$$

⁹ $(\underline{k}'' | V | \underline{k}'')$ means that the two $\underline{k}''$'s have the same magnitude. The angle between them is integrated out according to Eq. (26).

The integral involving V and standing in the denominator of Eq. (33) converges very slowly. A way of overcoming this evaluating Eq. (33) is described in the second appendix.

Born's approximation, the results of the calculations described in the previous paragraph, and the results of this Fredholm approximation are compared in Table I. The integrals are cut off at the same place when the exact and Fredholm approximations are compared. The contributions to Eq. (33) are also shown in Table II for the cases compared. Table III gives Born's and Fredholm's approximation (not cut off) for a wide range of energies and $\ell = 1 - 6$.

It may be seen that Born's approximation does not agree with the exact calculations or the Fredholm approximation. This is not really very surprising; it can even happen in the case of a two body potential which is non-monotonic¹⁰.

¹⁰Walter Kohn, Rev. Mod. Phys. 26, 292 (1954). See remark (b), section 13A.

VI. Analysis of Experimental Data and Conclusions.

Figure 1 shows that angular distributions calculated from the no polarization approximation and the Serber potential do not agree with experiment. The phase shifts used in preparing Figure 1 are given in Table IV.

At this point we have all the firm conclusions of our work, as mentioned in the introduction. Now we turn to a discussion of the other points mentioned there.

Christian and Gammel's² determination of S phase shifts from the experimental data was based on the assumption that Born's approximation is valid for $\ell \geq 1$. Without some such assumption, it is impossible to determine the 4S and 2S phase shifts from experiment; that is, for many sets of 4S and 2S phase shifts it is possible to fit the data by adjusting the phase shifts for $\ell \geq 1$.

To check their results, Christian and Gammel calculated the no polarization approximation for $\ell = 0$. They used a Gauss potential and a Gauss approximation to the deuteron wave function. For $E > 0$, the calculation was done rather crudely.

However, their use of Born's approximation for $\ell \geq 1$ and the no polarization approximation for $\ell = 0$ is inconsistent because the two do not give the same results for any value of ℓ , no matter what the energy, as Table III shows.

Nevertheless, this apparently inconsistent combination of approximations works. We cannot explain why it should, but we can show how closely the results of the calculations described in Section IV for $\ell = 0$ agree with Christian and Gammel's results about the 4S and 2S states.

Their results are summarized in their Fig. 6. The results for the 4S state are of most interest because the 2S state contributes so little to the scattering that next to nothing is learned about it from the experiments. The n-d 4S scattering length which they calculated the no polarization approximation agreed with experiment. They make an estimate of the difference between p-d and n-d scattering length which was

very rough but in agreement with the difference between the known n-d scattering lengths (known from zero energy experiments) and the p-d scattering lengths they found from their phase shift analysis. However, the n-d effective range resulting from their calculation did not agree with the p-d effective range they found from their phase shift analysis. In the case of p-p and n-p scattering, the effective ranges are about equal for the 1S state. Christian and Gammel thought that the inequality of the "observed" p-d and calculated n-d effective ranges was a result of the crudeness of their calculation for $E > 0$, and preferred to assume in analogy with the n-p and p-p results that the effective ranges for n-d and p-d are equal.

Our no polarization calculations for $\ell = 0$ agree with the p-d 4S scattering length and effective range resulting from Christian and Gammel's phase shift analysis in a remarkable way as shown in Fig. 2. We find for the scattering length 12.1×10^{-13} cm and they gave 12.5×10^{-13} cm. Our calculations also give an n-d 4S scattering length which agrees with experiment (we find 7.112×10^{-13} cm, and the experimental value is 6.2×10^{-13} cm; Christian and Gammel's gave an n-d 4S scattering length which was shorter than the experimental one), and the difference between the 4S p-d and n-d scattering lengths is

$$\frac{1}{a_4(p-d)} - \frac{1}{a_4(n-d)} = - .0577 \times 10^{13} \text{ cm}^{-1}$$

and Christian and Gammel's 4S p-d scattering length and the experimental 4S n-d scattering length give $-.0812 \times 10^{13} \text{ cm}^{-1}$ for this number.

The p-d and n-d effective ranges are still not equal. n-d angular distributions calculated with phase shifts from the no polarization approximation are shown in Fig. 3.

Finally, we have tried changing the phase shifts calculated in no polarization approximation to see which is most responsible for the lack of agreement shown in Fig. 1. We cannot get agreement by changing the 4S and 2S phase shifts. We can by increasing the magnitude of the P phase shifts, especially the 4P phase shift. The fits so obtained are shown in Fig. 1, and the phase shifts used in preparing in Fig. 1 are shown in Table IV.

Since increasing the 4P phase shift is in the direction of Born's approximation, we are led to the observation that it would be desirable if the effect of polarization was not very pronounced in the S states, and had the effect of making Born's approximation better than the no polarization approximation in the 4P state. We plan a calculation of the effect of polarization using the same potential we have used here. This should show whether or not such is the case.

APPENDIX I

A typical term in J_3 is

$$\frac{2}{\pi} J_3 = \frac{g}{2\pi} \int d\underline{r} d\underline{w} \exp(-i\underline{k}' \cdot \underline{w}) \frac{\exp(-\beta \underline{r})}{r} \frac{\exp(-\nu |\underline{w} + \frac{1}{2} \underline{r}|)}{|\underline{w} + \frac{1}{2} \underline{r}|} \quad (1A)$$

$$\exp(i\underline{k}'' \cdot (-\frac{1}{2} \underline{w} - \frac{3}{4} \underline{r})) \frac{\exp(-\alpha |\underline{w} - \frac{1}{2} \underline{r}|)}{|\underline{w} - \frac{1}{2} \underline{r}|}$$

Let

$$\underline{r} = \underline{z}$$

(2A)

$$\underline{w} - \frac{1}{2} \underline{r} = \underline{w}$$

where the $\underline{w}$ on the right of Eq. (2A) is not the same as the $\underline{w}$ on the left, but this should cause no confusion. Then

$$\frac{2}{\pi} J_3 = \frac{g}{2\pi} \int d\underline{w} d\underline{z} \exp(-i\underline{k}' \cdot (\underline{w} + \frac{1}{2} \underline{z})) \frac{\exp(-\beta \underline{z})}{z} \frac{\exp(-\nu |\underline{w} + \underline{z}|)}{|\underline{w} + \underline{z}|} \quad (3A)$$

$$\exp(-i\underline{k}'' \cdot (\frac{1}{2} \underline{w} + \underline{z})) \frac{\exp(-\alpha \underline{w})}{w}$$

The potential may be written

$$\frac{\exp(-\gamma|\underline{w} + \underline{z}|)}{|\underline{w} + \underline{z}|} = \frac{1}{2\pi^2} \int d\underline{K} \frac{\exp(i\underline{K} \cdot (\underline{w} + \underline{z}))}{K^2 + \gamma^2} \quad (4A)$$

This gives

$$\begin{aligned} \frac{2}{\pi} J_3 &= \frac{g}{4\pi^3} \int \frac{d\underline{K}}{K^2 + \gamma^2} \int \exp(i(\underline{K} - \underline{k}' - \frac{1}{2}\underline{k}'') \cdot \underline{w}) \frac{\exp(-\alpha w)}{w} d\underline{w} \\ &\quad \cdot \int \exp(i(\underline{K} - \frac{1}{2}\underline{k}' - \underline{k}'') \cdot \underline{z}) \frac{\exp(-\beta z)}{z} d\underline{z} \end{aligned} \quad (5A)$$

Carrying out the integrals over $\underline{w}$ and $\underline{z}$ gives

$$\frac{2}{\pi} J_3 = \frac{4g}{\pi} \int \frac{d\underline{K}}{\{K^2 + \gamma^2\} \{ \alpha^2 + K^2 + |\underline{k}' + \frac{1}{2}\underline{k}''|^2 - 2\underline{K} \cdot (\underline{k}' + \frac{1}{2}\underline{k}'') \} \{ \beta^2 + K^2 + |\frac{1}{2}\underline{k}' + \underline{k}''|^2 - 2\underline{K} \cdot (\frac{1}{2}\underline{k}' + \underline{k}'') \} } \quad (6A)$$

Using the result

$$\int_0^1 \frac{dx}{[d + (c - d)x]^2} = \frac{1}{cd} \quad (7A)$$

Eq. (6A) becomes

$$\begin{aligned} \frac{2}{\pi} J_3 &= \frac{4g}{\pi} \int \frac{d\underline{K}}{K^2 + \gamma^2} \int_0^1 \frac{dx}{\left[(\alpha^2 + K^2 + |\underline{k}' + \frac{1}{2}\underline{k}''|^2 + \{ \beta^2 - \alpha^2 + |\frac{1}{2}\underline{k}' + \underline{k}''|^2 - |\underline{k}' + \frac{1}{2}\underline{k}''|^2 \} x \right.} \\ &\quad \left. - 2\underline{K} \cdot (\underline{k}' + \frac{1}{2}\underline{k}'' + \frac{1}{2}(\underline{k}'' - \underline{k}')x) \right]^2 } \end{aligned} \quad (8A)$$

Let

$$\alpha^2 + |\underline{k}' + \frac{1}{2} \underline{k}''|^2 + \left\{ \beta^2 - \alpha^2 + \left| \frac{1}{2} \underline{k}' + \underline{k}'' \right|^2 - \left| \underline{k}' + \frac{1}{2} \underline{k}'' \right|^2 \right\} x = A^2$$

$$\underline{k}' + \frac{1}{2} \underline{k}'' + \frac{1}{2} (\underline{k}'' - \underline{k}') x = \underline{B} \quad (9A)$$

Then

$$\frac{2}{\pi} J_3 = \frac{4g}{\pi} \int \frac{d\underline{K}}{K^2 + \nu^2} \int_0^1 \frac{dx}{(A^2 + K^2 - 2K \cdot B)^2}$$

$$= 16g \int_0^\infty \frac{K^2 dK}{K^2 + \nu^2} \int_0^1 \frac{dx}{(A^2 + K^2)^2 - 4K^2 B^2} \quad (10A)$$

It can be verified that

$$(A^2 + K^2)^2 - 4K^2 B^2 > 0 \quad (11A)$$

The integral over K can be done by extending the limits to $-\infty$ to $+\infty$ (the integrand depends on K^2) and using the theory of residues. The roots of the denominator in Eq. (10A) are

$$K = \pm i\nu \quad K = \pm B + i\sqrt{A^2 - B^2} \quad (12A)$$

and it can be verified that

$$A^2 - B^2 > 0 \quad (13A)$$

The result of the integration is

$$\frac{2}{\pi} \frac{1}{8\pi} J_3 = \frac{1}{2} g \int_0^1 \frac{dx}{\sqrt{A^2 - B^2} (A^2 + \nu^2 + 2\nu \sqrt{A^2 - B^2})} \quad (14A)$$

We have

$$A^2 - B^2 = \alpha^2 + \left\{ \beta^2 - \alpha^2 + \frac{1}{4} (\underline{k}' - \underline{k}'')^2 \right\} x - \frac{1}{4} (\underline{k}' - \underline{k}'')^2 x^2 \quad (15A)$$

It can be verified that as $\nu \rightarrow \infty$, $J_3 \rightarrow J_1$ as it should.

The integral in Eq. (14A) is possible. First one completes the square in Eq. (15A); that is one introduces a new variable y which is a linear function of x such that

$$A^2 - B^2 = C^2 - y^2 \quad (16A)$$

where C stands for an expression involving $\alpha, \beta, |\underline{k}' - \underline{k}''|$. Then one introduces a new variable t ¹¹

¹¹ W. Grolner and N. Hofreiter, Integraltafel, Erster Teil, p. 35. (Wien und Innsbruck, Springer-Verlag, 1949).

$$y = \frac{2Ct}{t^2 + 1} \quad \sqrt{C^2 - y^2} = \frac{C(t^2 - 1)}{t^2 + 1}$$

$$dy = - \frac{2C(t^2 - 1)}{(t^2 + 1)^2} dt \quad t = \frac{\sqrt{C^2 - y^2} + C}{y} \quad (17A)$$

$$\int R(y, \sqrt{C^2 - y^2}) dy = - \int R \left(\frac{2Ct}{t^2 + 1}, \frac{C(t^2 - 1)}{t^2 + 1} \right) \frac{2C(t^2 - 1)}{(t^2 + 1)^2} dt$$

where R is the integrand in Eq. (14A) after the transformation Eq. (16A) (R is a rational function of $y, \sqrt{c^2 - y^2}$). This eliminates all square roots and gives a recognizable integral. Of course, it is necessary to keep up with the limits of the integral (see our ℓ_1 and ℓ_2 below).

The final result is as follows. Let

$$\alpha^2 = a \quad (18A)$$

$$\beta^2 - \alpha^2 + \frac{1}{4} (\underline{k}' - \underline{k}'')^2 = b \quad (19A)$$

$$\frac{1}{4} (\underline{k}' - \underline{k}'')^2 = c \quad (20A)$$

$$a + b^2/4c = \Delta^2 \quad (21A)$$

$$\ell_1 = 2 (\alpha + \Delta) \sqrt{c}/b \quad (22A)$$

$$\ell_2 = 2 (\beta + \Delta) \sqrt{c}/(2c - b) \quad (23A)$$

$$(\beta^2 - \alpha^2 + \left| \frac{1}{2} \underline{k}' + \underline{k}'' \right|^2 - \left| \underline{k}' + \frac{1}{2} \underline{k}'' \right|^2) \frac{b}{2c} + \nu^2 = \delta^2 \quad (24A)$$

$$(\beta^2 - \alpha^2 + \left| \frac{1}{2} \underline{k}' + \underline{k}'' \right|^2 - \left| \underline{k}' + \frac{1}{2} \underline{k}'' \right|^2) / \sqrt{c} = \rho \quad (25A)$$

$$\delta^4 - 4\nu^2 \Delta^2 - \rho^2 \Delta^2 = "ac - b^2" \quad (26A)$$

Then

$$\frac{2}{\pi} \frac{1}{8\pi} J_3 = \frac{g}{\sqrt{c}} \frac{1}{\sqrt{"ac-b^2"}} \left\{ \arctan \frac{\sqrt{"ac-b^2"}}{(\delta^2+2\nu\Delta)l_1-\rho\Delta} + \arctan \frac{\sqrt{"ac-b^2"}}{(\delta^2+2\nu\Delta)l_2+\rho\Delta} \right\}$$

(27A)

Again, it can be verified that as $\nu \rightarrow \infty$, $J_3 \rightarrow J_1$. Of course, J_3 consists of four such terms when the complete expression Eq. (29) for $\phi(r)$ is used.

APPENDIX II

The convergence of the integral can be improved by using

$$\frac{k''^2}{k''^2 - k_0^2} = \frac{k_0^2}{k''^2 - k_0^2} + 1 \quad (1B)$$

That is, it can be written as the sum of two integrals, the second of which is independent of the energy and can be evaluated once and for all either analytically or numerically, and the first of which converges more rapidly than the original integral (a constant k_0^2 stands in the numerator instead of k''^2).

Only the J_1 part of it causes trouble. As is well known, as long as the angle between the two $\underline{k}$'s is zero, J_1 has a value which is independent of their magnitude. At other angles, J_1 decreases rapidly with increasing magnitude of $\underline{k}$. Thus J_1 can have quite an angular dependence, and for $\ell \gg 1$, J_1 decreases quite slowly with increasing k . It is also necessary to use caution in evaluating Eq. (27) numerically. We actually used $\Delta\mu = .00125$ between $\mu = .95$ and $\mu = 1.0$.

It is not hard to evaluate the integral corresponding to the "1" in Eq. (1B) analytically in the n-d case, and this was done. It was also done numerically as a check on our code. Values of

$$\int dk'' (\underline{k}'' | J_1 | \underline{k}'')_\ell$$

are shown in Table V for n-d and p-d.

TABLE I

Values of $\frac{2}{\pi k} \tan^4 \delta$ for $E = 11.68$ Mev

ℓ	Exact	Born	Fredholm
0	1.0695	-.55185	.63249
1	.4284	.6778	.3899
2	-.2657	-.1492	-.2901

TABLE II

Contributions to Eq. 33 for $E = 11.68$ Mev Quartet State

ℓ	$\int \frac{k^2}{k^2 - k_0^2} (k V k)_\ell$	$\int k^2 (k O k)_\ell$	$\int k^2 (k_0 V k)_\ell (k O k_0)_\ell$	$\int \frac{k^2}{k^2 - k_0^2} (k_0 V k)_\ell (k V k_0)_\ell$
0	.58287	-1.3097	.75818	-.29895
1	.4280	.6475	.5362	.1851
2	0	-.3194	.09684	.0010

TABLE III

Phase Shifts Calculated from the Born and Fredholm Approximations.*

*The entries in the table are given in the following form:

 $q(p) = q \times 10^{-p}$; for example $1.5(2) = 0.015$.

a. n-d scattering, Born approximation

E_{lab} (Mev)	1.0	3.27	5.5	14.1	95.0
δ_1	1.734(1)	3.754(1)	4.434(1)	4.907(1)	3.354(1)
$\tan \delta_2$	-1.814(2)	-7.527(2)	-1.038(1)	-1.127(1)	4.386(2)
$\tan \delta_3$	2.137(3)	1.792(2)	3.157(2)	5.650(2)	9.458(2)
$\tan \delta_4$	-2.639(4)	-4.192(3)	-8.824(3)	-1.694(2)	1.702(2)
$\tan \delta_5$	4.715(5)	1.060(3)	2.701(3)	7.266(3)	2.483(2)
$\tan \delta_6$	-6.814(5)	-3.601(4)	-9.280(4)	-2.706(3)	6.100(3)
δ_1	-7.144(2)	-1.234(1)	-1.094(1)	7.431(3)	2.720(1)
$\tan \delta_2$	9.497(3)	4.341(2)	6.753(2)	1.236(1)	2.504(1)
$\tan \delta_3$	-1.058(3)	-8.541(3)	-1.405(2)	-1.390(2)	9.062(2)
$\tan \delta_4$	1.355(4)	2.134(3)	4.622(3)	1.177(2)	6.724(2)
$\tan \delta_5$	-2.429(5)	-5.252(4)	-1.321(3)	-2.871(3)	2.797(2)
$\tan \delta_6$	7.001(5)	2.373(4)	5.342(4)	1.613(3)	1.894(2)

b. n-d scattering, Fredholm approximation

$E_{\text{lab}} \text{ (Mev)}$		1.0	3.27	5.5	14.1	95.0
$s = 3/2$	δ_1	1.263(1)	2.154(1)	2.375(1)	2.874(1)	2.847(1)
	$\tan \delta_2$	-2.384(2)	-1.165(1)	-1.816(1)	-2.549(1)	2.896(2)
	$\tan \delta_3$	1.882(3)	1.366(2)	2.151(2)	3.314(2)	8.298(2)
	$\tan \delta_4$	-2.828(4)	-4.947(3)	-1.123(2)	2.563(2)	1.176(2)
	$\tan \delta_5$	4.601(5)	9.511(4)	2.231(3)	4.688(3)	2.126(2)
	$\tan \delta_6$	-6.827(5)	-3.807(4)	-1.045(3)	-3.696(3)	3.772(3)
$s = 1/2$	δ_1	-1.076(1)	-2.852(1)	-3.716(1)	-2.234(1)	4.033(1)
	$\tan \delta_2$	8.240(3)	3.477(2)	5.263(2)	1.034(1)	2.454(1)
	$\tan \delta_3$	-1.146(3)	-1.000(2)	-1.754(2)	-2.207(2)	9.697(2)
	$\tan \delta_4$	1.308(4)	1.945(3)	4.021(3)	9.633(3)	6.646(2)
	$\tan \delta_5$	-2.462(5)	-5.562(4)	-1.455(3)	-3.609(3)	2.778(2)
	$\tan \delta_6$	6.998(5)	2.320(4)	5.035(4)	1.356(3)	1.841(2)

c. p-d scattering, Born approximation

E_{lab} (MeV)	1.0	3.0	5.18	9.66	20.6	32.0	95
δ_1	1.020(1)	2.733(1)	3.567(1)	4.195(1)	4.382(1)	4.216(1)	3.198(1)
$\tan \delta_2$	-1.060(2)	-5.190(2)	-8.057(2)	-1.004(1)	-8.410(2)	-5.406(2)	4.167(2)
$\tan \delta_3$	1.249(3)	1.184(2)	2.384(2)	4.006(2)	5.796(2)	6.761(2)	8.984(2)
$\tan \delta_4$	-1.542(4)	-2.670(3)	-6.554(3)	-1.229(2)	-1.520(2)	-1.175(2)	1.616(2)
$\tan \delta_5$	2.756(5)	6.521(4)	1.965(3)	4.541(3)	8.195(3)	1.071(2)	2.359(2)
$\tan \delta_6$	-3.983(5)	-2.256(4)	-6.705(4)	-1.674(3)	-2.819(3)	-2.581(3)	5.795(3)
$S = 3/2$							
δ_1	-4.180(2)	-9.104(2)	-9.020(2)	-4.425(2)	6.766(2)	1.400(1)	2.590(1)
$\tan \delta_2$	5.551(3)	2.954(2)	5.147(2)	8.393(2)	1.358(1)	1.710(1)	2.379(1)
$\tan \delta_3$	-6.184(4)	-5.676(3)	-1.074(2)	-1.440(2)	-4.118(3)	1.383(2)	8.609(2)
$\tan \delta_4$	7.919(5)	1.356(3)	3.415(3)	7.156(3)	1.474(2)	2.305(2)	6.387(2)
$\tan \delta_5$	-1.420(5)	-3.235(4)	-9.638(4)	-2.083(3)	-2.062(3)	7.339(4)	2.657(2)
$\tan \delta_6$	4.092(5)	1.539(4)	3.896(4)	9.366(4)	2.077(3)	3.568(3)	1.800(2)
$S = 1/2$							

d. p-d scattering, Fredholm approximation

E_{lab} (MeV)	1.0	3.0	5.18	9.66	20.6	32.0	95
$S = 3/2$							
δ_1	7.829(2)	1.776(1)	2.115(1)	2.449(1)	2.854(1)	2.991(1)	2.717(1)
$\tan \delta_2$	-1.337(2)	-7.455(2)	-1.289(1)	-1.870(1)	-1.918(1)	-1.442(1)	2.891(2)
$\tan \delta_3$	1.122(3)	9.488(3)	1.724(2)	2.558(2)	3.652(2)	4.665(2)	7.920(2)
$\tan \delta_4$	-1.635(4)	-3.060(3)	-8.064(3)	-1.680(2)	-2.421(2)	-2.169(2)	1.137(2)
$\tan \delta_5$	2.699(5)	5.985(4)	1.675(3)	3.369(3)	5.091(3)	6.730(3)	2.030(2)
$\tan \delta_6$	-3.999(5)	-2.352(4)	-7.405(4)	-2.056(3)	-4.193(3)	-4.621(3)	3.656(3)
$S = 1/2$							
δ_1	-5.810(2)	-1.680(1)	-2.211(1)	-2.050(1)	5.443(2)	2.419(1)	3.589(1)
$\tan \delta_2$	4.932(3)	2.464(2)	4.176(2)	6.907(2)	1.220(1)	1.612(1)	2.341(1)
$\tan \delta_3$	-6.611(4)	-6.460(3)	-1.295(2)	-1.932(2)	-1.087(2)	8.999(3)	9.162(2)
$\tan \delta_4$	7.687(5)	1.258(3)	3.035(3)	6.031(3)	1.255(2)	2.070(2)	6.320(2)
$\tan \delta_5$	-1.436(5)	-3.387(4)	-1.045(3)	-2.413(3)	-2.937(3)	-3.477(4)	2.639(2)
$\tan \delta_6$	4.091(5)	1.514(4)	3.715(4)	8.375(4)	1.722(3)	3.042(3)	1.751(2)

TABLE IV

Phase Shifts used in preparing Figures 1 and 3.

$E(\text{Mev})$	$2 \delta_0^{3/2}$	$2 \delta_0^{1/2}$	$2 \delta_1^{3/2} + \phi_1$	$2 \delta_1^{1/2} + \phi_1$
5.18(p-d)	-2.9010 ^P	-2.0186 ^P	0.5617 ^F	-0.3035 ^F
	-2.9010 ^P	-2.0186 ^P	0.9653 ^A	-0.3884 ^A
9.66(p-d)	2.4836 ^P	-2.4876 ^P	0.5914 ^F	-0.3082 ^F
	2.4836 ^P	-2.4876 ^P	1.099 ^A	-0.4711 ^A
5.5(n-d)	2.3916 ^P	-1.8076 ^P	0.8869 ^B	-0.2188 ^B

The superscripts on the entries in the table have the following meaning: A(P phase shift increased to achieve agreement with experiment), B(phase shift calculated in Born approximation), F(phase shift calculated from Fredholm approximation), P(phase shift calculated in no polarization approximation). Phase shifts for $\ell = 2 - 6$ were calculated from the Fredholm approximation in the p-d cases, and from the Born approximation in the n-d case.

TABLE V

Values of $\int (\underline{k} | J_1 | \underline{k})_{\ell} dk$

	n-d	p-d
ℓ		
0	1.875	1.451
1	0.625	0.577
2	0.375	0.354
3	0.268	0.254
4	0.208	0.197
5	0.170	0.161
6	0.144	0.135

Fig. 1. p-d angular distribution calculated from the no polarization approximation are shown by solid curves. The dashed curves result from increasing the magnitude of the P phase shifts in order to achieve agreement with the experimental data (the circled points). The experiments are those of Rosen and Allred, Phys. Rev. 82, 777 (1951), and Allred, Armstrong, Bondelid, and Rosen, Phys. Rev. 88, 433 (1952). The phase shifts used in preparing this figure are shown in Table IV.

Fig. 2. " $k \cot \delta$ " plots for the 4S and 2S phase shifts for p-d and n-d scattering. The circled points are those found by Christian and Gammel² in their analysis of the experimental p-d scattering data.

Fig. 3. n-d angular distribution calculated from the no polarization approximation for $\ell = 0$ and Born's approximation for $\ell = 1 - 6$. It should be compared to Fig. 10, reference 2, to which it is very similar. The angular distribution is very insensitive to the value of the 2S phase shift.