97,536

PARTIAL WAVE ANALYSIS OF THE $(\pi^+\pi^+\pi^-)$ SYSTEM THROUGH THE REGION OF THE A_3 MESON*

G. Thompson, R. C. Badewitz, J. A. Gaidos, R. L. McIlwain,
K. Psler†, and R. B. Willmann

Purdue University, Physics Department, Lafayette, Indiana 47907

ABSTRACT

A recent spin parity analysis of the $(\pi^+\pi^+\pi^-)$ system formed opposite a proton and a coherent deuteron by incident $13 \text{ GeV}/c^2$ π^+ mesons, is extended to a three pion mass of $1.9 \text{ GeV}/c$. Relative proportions of the contributing partial waves are presented, from threshold, and the A_3 region is discussed in detail. Contrary to results with the $(3\pi)^-$ system, a change in phase is noted for the 2^- amplitude decaying to $f^0\pi^+$ via an S-wave.

The method employs the University of Illinois three body partial wave analysis program.

NOTICE

This report was prepared as an account of work sponsored by the United States Government. Neither the United States nor the United States Atomic Energy Commission, nor any of their employees, nor any of their contractors, subcontractors, or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness or usefulness of any information, apparatus, product or process disclosed, or represents that its use would not infringe privately owned rights.

MASTER

*Work supported in part by the U.S. Atomic Energy Commission.

†Now at Rutherford High Energy Laboratory, England.

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency Thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

DISCLAIMER

Portions of this document may be illegible in electronic image products. Images are produced from the best available original document.

1. INTRODUCTION

Increased statistics have enabled a low mass partial wave analysis [1] of the $\pi^+\pi^+\pi^-$ system produced opposite a proton and a coherent deuteron to be extended to a mass of 1.9 GeV/c² thus including the region of the A_3 resonance. Apart from some small evidence for 3π decay [2], past experiments [3,4,5] have favored the 100% mode $A_3 \rightarrow f^0\pi$ and it has been claimed that the A_3 is not a simple resonance but is a diffractive $f\pi$ effect similar to the A_1 in the $p\pi$ system. In a large experiment [5,6,7] with the negatively charged (3π) system, it has been found that, whereas there is a rapid relative phase change of the 2^+D partial wave through the A_2 region, the 1^+S ($p\pi$ decay) in the A_1 region and the 2^-S ($f\pi$ decay) in the A_3 region display negligible phase change, although both undergo a Breit-Wigner like change in intensity. Using the same maximum likelihood program [8], developed at the University of Illinois, this experiment has duplicated the A_1 and A_2 results and has demonstrated that the spin-parity make-up in the positively charged system opposite the proton is the same as in (3π)⁻.

The analysis assumes that the total amplitude is factorizable into production and decay terms such that the event distribution probability may be written in the form

$$W = \sum_{M_1 \eta} M_{M_1 \eta}^{J_1 P_1} \rho_{M_1 M_2 \eta}^{J_1 P_1 J_2 P_2} M_{M_2 \eta}^{J_2 P_2 *} \quad (1)$$

where the elements of the production density matrix, $M_{M_1 \eta}^{J_1 P_1}$, are fitted. The decay amplitude for a state of definite spin-parity J^P , z-component M and reflection parity η is expanded as a sum over intermediate two-body states, each having definite orbital angular momentum l between the π^+ and

$(\pi^+\pi^-)$ systems,

$$M_{\pi\pi}^{J^P} = \sum_{\ell} C_{\ell}^{J^P} M_{\pi\pi}^{J^P \ell}$$

The complex numbers $C_{\ell}^{J^P}$ form another set of parameters relating the phase and magnitude of the different angular momentum decay amplitudes. The decay is presumed to proceed by intermediate $\rho\pi$, $\rho\pi$, or $f_0\pi$ states except for an uncorrelated phase space background ("FLAT") having isotropic distribution in all variables. Experimentally, this latter amplitude has been found to have a contribution compatible with zero at all (3π) masses.

The decay amplitude is expressed as a function of the Dalitz-plot coordinates s_1, s_2 , ($s_{1,2} = M^2(\pi_{1,2}^+\pi^-)$); θ and φ the Gottfried-Jackson polar and azimuthal angles of the π^- ; and γ , the azimuthal angle of the $(\pi^+\pi^+)$ plane about the π^- direction in the 3π rest frame. Typical distributions in $M(\pi^+\pi^-)$ and the $\cos \theta, \varphi, \gamma$ are shown in fig. 1 for a 3π mass interval 1.5 to 1.8 GeV/c². The Jackson angle displays a clear dominance of $|\gamma_0^2|^2$ and the f^0 signal is very strong. The plots for the variables of the (3π) state opposite the coherent deuteron in fig. 2 show a very similar behavior.

2. EXPERIMENTAL DATA

The data for this analysis derive from separate exposures of the SLAC 82" bubble chamber with hydrogen (750,000 pictures) and deuterium (250,000 pictures), to an incident π^+ beam at 13 GeV/c. The event selection criteria are discussed elsewhere [1,9] and result in the following totals of four constraint events:

1. $\pi^+ p \rightarrow \pi^+ \pi^+ \pi^- p$ 40,303 events in hydrogen
2. $\pi^+ d \rightarrow \pi^+ \pi^+ \pi^- d$ 2,016 events in deuterium

For reaction (1) the observed $(3\pi)^+$ mass spectrum is shown in fig 3(a). A $\Lambda^{++}(1231)$ cut removes $\sim 8\%$ of events with (3π) masses 1.0 to 1.4 GeV/c² but $\sim 27\%$ of events in the region 1.4 to 1.9 GeV/c². In this analysis the $p\pi^+$ mass has been cut both for the experimental data and in the theoretical expression. Because of the size of this correction and the possible danger of tail effects, a number of different cuts were tried. The sensitivity of the results to these cuts is discussed later, but there was no significant change in any parameters in going from the large cut $M(p\pi^+) < 1.42$ GeV/c² to that finally adopted to maximise statistics, $1.18 \leq M(p\pi^+) \leq 1.38$ GeV/c². There is negligible contamination from Δ^0 and other N^* 's ($< 4\%$).

The $(3\pi)^+$ spectrum for reaction (2) is shown in fig. 3(b) and although the statistics are low, a great resemblance to fig. 3(a) is evident. In both plots the shaded region corresponds to events with at least one $(\pi^+ \pi^-)$ pair satisfying $1.17 \leq M(\pi^+ \pi^-) \leq 1.37$ GeV/c², and shows how the A_3 region is associated with increasing f^0 production. A previous analysis of the $f^0 \pi$ angular distribution [9] in the deuterium data has already suggested that this state has $J^P = 2^-$.

3. QUASI TWO-BODY PARAMETRIZATION

The (3π) decay matrix element contains a propagator for the $(\pi^+ \pi^-)$ system which is parametrized as a relativistic Breit-Wigner, with a simple angular momentum barrier factor inserted for the appropriate decay. For spin = 0,1,2 the masses and widths inserted were those of the ϵ^0 , ρ^0 , f^0 respectively. In separate tests in the A_1 - A_2 and the A_3

regions it was determined that none of the fitted parameters were sensitive to these masses and widths to within large limits. In general, those values were used which maximised the likelihood in a given region. The preferred parameters for the f^0 were $M(f) = 1274 \text{ MeV}/c^2$ $\Gamma(f) = 174 \text{ MeV}/c^2$, and for the ρ^0 $M(\rho) = 775 \text{ MeV}/c^2$ $\Gamma(\rho) = 135 \text{ MeV}/c^2$. The best mass and width for the ϵ increase as the (3π) mass increases and the corresponding (2π) phase space increases. The values used were $M(\epsilon) = 785 \text{ MeV}/c^2$, $\Gamma(\epsilon) = 400 \text{ MeV}/c^2$. In general there is a large negative correlation between the "FLAT" amplitude and the intensity of $O^-\pi$ ($\epsilon\pi$) decay.

4. ANALYSIS OF HYDROGEN DATA

The parameters of the fit are the C_{ℓ}^{JP} and ρ_{ℓ}^{JPJP} , presumed to be functions of the 3π mass (m) and the momentum transfer squared ($t' = t - t_{\min}$) from the beam to the 3π system. Initially, an interval $0.0 < |t'| < 0.5 \text{ (GeV}/c^2\text{)}$ was chosen for the proton data and a dependence $e^{At'}$ was imposed on the density matrix elements. The slope parameter A was determined as a function of mass separately. The maximum likelihood program was then used to fit the data in m intervals of $50 \text{ MeV}/c^2$ making up an energy independent analysis.

For the purpose of this paper, total spin J was limited ≤ 3 and $\ell \leq 2$. Within these restrictions, other states have been omitted after experimentally determining that they were negligible. For example, the fits required $< 1\%$ contributions from $\eta = -1$ states, implying almost complete natural parity exchange in a t channel model. Furthermore, the unnatural parity series appears principally in $M = 0$ states (e.g. the $M = 1$ state of $2^-\pi$ contributes $4.4 \pm 1.0\%$ of the total at the A_3 mass)

and the only sizeable member of the natural parity series, 2^+ , is produced in the $M = 1$ state.

Results of this analysis are displayed in fig. 4 where the number of events in a given bin of m have been multiplied by the diagonal terms of the density matrix to give the relevant number of "events" in that $J^P M (\eta = +1)$ state. This number is then corrected for the Δ^{++} cuts made. The dominance of 1^+S at low masses dies away very quickly, and the sharp rise and fall of $2^+D_{M=1}$ in the A_2 region is evident. In the region of the A_3 peak it is the 2^-S ($f\pi$) partial wave which undergoes the greatest change in intensity; however, there could also be some structure in 2^-P ($\rho\pi$). 1^+S and 1^+P intensities appear to be quite dissimilar as a function of mass, so it is unlikely that the high 2^-P level is a "following" effect due to the decay coherence assumption. In the $(3\pi)^-$ system the coherence has been checked and found to be unity within errors [5]. The contributions of spin 3 waves enter at about $1.3 \text{ GeV}/c^2$ and rise sufficiently slowly to be considered, along with the low partial waves, as being almost constant in the region 1.4 to $1.9 \text{ GeV}/c^2$.

The spin-parity make-up at the high mass region is shown in detail in fig. 5, summing over states of different l values and their interference contributions. The solid line in this figure represents the results from the reaction $\pi^- p \rightarrow \pi^- \pi^- \pi^+ p$ [5] in the same energy region normalized to the same overall total of events. Similar to the low mass analysis, the spin-parity make-up appears to be independent of the charge of the incident meson, although there is a suggestion that the 2^- intensity peaks a little earlier and sharper in this experiment. The states used for the high mass region are listed in Table (1) and require 36 parameters for an average number of 525 events per $50 \text{ MeV}/c^2$ mass bin.

5. DEUTERIUM RESULTS

The statistics for the deuterium data are more than an order of magnitude smaller and so preclude an analysis as a function of mass. However, in the mass interval 1.5 to 1.8 GeV/c² and the $|t'|$ interval 0.02 to 0.20 (GeV/c)² there are 307 events and since the 2⁺ partial wave is evidently absent (with an upper limit of 4% compared to the equivalent population in the proton data of $6.5 \pm 1.5\%$) the number of parameters may be reduced to 25. The lower limit of the $|t'|$ interval is chosen to ensure uniform detection efficiency of the recoiling deuteron as the events are derived solely from 4 prong topologies. Using the same (m, t') bin for the proton data (1759 ev) a comparison of the relative populations of the various spin-parity states is shown in fig. 6; in which is also displayed the relative proportions of the decay states, i.e. $\frac{|C_{\frac{1}{2}1}^{JP}|^2}{|C_{\frac{1}{2}2}^{JP}|^2}$. It may

be seen that, within errors, the two sets of data are entirely compatible, both being dominated by the large 2⁻S amplitude.

6. A₃ PARAMETERS

The "events" of the 2⁻S wave are displayed for clarity again in fig. 7 where the solid line indicates a fit to the intensity of an S-wave relativistic Breit-Wigner with $5 \pm 12\%$ constant background. This fit has a confidence level of 82% and returns the parameters $M(A_3) = 1.60 \pm .01$ GeV/c² and $\Gamma(A_3) = 0.31 \pm .04$ GeV/c², a lighter mass than obtained in the A₃⁻ analysis [5] ($M, \Gamma = 1.66 \pm .01, 0.27 \pm .06$ GeV/c²). The dotted line on the same figure illustrates the variation of the 2⁻P events (error bars

are of similar size). This will fit to a P-wave Breit-Wigner with $13 \pm 22\%$ background but the resulting width moves to $0.43 \pm .08 \text{ GeV}/c^2$. If this latter Breit-Wigner were relevant, it would imply a branching ratio for $A_3 \rightarrow \rho\pi$ of $48 \pm 14\%$, in contradiction to previous analyses [2,3,4,5].

To determine the differential cross section of A_3 production a 300 MeV/c^2 mass interval was chosen about $1.65 \text{ GeV}/c^2$ and within this the mass dependence of the density matrix elements inserted from the previous analysis i.e. a Breit-Wigner for 2^-S and a constant level for the other waves. Small $|t'|$ intervals were then taken and the 2^-S events plotted as in fig. 8. The solid line here is a fit to an exponential with slope parameter $A = 8.0 \pm 0.5 (\text{GeV}/c)^{-2}$, approximately the same value obtained at similar energies in the $(3\pi)^-$ system [5]. The differential cross section slopes of the other principal partial waves in the same mass interval are given in Table (1).

For comparison purposes, if the A_3^+ is defined as the 2^-S wave in the interval 1.5 to 1.8 GeV/c^2 and with $|t'| < 0.5 (\text{GeV}/c)^2$ the cross section is $28 \pm 5 \mu\text{b}$, compatible with results for A_3^- . Corrected for mass and t' limitations, this figure is $55 \pm 9 \mu\text{b}$, within errors of the result for A_3^+ at 11.7 GeV/c [4].

7. DENSITY MATRIX ELEMENTS OF THE A MESONS

The states used by the fitting program to expand the density matrix are eigenvectors of reflection parity [10] but as such are simply related to the usual $|J^P M\rangle$ representation by the relationship

$$|J^P M\rangle \propto [|J^P M\rangle + \eta_s (-1)^M |J^P -M\rangle]$$

with $s = P(-1)^{J+1}$ specifying the natural or unnatural spin parity series.

By fitting with all possible M and η states for a given J^P , the entire sub-matrix becomes observable. If, as in this experiment, the target is unpolarized the density matrix is diagonal with respect to reflection parity and there is a simple relationship between the representations [8].

In this way the results of Tables II, III, and IV were calculated using fits in broad intervals corresponding to the A_1 , A_2 , and A_3 enhancements. Each matrix is normalized, respectively, to the total 1^+ , 2^+ and 2^- contribution, mass dependences being imposed similar to the previous section. The results in the right hand columns give the elements for the interval $0.0 \leq |t'| \leq 0.5 \text{ (GeV/c)}^2$ using both the original definition of axes in the t -channel and also with a rotation of the coordinate system to the s -channel (helicity) frame where the z -axis is defined as the direction of motion of the 3π system in the overall center of mass.

Throughout, the contribution from $\eta = -1$ states is found to be compatible with zero. In the limit of high energy and forward scattering this corresponds to no exchange of unnatural parity states in a t -channel model.

The Tables also give the density matrix elements as a function of t' . In these fits, in the given intervals, the $\eta = -1$ states are suppressed to decrease the number of parameters. Hence the matrix elements are not all independent in the manner indicated in the footnotes.

Additionally, the elements ρ_{2n} are almost always compatible with zero in the t -channel system and have been constrained to be so for the 2^- fits, although not in the case of 2^+ where there are some small exceptions.

The Tables show that the A_1 and A_3 enhancements are formed in $M=0$

states and the A_2 in an $M=1$ state with a probability close to 100%. There is little evidence for any t' dependence except perhaps for a small decrease of $\rho_{00}^{2^-2^-}$ and corresponding increase in $\rho_{11}^{2^-2^-}$.

8. A PHASE SHIFT ANALYSIS

The current listing of the particle data tables [11] states that the A_3 is not a Breit-Wigner resonance. The principal evidence for this statement is an intensive investigation of the phase of the 2^-S wave in the negatively charged system. This was found to remain constant similar to the 1^+S phase behavior through the $A_1^\pm$ region and to be most unlike the 2^+D phase variation in the $A_2^\pm$ region [1,12]. Although there is an undeterminable overall phase, the interference density matrix element between two J^P states and the phase of the appropriate C_ℓ^{JP} gives the relative phase between two amplitudes.

In fig. 9 the phase difference $\delta_{JP} = (\text{phase } 2^-S) - (\text{phase } J^P\ell)$ is plotted for all other included $J^P\ell$ as a function of mass; for a genuine resonance in 2^-S and all other $J^P\ell$ phases constant in the mass region, one would expect an increase of phase of $\pi/2$ in the $\sim 300 \text{ MeV}/c^2$ FWHM through the A_3 mass region, by the same argument that the 1^+ exhibits no resonant structure.

If one examines the interferences with partial waves which are of secondary importance after 2^-S viz. 1^+P , 1^+S , 0^+S and 2^-P , three of the four show this magnitude of change. The approximately constant phase of 2^-S with respect to 2^-P could indicate that this latter is rotating with 2^-S and hence is also resonant. Of the other waves the phase vs. 3^+P shows a very large increase and that vs. 3^+D shows a small increase. The $(2^-S)-(2^+P)$ phase has very large errors but it is clear that there

is some very violent activity in this neighborhood, perhaps associated with an A_4 which has been claimed at $1.83 \text{ GeV}/c^2$ [2,13].

Of all the phase variations displayed, only that with respect to 0^-P shows a decrease. It should be noted from fig. 4 that 0^-P has a very small cross section. It has also been noted that the reflection of $\Delta^{++}(1231)$ comes in increasingly fast in this 3π mass region. As an experimental check in the mass region of $1.7 - 1.8 \text{ GeV}/c^2$, a variety of different width Δ cuts were utilized and the sensitivity of the results to these was tested. Except for the normalization factor necessary to correct for the cuts, most parameters stayed within one standard deviation throughout this procedure. The exceptions to this were the 3^+ population density (decreasing from 25 ± 3 to $14 \pm 4\%$ as the cut was extended outwards from no cut at all to $M(p\pi^+) < 1.42 \text{ GeV}/c^2$ and the 0^- population density (increasing from 4 ± 2 to $15 \pm 7\%$ for the same series of cuts). All 45 phase angles measured (not independent) remained within one standard deviation except for 10 which were measured with respect to 0^-S and 0^-P ; in particular the phase $(2^-S) - (0^-P)$ changed by as much as 114° . The contrary behavior of 0^-P given by this measurement is hence attributed to some residual Δ^{++} tail contamination coupled with a marked insensitivity of the likelihood function to the 0^-P phase (see next section).

To test the sensitivity of the data to phase changes two further fits were performed in $100 \text{ MeV}/c^2$ mass intervals centered at 1.45 and $1.75 \text{ GeV}/c^2$. In this region the average phase increase of the 2^-S amplitude, taken with respect to all other partial waves, is 68° . At both of these points, all parameters were then fixed except that the 2^-S amplitude was forced to rotate in 10° steps. The resulting effect on the likelihood is shown by

the solid parabolas in fig. 10. The vertical bars at 20° intervals show the increase in likelihood possible at a certain angle if the diagonal terms of the density matrix and those off-diagonal terms not associated with 2^- are then allowed to vary. It is clear that an angular change of $\sim 30^\circ$ results in a 7 to 15 drop in the likelihood - well within discrimination level. The dashed parabolas correspond to a similar test when all parameters remain fixed and the 0^-P amplitude is forced to rotate. The likelihood function is clearly very insensitive to the direction the 0^-P amplitude points and this, hence, makes a very poor comparison wave.

The conclusion that the 2^-S partial wave does, in fact, rotate in a positive sense in the Argand diagram is so different from previous analyses and has such far-reaching consequences that it would be imperative to confirm the same behavior in another reaction. Unfortunately the deuterium data sample is so small that this is very difficult. On the other hand there is no problem from d^+ contamination as there is from Δ^{++} so even though the number of events/parameter is very low the data was divided into three mass bins to get the 2^-S phase shift results of fig. 11. This is entirely analogous to fig. 8 and most of the behavior is similar but error bars are so large that a firm conclusion would be dangerous. Nevertheless, there is a general trend of an upward movement in most plots, and it is possible to say at least there is no contradiction of the proton results.

9. SUMMARY

In a high statistics partial wave analysis of the reaction $\pi^+p \rightarrow \pi^+\pi^+\pi^-p$ the general spin-parity composition of the 3π system in the high mass region

has been shown to be the same as in the reactions $\pi^- p \rightarrow \pi^- \pi^+ p$ and $\pi^+ d \rightarrow \pi^+ \pi^+ \pi^- d$.

The A_3 enhancement has been associated with the 2^-5 partial wave and mass, width, cross sections and t' dependence have been calculated under this assumption. There is, however, some evidence that 2^-P also has an enhancement which would imply an $A_3 \rightarrow \rho \pi^+$ decay rate of $48 \pm 14\%$.

Density matrix elements calculated for the 1^+ and 2^- enhancements show that they are produced in $M=0$ states and that the 2^- (A_2) is produced only in an $M=1$ state. Unnatural parity exchange is approximately zero and there is very little t' variation of any matrix elements.

In contradiction to all previous analyses a significant change of phase is found for the 2^-S amplitude which is of the right order of magnitude to associate the A_3 with a simple Breit-Wigner resonance. It has been demonstrated that the data is sensitive enough to measure this phase change and that possible residual contamination from $\Delta^{++}(1231)$ is not causing it. Although the statistics of the deuterium data are much smaller, they do not contradict this result.

Although a resonance would be a possible interpretation, it is not clear that confirmation of the phase change would invalidate a description of the A_3 as a diffractive enhancement [14].

REFERENCES

- [1] G. Thompson et al., to be published in Phys. Rev. (1973).
- [2] W. C. Harrison et al., Phys. Rev. Lett. 28, (1972) 775.
- [3] D. J. Crennell et al., Phys. Rev. Lett. 24, (1970) 781.
- [4] C. Caso et al., Nucl. Phys. B36, (1972) 349.
- [5] G. Ascoli et al., Phys. Rev. 7, (1973) 669.
- [6] G. Ascoli et al., Phys. Rev. Lett. 25, (1970) 962.
- [7] G. Ascoli et al., Phys. Rev. Lett. 26, (1971) 929.
- [8] Described in D. V. Brockway (Thesis 1970) COO-1195-197, AEC Report.
- [9] K. Paler et al., Phys. Rev. Lett. 26, (1971) 1675.
- [10] M. Jacob and G. C. Wilk, Am. Phys. 7, (1959) 404. As used in the Illinois program, there is a reversal of sign for η from the convention in this reference.
- [11] Review of Particle Properties, Rev. Mod. Phys. 45, (1973).
- [12] Y. M. Antipov et al., Experimental Meson Spectroscopy, (1972) 164.
- [13] Y. M. Antipov et al., Nucl. Phys. 1363, (1973) 141 and 153.
- [14] G. Ascoli et al., Illinois preprint ILL-(TH)-73-8, submitted to Phys. Rev. D (1973).

TABLE I

#	J	P	L	M	η	Decay	$A(\text{GeV}/c)^{-2}$
1)				"FLAT"		3π	
2)	0	-	S	0	+	$\pi\pi$	7.7 ± 1.8
3)	0	-	P	0	+	$\rho\pi$	3.7 ± 3.2
4)	1	+	P	0	+	$\pi\pi$	8.8 ± 2.0
5)	1	+	S	0	+	$\rho\pi$	10.1 ± 1.9
6)	2	-	S	0	+	$f\pi$	8.0 ± 0.5
7)	2	-	P	0	+	$\rho\pi$	5.5 ± 0.5
8)	3	+	D	0	+	$\rho\pi$	4.8 ± 0.8
9)	3	+	P	0	+	$f\pi$	8.9 ± 2.9
10)*	2	+	D	1	+	$\rho\pi$	2.8 ± 1.9
11)*	2	+	P	1	+	$f\pi$	1.9 ± 1.1
12)†	2	-	S	1	+	$f\pi$	
13)†	2	-	P	1	+	$\rho\pi$	
14)†	2	-	S	2	+	$f\pi$	
15)†	2	-	P	2	+	$\rho\pi$	
16)†	2	-	S	1	-	$f\pi$	
17)†	2	-	P	1	-	$\rho\pi$	
18)†	2	-	S	2	-	$f\pi$	
19)†	2	-	P	2	-	$\rho\pi$	

*The states 10), 11) were omitted for analysis of the deuterium data.

†The states 12) through 19) were used only for the 2^- density matrix evaluation.

Table II

 $1^+ (A_1)$ density matrix elementsMass Interval: 1.05 to 1.15 GeV/c^2

	$^+ t$ -channel axes, $ t' $ intervals $(\text{GeV}/c)^2$					$0.0 < t' < 0.5 (\text{GeV}/c)^2$	
$\rho_{ij}^{1^+1^+}$	0.0 - 0.05	0.05 - 0.1	0.1 - 0.2	0.2 - 0.3	0.3 - 0.7	t-channel	u-channel
ρ_{00}	.961 $\pm$.065	.955 $\pm$.092	.940 $\pm$.100	.936 $\pm$.123	.772 $\pm$.200	.952 $\pm$.039	.813 $\pm$.034
ρ_{11}	.019 $\pm$.017	.023 $\pm$.021	.030 $\pm$.026	.032 $\pm$.036	.114 $\pm$.101	.024 $\pm$.013	.093 $\pm$.012
ρ_{1-1}	-.019 $\pm$.017	-.023 $\pm$.021	-.030 $\pm$.026	-.032 $\pm$.036	-.114 $\pm$.101	-.024 $\pm$.013	.086 $\pm$.012
$\text{Re} \rho_{10}$	-.080 $\pm$.022	-.088 $\pm$.024	-.141 $\pm$.027	-.132 $\pm$.041	-.061 $\pm$.079	-.101 $\pm$.012	.211 $\pm$.012
$-\text{Im} \rho_{10}$	-.029 $\pm$.105	-.009 $\pm$.136	-.061 $\pm$.216	-.014 $\pm$.207	-.279 $\pm$.592	.053 $\pm$.076	.106 $\pm$.076
# Events	847	581	538	197	136	2271	2271

$^+$ In these fits $^--1$ states are constrained to be zero. i.e. $\rho_{11} = -\rho_{1-1}$.

Table III
 $Z^*(A_2)$ density matrix Elements

Mass Interval: 1.25 to 1.35 GeV/c²

	[†] t-channel axes, $ t' $ intervals (GeV/c) ²					0.0 < $ t' $ < 0.5 (GeV/c) ²	
$\rho_{ij}^{Z^*Z^*}$	0.0 - 0.05	0.05 - 0.1	0.1 - 0.2	0.2 - 0.3	0.3 - 0.7	t-channel	s-channel
ρ_{00}						.000±.051	.042±.042
ρ_{11}	.488±.199	.441±.186	.492±.067	.473±.066	.483±.050	.486±.046	.268±.036
ρ_{22}	.012±.247	.059±.167	.007±.078	.027±.068	.016±.048	.014±.046	.211±.037
ρ_{1-1}	.488±.199	.441±.186	.492±.067	.473±.068	.483±.050	.486±.046	.249±.036
ρ_{2-2}	-.012±.247	-.059±.167	-.007±.078	-.027±.068	-.016±.048	-.014±.046	-.211±.037
Re ρ_{10}						.000±.013	-.019±.019
Im ρ_{10}						.000±.050	-.005±.077
Re ρ_{20}						.000±.013	.003±.021
Im ρ_{20}						.000±.043	.003±.067
Re ρ_{21}	-.060±.127	.045±.127	.042±.055	.084±.042	.067±.042	.069±.033	.219±.026
Im ρ_{21}	.024±.349	.100±.359	-.018±.140	.053±.141	.032±.092	.012±.118	.030±.095
Re ρ_{2-1}	-.060±.127	.045±.127	.042±.055	.084±.042	.067±.042	.069±.033	.223±.026
Im ρ_{2-1}	.024±.349	.100±.359	-.018±.140	.053±.141	.032±.092	.012±.118	.034±.095
# Events	589	473	563	317	340	2201	2201

[†] In these fits $\eta = -1$ states are constrained to be zero, i.e. $\rho_{11} = \rho_{1-1}$; $\rho_{22} = -\rho_{2-2}$; $\rho_{21} = \rho_{2-1}$.

Table IV
 $2^- (A_3)$ Density Matrix Elements
Mass Interval: 1.5 to 1.8 GeV/c²

	[†] t-channel axes, $ t' $ intervals (GeV/c) ²					$0.0 < t' < 0.5$ (GeV/c) ²	
$\rho_{11}^{2^- 2^-}$	0.0 - 0.05	0.05 - 0.1	0.1 - 0.2	0.2 - 0.3	0.3 - 0.7	t-channel	s-channel
ρ_{00}	.952±.094	.892±.094	.932±.093	.827±.127	.767±.127	.909±.049	.325±.115
ρ_{11}	.024±.023	.054±.024	.034±.025	.087±.041	.116±.048	.043±.014	.283±.060
ρ_{22}						.003±.016	.055±.070
ρ_{1-1}	-.024±.023	-.054±.024	-.034±.025	-.087±.041	-.116±.048	-.043±.014	-.283±.060
ρ_{2-2}						.003±.016	.055±.070
Re ρ_{10}	.083±.034	.134±.029	.168±.038	.240±.053	.168±.053	.157±.016	.251±.040
Im ρ_{10}	-.016±.114	-.006±.087	-.023±.098	-.049±.128	.003±.136	-.029±.040	-.004±.040
Re ρ_{20}						.005±.014	.084±.017
Im ρ_{20}						.022±.043	.001±.075
Re ρ_{21}						-.004±.009	.118±.035
Im ρ_{21}						.006±.038	.003±.065
Re ρ_{2-1}						.004±.009	-.118±.035
Im ρ_{2-1}						-.006±.038	-.003±.065
# Events	1000	716	825	414	508	3328	3328

[†] In these fits $\eta = -1$ and $M = 2$ states are constrained to be zero. i.e. $\rho_{11} = -\rho_{1-1}$; and $\rho_{20} = 0$.

FIGURE CAPTIONS

1. Comparison of the fit to the data for events of the reaction $\pi^+p \rightarrow \pi^+\pi^+\pi^-p$ in intervals $1.5 \leq M_{\pi^+\pi^+\pi^-} \leq 1.8 \text{ GeV}/c^2$, $0.0 \leq |t'| \leq 0.5 \text{ (GeV}/c)^2$. (a) $\pi^+\pi^-$ mass spectrum, two entries per event (b) $\cos \theta$ (c) ϕ (d) γ (see text for definition of angles). The solid line is the result of a Monte Carlo event generation using the fitted parameters. The $\Delta^{++}(1231)$ is cut according to the selection $1.18 \leq M(p\pi^+) \leq 1.38 \text{ GeV}/c^2$.
2. Comparison of the fit to the data for events of the reaction $\pi^+d \rightarrow \pi^+\pi^+\pi^-d$ in the intervals $1.5 \leq M_{\pi^+\pi^+\pi^-} \leq 1.8 \text{ GeV}/c^2$, $0.02 \leq |t'| \leq 0.20 \text{ (GeV}/c)^2$. (a), (b), (c), and (d), and the solid line are as for fig. 1.
3. Mass spectrum of $\pi^+\pi^+\pi^-$ for events of (a) $\pi^+p \rightarrow \pi^+\pi^+\pi^-p$ and (b) $\pi^+d \rightarrow \pi^+\pi^+\pi^-d$. In (a) events with $1.12 \leq M(p\pi_g^+) \leq 1.32 \text{ GeV}/c^2$ are excluded. The shaded histograms have events in which at least one $\pi^+\pi^-$ pair satisfies $1.14 \leq M_{\pi^+\pi^-} \leq 1.36 \text{ GeV}/c^2$.
4. Events of reaction (1) in each $|J^P/M\rangle$ state as a function of $M(3\pi)^+$ in bins of $50 \text{ MeV}/c^2$ with $0.0 \leq |t'| \leq 0.5 \text{ (GeV}/c)^2$.
5. Spin-parity make-up of reactions (1) in the high mass region and $0.0 \leq |t'| \leq 0.5 \text{ (GeV}/c)^2$. The solid line shows the results from $\pi^+p \rightarrow \pi^+\pi^-\pi^+p$ [5] normalized to the same total of events.
6. Comparison of the spin-parity and decay state make-up of the $(3\pi)^+$ state produced opposite a proton and a coherent deuteron in the interval $1.5 \leq M_{\pi^+\pi^+\pi^-} \leq 1.8 \text{ GeV}/c^2$, $0.02 \leq |t'| \leq 0.20 \text{ (GeV}/c)^2$.

7. Events of the $2^{-}S$ (f_{π}) state as a function of $M_{\pi^{+}\pi^{+}\pi^{-}}$. The solid line is a fit to a relativistic S-wave Breit-Wigner with the parameters given in the text. The dashed line shows the variation of "events" in the $2^{-}P$ (p_{π}) state.
8. The differential cross section dN/dt' for "events" in the $2^{-}S$ state in the interval $1.15 \leq M_{\pi^{+}\pi^{+}\pi^{-}} \leq 1.8 \text{ GeV}/c^2$ fitted to $Be^{-At'}$.
9. The phase difference $\psi_{J^P L} = (\text{phase } 2^{-}S) - (\text{phase } J^P L)$ for all other contributing states as a function of $M_{\pi^{+}\pi^{+}\pi^{-}}$, for events of reaction (1) in $0.0 \leq |t'| \leq 0.5 \text{ (GeV}/c)^2$.
10. Variation of the likelihood () with phase angle of $2^{-}S$ amplitude (solid parabola) and $0^{-}P$ amplitude (dashed line) for two fits in the mass intervals $1.4 \leq M_{\pi^{+}\pi^{+}\pi^{-}} \leq 1.5$ and $1.7 \leq M_{\pi^{+}\pi^{+}\pi^{-}} \leq 1.8 \text{ GeV}/c^2$ with $0.0 \leq |t'| \leq 0.5 \text{ (GeV}/c)^2$.
11. The phase difference $\psi_{J^P L} = (\text{phase } 2^{-}S) - (\text{phase } J^P L)$ for all other contributing states as a function of $M_{\pi^{+}\pi^{+}\pi^{-}}$, for events of reaction (2) in $0.02 \leq |t'| \leq 0.20 \text{ (GeV}/c)^2$.

NUMBER OF EVENTS / (50 MeV/c²)

$M_{\pi^+\pi^-\pi^-} \text{ GeV}/c^2$