INFINITE COMPONENT WAVE EQUATIONS

Yoichiro Nambu

The Enrico Fermi Institute for Nuclear Studies
and the Department of Physics
The University of Chicago, Chicago, Illinois

LEGAL NOTICE

This report was prepared as an account of Government sponsored work. Neither the United States, nor the Commission, nor any person acting on behalf of the Commission:
A. Makes any warranty or representation, expressed or implied, with respect to the accuracy, completeness, or usefulness of the information contained in this report, or that the use of any information, apparatus, method, or process disclosed in this report may not infringe privately owned rights; or
B. Assumes any liabilities with respect to the use of, or for damages resulting from the use of any information, apparatus, method, or process disclosed in this report.

As used in the above, "person acting on behalf of the Commission" includes any employee or contractor of the Commission, or employee of such contractor, to the extent that such employee or contractor of the Commission, or employee of such contractor prepares, disseminates, or provides access to, any information pursuant to his employment or contract with the Commission, or his employment with such contractor.

Submitted to Physical Review

February 1967

Contract No. AT(11-1)-264
DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.
DISCLAIMER

Portions of this document may be illegible in electronic image products. Images are produced from the best available original document.
INFINITE COMPONENT WAVE EQUATIONS*

Yoichiro Nambu

The Enrico Fermi Institute for Nuclear Studies
and the Department of Physics
The University of Chicago, Chicago, Illinois

ABSTRACT

Continuing a previous work, we consider various models of relativistic wave equations having an infinite number of components. By combining a unitary representation of SO(4,2) and the ordinary finite (Dirac) representation of the Lorentz group, it is possible to construct equations which produce hydrogen-like mass spectra. However, they are also accompanied by redundant or unphysical solutions. In the non-relativistic limit, on the other hand, our equations can be shown to be mathematically equivalent to the Schrödinger equation for the hydrogen atom. This suggests that the method of infinite component wave equations may be a useful tool in exploring the physics of strong interactions. A general discussion is made about the principles and problems that will be relevant in pursuing such a program.

*This work supported in part by the U.S. Atomic Energy Commission.
I. INTRODUCTION

In the exploration of the nature of strongly interacting particles we are always plagued by our ignorance of the basic dynamical laws and an adequate mathematical tool to handle them. These two difficulties reinforce each other and make it hard to achieve a quantitative description of hadron properties after we have come to a fairly satisfactory qualitative understanding in terms of models, such as the quark model, and simple group theoretical arguments.

For this reason, setting up elaborate dynamical equations, for example a Bethe-Salpeter equation using the quark model, may not be very useful. First of all, we do not know the precise inter-quark dynamics, and secondly, it is difficult to solve the equation. Inasmuch as we are mainly interested in the properties of the solution, namely the mass spectrum, the internal structure revealed by form factors and scattering cross sections, etc., this is a roundabout approach.

In a recent paper\textsuperscript{1} we tried to strike an intermediate path by considering simple relativistic wave equations having an infinite number of components, which are easily soluble and yield information about the mass spectrum and the matrix elements of observables, and therefore serve as dynamical equations for describing the properties of complex systems. Two model equations were examined.
One is based on a set of multispinor fields $\psi_n^m(x)$, each regarded as an irreducible spinor representation of the homogeneous Lorentz group, while the other uses the set as a basis for a unitary infinite dimensional representation of the Lorentz group, of such a nature that it actually constitutes an irreducible unitary representation of the larger group $SO(4,2)$. In the former, the mass spectrum and degeneracy structure simulate the real hydrogen-like atom; but otherwise the equation is unphysical essentially because it does not admit a positive definite probability density. In the latter, we do not seem to run into obvious conflict with quantum mechanical interpretation, but there are still some odd features, namely that the discrete hydrogen-like mass spectrum is inverted in order, and allows only one sign of frequency, contrary to ordinary relativistic wave equations which always admit both signs.

The concept of infinite component wave equations is not new. Already in 1932 Majorana was led to this type of equation, which was later rediscovered by Gelfand and Yaglom. Our second example is a slight generalization of this Majorana-Gelfand-Yaglom equation. As representations of the Lorentz and Poincaré groups, the general structure of infinite component wave equations was studied in detail by Gelfand, Naimark, and others. An alternative approach to this problem, dating back to Dirac, is to use continuous variables rather than discrete ones. This looks more directly related to the description of composite particles, whether the continuous variables are regarded as relative internal
coordinates of a bound system, or taken in a more abstract sense as in the bilocal theory of Yukawa\textsuperscript{7} and subsequent works of many people. In particular, in a series of papers Takabayashi\textsuperscript{8} has made a transition from the continuous to the discrete basis of representation, and considered simple wave equations and their mass spectra. His equations are based on a harmonic oscillator type model which is viewed as a geometrical interpretation of SU(3) symmetry.\textsuperscript{9}

With the apparent success of the quark model and SU(6) symmetry, people have been led to appreciate the use of non-compact groups such as SL(6) and U(6,6) and their representations as a means of characterizing and systematizing the properties of hadrons without reference to their dynamical origin. This program has been vigorously pursued from a formal mathematical point of view by a number of people,\textsuperscript{10} For the justification of the substitution of dynamics with group theory one usually quotes two conspicuous examples, the isotopic harmonic oscillator and the hydrogen atom. However, the demonstration has been less than complete in the case of the hydrogen atom.

Our example equations, although written down merely on the basis of simplicity, seem to possess many features of the hydrogen atom at least on the surface. This accident encourages us to embark on a systematic study of infinite component wave equations. Our main purposes in the present paper are to search for wave equations which simulate more closely the real hydrogen-like atom, and to determine how deep the similarities are. Our
attempt is only partially successful as far as translating the hydrogen atom into a relativistic form since the class of equations we find still suffer from various diseases if we want to regard them as a bona fide field theory. These diseases include unphysical redundant solutions, especially those with space-like four-momenta; a tendency to lack anti-particle (negative frequency) solutions (thereby spoiling the TCP theorem); and indefinite sign of energy, either in c-number or quantized theory. The loosening of spin-statistics connection has been pointed out by Fronsdal,¹⁰ and by Feldman and Matthews¹¹ recently. But a systematic examination of these problems is not our concern here.

In the case of the non-relativistic Schrödinger hydrogen atom, on the other hand, the mathematical equivalence of our formalism to the conventional one is actually almost complete, as has been recently shown by Barut and Kleinert,¹² and by Fronsdal¹³ recently. In fact, we can carry out the transition from one to the other by finding a mapping of SO(4,2) algebra onto functions of continuous variables. Their same mapping formulas also serves us in such purposes as computing analytically the form factors in relativistic models, and casting the Bethe-Salpeter equation into our discrete form.
II. GENERALIZED MAJORANA EQUATION WITH SO(4) SYMMETRY

In this section we recapitulate and discuss the results of A. Since we shall be dealing with a number of model equations, it is convenient to label each of them by a serial number. Thus the two examples in A will be called No. 1 and No. 2.

We take a set of spinors \( \{ \psi_n^m \} \), \( n, m = 1, 2, \ldots \), with \( n \) and \( m \) lower and upper spin indices, each being symmetric under interchange of indices among themselves. Thus each \( \psi_n^m \) is an irreducible representation \( D(\frac{n}{2}, \frac{m}{2}) \) of the group \( SU(2) \times SU(2) \) or \( SO(4) \). This choice corresponds to the fact that we shall be dealing with a system which exhibits an \( SO(4) \) degeneracy structure.

The spin part of the rotation group \( SO(3) \) in real space is identified with a subgroup of the above \( SO(4) \). We then extend \( SO(4,1) \) which contains the Lorentz group \( SO(3,1) \) in such a way that it can be realized within the set \( \{ \psi_n^m \} \). One way to do this is by regarding each \( \psi_n^m \) as the usual spinor representation of \( SO(3,1) \sim SL(2, \mathbb{C}) \). Another way is to take a unitary representation. In either case, the set is large enough to accommodate not only the generators of \( SO(3,1) \) but also Lorentz scalar \( S \) and vector (current) \( \Gamma_\mu \) operators, as well as space reflection. In terms of these one can write down a wave equation

\[
\left[ \Gamma_\mu p^\mu + (S-\alpha)e^x \right] \Psi(x) = 0 \tag{1}
\]

where \( \Psi(x) = \{ \psi_n^m(x) \} \), and \( \alpha \) and \( e^x \) are \( c \)-number parameters.
The first example (Model No. 1), based on an infinite sum of representations, need not be discussed here for the reason already mentioned. In the case of unitary representations, we have considered a special case realized on a subset

\[ S_{\pm k} = \{ \psi_{n}^{m} \}, \quad n-m = \text{const.} = \pm k. \]

It turns out that this set not only constitutes an irreducible unitary representation of \( SO(4,1) \) but also that of \( SO(4,2) \), or more precisely, its covering group \( SU(2,2) \). Here we will arrange the six dimensions 0, 1, 2, ..., 5 with the metric \(+ + - - - +\), and identify the Minkowski subspace with \( 123 \) (space) and 0 (time).

The 15 generators \( M_{\alpha\beta} \) of \( SO(4,2) \) are defined in such a way that \( 1 + i \epsilon^{\alpha\beta} \epsilon_{\alpha\beta} \) is the infinitesimal transformation, and

\[ M_{\alpha\beta} = -M_{\beta\alpha} \]
\[ M_{\alpha}^{\beta} = M_{\alpha}^{\gamma} g^{\gamma\beta}, \quad g^{\alpha\beta} = g_{\alpha\beta} = \delta_{\alpha\beta} \times (1,-1,-1,-1,1) \]
\[ M_{\alpha}^{\beta} = -M_{\beta}^{\alpha}, \quad \alpha, \beta \epsilon (1, \ldots 4) \text{ or } \alpha, \beta \epsilon (0, 5) \]
\[ M_{\alpha}^{\beta} = M_{\beta}^{\alpha}, \quad \alpha(\beta) \epsilon (1, \ldots 4), \beta(\alpha) \epsilon (0, 5) \quad (2) \]

They satisfy the commutation relations

\[ -i\left[ M_{\alpha\beta}, M_{\gamma\delta} \right] = g_{\beta\gamma} M_{\alpha\delta} + g_{\delta\alpha} M_{\beta\gamma} - g_{\alpha\gamma} M_{\beta\delta} - g_{\beta\gamma} M_{\alpha\delta} \quad (3) \]

Under the restriction to \( SO(3,1) \), these 15 generators break up into a 6-vector \( (M_{ik}, M_{0i}, i = 1,2,3) \), a 4-vector \( (M_{51}, M_{50}) \), a 4-vector \( (M_{41}, M_{40}) \) and a scalar \( (M_{54}) \).

In our representation space \( S_{k} \), the generators have the following expression
\[ Z_{M_0} = (a^+ a + b^+ b + 2) \equiv N, \quad C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \] (4)

in terms of the two-component Bose operators \( a^+, a \) and \( b^+, b \) which change \( \psi_n^m \) into \( \psi_{n+1}^{-m} \) and \( \psi_{n+1}^{-m+1} \) respectively. If we combine \( a \) and \( b \) into a 4-component Dirac spinor \( \xi = (a, b) \), and use two sets of Pauli matrices \( \sigma_i \) and \( \rho_i \), Eq. (4) can be seen to consist of 7 bilinear forms \( \xi^\dagger \xi, \xi^\dagger \sigma_i \xi, \xi^\dagger \rho_j \sigma_i \xi \) and 8 quadratic (and necessarily symmetric) forms \( \xi^\dagger \xi, \xi^\dagger \sigma_i \xi^\dagger \sigma_i \xi \) and h.c. (\( \bar{C} = \rho_2 \sigma_2 \)), whose algebra (the algebra of 15 Dirac matrices) is equivalent to that of the non-compact group \( SU(2,2) \) of 4 x 4 matrices that leaves \( \xi^\dagger \rho_j \xi = \xi^\dagger C \xi \) invariant. The parity, or reflection in the subspace (123), may be defined within the pair \( S_{\pm k} \) by the operation \( S_{\pm k} \rightarrow RS_{\pm k} \sim S_{\mp k} \) such that

\[
\begin{align*}
a &\rightarrow RaR^{-1} = -ib, \quad b \rightarrow RbR^{-1} = -ia, \\
a^+ &\rightarrow ib^+, \quad b^+ \rightarrow ia^+, \\
R &\Rightarrow \exp [i\pi/2(a^+b + b^+a)], \\
R^2 &= (-i)^{n+m}. \quad (5)
\end{align*}
\]

The case \( k = 0 \) is unique in that it does not require doubling of the representation.
An important property of this degenerate unitary representation $S_k$ is that, with respect to the compact subgroup $SO(4) \times SO(2)$, it reduces to a sum

$$\sum_{n-m=k} D_{SO(4)}\left(\frac{n}{2}, \frac{m}{2}\right) \times D_{SO(2)}\left(\frac{n}{2} + \frac{m}{2} + 1\right).$$

Thus, to a given eigenvalue of $M_{50} = N$, there corresponds a unique irreducible representation in the complementary space $(1, \ldots, 4)$.

Our model equation No. 2 was obtained by equating in Eq. (1)

$$\Gamma_{\mu} = 2 M_{5\mu}, \quad \mu = 1, 2, 3, 0$$

$$S = 2 M_{54}$$

(In general one could take for $\Gamma_{\mu}$ a linear combination of $M_{5\mu}$ and $M_{4\mu}$. But, as will be clear from what follows, it can be reduced to either pure $M_{5\mu}$, pure $M_{4\mu}$, or $M_{5\mu} \pm M_{4\mu}$ by a rotation. The last two do not give a discrete spectrum.) Introducing the 6-dimensional notation $q^{\alpha} = (p_0, p, \kappa, 0)$ and $\Gamma_{\alpha} = M_{5\alpha}$, we have then

$$(\Gamma_{\alpha} q^{\alpha} - \kappa \alpha) \psi = 0 \quad (7)$$

Since the spaces $S_{\pm k}$ are decoupled, we first note that all solutions come in pairs of opposite parities except when $k = 0$. Now if $q^{\alpha}$ is time-like, i.e.,

$$q_{\alpha} q^{\alpha} = p_0^2 - p^2 - \kappa^2 = m^2 - \kappa^2 > 0,$$

we can go to a "rest frame" in which

$$q^{\alpha} = (\text{sgn}(n_i) \sqrt{m^2 - \kappa^2}, 0, 0)$$
and obtain the eigenvalues

\[ \Gamma_0 \left( m^2 - \kappa^2 \right) \frac{\nu}{x} \text{sgn}(m) = N \left( m^2 - \kappa^2 \right) = \kappa \alpha \]

or

\[ m = |\kappa|(1 + \alpha^2/N^2)^{1/2} \text{sgn}(\kappa \alpha). \] (8)

When \( q_\alpha q^\alpha < 0 \) we can take a frame in which only one of the space-like components, e.g., \( \Gamma_4 \), survives, so that

\[ \pm \Gamma_4 \left( \kappa^2 - m^2 \right) \frac{\nu}{x} = \kappa \alpha. \] (9)

Since \( \Gamma_4 \) is a non-compact generator, its eigenvalues are continuous, running between \(-\infty\) and \(+\infty\), which means \( \kappa^2 > m^2 > -\infty\).

The corresponding eigenfunctions are non-normalizable. Formally Eq. (9) is obtained from Eq. (8) by analytic continuation of \( N^2 \) to negative values. (The special case \( q_\alpha q^\alpha = 0 \) belongs to the end of the spectrum (8), and likewise has only one sign, although the eigenfunction is non-normalizable.)

Thus, Eq. (7) possesses a discrete inverted hydrogen-like spectrum \( m > \kappa \) followed by a continuous part of both signs \( |m| < \kappa \), which extends into imaginary \( m \), or space-like 4-momentum. \(^{14}\) The continuous part must be regarded as physically relevant since under an external field, transition between discrete and continuous parts will take place in general.

We see from the foregoing that this model equation possesses still many unphysical features which it shares with the original Majorana equation. The only improvement is that the discrete mass spectrum does not come down to zero. The difficulties are:
1) The mass spectrum is inverted; 2) it is not symmetric between positive and negative values; 3) it contains possibly dangerous space-like solutions. On the other hand, it has also some interesting features. Namely, it has the hydrogen-like degeneracy, i.e., \( SO(4) \) type for the discrete states, and \( SO(3,1) \) type for the continuum. This can be seen by noting that in its "6-dimensional rest frame", Eq. (7) contains only the generator of a 2-dimensional subspace \((05)\) or \((45)\), and therefore commutes with the generators of the complementary space. The symmetry is "dynamical", in the sense that the symmetry subspace is cut out from the 6-space in a different way for each mass level. For the same reason, it also follows that the orthogonality of eigenfunctions does not hold with respect to the norm \( \psi^+ \psi \) which forms the basis of the unitary representation, but it holds only with respect to the charge (density) \( \psi^+ \Gamma_0 \psi \) which is the physically conserved quantity. These two things are not equivalent; a mass eigenstate is not an eigenstate of \( \Gamma_0 \). Nevertheless, \( \Gamma_0 \) has no off-diagonal elements since the eigenfunctions are not orthogonal in the usual sense.

In A, we computed the magnetic moment under the minimal electromagnetic interaction, and found the \( g \) factor to be negative, unlike the hydrogen atom. On the other hand, the form factors show reasonable behavior. For the scalar and vector vertices, we find in fact
\[ \psi^+(p') \psi(p) = F(t) = \frac{4(m^2 - \kappa^2)}{4(m^2 - \kappa^2) - t} \]
\[ t = (p' - p)^2, \]
\[ \psi(p') \gamma_\mu \psi(p) = \frac{[(p_\mu + p'_\mu)/2m]}{2m} F(t)^2 \]  

for the ground state \( N = 2 \). This can be derived by using the techniques of Section IV, but its general property may be inferred by observing that \( F(t) \) must be a function of

\[ q_\alpha q^\alpha = p_\mu p^\mu - \kappa^2 = t/2 + m^2 - \kappa^2 = (m^2 - \kappa^2) \cosh \mathcal{G}, \]

where \( \mathcal{G} \) is the hyperbolic angle between \( q \) and \( q' \). \( F(t) \) is then the elementary spherical function \( 1/\cosh^2(\mathcal{G}/2) \) associated with our representation.

III. RELATIVISTIC MODELS WITH HYDROGEN-LIKE SPECTRA

We try now to improve our Model No. 2 so as to obtain a correct hydrogen-like behavior with respect to level ordering, form factors, etc. Our previous examples were first order wave equations. This is not only due to our search for simple soluble equations. The algebra of \( SO(4,2) \) happens to contain 4-vector generators so that it is indeed possible to write a first order equation. We shall therefore restrict ourselves at first to first order equations. By means of elimination, first order equations can be converted into second order equations, if necessary. The search for general first order equations will be done by taking a product of infinite dimensional and finite dimensional
representations, i.e., regarding $\Psi$ to have components $\psi_{n,s}^m$, where $s$ is a spinor index of finite order. An underlying argument is that the conventional field theory for finite spin is known to work. Especially for spin 0, 1/2 and 1 cases the degree of singularity (or growth) of form factors associated with the finite spin is not serious, and we may expect enough compensation from the unitary part. Furthermore, the unitary representation can be interpreted as describing the internal orbital motion of particles (see later Sections). It will then be quite proper to introduce intrinsic spins of the constituents as separate variables. We can adopt the Dirac and Duffin-Kemmer formalism to handle the finite spin part. We discuss here a few different models as typical examples. Other examples will be found in the Appendix.

Model No. 3

\[ L\Psi = \left( \Gamma_\mu p^\mu + S(x\gamma_\mu p^\mu + \kappa \gamma_5 N_5) - \kappa \alpha \right) \overline{\Psi} = 0 \]  

(11)

$\Gamma_\mu$ and $S$ are the same as in Eq. (6). The pseudoscalar $N_5 = n - m = k(\neq 0)$ is a constant to within a sign in the representation space $S_{\pm k}$. The Dirac $\gamma$'s are standard ones, with $\gamma_0$ being Hermitian, and $\gamma_1, \gamma_5 = \gamma_0 \gamma_1 \gamma_2 \gamma_3$ anti-Hermitian. This equation follows from a Lagrangian $L\Psi$, where the adjoint function to $\Psi$ must be defined by $\overline{\Psi} = \Psi^\dagger \gamma_0$. The parameters of the equation are $\kappa$, $\alpha$, and $x$, which we will take to be positive.

In order to diagonalize Eq. (11) in the rest frame
$p^\mu = (m, 0)$, we first make a non-unitary rotation in the Dirac space:

$$\psi = \exp[\frac{i}{2} \gamma_0 \gamma_5] \psi_1 = U_1 \psi_1, \quad \overline{\psi} = \overline{\psi}_1 U_1^{-1},$$

$$\theta \phi_1 = \kappa \frac{k}{x m}$$

which leads to

$$[\Gamma^0 m + S \gamma_0 \left( x^2 m^2 - \kappa^2 k^2 \right) \text{sgn}(m) - \kappa \alpha] \psi_1 = 0$$

We have assumed that $x^2 m^2 > \kappa^2 k^2$. The next transformation is a unitary one in the $(0^4)$ plane

$$\psi_1 = \exp[\frac{i}{2} \gamma_0 M_0 \gamma_4] \psi_2 = U_2 \psi_2, \quad \overline{\psi}_1 = \overline{\psi}_2 U_2^{-1}$$

$$\theta \phi_2 = \left( x^2 m^2 - \kappa^2 k^2 \right) \text{sgn}(x m) / m$$

assuming $|m| > \left( x^2 m^2 - \kappa^2 k^2 \right)^{1/2}$, and we find

$$[\Gamma^0 \left( \kappa^2 k^2 - (x^2 - 1)m^2 \right) \text{sgn}(m) - \kappa \alpha] \psi_2 = 0$$

Thus

$$m = \kappa (k^2 - \alpha^2 / N^2)^{1/2} (x^2 - 1)^{1/2}$$

$$= m_0 (1 - \alpha^2 / k^2 m^2)^{1/2}, \quad m_0 = \kappa k / (x^2 - 1)^{1/2}$$

This equation is correctly hydrogen-like and consistent with the rotations (12) and (14), if we require $x, \alpha$ and $k$ to be such that

$$x^2 - 1 > 0, \quad 1 - x^2 \alpha^2 / k^2 (|k| + 2)^2 > 0$$

[Note that $N = |k| + 2, \quad |k| + 4, \ldots$.]

The continuum part of the spectrum above $m_0$ can be obtained
in a similar way by rotating Eq. (13) into \( S \) direction instead of \( \Gamma_0 \). As was already discussed in Section 2, however, the continuum part consists of both signs since \( S \) takes all real eigenvalues. Unlike Model No. 2, on the other hand, there are no space-like solutions as we can check easily by diagonalizing the equation under the assumption that \( p^\mu \) is space-like.

The difficulties with this equation are that the discrete spectrum has only positive eigenvalues as in Model No. 2, and that each eigenvalue is four-fold degenerate, not counting the \( O(4) \) degeneracy of spins. The latter situation is due to the two signs \( \gamma_0 \) and \( k \) can take in Eq. (15), which means that half of them have opposite parity from the other.

The current operator that follows from Eq. (11) is

\[
J_\mu = \overline{\psi} (\Gamma_\mu + xS\gamma_\mu) \psi
\]  

(18)

After carrying out the transformations (12), (13), we find that \( j_0 \) changes sign with \( \gamma_0 \). Hence the energy operator also changes sign with \( \gamma_0 \) (for a fixed \( m > 0 \)). Certainly this is a serious difficulty in setting up a physically consistent Lagrangian formalism. We find, keeping only the diagonal part,

\[
J_0 = \psi_2^+ \gamma_0 \Gamma_0 (\text{ch}_2 \phi_2 - \text{sh}_2 \phi_2 \text{ch}_1 \phi_1) \psi_2
\]

\[
= [(1-x^2) m N^2/\kappa a] \psi_2^+ \gamma_0 \psi_2
\]

Hence the "charge" \( J_0 \) changes sign with \( \gamma_0 \), and so does the energy \( T_{00} = mj_0 \) in spite of \( m \) being \( > 0 \).
Eq. (11) was meaningful only for \( k \neq 0 \). When \( k = 0 \), or \( \Psi = \{ \psi^1_n \} \), we simply double the space: \( \Psi \rightarrow (\psi^{(1)}, \psi^{(2)}) \), and write

\[ L\Psi = \tau_3 (\Gamma_\mu p^\mu - \kappa \alpha) + S(x\gamma_\mu p^\mu + \kappa \tau_1) \Psi = 0 \]  

(19)

where the \( \tau \) matrices operate in the space \( (\psi^{(1)}, \psi^{(2)}) \). Again this can be diagonalized by means of two rotations

\[ \exp[\phi_1 \gamma_0 \tau_2/2] \text{ and } \exp[i \phi_2 \tau_3 \gamma_0 M_\alpha] \],

and leads to the solution (16) (with \( \phi \) replaced by 1). The 4-fold degeneracy corresponds to \( \tau_3, \gamma_0 = \pm 1 \).

The models considered above are closely related to the quadratic equations found by Fronsdal.\(^{13}\)

For example, Eq. (13) can be squared to give

\[ \left\{ \left[ S^{-1}(\Gamma_\mu m - \kappa \alpha) \right]^2 - x^2 m^2 + \kappa^2 k^2 \right\} \Psi = 0 \]

or

\[ L\Psi = \left\{ \left[ S^{-1}(\Gamma_\mu p^\mu - \kappa \alpha) \right]^2 - x^2 p_\mu p^\mu + \kappa^2 k^2 \right\} \Psi = 0 \]  

(20)

We may then drop the Dirac spin indices, and obtain Fronsdal's equation. The Lagrangian has to be defined by \( \Psi^\dagger S L\Psi \) for reasons of Hermiticity.

Although the Dirac indices may be eliminated in this way, the 4-fold degeneracy and associated difficulties of the solution remain unchanged.

**Model No. 4**

\[ L\Psi = \left[ \Gamma_\mu p^\mu + (S-\alpha)(x\gamma_\mu p^\mu + \kappa \gamma_5 N_5) \right] \Psi = 0 \]  

(21)
The discrete solution is

$$\Gamma_0[k^2 - (x^2 - 1)\frac{m^2}{N^2}]^{1/2} \text{sgn}(m) - \alpha \gamma_0[x^2 - \frac{\alpha^2}{N^2}]^{1/2} \times \text{sgn}(m) = 0$$

or

$$m = \pm \kappa |k| (1 + \frac{a^2}{N^2})/(x^2 - 1 + \frac{x^2 \alpha^2}{N^2})]^{1/2},$$

$$\gamma_0 = +1, \ x^2 \geq 1.$$ (22)

This is also hydrogen-like, and besides has the symmetric degeneracy pattern $m, m, -m, -m$.

The continuum part, on the other hand, is found to consist of upper and lower regions:

$$|m| > \kappa |k|/\left(\frac{x^2 - 1}{x^2} - \frac{\kappa^2}{x^2} > m^2\right)$$

and

$$0 < |m| < \kappa |k|/\left(\kappa^2 - x^2 m^2 > 0\right),$$

each $m$ being 4-fold degenerate. Beyond this there is also the space-like solution corresponding to imaginary $m$.

In spite of the unphysical lower and imaginary continua, this model possesses one feature. If we evaluate the current

$$j_{\mu} = \bar{\psi}[\gamma_\mu + x(S-u)\gamma_{\mu}]\psi,$$

we find that it has a definite sign for the discrete part of the spectrum. This suggests the possibility of quantizing the field according to Fermi statistics, and thus satisfying positive definiteness of energy (for the physical part) irrespective of integer or half-integer spin.
We next consider a second order equation.

Model No. 5.\textsuperscript{17}

Take the space $S_0$ and put

$$\left(\Gamma^\mu p_\mu + \frac{1}{\kappa} S p_\mu p^\mu - \alpha \gamma_5 p^\mu\right)\Psi = 0$$

(23)

The massive solution has the spectrum

$$m = \pm \kappa \left(1 - \alpha^2 / N^2\right)^{1/2}, \quad (\gamma_0 = +1)$$

$$|m| > \kappa, \quad (S m \gamma_0 > 0)$$

(24)

for the discrete and continuum states, which is symmetric in sign, but otherwise similar to Model 3. In addition, however, there exists the obvious massless solution $p_\mu = 0$. In a typical case, this means

$$[(\Gamma_0 - \Gamma_3) - \alpha (\gamma_0 - \gamma_3)] \Psi = 0$$

or

$$(\gamma_0 - \gamma_3) \Psi = (\Gamma_0 - \Gamma_3) \Psi = 0$$

(25)

as can be seen by squaring the first equation. The eigenvalues of $\Gamma_0 - \Gamma_3$ are continuous and $\geq 0$, so Eq. (25) corresponds to the end point of continuum.

Except for the massless solution, the spectrum is right for a hydrogen-like system: The degeneracy is $m$, $-m$ and $m$, $m$, $-m$, $-m$ for the discrete and continuous part respectively. It may be allowed to interpret this as corresponding to the discrete and continuous parts of the system $p^\pm e^{\pm}$ combined (where $p$ is a scalar in this case).
As for the current
\[ j_\mu = \bar{\psi}(\Gamma_\mu - \alpha \gamma_\mu + \frac{2}{\kappa} S \rho_\mu)\psi, \]

it turns out that the charge has definite sign for the discrete and continuous parts.\(^1\) It would seem possible, therefore, to quantize the field correctly according to Fermi statistics and make the energy positive definite.

Summarizing this section, we have considered linear as well as quadratic equations in the product space of \( S \) and Dirac spinor. All of them have hydrogen-like discrete and continuous spectra, but suffer from some unwanted features such as the lack of symmetry in positive and negative frequencies, the presence of redundant solutions—including space-like solutions,\(^2\) and the indefiniteness of energy sign. Relatively speaking, Model 5 may be the most satisfactory, but the significance of massless solutions remains to be clarified.

IV. ALTERNATIVE REPRESENTATIONS OF SO(4,2) ALGEBRA

We develop here representations of the SO(4,2) algebra in terms of continuous variables. First we note that the set \( S_{\pm k} \) can be generated from the ground state \( \psi_{k}^0 \) or \( \psi_{0}^k \) by repeated applications of \( X_\mu^+ \), where

\[ X_\mu^+ = a^+ \sigma_\mu C b^+ \, , \quad X^\mu = a C \sigma^\mu b = (X_\mu^+)^+, \]

\[ \sigma_\mu = (\sigma_0 = 1, \sigma_1, \sigma_2, \sigma_3), \quad \sigma^\mu = (\sigma_0, -\sigma_1, -\sigma_2, -\sigma_3) \]

\( (27) \)
which satisfy

\[ X_\mu^+ X_\mu^+ = X_\mu^\mu = 0 \]  

(28)

In view of Eq. (4), this means also

\[ \sum_{i=1}^{4} (M_{0i} M_0^i + M_{5i} M_5^i) = 0 \]

\[ \sum_{i=1}^{4} (M_{0i} M_5^i + M_{5i} M_0^i) = 0 \]

(29)

For simplicity we restrict ourselves to the case \( k = 0 \) below.

Let us set up a correspondence between \( S_0 \) and a space \( F = \{ f(x_\mu) \} \) of functions of four variables such that

\[ \psi^0_0 = \psi^0 \rightarrow 1, X_\mu^+ \rightarrow x_\mu \]  

(30)

\( F \) is then a set of polynomials in \( x_\mu \).

For the time being, we assume \( x_\mu \) to be independent. \( X_\mu^+ \) and \( X_\mu \) generate the algebra

\[ [X_\mu^\mu, X_\mu^+] = 2N = a^+ a + b^+ b + 2 \]  

(no summation)

\[ [X_0, X_1^+] = 2 \epsilon_1 = a^+ \sigma_1 a - b^+ \sigma_1 b \]

\[ [X_i, X_j^+] = -2i \epsilon_{ijk} L_k = -i \epsilon_{ijk} (a^+ \sigma_k a + b^+ \sigma_k b), (i \neq j) \]

\[ [X_\mu^+, N] = -X_\mu^+, [X_\mu^+, N] = X_\mu. \]  

(31)

These relations can be translated into the space \( F \) by the assignment
Here \( a \) is an arbitrary constant. If, however, we impose the condition (29) or that \( f(x_{\mu}) \) is a function of a null-vector:

\[
x_{\mu}x^{\mu} = x_{0}^{2} - x_{1}^{2} = 0,
\]

such a condition must be compatible with Eq. (32). In other words, for any operator \( 0 \) in (32), we must have \([x_{\mu}x^{\mu}, 0]f = 0\) if \( x_{\mu}x^{\mu} = 0\). This is true only if

\[
a = 1,
\]

which will be assumed from now on. With Eqs. (4), (28), (30), (33), we can complete the mapping of the space \( S_{0} \) onto \( F \). For computing the norm of a vector in \( F \) we essentially follow the prescription: Write first

\[
(\psi, \psi) = (f(x^{+})\psi_{0}, f(x^{+})\psi_{0}) = (\psi_{0}, f^{*}(x)f(x^{+})\psi_{0})
\]

and then translate \( x, x^{+} \) and \( \psi_{0} \) using Eqs. (30) and (31). An analytic definition is then

\[
(\psi, \psi) \rightarrow \frac{1}{(2\pi i)^{4}} \int \cdots \int \frac{dz_{0} \cdots dz_{3}}{z_{0} \cdots z_{3}} f^{*}(x)f(x^{+})
\]

where the \( Z \)'s are complex extensions of the \( x \)'s, and the contours taken around zero. This formula is often convenient in practical calculations.
We next show that there exists another mapping 
\[ S_0 \rightarrow G = \mathcal{O}\{f(x_\mu)\}, \]
in which the norm can be defined as an integral over the real axis with a simple weight function:

\[ (\psi, \psi) \rightarrow (f, f) = \int \int f(x_\mu)^* f(x_\mu) \frac{dx_0 \cdots dx_3}{x_\lambda x} \]  

(36)

It is realized by retaining the correspondence (32) but redefining the generators as

\[ \frac{1}{2}(x_0^+ + x_0) = M_{40} \text{ or } M_{45} \]
\[ \frac{1}{2}(x_0^+ - x_0) = M_{50} \text{ or } M_{50} \]
\[ \frac{1}{2}(x_1^+ + x_1) = M_{51} \text{ or } M_{41} \]
\[ \frac{1}{2}(x_1^+ - x_1) = M_{51} \text{ or } -M_{01} \]
\[ -ik_i = M_{01} \text{ or } -M_{01} \]
\[ L_1 = \varepsilon_{ijk} M_{jk} \]
\[ -iN = M_{54} \text{ or } -M_{04} \]  

(37)

(The two choices are mathematically equivalent as they correspond to the interchange 0 \(\leftrightarrow\) 5, but can make a difference in physical interpretation.)

It can be readily verified that these operators are self-adjoint in the sense \((g, Of) = (0g, f)\) when the scalar product is taken according to Eq. (36).

We have actually not made the restriction (29) on the generators. Therefore we are dealing with a different \(\mathcal{O}\).
representation from Eq. (4). From Eqs. (32) and (37) we derive the formula

\[ x_\mu x^\mu D_\lambda D^\lambda = \sum_\alpha (M_{a\alpha} M_{5\alpha} + M_{4\alpha} M_{4\alpha}) + 2 \]

\[ x_\mu x^\mu = \sum_\alpha (M_{5\alpha} + M_{4\alpha})(M_{5\alpha} + M_{4\alpha}) \]

\[ x_\mu x^\mu (D_\lambda D^\lambda)^2 = \sum_\alpha (M_{5\alpha} - M_{4\alpha})(M_{5\alpha} - M_{4\alpha}) \] (38)

Next we proceed to impose the condition

\[ x_\mu x^\mu = 0, \]

or

\[ x_0 = \pm (x_1 x_1)^{1/2} = \pm r \] (39)

Since this is consistent with the commutator algebra, we can make the substitution: \( f(x_\mu) \rightarrow f(x_1, x_0(x_1)) = g(x_1) \). Accordingly we can also drop \( D^0 \) and replace \( x_0 \) by \( \pm r \) in Eq. (32) without affecting the commutation relation. The result is

\[ x_0^+ = \pm r = M_{40} + M_{50} \text{ or } M_{45} + M_{50} \]

\[ x_0^+ = \pm r\Delta = M_{40} - M_{50} \text{ or } M_{45} - M_{50} \]

\[ x_1^+ = x_1 = M_{41} + M_{51} \text{ or } M_{41} - M_{01} \]

\[ x_1 = x_1\Delta + 2x_k D^k D_1 + 2D_1 = M_{51} - M_{41} \text{ or } -(M_{01} + M_{41}) \]

\[ \pm iK_1 = \pm i \pm r \Delta D_1 = M_{01} \text{ or } M_{51} \]

\[ -iN = -i(x_1 D_1^\dagger + 1) = M_{54} \text{ or } -M_{04} \]

\[ (\Delta = D_1 D_1^\dagger = D_1 D_1^\dagger) \] (40)
The metric that replaces Eq. (36) for \( g(x_1) \) is

\[
(g,g) = \pm \int \cdots \int g^*(x_1) g(x_1) \frac{dx_1 dx_2 dx_3}{r}
\]

as may be understood from the fact

\[
\delta(x_0) \delta(x_\mu x^\mu) d^4x \sim d^3x/2r.
\]

Finally we observe that

\[
(g, M_{50} g) = \pm \frac{1}{2} \int \cdots \int g^*(1-\Delta) g d^3x
\]

\[
= \pm \frac{1}{2} \int \cdots \int (g^* g + \nabla_1 g^* \nabla_1 g) d^3x
\]

Remembering that \( M_{50} = N \) defined by Eq. (4) has positive eigenvalues, we conclude that the + sign has to be taken for Eqs. (40) and (41) to be equivalent to the original representation (4).

The representation (40) is essentially the same as that found by Fronsdal.\(^\text{13}\) He arrived at this result by starting from the Fock representation of the hydrogenic Schrödinger equation. Barut and Kleinert\(^\text{12}\) had also discovered the use of SO(4,2) in the hydrogen problem, identified the relevant representation and written down observables in terms of the generators. The next section deals with a re-examination of this problem.

V. NON-RELATIVISTIC HYDROGEN ATOM

Eq. (40) contains all the necessary operators for writing down a Schrödinger equation for the hydrogen atom if we identify the variables \( x_i \) with the actual spatial coordinates. To exhibit
the significance of our procedure clearly, it is better to treat the case as a two-body problem. We will thus write down the Hamiltonian

\[ H = \frac{1}{2m_1} [p^{(1)} - e_1 A(r^{(1)})]^2 \]

\[ + \frac{1}{2m_2} [p^{(2)} - e_2 A(r^{(2)})]^2 \]

\[ + \frac{e_1 e_2}{|r^{(1)} - r^{(2)}|} + e_1 \varphi(r^{(1)}) + e_2 \varphi(r^{(2)}) \]  

(42)

for a generalized Hydrogen-like system in an external electromagnetic field. Introducing the new coordinates and momenta

\[ C_1 r^{(1)} + C_2 r^{(2)} = X, \quad C_1 + C_2 = 1 \]

\[ r^{(1)} - r^{(2)} = r \]

\[ p^{(1)} + p^{(2)} = p \]

\[ C_2 p^{(1)} - C_1 p^{(2)} = p \]  

(43)

we transform Eq. (42) into

\[ H = \frac{1}{2m_1} [C_1 p + p - e_1 A(X + C_2 r)]^2 \]

\[ + \frac{1}{2m_2} [C_2 p - p - e_2 A(X - C_1 r)]^2 \]

\[ + e_1 \varphi(X + C_2 r) + e_2 \varphi(X - C_1 r) + \frac{e_1 e_2}{r} \]  

(44)
The relative coordinates $r$ may now be identified with the representation variables of Eq. (40), and the operators $r$, $r$, $p$, $p^2$ that appear in Eq. (44) replaced by the generators of $SO(4,2)$. An awkward point, however, is that $r$ appears in the external fields $A$ and $\varphi$. This cannot be avoided in general, but in the limiting case $m_2 \gg m$, and $\varphi = 0$ we can choose $C_1 = 1$, $C_2 = 0$ to obtain

$$H = \frac{1}{2m_1} [P + p - e_1 A(X)]^2 + \frac{e_1 e_2}{r}$$

(45)

[Note that the usual center of mass transformation would correspond to $C_1 = 0$, $C_2 = 1$.] Multiplying (45) by $r/r_0$ from the left,

$$\frac{r}{r_0} H = -\frac{1}{2m_1} \frac{r}{r_0} [P - e_1 A(x)]^2 - \frac{1}{m_1 r_0} \frac{r}{r_0} p_2 [P - e_1 A(x)]$$

$$+ \frac{1}{2m_1} \frac{r}{r_0} p^2 + \frac{e_1 e_2}{r_0}$$

(46)

where $r_0$ is an arbitrary scale factor. Identifying $r_1/r_0$ with the dimensionless variables in Eq. (40), this leads to the eigenvalue equation

$$\left\{ \frac{1}{r_0} (M_{50} + M_{45})[E - \frac{1}{2m_1} (P - e_1 A(X))^2]$$

$$- \frac{1}{2m_1 r_0} (M_{50} - M_{45}) - \frac{1}{m_1} M_{51} [P_1 - e_1 A_1(X)]$$

$$- \frac{e_1 e_2}{r_0} \right\} \psi = 0,$$

(47)
adopting the second assignment of generators.

We have thus succeeded in transforming the Schrödinger equation into the form of an infinite component equation in which only the generators $M_{50}^a, M_{40}^a, M_{01}^a$ appear. Clearly it also preserves the norm in view of Eq. (41) and $H = \frac{1}{r} (rH)$.

Eq. (47) was first obtained by Fronsdal. In case $A = 0$, it can be easily diagonalized by the previous techniques and yields the well-known energy spectrum

$$E = \frac{\mathbf{p}^2}{2m_1} - \frac{4(e_1 e_2)^2 m_1}{N^2}, \quad N = 2, 4, 6, \ldots$$

(\(r_0\) drops out from the result.) The first term is the kinetic energy of the whole system in motion, and the second term is the binding energy. Of course the continuum $E - \mathbf{p}^2/2m_1 > 0$ also follows from the equation. Besides, Eq. (47) satisfies the gauge principle when a transverse field is switched on. Thus the system behaves as if it were in a single particle and its internal dynamics has been replaced by the abstract algebra of $SO(4,2)$.

It is instructive to re-examine the previous relativistic models in the light of the present results. For example, Model 2, Eqs. (6) and (7), can be translated into the "Schrödinger representation"

$$(2M_{50}^0 + 2M_{51}^1 + 2M_{54}^\kappa - \kappa \alpha)\psi \rightarrow [(P_0 - \kappa r/r_0 + (P_0 + \kappa) r_0 r p^2 - 2r p \cdot p - \kappa \alpha)\psi (r)$$
or
\[
\left[ (P_0 + \kappa) r_0 + (P_0 - \kappa) r_0 p^2 - 2 P \cdot p - \kappa a / r \right] \psi (r) = 0 \quad (49)
\]

Setting \( r_0 = 1 / (P_0 + \kappa) \) and \( p - P \rightarrow p' \), this becomes

\[
\left[ P_0^2 - P^2 - \kappa^2 + (p')^2 - \kappa a / r \right] \psi (r) = 0 \quad (50)
\]

It is a Klein-Gordon type equation where the internal energy is given by the hydrogenic Hamiltonian (the last term) but with the wrong sign. In the small binding non-relativistic limit, the system looks as if made up of two particles with total mass = \( \kappa \) and reduced mass = \(-\kappa\).

In a similar way, other models may be treated. In the non-relativistic limit of Model 3 we may start from Eq. (13). (However, the formulas in this section apply only to the space \( S_0 \).) We write

\[
L \psi = \left[ T P + S \left( x^2 (P_0^2 - P^2) - \kappa^2 \right) / 2 - \kappa a / r \right] \psi = 0
\]

and expand it thus:

\[
\left[ \Gamma_0 (m_0 + \epsilon) + S (m_0 + x^2 \epsilon - x^2 p^2 / 2m_0) (x^2 - 1)^{\frac{1}{2}} m_0 a / r \right] \psi = 0,
\]

\[
\kappa = (x^2 - 1)^{\frac{1}{2}} m_0, \quad p_0 = m_0 + \epsilon, \quad \epsilon \ll m_0. \quad (52)
\]

This becomes in the Schrödinger representation

\[
\left[ -\epsilon + \frac{x^2}{x^2 - 1} \frac{1}{2m_0} (P - \frac{2m_0 r_0}{x^2} p)^2 - \frac{m_0 r_0}{(x^2 - 1)^{\frac{1}{2}}} \frac{a}{r} \right] \psi (r) = 0 \quad (53)
\]

In the presence of an external field, we replace \( P \) and \( \epsilon \) by \( P - eA(x) \) and \( \epsilon - e\varphi (x) \). Comparing Eq. (53) with Eq. (44) we
find then that they are equivalent if we choose

\[ c_1 = 0, \quad c_2 = 2m_0 r_0 / x^2 = 1, \]

\[ m_1 = m_0 / x^2, \quad m_2 = m_0 (x^2 - 1) / x^2, \quad m_1 + m_2 = m_0 \]

\[ e_2 = e, \quad e_1 = 0, \quad \alpha = 2e_1 e_2 (m_1 / m_2)^{1/2} \]

\[ = 2e_1 e_2 / (x^2 - 1)^{1/2}. \] (54)

We can keep \( \alpha \) finite if we regard this as a limit \( e_1 \to 0 \) and \( m_2 \to 0 \), keeping \( e_1^2 / m_2 \) finite.

In the case of Model 5, the equation corresponding to (49) is

\[ [2M_{50} P^0 + 2M_{51} P^1 + 2M_{54} (P_0^2 - P^2) / \kappa - \alpha (\gamma_0 P^0 + \gamma_1 P^1)] \psi = 0 \] (55)

Replacing the last term by \( -\alpha (P_0^2 - P^2)^{1/2} \), we find in the lowest approximation

\[ (\epsilon + \frac{1}{\kappa} (P - p)^2 + \frac{1}{\kappa} p^2 - \frac{\alpha}{\kappa}) \psi(r) = 0 \] (56)

so that

\[ m_1 = m_2 = \kappa / 2, \quad e_1 = 0, \quad e_2 = e \]

but

\[ \alpha = e_1 e_2 \neq 0, \] (57)

which is not exactly a hydrogen-like situation.

We close this section with a reference to Bethe-Salpeter equations. It is a natural idea that the Bethe-Salpeter equation might also be amenable to transformation into a discrete representation. In fact, Eqs. (32) and (38) provide the
necessary formulas for this purpose. Consider, for example, two scalar particles interacting via a scalar photon. The B-S equation reads, in the differential form,

\[ \{(\frac{p}{2} + p)^2 - \mu^2\}(\frac{p}{2} - p)^2 - \mu^2\} - \frac{\sigma^2}{4\pi^2 x^2} \psi(x_\mu) = 0, \]

or in the rest frame,

\[ [x^2(p^2)^2 + 2x^2p^2(\frac{m^2}{4} - \mu^2) + x^2(\frac{m^2}{4} - \mu^2)^2 \]

\[ - m^2x^2p_0^2 - \frac{\sigma^2}{4\pi^2}] \psi(x_\mu) = 0 \]

\[ (x^2 = x_\mu x^\mu, \quad p^2 = p_\mu p^\mu = -D_\mu D^\mu) \]

For the first three terms we have the ready-made formulas (38). Equating the necessary scale parameter \( r_0^2 \) with \( 1/(\frac{m^2}{4} - \mu^2) \), the sum of them simply become

\[ -(m^2 - 4\mu^2)(M_4\alpha M^{\alpha}_4 + 1), \quad (m^2 - 4\mu^2 < 0, \text{ bound states}) \]

\[ -(m^2 - 4\mu^2)(M_5\alpha M^{\alpha}_5 + 1), \quad (m^2 - 4\mu^2 > 0, \text{ continuum}) \]

The fourth term, on the other hand, cannot be translated so simply. We have to express \( D_0 \) from \( X_0 \) of Eq. (32):

\[ 2(x_\lambda D^{\lambda} + 1) D_0 - x_0 D_\lambda D^{\lambda} = X_0 \]

and making the substitution (37). Although the equation cannot be solved easily except when \( m^2 = 0 \), this method provides another way of looking at the B-S equation.
VI. FORMULATION OF THE GENERAL PROBLEM

The main lessons we have learned from the results of the previous sections may be summarized as follows. 1) It is possible to set up infinite-component simple model equations which simulate a composite, hydrogen-like system in many respects. But so far they do not satisfy all the requirements for being a physically meaningful Lagrangian field theory, except possibly Model 5. 2) In the non-relativistic sense, however, there are no basic difficulties. In fact, the Schrödinger equation can be transformed into our type of equation and vice versa. One limitation to this is that as a genuine two-body problem the correspondence is only approximate when external fields are introduced.

As a field theory, the infinite component fields need not be regarded as approximate and phenomenological substitutes for dynamical equations for composite systems. They may be fundamental in their own right. But there are many novel properties we have yet to understand in this type of theory. For example, we may ask: a) What becomes of the spin-statistics connection in general (which seems to have been loosened)? b) What are the implications of the positive-negative asymmetry of mass spectrum vis-à-vis P, C and T invariance (or non-invariance)? c) Are the space-like solutions really dangerous? d) Do b) and c) imply breakdown of local commutativity and causality between fields? If so, is it possible that the currents (bilinear forms of fields) still satisfy causality? Although some of these problems
have been examined by Feldman and Matthews\textsuperscript{11} and Fronsdal,\textsuperscript{10} much remains to be done.

Assuming that these difficulties can be overcome or ignored, at least in suitable model equations, we may set down the working principles to be followed in applying our method to other complex systems, in particular to hadron physics and nuclear physics.\textsuperscript{21}

First we take a definite model (like the quark model or the harmonic oscillator model) with a certain compact symmetry group $G_0$ (considered as rest symmetry) and degeneracy structure represented by a particular set $S$ of representations. $G_0$ must contain the SU(2) or O(3) group corresponding to internal rotations of a system. We enlarge the group $G_0$ to $G$ in such a way that a) $S$ is its unitary representation (or a product of unitary and simple finite representations), b) $G$ contains the (internal) Lorentz group, and c) there exist among the generators of $G$ also Lorentz four-vectors $\Gamma_{\mu}$, symmetric tensors $\Gamma_{\mu\nu}$, etc.

We can then couple $\Gamma_{\mu}$, $\Gamma_{\mu\nu}$, etc., with the external momentum to write down a first order or second order wave equation. In any case, we may assume (at least as a simplest possibility) that the wave equation is linear or quadratic in all these generators and momenta.

In practice we may try the ladder operator technique, as we have done here, to generate $G$ out of $G_0$, which will also determines the type of the representation $S$. For example, the Majorana equation has $G_0 = SO(3)$, $G = SO(3,2)$ and $S = D(0) + D(1) + \ldots$ or $D(\frac{1}{2}) + D(\frac{3}{2}) + \ldots$ of SO(3) (of course the reflection group must also be built into this). Our Model 2 has $G_0 = SO(4)$, $G = SO(4,2)$,
S = \sum D(n, \frac{1}{2} + n). In other models, \( G_0 = SO(4) \times SU(2), G = SO(4,2) \times SL(2,C) \), and \( S \) is the product of finite (Dirac) and infinite representations.

When \( G_0 = SU(3) \), considered as the symmetry group of the harmonic oscillator, the corresponding \( S \) consists of \( D(n,0), n=0, 1, 2, \ldots \) of \( SU(3) \). Use of the three-component ladder operators \( a_1, a_1^+ \) leads to an enlarged group \( S_p(6,R) \), with nine compact generators \( a_1^+ a_k \) and twelve non-compact generators \( a_1^+ a_k, a_1 a_k^+ \). This group, however, does not contain the Lorentz group. The correct seems to be \( G = SU(3,1) \), with fifteen generators \( a_1^+ a_k \) and \( (N+C)^{\frac{1}{2}} a_1, a_1^+ (N+C)^{\frac{1}{2}}, (N = a_1^+ a_1) \). These make up the Lorentz generators and a symmetric traceless tensor \( \sum_{\mu\nu} \) so that a second order wave equation can be written down.\(^{23}\)

The existence of vector or tensor operators serves the purposes of giving a mass spectrum and bringing these operators into a well defined algebra. From the foregoing results it is clear that the infinite representation is suitable for characterizing in an abstract way the internal orbital motion of a system. Namely, the rest symmetry group \( G_0 \) reflects how the internal orbital motion is organized, i.e., how the rotational and radial degrees of freedom can be excited. The intrinsic spin of the constituents will then be multiplied into this orbital wave function. Thus the observables consist of generators of the orbital as well as the intrinsic spin algebra. Of course, the strength of our approach lies in that we do not have to always draw a line between orbital and intrinsic spin. The latter may
also belong to an infinite dimensional representation (as in
the SL(6) or U(6,6) models). The orbital representation space,
at the same time, need not always be introduced for each rela-
tive coordinate or constituent particle. The key factor that
determines our space S is the energy level structure and its
degeneracy, exact or approximate, especially for the low lying
excitations.

The method of infinite component equations, considered in
the above sense, is a rather phenomenological and simplified
scheme of characterizing a complex system in toto, albeit in an
approximate fashion but without missing its essential features.
It may also be regarded as a way of realizing the algebra of cur-
cents (observables) that follows from the conventional field
theory. In fact the group G and space S may be chosen in such
a way as to satisfy the algebra. On the other hand, it is con-
ceivable that the wave equation has more dynamical contents
(since it specifies not only commutators, but also anticommu-
tators, etc., which depends on a specific representation), and yet
at the same time may satisfy the current algebra only approxi-
mately (since we imbed observables in an algebra of finite
dimensions and take only a few irreducible representations).

One of the deficiencies of the wave equation method is the
lack of a clear cut principle for choosing an equation, except
that it is expected to produce at least a qualitatively correct
level scheme. Besides, we do not know yet whether such an
equation can be realized in general as a field theory, or merely
as an S-matrix theory. One may perhaps conjecture that it will fall somewhere in between, in the sense that a Lagrangian formalism can be set up, and quasi-local interactions with external as well as among infinite component fields introduced according to a simple prescription, which can reproduce complicated processes sufficiently well in Born approximation, but it may not quite qualify for being a complete and consistent field theory.

The author expresses his gratitude to Profs. A.O. Barut, C. Fronsdal and T. Takabayashi for very enlightening communications.
APPENDIX A

SOME MORE MODELS

Model No. 6

\[ [\tau_3 \Gamma_\mu p^\mu + S(x\gamma_\mu p^\mu + \tau_1 \kappa) - \tau_3((x^2-1)^{1/2} \gamma_\mu p^\mu - \kappa)\alpha] \psi = 0 \]  
(A1)

This is a modification of Eq. (19). A new feature is that the spectrum is linear in \( m \):

\[ m = \frac{\kappa}{(x^2-1)^{1/2}} \frac{1-\alpha^2/N^2}{1+\alpha^2/N^2} \]

\( \gamma_0, \tau_3 = \pm 1 \)  
(A2)

for the discrete part, with the degeneracy \( m, m, m, m \). The continuum extends over the ranges \( |m| > \kappa/(x^2-1)^{1/2} = m_0 \). In addition, there is an infinite degeneracy of all spins at \( m = m_0 \). The energy takes both signs.

Model No. 7

Modify Eq. (11) with

\[ L\psi = [\Gamma_\mu p^\mu + S(x\gamma_\mu [N_5, \gamma_\mu p^\mu] - \alpha\eta) \gamma_\mu p^\mu] \psi = 0 \]  
(A3)

for the case \( |N_5| = |k| = \text{even} \), where \( \eta \) is the operator with the properties

\[ \eta: \ a \to b, \ b \to a, \]

\[ [\Gamma_\mu, \eta] = 0, \ [S, \eta] = 0, \ [N_5, \eta] = 0, \]

\[ R\eta R^{-1} = (-1)^N_5 \eta, \ \eta^2 = 1. \]  
(A4)
When \(|k|\) = odd, replace \(\eta\) with \(iN\eta\). Eq. (21) may be diagonalized to give
\[
\Gamma_0[\kappa^2-(x^2-1)m^2]^{1/2} \text{sgn}(m) - \alpha \gamma_0 m = 0
\]
or
\[
m = \pm \kappa/(x^2-1 + \alpha^2/N^2)^{1/2},
\]
\[
\gamma_0 \eta = 1. \quad \text{(A5)}
\]
Thus we have a 4-fold solution \(m, m, -m, -m\) corresponding to \(\eta = \gamma_0 = \pm 1\). The continuum part now extends above \(\kappa/(x^2-1)^{1/2}\) only, though there still exist imaginary solutions. Eq. (24) resembles very closely the solution of the relativistic Coulomb problem, except that we do not have spin-orbit splitting. The energy again remains indefinite.

For the case \(k = 0\), an analogous equation takes the form
\[
L\Psi = \left[\tau_3(\Gamma_\mu p^\mu - \alpha \gamma_\mu p^\mu) + S(x\gamma_\mu p^\mu + \kappa \tau_2)\right]\Psi = 0 \quad \text{(A6)}
\]
with the same spectrum as Eq. (24), but
\[
\gamma_0 = +1, \quad \tau_3 = \pm 1.
\]
An interesting point in this model is that it admits a Pauli-Gursey group which accounts for the degeneracy structure. The Pauli-Gursey transformation for Eq. (24) is defined by
\[
\delta \Psi = u \Psi^+, \quad \delta \Psi^+ = u^* \Psi,
\]
\[
u = e^{i\alpha} u_0, \quad (\alpha \text{ arbitrary})
\]
\[
u_0 = \gamma_5 \tau_2 G, \quad G = \tau_2 C \gamma \mathcal{C}
\]
\[
u^\dagger = -u, \quad u^* u = 1. \quad \text{(A7)}
\]
Here $\mathcal{C}$ is the charge conjugation ($-\rho_2^c \sigma_2$) in the Dirac space, and $\mathcal{C}$ is its analog for the infinite representation:

$$\mathcal{C} = \exp \left[ \frac{i}{\hbar} \mathbf{N} \right] \exp \left[ \frac{i}{\hbar} (a^+ \mathbf{\sigma}_2 a - b^+ \mathbf{\sigma}_2 b) \right]$$

$$\mathcal{C}^2 = 1, \quad \mathcal{C}^T = (-1)^N \mathcal{C} = \mathcal{C}, \quad \mathcal{C}_a \mathcal{C}^{-1} = iCa, \quad \mathcal{C}_b \mathcal{C}^{-1} = -iCb,$$

$$\mathcal{C} T \mathcal{C}^{-1} = T, \quad \mathcal{C} S T \mathcal{C}^{-1} = S.$$ (A8)

The Lagrangian $\mathcal{Y}^+ \mathcal{L} \mathcal{Y}$ remains invariant under (26) if $\mathcal{Y}$ is quantized according to Fermi statistics, because $\mathcal{L} \mathcal{Y} \mathcal{L} \mathcal{Y} + h.c.$ is a symmetric quadratic form and hence has to vanish. The ordinary gauge transformation and Eq. (26) form a SU(2) group, as can be seen by writing $\mathcal{X} = (\mathcal{Y}, \mathcal{Y}^+)$ and the three generators of this group as

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & u_0 \\ -u_0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & iu_0 \\ -iu_0 & 0 \end{pmatrix}.$$ (A9)

Similar transformations can be defined for Eq. (21) with general $k$, if we maintain the conventional relation between spin and statistics. The symmetry group for integral spin case, however, is SU(1,1) instead of SU(2).

**Model No. 8**

This is a variation of Model 5, without the Dirac space

$$[\Gamma_\mu p^\mu + (S-\alpha)p_\mu p^\mu] \psi = 0$$ (A10)
When $p_\mu p^\mu \neq 0$, this leads to

$$[\Gamma_0 \kappa + (M-\alpha)m] \Psi = 0$$

or

$$m = \kappa/(1 + \alpha^2 m^2)^{1/2}$$  \hspace{1cm} (All)

which is similar to the Model 7 case. In addition, however, there is a massless family of solutions satisfying $\Gamma_\mu p^\mu \Psi = 0$, as in Model 5.
APPENDIX B

COMPUTATION OF FORM FACTORS

We show here a method of computing form factors which is based on Eqs. (4) and (27)-(35). We demonstrate it to derive Eq. (10) in Model 2. The scalar form factor between the same levels is given by

\[ F = \langle \Psi(p), \Psi(0) \rangle = \langle \Psi(0), e^{i\sigma_{M03}} \Psi(0) \rangle \]

if the momentum is in the direction \( 3 \). Going to the 6-dimensional rest frame, (B1) further reduces to

\[ \langle \Psi_1 e^{-i\theta_{M04}} e^{i\sigma_{M03}} e^{-i\theta_{M04}} \Psi_1 \rangle \]

\[ \langle \Psi_1 \exp[i\sigma(M_{03} c \theta - M_{03} s \theta)] \Psi_1 \rangle \]

\( \theta = \kappa/m \)

According to Eqs. (4), (27), (31) and (32), we can make the replacement

\[ 2M_{03} = x^+_3 + \lambda^3 \rightarrow x^+_3 + x^+_3 \lambda^\lambda - 2x^\lambda \lambda^\lambda D_3 - 2D_3 \]

\[ 2M_{43} = 2K_3 \rightarrow 2(x^0_3 D^0 + x^0_0 D^3) \]

The eigenfunction \( \Psi_1 \) is a polynomial \( f(x_\mu) \). Let us consider instead the Laplace kernel \( \exp[\kappa^\mu x_\mu] \). When applied to it, we may replace (B3) by
acting on \( \exp[\mathbf{k}^\mu x_\mu] \). Since Eq. (B5) is linear in the derivatives, the exponential form (B2) may be computed explicitly by a suitable change of variables. By expanding the kernel in powers of \( \mathbf{k}^\mu \) before and after the operation, and comparing the coefficients, we find how a monomial \( x^n \) is transformed by the operation.

For the ground state, we need keep only the variable \( \mathbf{k}^0 \equiv \Delta \) and \( \mathbf{k}^3 \equiv t \). Thus

\[
2M_{03} \; \text{ch}\theta - 2M_{43} \; \text{sh}\theta
\]

\[
= \left\{ (1 + s^2 + t^2) \text{ch}\theta + 2s \text{sh}\theta \right\} D_t + 2t \text{ch}\theta
\]

\[
+ \left\{ 2st \text{ch}\theta - 2t \text{sh}\theta \right\} D_s
\]

\[
= \left\{ (1 + \xi^2) \text{ch}\theta + 2\xi \text{sh}\theta \right\} D_\xi + \xi \text{ch}\theta
\]

\[
- \left\{ (1 + \eta^2) \text{ch}\theta + 2\eta \text{sh}\theta \right\} D_\eta + \eta \text{ch}\theta \right\}
\]

where \( s + t = \xi \), \( s - t = \eta \). A further transformation turns this into
42.

\[ [D_p - (\cot p + \text{sh} \theta)] - [D_q - (\cot q + \text{sh} \theta)] \]

\[ = \exp [\ln \sin p \sin q - (p+q) \text{sh} \theta] (D_p-D_q) \times \exp [-\ln \sin p \sin q + (p+q) \text{sh} \theta], \]

\[ p = \frac{1}{24} \ln (\xi_1-\xi_2)/(\xi_1-\xi_2), \quad q = \frac{1}{24} \ln (\eta_1-\eta_2)/(\eta_1-\eta_2) \]

\[ \frac{\xi_1}{\eta_1} = -\text{th} \theta \pm i \text{sech} \theta. \quad (B6) \]

The expectation value (B2) is then

\[ \exp [i \mathcal{J} (M_{03} \text{ch} \theta - M_{03} \text{sh} \theta)] \exp [\ell_{\mu} x_{\mu}] \]

\[ \rightarrow \exp [\ln \sin p \sin q - (p+q) \text{sh} \theta] \exp [i \frac{\mathcal{J}}{2} \left( \frac{\partial}{\partial p} - \frac{\partial}{\partial q} \right)] \times \exp [-\ln \sin p \sin q + (p+q) \text{sh} \theta] \]

\[ = \exp [\ln \sin p \sin q - \ln \sin (p + i \frac{\mathcal{J}}{2}) \sin (q - i \frac{\mathcal{J}}{2})] \quad \text{at} \quad p = q = a, \]

\[ a = \frac{1}{24} \ln (\xi_1/\xi_2) = \frac{1}{2} \ln \frac{\text{sh} \theta - i}{\text{sh} \theta + i} \]

which is equal to

\[ \frac{1}{\text{ch}^2 \theta \text{sh}^2 \frac{\mathcal{J}}{2} + 1} = \frac{1}{1 - t/4(m^2-\kappa^2)} \quad (B7) \]

Similarly the vector form factor (\(\Psi(p), \Gamma_{\mu} \Psi(0)\)) may be calculated, and yields the results (10) of Section II.
REFERENCES

1. Y. Nambu, Prog. Theor. Phys., to be published, referred to as A hereafter.

2. E. Majorana, Nuovo Cimento 9, 335 (1932).


14. The existence of space-like solutions in the Majorana equation was pointed out by Majorana⁴ and by V. Bargmann, Math. Rev. 10, 583 (1949).

15. The Dirac and Duffin-Kemmer algebras are finite dimensional realizations of SU(2,2), but probably this fact is accidental to the hydrogen atom problem.

16. The Duffin-Kemmer type equations are not treated in this paper. In general, they lead to additional spurious solutions which are not present (or whose masses are pushed to $\infty$) in the ordinary Duffin-Kemmer equation.

17. A somewhat related equation was considered by Barut and Kleinert (preprint). They discuss only physically relevant parts of the spectrum.

18. This can be seen from Eq. (40). $\Gamma_0 - \Gamma_3 \sim M_{50} - M_{53}$

$\sim M_{50} - M_{54} \sim p$ as far as the spectrum is concerned.

19. We need some care in the continuum case. Instead of a pure eigenstate of $S$ (after rotation), a wave packet of finite norm should be considered. Though both $S$ and $\Gamma_0$ have diagonal elements, we may eliminate $S$ in $j_0$ by means of the wave equation. The coefficient of $\Gamma_0$ then determines the sign as $\langle \Gamma_0 \rangle$ can be made arbitrarily large.

20. In general this is inevitable. It derives from the fact that the Hamiltonian defined (from a linear equation) by $i \frac{\partial \Psi}{\partial t} = H \Psi$ is not Hermitian as $\Gamma_0^{-1}$ does not commute with $\vec{P} \cdot \vec{k}$. Thus there may be complex energy for some $p$. 

22. This problem is being investigated in collaboration with S. P. Rosen.

23. To achieve the familiar equidistant spectrum, one needs a fourth or third order equation

\[ \left[ \Gamma_{\mu\nu} p^\mu p^\nu + \alpha p^2 + \beta (p^2)^2 \right] \Psi = 0 \]

or

\[ \left[ \Gamma_{\mu\nu} p^\mu p^\nu + \alpha p^2 + \beta p^2 \gamma_5 p^\mu \right] \Phi = 0 \]

They have space-like (and discrete!) solutions too, but no massless solutions. Other possibilities are under study.