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FORMULAS PERTAINING TO WEAK-SHOCKS

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ABSTRACT

Certain solutions of the cylindrical wave-equation, such as the Possio integral, which are needed in the investigation of 2-dimensional weak-shocks are determined in this paper.

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FORMULAS PERTAINING TO WEAK-SHOCKS

1. Introduction. Certain fundamental formulas are required when treating a group of problems having to do with the reflection and diffraction of weak-shocks. For instance there is the Possio integral which is of such importance in the investigation of aerodynamical and blast problems [1]. As the desired formulas are not readily available, the writer has had to determine them by derivations. All of these formulas are obtainable as special solutions of the following problem for the cylindrical wave-equation. Determine that solution of the wave-equation

$$(1.1) \quad \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = f(x, y, t), \quad (y > 0)$$

which assumes the given initial-conditions

$$(1.2) \quad u(x, y, 0) = U(x, y), \quad u_t(x, y, 0) = V(x, y)$$

and the given boundary-condition

$$(1.3) \quad u_y(x, 0, t) = W(x, t).$$

The explicit solution of this problem is given in equation (6.13). In section 7, the Possio integral is obtained from the general solution by taking $U = V = f = 0$. In section 8, we show that the Possio integral,

$$(1.4) \quad u(x_0, y_0, t_0) = -\frac{1}{\pi} \int_{x_0 - \sqrt{t_0^2 - y_0^2}}^{x_0 + \sqrt{t_0^2 - y_0^2}} \frac{dx}{\sqrt{t_0^2 - y_0^2}} \int_0^{t_0 - \sqrt{(x-x_0)^2 + y_0^2}} \frac{W(x, t) dt}{\sqrt{(t-t_0)^2 - (x-x_0)^2 - y_0^2}},$$

actually does satisfy the boundary-condition (1.3), the other condition (1.2) being evidently true.

In future work dealing with applications, this paper will be used as a reference. We now give a partial indication of the desirability of studying the above problem. Using the customary notation and assuming the constancy of entropy, the non-linear 2-dimensional equations of gas dynamics are:

$$(1.5) \quad \rho_t + (\rho u)_x + (\rho v)_y = 0,$$

and

$$(1.6) \quad \begin{aligned} u_t + uu_x + vu_y + \rho^{-1} p_x &= 0, \\ v_t + uv_x + vv_y + \rho^{-1} p_y &= 0. \end{aligned}$$

Now, letting $\sigma = \ln \rho$ and $dp = c^2 d\rho$ (c constant), the equations (1.5) and (1.6) imply

$$(1.7) \quad \frac{\partial^2 \sigma}{\partial t^2} = c^2 \left(\frac{\partial^2 \sigma}{\partial x^2} + \frac{\partial^2 \sigma}{\partial y^2} \right),$$

provided that the second-order terms are neglected which is permissible for weak-shocks. The equation (1.7) is essentially (1.1) with $f = 0$. Besides, if the gas is confined by a solid wall, say $y = 0$, then v , the normal component of gas velocity at this wall, must be identically zero. Thus, because of (1.6), $0 = \rho^{-1} p_y = c^2 \sigma_y$ on the wall. This condition $\sigma_y(x, 0, t) = 0$ is (1.3) with $W = 0$.

2. Domain of Influence for Solution.

FIGURE I

Let x', y', t' denote axes respectively parallel to the Cartesian coordinate axes x, y, t , and intersection at the point $(x_0, y_0, 0)$. Now, let G denote the finite 3-dimensional open region which is bounded by the (characteristic) cone

$$(2.1) \quad C: (t-t_0)^2 - (x-x_0)^2 - (y-y_0)^2 = 0 (t \leq t_0),$$

the plane

$$(2.2) \quad t = 0,$$

which will bear the initial-data (1.2), and the plane

$$(2.3) \quad y = 0$$

which will bear the boundary-data (1.3). Henceforth, C will denote the conical portion of the boundary of G . The portion of the boundary of G lying on (2.2) will be denoted by Ω_1 , while that portion lying on (2.3) will be denoted by Ω_2 . The explicit formula (6.14), for the solution of our problem, (1.1) - (1.3), will show that the value of the solution at (x_0, y_0, t_0) is dependent only upon the values of f on G , the initial-values on Ω_1 , and the boundary-values on Ω_2 . These regions are said to be the regions of influence for the solution at (x_0, y_0, t_0) . That is, the value of the solution at (x_0, y_0, t_0) depends only upon the initial values on Ω_1 , the boundary-values on Ω_2 , and the values of f on G .

3. The Riemann-Functions. The discussion of the next three sections is that given by H. Lewy [2] in his treatment of the initial-value problem. With trivial modifications his treatment will yield the solution of our problem except that there arises a complication which is resolved in section 6 by means of a method of images.

An essential step in his method is that of being able to determine three (Riemann) functions

$$(3.1) \quad w_i = w_i(x-x_0, y-y_0, t-t_0) \quad (i = 1, 2, 3)$$

which satisfy the wave-equation

$$(3.2) \quad L[w_i] = \frac{\partial^2 w_i}{\partial t^2} - \frac{\partial^2 w_i}{\partial x^2} - \frac{\partial^2 w_i}{\partial y^2} = 0 \quad (i = 1, 2, 3)$$

inside of the cone $(t-t_0)^2 = (x-x_0)^2 + (y-y_0)^2$ ($t < t_0$), and which assume on the surface of the cone the boundary-values

$$(3.3) \quad w_1 = -\frac{\partial x}{\partial \nu}, \quad w_2 = -\frac{\partial y}{\partial \nu}, \quad w_3 = -\frac{\partial t}{\partial \nu},$$

where $(\frac{\partial x}{\partial \nu}, \frac{\partial y}{\partial \nu}, \frac{\partial t}{\partial \nu})$ are respectively the x, y, t direction cosines of the exteriorly directed normal to C . One easily verifies that these functions are given by

$$(3.4) \quad w_1 = \frac{(x-x_0)}{\sqrt{2}} \left[\frac{(t-t_0) + \sqrt{(t-t_0)^2 - r^2}}{r^2} \right],$$

$$w_2 = \frac{(y-y_0)}{\sqrt{2}} \left[\frac{(t-t_0) + \sqrt{(t-t_0)^2 - r^2}}{r^2} \right],$$

$$w_3 = \frac{1}{\sqrt{2}},$$

where

$$(3.5) \quad r = \sqrt{(x-x_0)^2 + (y-y_0)^2}.$$

4. A Green's Formula. Denote the direction cosines of the exteriorly directed normal to G^* , the boundary of G , by $(\frac{\partial x}{\partial \nu}, \frac{\partial y}{\partial \nu}, \frac{\partial t}{\partial \nu})$, and on G^* define the differential operator

$$(4.1) \quad \frac{\partial}{\partial \tau} \equiv \frac{\partial t}{\partial \nu} \frac{\partial}{\partial t} - \frac{\partial x}{\partial \nu} \frac{\partial}{\partial x} - \frac{\partial y}{\partial \nu} \frac{\partial}{\partial y}.$$

Then, recalling (3.2),

$$(4.2) \quad \iiint_G \left\{ vL[u] - uL[v] \right\} dG = \iint_{G^*} \left\{ v \frac{\partial u}{\partial \tau} - u \frac{\partial v}{\partial \tau} \right\} dG^*,$$

where $dG = dx dy dt$ is the element of volume on G and dG^* is the element of surface area on G^* . Now, adding together the three equations which result by successively and respectively replacing u and v in (4.2) by

u_x and w_1 , u_y and w_2 , u_t and w_3 , one finds that

$$(4.3) \quad \iiint_G \left\{ w_1 L[u_x] + w_2 L[u_y] + w_3 L[u_t] \right\} dG - \iiint_G \left\{ u_x L[w_1] + u_y L[w_2] + u_t L[w_3] \right\} dG =$$

$$\iint_{G^*} \left\{ w_1 \frac{\partial u_x}{\partial \tau} + w_2 \frac{\partial u_y}{\partial \tau} + w_3 \frac{\partial u_t}{\partial \tau} \right\} dG^* - \iint_G \left\{ u_x \frac{\partial w_1}{\partial \tau} + u_y \frac{\partial w_2}{\partial \tau} + u_t \frac{\partial w_3}{\partial \tau} \right\} dG^*.$$

Because of (1.1) and (3.2), (4.3) is expressible as

$$(4.4) \quad \iint_{G^*} \left\{ w_1 \frac{\partial u_x}{\partial \tau} + w_2 \frac{\partial u_y}{\partial \tau} + w_3 \frac{\partial u_t}{\partial \tau} \right\} dG^* - \iint_{G^*} \left\{ u_x \frac{\partial w_1}{\partial \tau} + u_y \frac{\partial w_2}{\partial \tau} + u_t \frac{\partial w_3}{\partial \tau} \right\} dG^* = \\ \iiint_G \left\{ w_1 f_x + w_2 f_y + w_3 f_t \right\} dG.$$

This is the Green's formula which will yield the solution of our problem.

5. Application of Green's Formula. As the (characteristic) cone C can be parametrized in the form

$$(5.1) \quad x - x_0 = \frac{s}{\sqrt{2}} \cos \theta, \quad y - y_0 = \frac{s}{\sqrt{2}} \sin \theta, \quad t - t_0 = -\frac{s}{\sqrt{2}},$$

the following relations must hold on its surface:

$$(5.2) \quad \frac{\partial x}{\partial \nu} = \frac{\cos \theta}{\sqrt{2}} = -w_1, \quad \frac{\partial y}{\partial \nu} = \frac{\sin \theta}{\sqrt{2}} = -w_2, \quad \frac{\partial t}{\partial \nu} = \frac{1}{\sqrt{2}} = w_3,$$

$$(5.3) \quad \frac{\partial}{\partial \tau} = \frac{\partial t}{\partial \nu} \frac{\partial}{\partial t} - \frac{\partial x}{\partial \nu} \frac{\partial}{\partial x} - \frac{\partial y}{\partial \nu} \frac{\partial}{\partial y} = w_1 \frac{\partial}{\partial x} + w_2 \frac{\partial}{\partial y} + w_3 \frac{\partial}{\partial t} = \frac{\partial}{\partial s},$$

$$(5.4) \quad dG^* = \frac{s}{\sqrt{2}} ds d\theta.$$

Consequently, using (5.2) - (5.4), the Green's formula (4.4) becomes

$$(5.5) \quad \iint_{\Omega_1 + \Omega_2} \left\{ w_1 \frac{\partial u_x}{\partial \tau} + w_2 \frac{\partial u_y}{\partial \tau} + w_3 \frac{\partial u_t}{\partial \tau} \right\} dG^* - \iint_{\Omega_1 + \Omega_2} \left\{ u_x \frac{\partial w_1}{\partial \tau} + u_y \frac{\partial w_2}{\partial \tau} + u_t \frac{\partial w_3}{\partial \tau} \right\} dG^* \\ + \frac{1}{\sqrt{2}} \iint_C s \frac{\partial^2 u}{\partial s^2} ds d\theta = \iiint_G \left\{ w_1 f_x + w_2 f_y + w_3 f_t \right\} dG.$$

Let $\rho(\theta)$ denote the distance from the vertex of C to the point where the line (5.1) intersects either the region Ω_1 or Ω_2 . Using this notation and applying integration by parts to the third integral of (5.5), one gets

$$(5.6) \quad \sqrt{2} \pi u(x_0, y_0, t_0) = \frac{1}{\sqrt{2}} \int_0^{2\pi} \left[u - s \frac{u}{s} \right]_{s=\rho(\theta)} d\theta + \iiint_G \left\{ w_1 f_x + w_2 f_y + w_3 f_t \right\} dG \\ + \iint_{\Omega_1} \left\{ w_1 \frac{\partial u_x}{\partial t} + w_2 \frac{\partial u_y}{\partial t} + w_3 \frac{\partial u_t}{\partial t} - u_x \frac{\partial w_1}{\partial t} - u_y \frac{\partial w_2}{\partial t} \right\} d\Omega_1 \\ - \iint_{\Omega_2} \left\{ w_1 \frac{\partial u_x}{\partial y} + w_2 \frac{\partial u_y}{\partial y} + w_3 \frac{\partial u_t}{\partial y} - u_x \frac{\partial w_1}{\partial y} - u_y \frac{\partial w_2}{\partial y} \right\} d\Omega_2.$$

Invoking the initial and boundary conditions, (1.2) and (1.3), the equation (5.6) becomes

$$\begin{aligned}
 (5.7) \quad u(x_0, y_0, t_0) = & \frac{1}{2\pi} \int_0^{2\pi} [u-s \ u_s]_{s=\rho(\theta)} d\theta + \frac{1}{\sqrt{2}\pi} \iiint_G \left\{ w_1 f_x + w_2 f_y + w_3 f_t \right\} dG \\
 & + \frac{1}{\sqrt{2}\pi} \iint_{\Omega_1} \left\{ w_1 v_x + w_2 v_y + w_3 (u_{xx} + u_{yy}) - u_x \frac{\partial w_1}{\partial t} - u_y \frac{\partial w_2}{\partial t} \right\} dG^* \\
 & - \frac{1}{\sqrt{2}\pi} \iint_{\Omega_2} \left\{ w_1 w_x + w_2 u_{yy} + w_3 w_t - u_x \frac{\partial w_1}{\partial y} - w \frac{\partial w_2}{\partial y} \right\} dG^*.
 \end{aligned}$$

The solution of the general problem, (1.1) - (1.3), must be given by the formula (5.7). However, since the values of u_x and u_{yy} are not known on Ω_2 , the problem now confronts us of eliminating these unknown values from (5.7). This will be accomplished in the next section by a method of images. However, before taking up these considerations, we shall now deduce from (5.7) the solution of the initial-value problem, (1.1) and (1.2). Since in this case $\rho = \sqrt{2} \ t_0$ and $\Omega_2 = 0$, (5.7) becomes

$$\begin{aligned}
 (5.8) \quad u(x_0, y_0, t_0) = & \frac{1}{2\pi} \int_0^{2\pi} [u-s \ u_s]_{s=\sqrt{2} \ t_0} d\theta + \frac{1}{\sqrt{2}\pi} \iiint_G \left\{ w_1 f_x + w_2 f_y + w_3 f_t \right\} dG \\
 & + \frac{1}{\sqrt{2}\pi} \iint_{\Omega_1} \left\{ w_1 v_x + w_2 v_y + w_3 (u_{xx} + u_{yy}) - u_x \frac{\partial w_1}{\partial t} - u_y \frac{\partial w_2}{\partial t} \right\} dG^*,
 \end{aligned}$$

where Ω_1 is now the interior of the circle $(x-x_0)^2 + (y-y_0)^2 = t_0^2$ lying in the plane $t = 0$. Again using integration by parts, and the fact that on the boundary of Ω_1 $dx = -t_0 \sin \theta \ d\theta$, $dy = t_0 \cos \theta \ d\theta$, the equation

(5.8) transforms into the following completely defined formula

$$(5.9) \quad u(x_0, y_0, t_0) = \frac{1}{2\pi} \int_0^{2\pi} U(x_0 + t_0 \cos \theta, y_0 + t_0 \sin \theta) d\theta + \frac{1}{2\pi} \int_0^{2\pi} \int_0^{t_0 \sqrt{t_0^2 - r^2}} \frac{1}{\sqrt{t_0^2 - r^2}} U_r dr d\theta$$

$$+ \frac{1}{2\pi} \int_0^{2\pi} \int_0^{t_0} \frac{V(x_0 + r \cos \theta, y_0 + r \sin \theta) r dr d\theta}{\sqrt{t_0^2 - r^2}} + \frac{1}{\sqrt{2\pi}} \iiint_G \left\{ w_1 f_x + w_2 f_y + w_3 f_t \right\} dG.$$

6. A Method of Images. We shall presently apply the Green's formula (4.4)

FIGURE II

to a region G_2 , which, as depicted in Figure II, is contained in the region of Figure I. This region is defined by means of the inequalities

$$(6.1) \quad G_2: 0 \leq (t-t_0)^2 - (x-x_0)^2 - (y+y_0)^2$$

$$0 \leq y$$

$$0 \leq t \leq t_0.$$

The conical portion of the boundary of G_2 , namely, $\bar{C}$, is the image in the plane $y = 0$ of the corresponding

portion of the cone (2.1), while the other surfaces bounding G_2 are Ω_2 and $\bar{\Omega}_1$, the latter being contained in Ω_1 .

In order to apply the Green's formula (4.4) to the region G_2 , it is necessary to take the Riemann-functions of (3.4) with the y_0 replaced by

$-y_0$. The resulting functions are denoted by

$$(6.2) \quad \bar{w}_i = w_i(x-x_0, y+y_0, t-t_0) \quad (i=1,2,3).$$

Consequently, applying (4.4) to the region G_2 ,

$$(6.3) \quad \frac{1}{\sqrt{2}} \iint_{\bar{C}} s \frac{\partial^2 u}{\partial s^2} ds d\phi + \iint_{\bar{H}_1 + \bar{H}_2} \left\{ \bar{w}_1 \frac{\partial u_x}{\partial \tau} + \bar{w}_2 \frac{\partial u_y}{\partial \tau} + \bar{w}_3 \frac{\partial u_t}{\partial \tau} - u_x \frac{\partial \bar{w}_1}{\partial \tau} - u_y \frac{\partial \bar{w}_2}{\partial \tau} \right\} G_2^* -$$

$$\iiint_{G_2} \left\{ \bar{w}_1 f_x + \bar{w}_2 f_y + \bar{w}_3 f_t \right\} dG_2,$$

the conical surface $\bar{C}$ lying on the cone

$$(6.4) \quad x-x_0 = \frac{s}{\sqrt{2}} \cos \phi, \quad y+y_0 = \frac{s}{\sqrt{2}} \sin \phi, \quad t-t_0 = -\frac{s}{\sqrt{2}}.$$

Since for $y = 0$

$$(6.5) \quad \bar{w}_1 = w_1, \quad \bar{w}_2 = -w_2, \quad \bar{w}_3 = w_3, \quad \frac{\partial w_1}{\partial y} = -\frac{\partial w_1}{\partial y},$$

it follows that (6.3) is expressible as

$$(6.6) \quad \frac{1}{\sqrt{2}} \int_0^{\phi_1} [s u_s - u] d\phi - \frac{1}{\sqrt{2}} \int_0^{\phi_1} [s u_s - u] d\phi$$

$$\quad \quad \quad s=\sqrt{2} t_0 \quad \quad \quad s=\rho(\phi)$$

$$+ \iint_{\bar{H}_1} \left\{ u_x \frac{\partial w_1}{\partial t} + u_y \frac{\partial w_2}{\partial t} - v_x \bar{w}_1 - v_y \bar{w}_2 - u_{tt} \bar{w}_3 \right\} dG_2^*$$

$$\begin{aligned}
 & + \iint_{\Omega_2} \left\{ w_x w_1 - u_{yy} w_2 + w_t w_3 + u_x \frac{\partial w_1}{\partial y} - w \frac{\partial \bar{w}_2}{\partial y} \right\} d\Omega_2^* \\
 & - \iiint_{G_2} \left\{ f_x \bar{w}_1 + f_y \bar{w}_2 + f_t \bar{w}_3 \right\} dG_2 = 0,
 \end{aligned}$$

where

$$(6.7) \quad \phi_0 = \arcsin \left(\frac{y_0}{t_0} \right), \quad \phi_1 = \pi - \phi_0.$$

Dividing (6.6) by $\sqrt{2}\pi$ and subtracting the resulting equation from

(5.7) yields

$$\begin{aligned}
 (6.8) \quad u(x_0, y_0, t_0) &= \frac{1}{2\pi} \int_0^{2\pi} [u-s u_s] d\theta - \frac{1}{2\pi} \int_{\phi_0}^{\phi_1} [u-s u_s] d\phi + \frac{1}{2\pi} \int_{\phi_0}^{\phi_1} [u-s u_s] d\phi \\
 & \quad s=\rho(\theta) \quad s=\rho(\phi) \quad s=\sqrt{2} t_0 \\
 & + \frac{1}{\sqrt{2}\pi} \iiint_G \left\{ w_1 f_x + w_2 f_y + w_3 f_t \right\} dG + \frac{1}{\sqrt{2}\pi} \iiint_{G_2} \left\{ \bar{w}_1 f_x + \bar{w}_2 f_y + \bar{w}_3 f_t \right\} dG_2 \\
 & + \frac{1}{\sqrt{2}\pi} \iint_{\Omega_1} \left\{ v_x w_1 + v_y w_2 + (u_{xx}, u_{yy}) w_3 - u_x \frac{\partial w_1}{\partial t} - u_y \frac{\partial w_2}{\partial t} \right\} d\Omega_1 \\
 & + \frac{1}{\sqrt{2}\pi} \iint_{\bar{\Omega}_1} \left\{ v_x \bar{w}_1 + v_y \bar{w}_2 + (u_{xx} + u_{yy}) \bar{w}_3 - u_x \frac{\partial \bar{w}_1}{\partial t} - v_y \frac{\partial \bar{w}_2}{\partial t} \right\} d\bar{\Omega}_1 \\
 & - \frac{1}{\sqrt{2}\pi} \iint_{\Omega_2} \left\{ w_x w_1 + u_{yy} w_2 + w_t w_3 - u_x \frac{\partial w_1}{\partial y} - w \frac{\partial w_2}{\partial y} \right\} d\Omega_2
 \end{aligned}$$

$$+ \frac{1}{\sqrt{2\pi}} \iint_{\Omega_2} \left\{ -W_x w_1 + u_{yy} w_2 - W_t w_3 - u_x \frac{\partial w_1}{\partial y} + W \frac{\partial \bar{w}_2}{\partial y} \right\} d\Omega_2.$$

An easy calculation shows that

$$(6.9) \quad \frac{1}{2\pi} \int_0^{2\pi} [u-s u_s] d\theta = \frac{1}{2\pi} \int_{-\sin^{-1}(\frac{y_0}{t_0})}^{\pi+\sin^{-1}(\frac{y_0}{t_0})} [u-s u_s] d\theta + \frac{1}{2\pi} \int_{\pi+\sin^{-1}(\frac{y_0}{t_0})}^{2\pi-\sin^{-1}(\frac{y_0}{t_0})} [u-s u_s] d\theta,$$

where the second integral has the arguments $\{x = x_0 + t_0 \cos \theta, y = y_0 + t_0 \sin \theta, t = 0\}$ and the third integral the arguments $\{x = x_0 - y_0 \cot \theta, y = 0, t = t_0 + y_0 \csc \theta\}$.

Moreover,

$$(6.10) \quad \frac{1}{2\pi} \int_{\theta_0}^{\theta_1} [u-s u_s] d\theta = \frac{1}{2\pi} \int_{\sin^{-1}(\frac{y_0}{t_0})}^{\pi-\sin^{-1}(\frac{y_0}{t_0})} [u-s u_s] d\theta,$$

the arguments of the second integral being $\{x = x_0 - y_0 \cot \theta, y = 0, t = t_0 - y_0 \csc \theta\}$.

$$\text{On } (t-t_0)^2 - (x-x_0)^2 - (y-y_0)^2 = 0,$$

$$(6.11) \quad -s u_s = (t_0 - t) u_t - (x - x_0) u_x - (y - y_0) u_y;$$

$$\text{while on } (t-t_0)^2 - (x-x_0)^2 - (y+y_0)^2 = 0,$$

$$(6.12) \quad -s u_s = (t_0 - t) u_t - (x - x_0) u_x - (y + y_0) u_y.$$

Using the last four equations, (6.8) reduces to the final form

$$(6.13) \quad u(x_0, y_0, y_0) = \frac{y_0}{\pi} \int_{\pi+\sin^{-1}(\frac{y_0}{t_0})}^{2\pi-\sin^{-1}(\frac{y_0}{t_0})} W(x_0 - y_0 \cot \theta, t_0 + y_0 \csc \theta) d\theta$$

$$+ \frac{1}{2\pi} \int_{-\sin^{-1}(\frac{y_0}{t_0})}^{\pi + \sin^{-1}(\frac{y_0}{t_0})} [U(x_0 + t_0 \cos \theta, y_0 + t_0 \sin \theta) + t_0 (V(x_0 + t_0 \cos \theta, y_0 + t_0 \sin \theta)$$

$$- \cos \theta U_x - \sin \theta U_y) d\theta$$

$$+ \frac{1}{2\pi} \int_{\sin^{-1}(\frac{y_0}{t_0})}^{\pi - \sin^{-1}(\frac{y_0}{t_0})} [U(x_0 + t_0 \cos \theta, -y_0 + t_0 \sin \theta) + t_0 (V(x_0 + t_0 \cos \theta, -y_0 + t_0 \sin \theta)$$

$$- \cos \theta U_x - \sin \theta U_y) d\theta$$

$$+ \frac{1}{\sqrt{2\pi}} \iint_{\Omega_1} \left\{ v_x w_1 + v_y w_2 + (u_{xx} + u_{yy}) w_3 - u_x \frac{\partial w_1}{\partial t} - u_y \frac{\partial w_2}{\partial t} \right\} d\Omega_1$$

$$+ \frac{1}{\sqrt{2\pi}} \iint_{\bar{\Omega}_1} \left\{ v_x \bar{w}_1 + v_y \bar{w}_2 + (u_{xx} + u_{yy}) \bar{w}_3 - u_x \frac{\partial \bar{w}_1}{\partial t} - u_y \frac{\partial \bar{w}_2}{\partial t} \right\} d\bar{\Omega}_1$$

$$- \frac{1}{\sqrt{2\pi}} \iint_{\Omega_2} \left\{ 2w_x w_1 + 2w_t w_3 - w \frac{\partial w_2}{\partial y} - w \frac{\partial \bar{w}_2}{\partial y} \right\} d\Omega$$

$$+ \frac{1}{\sqrt{2\pi}} \iiint_G \left\{ w_1 f_x + w_2 f_y + w_3 f_t \right\} dG + \frac{1}{\sqrt{2\pi}} \iiint_{G_2} \left\{ \bar{w}_1 f_x + \bar{w}_2 f_y + \bar{w}_3 f_t \right\} dG_2 .$$

7. The Possio Integral. It is defined implicitly as being the solution of the problem (1.1) - (1.3), with $U = V = f = 0$. In this case the solution (6.13) reduces to the formula

$$(7.1) \quad u(x_0, y_0, t_0) = \frac{y_0}{\pi} \int_{\pi + \sin^{-1}(\frac{y_0}{t_0})}^{2\pi - \sin^{-1}(\frac{y_0}{t_0})} W(x_0 - y_0 \cot \theta, t_0 + y_0 \csc \theta) d\theta$$

$$- \frac{1}{\sqrt{2\pi}} \iint_{\Omega_2} \left\{ 2 W_x w_1 + 2 W_t w_3 - W \frac{\partial w_2}{\partial y} - W \frac{\partial \bar{w}_2}{\partial y} \right\} d\Omega_2 .$$

Now, using integration by parts, (7.1) becomes

$$(7.2) \quad u(x_0, y_0, t_0) = \frac{y_0}{\pi} \int_{\pi + \sin^{-1}(\frac{y_0}{t_0})}^{2\pi - \sin^{-1}(\frac{y_0}{t_0})} W(x_0 - y_0 \cot \theta, t_0 + y_0 \csc \theta) d\theta + \int_{\text{Boundary } \Omega_2} \left[\frac{\sqrt{2}}{\pi} W w_1 dt - \frac{W}{\pi} dx \right]$$

$$+ \frac{1}{\sqrt{2\pi}} \iint_{\Omega_2} W \left[2 \frac{\partial w_1}{\partial x} + \frac{\partial w_2}{\partial y} + \frac{\partial \bar{w}_2}{\partial y} \right] d\Omega_2 .$$

y=0

Using the fact that $W = 0$ on the line boundary of Ω_2 and that the parametric form of the curved boundary of Ω_2 is $\{x-x_0 = Y_0 \cot \theta, t-t_0 = y_0 \csc \theta\}$, one readily shows that the second integral in (7.2) is the negative of the first. Thus,

$$(7.3) \quad u(x_0, y_0, t_0) = \frac{1}{\sqrt{2}\pi} \iint_{\Omega_2} W \left[2 \frac{\partial w_1}{\partial x} + \frac{\partial w_2}{\partial y} + \frac{\partial \bar{w}_2}{\partial y} \right] d\Omega_2 \Big|_{y=0}.$$

Hence, as

$$(7.4) \quad \left[2 \frac{\partial w_1}{\partial x} + \frac{\partial w_2}{\partial y} + \frac{\partial \bar{w}_2}{\partial y} \right]_{y=0} = \frac{-\sqrt{2}}{\sqrt{(t-t_0)^2 - (x-x_0)^2 - y_0^2}},$$

the final form for the Possio integral is

$$(7.5) \quad u(x_0, y_0, t_0) = -\frac{1}{\pi} \iint_{\Omega_2} \frac{W d\Omega_2}{\sqrt{(t-t_0)^2 - (x-x_0)^2 - y_0^2}},$$

which is the formula (1.4).

8. Boundary values assumed by Possio Integral. Let $X_1 = x_0 - \sqrt{t_0^2 - y_0^2}$ and $X_2 = x_0 + \sqrt{t_0^2 - y_0^2}$. Then, by applying integration by parts, (1.4) becomes

$$(8.1) \quad u(x_0, y_0, t_0) = \frac{1}{\pi} \int_{X_1}^{X_2} W(x, t_0 - \sqrt{(x-x_0)^2 + y_0^2}) \ln [\sqrt{(x-x_0)^2 + y_0^2}] dx \\ - \frac{1}{\pi} \int_{X_1}^{X_2} \int_0^{t_0 - \sqrt{(x-x_0)^2 + y_0^2}} W_t(x, t) \ln [(t_0 - t) + \sqrt{(t-t_0)^2 - (x-x_0)^2 - y_0^2}] dx dt.$$

Differentiating $u(x_0, y_0, t_0)$ with respect to y_0 yields

$$(8.2) \quad \frac{\partial u}{\partial y_0}(x_0, y_0, t_0) = \frac{1}{\pi} \int_{X_1}^{X_2} \frac{y_0 W(x, t_0 - \sqrt{(x-x_0)^2 + y_0^2}) dx}{(x-x_0)^2 + y_0^2} \\ + \frac{1}{\pi} \int_{X_1}^{X_2} \int_0^{t_0 - \sqrt{(x-x_0)^2 + y_0^2}} \frac{y_0 W_t(x, t) dx dt}{[(t_0 - t) + \sqrt{(t-t_0)^2 - (x-x_0)^2 - y_0^2}][\sqrt{(t-t_0)^2 - (x-x_0)^2 - y_0^2}]}.$$

As the double-integral is integrable, it follows that

$$(8.3) \quad \lim_{\substack{y_0 \rightarrow 0 \\ y_0 > 0}} \frac{\partial u(x_0, y_0, t_0)}{\partial y_0} = \lim_{\substack{y_0 \rightarrow 0 \\ y_0 > 0}} \frac{1}{\pi} \int_{X_1}^{X_2} \frac{y_0 W(x, t_0 - \sqrt{(x-x_0)^2 + y_0^2}) dx}{(x-x_0)^2 + y_0^2}.$$

We rewrite this equation in the form

$$(8.4) \quad \lim_{\substack{y_0 \rightarrow 0 \\ y_0 > 0}} \left[\frac{\partial u(x_0, y_0, t_0)}{\partial y_0} - W(x_0, t_0) \right] = \lim_{\substack{y_0 \rightarrow 0 \\ y_0 > 0}} \frac{1}{\pi} \int_{X_1}^{X_2} \frac{y_0 [W(x, t_0 - \sqrt{(x-x_0)^2 + y_0^2}) - W(x_0, t_0)] dx}{(x-x_0)^2 + y_0^2} \\ + \lim_{\substack{y_0 \rightarrow 0 \\ y_0 > 0}} \frac{1}{\pi} \int_{X_1}^{X_2} \frac{y_0 W(x_0, t_0) dx}{(x-x_0)^2 + y_0^2} - W(x_0, t_0)$$

Making the sufficient assumption that W_x and W_t are continuous, it follows that the limit of the first integral in (8.4) is zero. Furthermore, as

$$(8.5) \quad \lim_{\substack{y_0 \rightarrow 0 \\ y_0 > 0}} \frac{y_0}{\pi} \int_{x_1}^{x_2} \frac{dx}{(x-x_0)^2 + y_0^2} = 1,$$

it follows that

$$(8.6) \quad \lim_{\substack{y_0 \rightarrow 0 \\ y_0 > 0}} \left[\frac{\partial u(x_0, y_0, y_0)}{\partial y_0} - W(x_0, t_0) \right] = 0.$$

This relation shows that on the plane $y = 0$ the normal derivative of the Possio integral is W ; that is, $u_y(x, 0, t) = W(x, t)$.

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