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TRANSPORT THEORY AND SPECTRAL PROBLEMS

by

G. Milton Wing

University of New Mexico

Consultant to Sandia Corporation

May 1959

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ABSTRACT

This paper considers a simple model of time-independent neutron transport on a line as a stochastic process, using the method of invariant imbedding. Nonlinear equations for the expected values (flux) are also obtained and solved, the results are compared with the ordinary linear theory, and possible advantages of the new formulation are cited. Generalizations to a large class of transport problems are discussed. The nonlinear time-dependent operator for transport in one dimension is considered in detail. It has a pure point spectrum, and expansion theorems can be proved. These results contrast with those for isotropic one-velocity neutron transport in the infinite slab. Here there are only a finite number of points in the point spectrum, with a half-plane in the continuous spectrum. Approximations to the eigenvalues and eigenfunctions for the slab case, as well as extensions to the multivelocity problem, are mentioned. There is a brief discussion of recent spectral and expansion theorems for very general geometries.

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PREFACE

The material contained in this paper was presented on April 23, 1959, as an invited address at the Symposium on Nuclear Reactor Theory, sponsored by the American Mathematical Society, with the financial aid of the Office of Ordnance Research. This paper will eventually appear, in a slightly modified form, in the Proceedings of the Symposium.

TRANSPORT THEORY AND SPECTRAL PROBLEMS

Introduction

This paper concerns the problem of neutron transport for certain simplified models. There is no recourse to diffusion theory or other approximate methods. The first two sections consider nonlinear formulations of some time-independent problems. The third section analyzes a nonlinear time-dependent case in detail. For other time-dependent considerations the more classical linear transport theory is used. Section IV discusses the spectrum of the operator for one-velocity transport in the infinite slab. The information obtained leads to a complete solution of this problem. The last section briefly mentions the situation for more complicated physical problems.

I. A One-Dimensional Time-Independent Transport Problem

Consider the following very simple neutron transport model. A homogeneous one-dimensional rod consists of fixed nuclei with which neutrons moving to right or left may collide. Such a collision always gives rise to two neutrons, one moving to the right, the other to the left. When a neutron passes through a length Δ of the rod, such a fission collision occurs with probability $\lambda^{-1}\Delta + o(\Delta)$. No other collisions are possible.

Suppose the rod is of length x , situated on the interval $0 \leq z \leq x$. Let one neutron--the trigger particle--enter the rod at $z = x$. What is the probability $p_n(x)$ that in all time exactly n neutrons emerge from $z = x$?

This problem is easily approached by using the theory of invariant imbedding (see References 1-5). Consider, instead of the given rod, one of length $x + \Delta$, and pose the same problem for this new rod (see Figure 1).

Figure 1.

The neutron behavior in the new rod may be described as follows:

1. The trigger neutron passes through $(x, x + \Delta)$ without collision and gives rise to a total of n neutrons emerging from $(0, x)$, all of which pass through $(x, x + \Delta)$ without collision.

2. The trigger neutron passes through $(x, x + \Delta)$ without collision, gives rise to a total of k , $k < n$, neutrons emerging from $(0, x)$, and one of these collides with a nucleus in $(x, x + \Delta)$. The remaining $(k - 1)$ neutrons emerge at $z = x + \Delta$, together with the right-moving one produced in the fission. The left-moving fission neutron acts as a trigger neutron for $(0, x)$ and produces $(n - k)$ particles from $z = x$, all of which pass through $(x, x + \Delta)$ without incident.

3. The trigger neutron collides with a nucleus in $(x, x + \Delta)$ upon entrance. The resultant right-moving particle emerges at $x + \Delta$, while the left-moving one acts as a trigger for $(0, x)$ and produces $(n - 1)$ neutrons from $z = x$, none of which has a collision in $(x, x + \Delta)$.

4. There is more than one collision in $(x, x + \Delta)$.

The great advantage of the method under discussion is that all complicated collision processes can be lumped into Item 4 above, the probability of such events being $o(\Delta)$. A careful accounting of cases now yields the formula, for $n \geq 1$,

$$\begin{aligned}
 p_n(x + \Delta) = & (1 - \lambda^{-1}\Delta)p_n(x)(1 - \lambda^{-1}\Delta)^n \\
 & + (1 - \lambda^{-1}\Delta) \sum_{k=1}^n p_k(x)(k\lambda^{-1}\Delta)p_{n-k}(x)(1 - \lambda^{-1}\Delta)^{n-k} \\
 & + \lambda^{-1}\Delta p_{n-1}(x)(1 - \lambda^{-1}\Delta)^{n-1} + o(\Delta), \quad (1.1)
 \end{aligned}$$

which, on letting $\Delta \rightarrow 0$, becomes

$$\lambda p'_n(x) = -(n + 1)p_n(x) + \sum_{k=1}^n k p_k(x)p_{n-k}(x) + p_{n-1}(x). \quad (1.2a)$$

The special case $n = 0$ is readily settled to result in

$$\lambda p'_0(x) = -p_0(x), \quad (1.2b)$$

as are the initial conditions,

$$p_0(0) = 1, \text{ and}$$

$$p_n(0) = 0, \quad n \geq 1. \quad (1.2c)$$

Equations 1.2 have been solved numerically and some of the results are shown in Figure 2.

Figure 2.

A complete analytical study of the system expressed by Equations 1.2 seems very difficult and has not been made. However, it is of interest to consider the expected number $E(x)$ of neutrons emergent in all time from the rod at $z = x$. By definition,

$$E(x) = \sum_{n=0}^{\infty} n p_n(x) . \quad (1.3)$$

An equation satisfied by $E(x)$ is readily derived by multiplying Equation 1.2a by n , adding, and manipulating the resulting expressions. The result is

$$E'(x) = \lambda^{-1} [1 + E^2(x)] , \text{ and} \\ E(0) = 0 , \quad (1.4)$$

to produce

$$E(x) = \tan (\lambda^{-1} x) . \quad (1.5)$$

Note that $E(x)$ becomes infinite at $x = \lambda\pi/2$. Hence, $\lambda\pi/2$ is the critical length of the rod. It is interesting to consider higher moments of the distribution, and in this study the use of the moment-generating function proves convenient (see Reference 5).

Equations 1.2 and 1.4 are evidently nonlinear. The common formulation of the transport problem results in linear expressions. It is appropriate to investigate the usual transport equation for the model under discussion.

Let $u_+(z)$ be the density of neutrons moving to the right in the rod, $u_-(z)$ the density of those moving to the left. Then (see Reference 5)

$$\lambda \frac{du_+}{dz} + u_+ = u_+ + u_- , \\ -\lambda \frac{du_-}{dz} + u_- = u_+ + u_- , \text{ and} \\ u_+(0) = 0, \quad u_-(x) = 1 . \quad (1.6)$$

Equations 1.6 involve the solution of a two-point boundary value problem, while Equation 1.4 is merely a one-point problem. The advantages of the latter, particularly in computation, are obvious. In the simple case under discussion, the two-point boundary value problem is readily solved to give

$$u_+(z) = \frac{\sin(\lambda^{-1}z)}{\cos(\lambda^{-1}x)}, \text{ and}$$

$$u_-(z) = \frac{\cos(\lambda^{-1}z)}{\cos(\lambda^{-1}x)}. \quad (1.7)$$

Thus $E(x) = u_+(x) = \tan(\lambda^{-1}x)$, as before.

II. The Flux Equations in More General Time-Independent Cases

The method of invariant imbedding may be used to obtain the flux equation (1.4) directly. Noting again the possibilities that occur when a trigger neutron enters the rod (page 6) readily results in

$$\begin{aligned} E(x + \Delta) &= (1 - \lambda^{-1}\Delta)E(x)(1 - \lambda^{-1}\Delta) \\ &\quad + (1 - \lambda^{-1}\Delta)E(x)(\lambda^{-1}\Delta)[E(x) + 1] \\ &\quad + (\lambda^{-1}\Delta)[E(x) + 1] + o(\Delta), \end{aligned} \quad (2.1)$$

leading to Equation 1.4.

The question of applying the methods described to a wider class of problems at once arises. Can equations analogous to Equation 1.4 be found for more complicated geometries and when neutron velocities are allowed to change? Up to the present, this investigation has been carried to the point where nonlinear flux equations can be obtained provided the geometries involved are sufficiently symmetric (see Reference 6). Neutrons considered may be in different states, these describing energy, direction, etc., with transition from one state to another occurring with assigned probability upon collision. Stochastic equations, from which expressions for the flux may be obtained by taking expected values, have also been derived. In general, these are extremely complicated.

It is possible, too, to find functional equations relating the nonlinear formulations to the more familiar linear ones. The interested reader is referred to Reference 7.

III. A One-Dimensional Time-Dependent Problem

Consider here the transport model of Section I, but now include time dependence.

Write

$U(x, t)$ = the expected number of neutrons emerging from $z = x$ between time zero and time t due to a trigger neutron entering at x at time zero;

$u(x, t)dt$ = the expected number of neutrons emerging between t and $t + dt$ and arising in the same manner,

v = constant neutron speed.

Observe that $u(x, t) = \partial U / \partial t$.

To obtain an equation for $U(x, t)$, begin with the fact that the probability is $(1 - \lambda^{-1}\Delta)$ that a neutron traversing $(x, x + \Delta)$ continues without incident, and the probability is $\lambda^{-1}\Delta$ that the neutron causes a fission. (All statements are correct to order $o(\Delta)$.)

The first type of neutron produces a flux from $(0, x)$ of $u(x, s)ds$ in the time interval $(s, s + ds)$. Of this flux an amount $(1 - \lambda^{-1}\Delta)u(x, s)ds$ continues through $(x, x + \Delta)$ without collision, while the remaining portion $(\lambda^{-1}\Delta)u(x, s)ds$ produces fission neutrons going in both directions. Those going to the left result in a flux $U(x, t - s)(\lambda^{-1}\Delta)u(x, s)ds$ in the time interval $(t - s)$ that remains to time t .

The second type of neutron simply gives rise to a total flux of $U(x, t)$ plus the right-moving fission neutron produced in $(x, x + \Delta)$. Thus

$$U\left(x + \Delta, t + \frac{2\Delta}{v}\right) = (1 - \lambda^{-1}\Delta) \left\{ \int_0^t u(x, s)ds + \int_0^t u(x, s)(\lambda^{-1}\Delta)U(x, t - s)ds \right\} \\ + \lambda^{-1}\Delta \left\{ U(x, t) + 1 \right\} + o(\Delta),$$

giving

$$\frac{\partial U}{\partial x} + \frac{2}{v} \frac{\partial U}{\partial t} = \lambda^{-1} \int_0^t u(x, s)U(x, t - s)ds + \lambda^{-1}. \quad (3.2a)$$

Boundary and initial conditions are

$$U(x, 0) = 0, \quad U(0, t) = 0. \quad (3.2b)$$

It is evident that scale changes can be made so that $\lambda = v = 1$. Differentiation of Equation 3.2a with respect to time then yields

$$\frac{\partial u}{\partial x} + 2 \frac{\partial u}{\partial t} = \int_0^t u(x, s) u(x, t-s) ds, \quad (3.3a)$$

with

$$u(x, 0) = 1/2, \quad u(0, t) = 0. \quad (3.3b)$$

Equations 3.3 have been examined in rather complete detail in Reference 8. There is a unique solution $u(x, t)$ in any finite rectangle in the (x, t) plane with the bound

$$|u(x, t)| \leq t^{-1} I_1(t), \quad t \geq 0, \quad x \geq 0,$$

where $I_1(t)$ is the Hankel function of first kind. The flux $U(x, t)$ can be considered by Laplace transform methods, and the result is

$$U(x, t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{pt} \sin(x\sqrt{1-p^2}) dp}{p \left\{ \sqrt{1-p^2} \cos(x\sqrt{1-p^2}) + p \sin(x\sqrt{1-p^2}) \right\}}, \quad (3.4)$$

$$c > 1.$$

The behavior of the denominator in the integrand of Equation 3.4 can be examined in explicit detail, and the following behavior of $U(x, t)$ is found:

(a) for $x > \pi/2$,

$$U(x, t) = \frac{e^{p_0 t}}{p_0} \frac{1 - p_0^2}{1 + x p_0} + R(x, t); \quad (3.5a)$$

(b) for $x = \pi/2$,

$$U(x, t) = t + R(x, t); \quad (3.5b)$$

(c) for $0 < x < \pi/2$,

$$U(x, t) = \tan x + R(x, t). \quad (3.5c)$$

Here $R(x, t)$ in each case is small relative to the given term for large values of t . Further, p_0 is the largest real root of the denominator of the integrand of Equation 3.4, all of whose roots with positive real parts can be shown to be real. It is to be observed that Equation 3.5c agrees with the results of Section I.

The analysis of Equation 3.4 may evidently be carried out more completely, leading to an infinite series expansion for $U(x, t)$. Such matters are considered more carefully in the remaining sections.

IV. Time-Dependent Neutron Transport in the Infinite Slab

The remainder of this paper returns to the more classical linear formulation of the transport problem. This section considers the case of one-velocity isotropic transport in an infinite slab of material extending from $x = -a$ to $x = +a$. Surrounding the slab is a vacuum, so that a neutron which escapes from a face $x = \pm a$ never re-enters. The equation describing this situation is (see Reference 9):

$$\frac{1}{v} \frac{\partial N}{\partial t} + \mu \frac{\partial N}{\partial x} + \sigma N = \frac{c}{2} \int_{-1}^1 N(x, \mu', t) d\mu', \quad (4.1a)$$

where v is the constant neutron speed, σ the constant cross section for collision, and c/σ the average number of neutrons emerging from a collision. As usual, μ is the cosine of the angle between the direction of motion of the particle and the positive x -direction, and $N(x, \mu, t)$ is the density of neutrons at x , going in direction μ , at time t .

Consider an initial distribution $f(x, \mu)$ of neutrons in the slab and ask for the time history of the distribution. Thus Equation 4.1a is subject to the conditions:

$$N(-a, \mu, t) = 0, \quad \mu > 0, \quad t > 0; \quad (4.1b)$$

$$N(a, \mu, t) = 0, \quad \mu < 0, \quad t > 0; \text{ and} \quad (4.1c)$$

$$N(x, \mu, 0) = f(x, \mu), \quad |x| \leq a, \quad |\mu| \leq 1. \quad (4.1d)$$

Equation 4.1a may be written in the form (choosing $v = 1$, for convenience)

$$\frac{\partial \phi}{\partial t} = A\phi, \quad (4.2)$$

where

$$\phi(x, \mu, t) = e^{\sigma t} N(x, \mu, t), \quad (4.3)$$

and A is the operator

$$A \cdot = -\mu \frac{\partial}{\partial x} \cdot + \frac{c}{2} \int_{-1}^1 \cdot d\mu'. \quad (4.4)$$

Again,

$$\phi(-a, \mu, t) = 0, \quad \mu > 0, \quad t > 0; \quad (4.5a)$$

$$\phi(a, \mu, t) = 0, \quad \mu < 0, \quad t > 0; \quad (4.5b)$$

$$\phi(x, \mu, 0) = f(x, \mu), \quad |x| \leq a, \quad |\mu| \leq 1. \quad (4.5c)$$

The form of Equation 4.2 is typical of the linear formulation of transport problems, the operator A becoming more complicated as geometries become complex, anisotropies are introduced, etc. This form, together with the physical background of the problem, has long suggested an expansion of the type

$$\phi(x, \mu, t) = \sum_{n=1}^{\infty} \psi_n(x, \mu) e^{\lambda_n t}, \quad (4.6)$$

where λ_n and ψ_n are the eigenvalues and eigenfunctions of the operator A :

$$\lambda_n \psi_n = A\psi_n. \quad (4.7)$$

Indeed, the series arising from Equation 3.4 is clearly of the type of Equation 4.6.

Several years ago J. Lehner and the author tried to establish rigorously the validity of Equation 4.6 for the problem expressed by Equations 4.2-4.5. Surprisingly enough it was found that such an expansion is not possible for this case. The findings of that investigation will now be discussed briefly.

Let $\mathcal{H}$ be the Hilbert space of all functions $g(x, \mu)$ defined and square integrable on $|x| \leq a, |\mu| \leq 1$:

$$\int_{-a}^a dx \int_{-1}^1 |g(x, \mu)|^2 d\mu < \infty ,$$

and let the domain $\mathcal{D}(A)$ of the operator A be the linear manifold of those functions $g \in \mathcal{H}$ which satisfy

$$g(a, \mu) = 0, \mu < 0 ,$$

$$g(-a, \mu) = 0, \mu > 0 .$$

We are led to discuss the eigenvalues, λ , and eigenfunctions, ψ , of the system,

$$\begin{aligned} A\psi &= -\mu \frac{\partial \psi}{\partial x} + \frac{c}{2} \int_{-1}^1 \psi(x, \mu') d\mu' \\ &= \lambda \psi, \end{aligned}$$

where $\psi \in \mathcal{D}(A)$.

If the inner product

$$(g, h) = \int_{-a}^a dx \int_{-1}^1 g(x, \mu) \overline{h(x, \mu)} d\mu$$

is introduced, it is found that the operator A is not self-adjoint. In fact, the adjoint operator, A^* , is given by

$$A^* \cdot = +\mu \frac{\partial}{\partial x} \cdot + \frac{c}{2} \int_{-1}^1 \cdot d\mu' , \quad (4.9)$$

with domain $\mathcal{D}(A^*)$ consisting of those functions g in $\mathcal{H}$ satisfying

$$g(a, \mu) = 0, \quad \mu > 0 ,$$

$$g(-a, \mu) = 0, \quad \mu < 0 .$$

Because of the presence of the term $\mu(\partial/\partial x)$, the operator A is not bounded.

These two unpleasant properties of A make its study rather difficult. Complex variable methods reveal that there are no eigenvalues of Equation 4.8 with $\text{Re} \lambda < 0$. Fortunately, when $\text{Re} \lambda \geq 0$, Equation 4.8 may be transformed into the integral equation (see Reference 9),

$$\frac{2}{c} \Phi(x) = \int_{-a}^a E(\lambda|x-y|) \Phi(y) dy , \quad (4.10)$$

where

$$\Phi(x) = \int_{-1}^1 \psi(x, \mu') d\mu' ,$$

and

$$E(u) = \int_1^\infty e^{-ut} \frac{dt}{t} .$$

The eigenvalue λ occurs in Equation 4.10 in a peculiar nonlinear fashion. The study of the equation is facilitated by considering $2/c$ as an eigenvalue dependent upon the kernel E with λ as a parameter. Taking this approach, one finds that there is always one but never more than a finite number of real values λ satisfying Equation 4.10. It is not difficult to show that λ cannot be complex.

We are hence confronted with the fact that an expansion of the type in Equation 4.6 is impossible for the slab problem, simply because there are only a finite number of eigenvalues. What kind of expansion theorems can then be expected?

An answer to this question requires a complete analysis of the spectrum A . The results are as follows:

The point spectrum of A consists of a nonempty finite set of real numbers.

The continuous spectrum of A consists of the entire half plane $\text{Re } \lambda \leq 0$.

The residual spectrum of A is empty.

The remainder of the spectral plane is resolvent set.

To gain further insight into the problem, turn to the theory of semigroups. The formal solution of Equation 4.2 is

$$\begin{aligned}\phi(x, \mu, t) &= e^{At} f(x, \mu) \\ &= T(t)f(x, \mu) .\end{aligned}\tag{4.11}$$

A more complete study of A shows that the Hille-Yosida theorem applies and does indeed insure the existence of a semigroup of bounded operators $T(t)$ providing the unique solution to the problem posed. The solution may be represented in the form of the Laplace integral,

$$\begin{aligned}\phi(x, \mu, t) &= \lim_{\omega \rightarrow \infty} \frac{1}{2\pi i} \int_{\gamma - i\omega}^{\gamma + i\omega} e^{\lambda t} (R_{\lambda} f) d\lambda , \\ t > 0, \quad \gamma > \text{largest eigenvalue of } A.\end{aligned}\tag{4.12}$$

Here, $R_{\lambda} = (\lambda - A)^{-1}$ is the resolvent operator, which may be obtained in explicit form.

It is possible to shift the path of integration to the left of all points in the point spectrum of A , but not to the left of the origin. Carrying out this analysis[†] results in

$$\phi(x, \mu, t) = \sum_{j=1}^m (f, \psi_j^*) \psi_j(x, \mu) e^{\lambda_j t} + \zeta(x, \mu, t) ,\tag{4.13}$$

[†]For a discussion of a possible exceptional case, overlooked in Reference 10, see Reference 11.

where

$$\zeta(x, \mu, t) = \lim_{\omega \rightarrow \infty} \frac{1}{2\pi i} \int_{\gamma_1 - i\omega}^{\gamma_1 + i\omega} e^{\lambda t} (R_\lambda f) d\lambda,$$

$$0 < \gamma_1 < \text{smallest } \lambda_j.$$

In Equation 4.13 the λ_j 's are the eigenvalues of A , and the ψ_j 's are the corresponding eigenfunctions. It can be shown that A^* has the same eigenvalues; its eigenfunctions are ψ_j^* .

Thus, for the slab problem, the result is the expansion indicated by Equation 4.13, not the type (Equation 4.6) anticipated. The properties of $\zeta(x, \mu, t)$ are of interest, of course. By requiring that the initial distribution $f(x, \mu)$ satisfies the extra conditions

$$f \in \mathcal{D}(A), \quad \left\| \frac{\partial f}{\partial x} \right\| < \infty,$$

$$Af \in \mathcal{D}(A), \quad \left\| \frac{\partial(Af)}{\partial x} \right\| < \infty,$$

we find

$$\lim_{t \rightarrow \infty} \zeta(x, \mu, t) = 0$$

for almost all fixed values of (x, μ) .

Other estimates of $\zeta(x, \mu, t)$ are obtainable. Thus,

$$\|\zeta(x, \mu, t)\| \leq kt^3$$

for some number k and sufficiently large values of t . Moreover, if

$$f(x, \mu) = \sum_{j=1}^m (f, \psi_j^*) \psi_j(x, \mu) + g(x, \mu),$$

and we consider $\zeta = \zeta[g]$, then, for each value of $\epsilon > 0$ we have $c(\epsilon) > 0$, such that

$$\|\zeta[g]\| > c(\epsilon)e^{-\epsilon t}.$$

The treatment of the slab problem as just described gives no hint of the number of eigenvalues λ or of their distribution as functions of c , σ , and a , nor does it suggest the shapes of the eigenfunctions themselves. Methods have been devised to approximate these quantities (Reference 12). Rough bounds on the eigenvalues are not hard to obtain, but good estimates require modern computing machines.

The investigation of the transport equation for the slab geometry has been carried out for the n -velocity group model (see Reference 11). Results are not as complete as those described above. One writes the equation in vector-matrix form as

$$\bar{v}^{-1} \frac{\partial}{\partial t} N = -\mu \frac{\partial}{\partial x} N - \bar{\sigma} N + \frac{1}{2} \bar{C} \int_{-1}^1 N d\mu', \quad (4.14)$$

where $\bar{v}$ is an n -by- n diagonal velocity matrix, $\bar{\sigma}$ is a similar cross-section matrix, N is a column vector of functions describing the neutron distribution in each velocity group, and $\bar{C}$ is an n -by- n matrix of nonnegative transfer coefficients. Boundary and initial conditions are the usual ones (Equations 4.1b-d) where now f is also a column vector.

Results concerning Equation 4.14 depend very strongly upon the properties of the matrix $\bar{C}$. For a set of theorems nearly so complete as the theorems obtained in the one-velocity case, it has been necessary to assume that there exists a diagonal matrix $\bar{D}$ such that $\bar{D} \bar{C} \bar{D}^{-1}$ is a positive symmetric operator. This is actually the case for many problems of physical interest.

V. Some Results for Bounded Geometries

Equation 4.1a has a singular behavior at $\mu = 0$. This direction, parallel to the slab faces, is a very special one. A neutron proceeding in this direction may move for an arbitrarily long time within the system without a collision. In bounded regions such an event is impossible, provided the neutron velocity cannot be arbitrarily small. Here there is a time interval τ such that in any longer interval every neutron either leaves the system or suffers a collision. It has been thought for some time that the unexpected results discussed in Section IV were connected with this

phenomenon. Recent theorems tend to confirm these suspicions (see Reference 13). These theorems shall be described briefly here.

Let $\bar{D}$ and $\bar{V}$ be bounded measurable sets in position space, (x_1, x_2, x_3) , and velocity space, (v_1, v_2, v_3) , respectively. Let $\bar{V}$ be bounded away from zero as well. Consider the space $\mathcal{N}$ of square integrable functions over $\bar{D} \times \bar{V}$. The transport operator A for this case is given by

$$AN = -v \cdot \text{grad}_x N - \sigma(x, v)N + KN, \quad (5.1)$$

where σ is measurable, $0 \leq \sigma \leq M$, K is a bounded linear scattering operator, and N is in $\mathcal{N}$. Assume that a particle leaving $\bar{D}$ cannot re-enter. Then the problem

$$\frac{\partial N}{\partial t} = AN, \quad (5.2)$$

$$N(x, v, 0) = f(x, v)$$

has a unique solution

$$\begin{aligned} N(x, v, t) &= e^{At} f(x, v) \\ &= T(t)f(x, v). \end{aligned} \quad (5.3)$$

Under certain conditions on the scattering operator K , the operator $T(t)$ is completely continuous for $t > 3\tau$, τ as described in the previous paragraph. For example, K satisfies these conditions in the n -velocity group theory ($n \geq 1$). Hence the spectrum of A is discrete. It is not necessary for an eigenvalue to exist, however. (The case $K = 0$ is a trivial example.) When eigenvalues do exist, expansion theorems of the expected type (see Equation 4.6) hold asymptotically for large values of t . When these expansions are infinite, questions of actual convergence are still open.

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