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Subject: Viscous Compressible Fluid
Flow in a Network of
Narrow Channels

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Introduction

In estimating the pressure distribution for compressible fluid flow in long, very narrow channels formed by gaps between identical prismatic rods, it is important to realize that there is a range of channel thickness below which the flow is controlled by viscous shear stresses in the fluid rather than by conversion of pressure head to velocity head (for any given circumstances it is rather simple to calculate the flow under both assumptions, the lesser flow being the better approximation to the true flow.) This shear stress controlled range is of importance since in it there can be an appreciable time lag from the time a flow channel reaches a certain configuration and the time the equilibrium pressure distribution for that configuration arrives. This time lag of course implies the possibility of self-excited vibrations of the fluid flow-mechanical system. This viscous flow will be considered herein.

While the prismatic rods are discrete and have discrete flow paths around and along them, in the case where there are a large number of such rods we would seem to be justified in the assumption of an infinite number of rods, hence the use of distributed properties. Such a model can of course offer calculation advantages.

Nomenclature

R, θ, x	radial, circumferential and axial coordinates in., rad, in
P	gas pressure #/ sq. in.
h	flow channel thickness, in.
μ	gas viscosity #Sec/sq.in.
R	gas constant, in ³ /°R
T	temperature, °R
M	no. of prisms/sq.in.
l	circumference of cross section of each prism, in.
t	time, sec.
P_s	supply pressure to flow path
P_A	ambient pressure after leakage path
P_B	bundling pressure on outside diameter

Derivation

The viscous volumetric flow of a fluid in a very narrow channel is given by *

$$q_x = -\frac{h^3}{12\mu} \frac{\partial P}{\partial x} \quad \text{C.U. IN. / SEC. / IN.}$$

If the fluid is a gas, the weight flow is then given by

$$-\frac{Ph^3}{RT\mu} \frac{\partial P}{\partial x} \quad \text{\#/SEC. / IN.}$$

Considering the radially symmetrical flow through narrow channels in a volume formed by equally spaced identical prismatic bars, we have for the flow along the axis of the bars

$$q_x = -\frac{P}{RT\mu} \frac{h^3}{\partial x} \left(\frac{Ml}{2}\right) R d\theta dR$$

where $\left(\frac{Ml}{2}\right)$ gives the linear width of flow channel per square inch.

Due to the prisms, the length of the radial flow path per inch of radius is approximately

$$\frac{\sqrt{\pi}}{4} l \sqrt{M}$$

if the prisms are so arranged that flow must go over half the circumference of a prism before reaching the equivalent position on the next prism. The radial flow is then given by

$$q_r = -\frac{P}{RT\mu} \frac{h^3}{\partial R} \left(\frac{4}{\sqrt{\pi} l \sqrt{M}}\right) R d\theta \sqrt{M} dx$$

OR

$$q_r = -\frac{4}{\sqrt{\pi} l} \frac{P}{RT\mu} \frac{h^3}{\partial R} R d\theta dx$$

Conservation of mass requires that

$$\frac{\partial}{\partial x} q_x dx + \frac{\partial}{\partial R} q_r dR = -\frac{\partial}{\partial t} (\text{Stored weight of gas})$$

The stored weight of gas equals

$$\frac{P}{RT} (R d\theta dR \frac{Ml}{2}) h dx$$

Hence

$$\begin{aligned} \frac{\partial}{\partial x} \left[-\frac{Ph^3}{RT\mu} \frac{\partial P}{\partial x} R d\theta dR \frac{Ml}{2} \right] dx + \frac{\partial}{\partial R} \left[-\frac{4}{\sqrt{\pi} l} \frac{P}{RT\mu} \frac{h^3}{\partial R} R d\theta dx \right] dR = \\ -\frac{\partial}{\partial t} \left[\frac{P}{RT} R d\theta dR \frac{Ml}{2} h dx \right] \end{aligned}$$

*See any standard lubrication text

$$\text{or } \frac{\partial}{\partial x} \left[\frac{Rh^3 P \partial P}{12\mu RT \partial x} \right] + \frac{8}{\sqrt{\pi} M L^2} \frac{\partial}{\partial R} \left[\frac{Rh^3 P \partial P}{12\mu RT \partial R} \right] = \frac{\partial}{\partial t} \left[\frac{RhP}{RT} \right] \quad (1)$$

It may be noted that the temperature and viscosity of the gas T and μ have been assumed constant across the channel thickness but not along the channel in either direction.

Steady State Pressure Distribution

One of the items of interest is the steady state solution of (1), where the right hand side is set equal to zero. While the resulting equation is still non-linear in P , it may be observed that if h and μ are not a function of P , the equation is linear in P^2 , thereby permitting an analogy with heat flow (for example), and the use of existing computer codes to calculate an approximate solution. It would also appear that for certain assumptions of the nature of the variation of h, T with R and X , analytical solutions are readily obtained (see Appendix 1). The results of Appendix 1 indicate that the effect of temperature increase in the axial direction is the same qualitatively, and of the same order quantitatively as the effect of converging flow paths in the axial direction.

Transient Pressure Distributions and the Possibility of Self Excited Pressure Fluctuations

The real interest in equation (1) is due to the fact that in theory at least, it permits the calculation of transient pressure distributions, hence the possibility of determining the existence of self-excited pressure fluctuations. We expect that by considering small pressure perturbations around a position of static equilibrium, and the equations of elasticity and motion of the prismatic rods (suitably reduced to a small number of degrees of freedom), the stability (in a dynamical as well as a statical sense) of the system can be computed. While the use of a perturbation method makes the unknown pressures appear linearly, the equations will have variable coefficients and will be quite difficult to handle, thereby implying the necessity of a rather complicated computer solution. Should no easy, reasonably accurate stability criterion appear however, it may be necessary to follow such a course.

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APPENDIX (1)

For certain forms of the function $\frac{h^3}{\mu T}$ the steady state case can be solved in closed form.

$$\text{Let } \frac{h^3}{\mu T} = H(x) \quad (A1)$$

Then for the steady state case (1) reduces to

$$\frac{1}{H} \frac{\partial}{\partial x} \left(H \frac{\partial P^2}{\partial x} \right) + \frac{C_o^2}{R} \frac{\partial}{\partial R} \left(R \frac{\partial P^2}{\partial R} \right) = 0 \quad (A2)$$

Separating variables in the usual way

$$P^2 = f(R) g(x) \quad (A3)$$

$$g'' + \frac{H'}{H} g' + \lambda^2 g = 0 \quad (A4)$$

$$f'' + \frac{1}{R} f' - \left(\frac{\lambda}{C_o} \right)^2 f = 0 \quad (A5)$$

When

$$\lambda^2 = 0, \quad \frac{\partial}{\partial x} (H g') = 0$$

$$g_o = P_s^2 - (P_s^2 - P_A^2) \frac{\int_0^x \frac{dx}{H}}{\int_0^L \frac{dx}{H}} \quad (A6)$$

Which satisfies the B.C.

$$P^2(0, R) = P_s^2; \quad P^2(L, R) = P_A^2$$

When the bundling pressure $P_b(x, R_o)$ is 'ideal' (in that the flow is one dimensional), equation (A6) gives the entire solution and can be readily integrated either formally or numerically. Note especially that only the combination $\frac{h^3}{\mu T}$ determines the pressure distribution.

One particular case of interest is the isothermal (cold flow) case with constant leakage gap thickness h . For this case we have from (A6);

$$g_o = P_s^2 - (P_s^2 - P_A^2) \frac{x}{L} \quad (A7)$$

The solution to (A5) is

$$f(R) = A J_o \left(i \frac{\lambda R}{C_o} \right) + B N_o \left(i \frac{\lambda R}{C_o} \right)$$

From (A4) we have, since $H' = 0$

$$g(x) = C \sin \lambda x + D \cos \lambda x$$

Since the boundary conditions are

$$P^2(0, R) = P_s^2; \quad P^2(L, R) = P_A^2 \quad (A8)$$

$$P^2(x, R_o) = P_b^2; \quad \frac{\partial P^2}{\partial R} \Big|_{x, R_o} = 0$$

APPENDIX (1)

We have

$$D = B = 0; \lambda L = N\pi, N = 1, 2, 3, \dots$$

$$P_B^2 = P_S^2 - (P_S^2 - P_A^2) \frac{x}{L} + \sum_{N=1}^{\infty} \left\{ A_N J_0 \left(i \frac{\lambda_N R_0}{C_0} \right) \sin N\pi \frac{x}{L} \right\}$$

A_N may be calculated from this equation giving

$$P^2 = P_S^2 - (P_S^2 - P_A^2) \frac{x}{L} + \sum_{N=1}^{\infty} \left\{ 2 \int_0^1 \left[P_B^2 - P_S^2 + (P_S^2 - P_A^2) \frac{x}{L} \right] \sin N\pi \frac{x}{L} d \left(\frac{x}{L} \right) \frac{J_0 \left(i \frac{N\pi R}{C_0 L} \right)}{J_0 \left(i \frac{N\pi R_0}{C_0 L} \right)} \sin N\pi \frac{x}{L} \right\} \quad (A9)$$

Another particular case of interest, and one for which a solution is readily obtained occurs when $\frac{h^B}{\mu T} = H_0 e^{\alpha x}; \alpha = \frac{1}{L} \ln \frac{H_1}{H_0}$

$$(A6) \text{ THEN GIVES } g_0 = P_S^2 - (P_S^2 - P_A^2) \left[\frac{1 - \left(\frac{H_0}{H_1} \right)^{\frac{x}{L}}}{1 - \left(\frac{H_0}{H_1} \right)} \right] \quad (A10)$$

The solution to (A5) remains the same, but (A4) now gives

$$g = e^{-\frac{\alpha x}{2}} A \sin \left\{ \left(1 - \frac{\alpha^2}{4\lambda^2} \right)^{1/2} \lambda x + \phi \right\}$$

From the B.C., (A8) we have

$$\phi = 0; \lambda_N = \sqrt{\left(\frac{\alpha}{2} \right)^2 + \left(\frac{N\pi}{L} \right)^2}$$

$$\text{Hence } P_B^2 = P_S^2 - (P_S^2 - P_A^2) \left[\frac{1 - e^{-\alpha x}}{1 - e^{-\alpha L}} \right] + \sum_{N=1}^{\infty} A_N J_0 \left(i \frac{\lambda_N R_0}{C_0} \right) e^{-\frac{\alpha x}{2}} \sin N\pi \frac{x}{L}$$

A_N is calculated in the usual way, giving

$$P^2 = P_S^2 - (P_S^2 - P_A^2) \left[\frac{1 - \left(\frac{H_0}{H_1} \right)^{\frac{x}{L}}}{1 - \left(\frac{H_0}{H_1} \right)} \right] + \sum_{N=1}^{\infty} \left\{ 2 \int_0^1 \left[P_B^2 - P_S^2 + (P_S^2 - P_A^2) \left(\frac{1 - \left(\frac{H_0}{H_1} \right)^{\frac{x}{L}}}{1 - \left(\frac{H_0}{H_1} \right)} \right) \right] \left(\frac{H_1}{H_0} \right)^{\frac{x}{L}} \sin N\pi \frac{x}{L} d \left(\frac{x}{L} \right) \frac{J_0 \left(i \frac{N\pi R}{C_0 L} \sqrt{1 + \left(\frac{\ln H_1/H_0}{2\lambda R} \right)^2} \right)}{J_0 \left(i \frac{N\pi R_0}{C_0 L} \sqrt{1 + \left(\frac{\ln H_1/H_0}{2\lambda R_0} \right)^2} \right)} \left(\frac{H_0}{H_1} \right)^{\frac{x}{L}} \sin N\pi \frac{x}{L} \right\} \quad (A11)$$

$$l = 18 a$$

$$M = \frac{2}{21\sqrt{3}} a^2$$

Typical Cross Section of Prism