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PLASMA CONFINEMENT AT UNIFORM TEMPERATURE

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ABSTRACT

The hydromagnetic equations are solved to determine the properties of a plasma confined in the steady state by a magnetic field. The steady state is presumed to be realized by injecting neutral particles which subsequently become ionized and replace the particles lost by diffusion. Assuming uniform temperature throughout the plasma makes possible analytic determination of the macroscopic quantities of interest. These include the electric and magnetic fields, current density, particle density, and macroscopic velocity. In the first case treated, the region of confinement lies inside a circular cylinder of infinite length, and in the second case within the space between two parallel planes. A comparison is made between uniformly distributed injection and concentrated injection, and an estimate is given of the decay time after injection ceases.

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1. INTRODUCTION

A plasma surrounded by a region in which the pressure of the ionized gas is zero, the pressure of the gas being balanced by magnetic pressure, is said to be confined.^{1a} It is a well-known fact that an ionized gas confined by a magnetic field always drifts out of the region of confinement. Consequently, in order to secure a steady-state condition plasma must be supplied at the same rate at which it is lost by drift; otherwise the system will be in a transient state.

In what follows it will be assumed that the system has reached a steady state, and that the lost plasma is supplied by injecting neutral atoms that subsequently become ionized and reach the equilibrium temperature of the plasma. When neutral atoms are injected, the usual assumption of continuity of (ionized) matter is no longer valid, and the equation of continuity

$$\frac{\partial n}{\partial t} = -\vec{\nabla} \cdot (n\vec{v})$$

must be modified to read

$$\frac{\partial n}{\partial t} = -\vec{\nabla} \cdot (n\vec{v}) + Q(\vec{r})$$

^{1a} Definition of confinement given by L. Spitzer, Jr., Physics of Fully Ionized Gases, Interscience Publishers, Inc., New York, 1956, p 25.

where n is the total number of electrons and ions per cc, $\vec{v}$ is the macroscopic velocity, and $Q(\vec{r})$ is the rate of ionization of the neutral atoms at the position $\vec{r}$. With this modification the steady-state hydromagnetic equations may be written

$$0 = \vec{\nabla} \times \vec{H} - 4\pi\vec{J} \quad (1.1)$$

$$0 = \vec{\nabla} \times \vec{E} \quad (1.2)$$

$$0 = -\eta \vec{J} + \vec{E} + \vec{v} \times \vec{H} + \frac{c}{e} \vec{\nabla} p \quad (1.3)$$

$$0 = -\vec{\nabla} \cdot (n\vec{v}) + Q(\vec{r}) \quad (1.4)$$

$$0 = \vec{J} \times \vec{H} - \vec{\nabla} p \quad (1.5)$$

$$0 = p - nkT \quad (1.6)$$

where electromagnetic units are used for the electric field $\vec{E}$, magnetic field $\vec{H}$, current density $\vec{J}$, and resistivity η , p is the combined pressure of electrons and ions in dynes/cm², T is the temperature of both electrons and ions in degrees Kelvin, c is the velocity of light (3×10^{10} cm/sec), e is the electronic charge (4.80×10^{-10} esu.), and k is Boltzmann's constant (1.38×10^{-16} ergs/°K).

The equations given above are essentially Maxwell's equations (1.1, 1.2), the linearized equations of transfer for a perfect plasma (1.3, 1.5), the equation of continuity (1.4), and the equation of state (1.6).^{1b} All terms having an explicit time derivative ($\partial/\partial t$) as a factor have been omitted. Equations (1.3), (1.4) and (1.6) are valid only if the electron and ion temperatures are equal.² If the divergence of the current density is not zero, Eq. (1.4) contains another term, discussed in Appendix A.

^{1b} The approximations that must be made to obtain the linearized equations (1.3) and (1.5) from the Boltzmann equation are discussed by Spitzer, loc. cit., p 20.

² The last term in Eq. (1.3) is associated with the electrostatic potential developed within a confined plasma; it is omitted altogether in the macroscopic equations used by H. Alfvén in Cosmical Electrodynamics, Clarendon Press, Oxford, 1950.

2. CONFINEMENT WITHIN A CIRCULAR CYLINDER

Analytic solutions of the steady-state equations can be found for certain configurations of the region of confinement, and, although none of the configurations to be considered here are finite in extent, it seems worthwhile to discuss the solutions, because they exhibit the mechanism by which a magnetic field inhibits the flow of plasma out of the region of confinement. The first case to be considered is that in which the region of confinement lies inside a circular cylinder of infinite length. In the absence of plasma the magnetic field inside the cylinder is supposed to be the uniform value H_0 pointing along the axis of the cylinder. Such a field could be produced by a current sheet at the wall of the cylinder carrying current in the azimuthal (θ) direction.

2.1 Equations in Cylindrical Coordinates

Taking the usual cylindrical coordinates (r, θ, z), Eqs. (1.1) through (1.6) can be written

$$0 = \frac{dH}{dr} + 4\pi J \quad (2.1)$$

$$0 = E + \frac{c}{en} \frac{dp}{dr} \quad (2.2)$$

$$0 = \eta J + vH \quad (2.3)$$

$$0 = -\frac{1}{r} \frac{d}{dr} (r n v) + Q(r) \quad (2.4)$$

$$0 = JH - \frac{dp}{dr} \quad (2.5)$$

$$0 = p - n k T \quad (2.6)$$

where we have eliminated all variations with respect to θ and z , and have

retained only those terms required to satisfy the boundary conditions in which we are interested. Note the replacement of H_z by H , J_θ by J , E_r by E and v_r by v . The azimuthal component of electric field E_θ is not required for the steady-state problem. The radial component of electric field E_r associated with the last term in Eq. (1.3) has been written separately in Eq. (2.2); in this latter equation the radial component of current density is taken to be zero because no such component of current can exist in the steady state.

From Eq. (2.1) the current density may be written

$$J = -\frac{1}{4\pi} \frac{dH}{dr}$$

Substituting this expression for J into Eq. (2.3) the macroscopic velocity is found to be

$$v = \frac{\eta}{4\pi} \frac{1}{H} \frac{dH}{dr} \quad (2.7)$$

Integration of Eq. (2.5), with J substituted from Eq. (2.1), yields the familiar equation

$$p = \frac{H_o^2}{8\pi} - \frac{H^2}{8\pi} \quad (2.8)$$

where by definition of the confinement problem the pressure must reach zero at the boundary where $H = H_o$. Combining Eqs. (2.6) and (2.8) the particle density becomes

$$n = \frac{H_o^2}{8\pi kT} \left[1 - \left(\frac{H}{H_o} \right)^2 \right] \quad (2.9)$$

Up to this point the current density, macroscopic velocity, gas pressure, and particle density have all been expressed in terms of the magnetic field and magnetic field gradient. The distribution of the magnetic field itself depends on the function $Q(r)$ appearing in Eq. (2.4). The integral of that equation is

$$r n v = \int_0^r r Q(r) dr + \frac{q_0}{2\pi} \quad (2.10)$$

where q_0 is a constant of integration. Substituting the expression for velocity obtained in Eq. (2.8) the particle density may be written

$$n = \frac{4\pi}{\eta r} \frac{H}{(dH/dr)} \left(\int_0^r r Q(r) dr + \frac{q_0}{2\pi} \right) \quad (2.11)$$

Combining this result with Eqs. (2.6) and (2.8) yields

$$\frac{4\pi k T H}{\eta r \frac{dH}{dr}} \left(\int_0^r r Q(r) dr + \frac{q_0}{2\pi} \right) = \frac{H_0^2}{8\pi} - \frac{H^2}{8\pi} \quad (2.12)$$

which can be rearranged to read

$$\frac{32\pi^2 k T}{H_0^2 \eta r} \left(\int_0^r r Q(r) dr + \frac{q_0}{2\pi} \right) dr = \frac{dH}{H} - \frac{1}{H_0^2} H dH \quad (2.13)$$

The right side of this equation can be integrated immediately, while the left side can be integrated for a wide variety of functions $Q(r)$. It can also be integrated for a variety of temperature distributions $T(r)$, as well as variations in resistivity $\eta(r)$. Two cases will now be treated; in the first case $Q(r)$ is constant, and in the second, $Q(r)$ is essentially the δ -function. The temperature and resistivity are both assumed constant. The assumption of constant resistivity is consistent with the assumption of constant temperature.^{1c}

^{1c} See page 84, Spitzer, loc. cit.

2.2. Uniform Injection

If plasma is supplied uniformly over the interior of the cylinder by appropriate injection, $Q(r)$ will be a constant, Q_o . Eq. (2.10) then becomes

$$rnv = \frac{Q_o r^2}{2} \quad (2.14)$$

where the constant of integration q_o must be zero in order that rnv vanish at the origin. Thus

$$n = \frac{Q_o r}{2v} \quad (2.15)$$

and using Eq (2.7),

$$n = \frac{2\pi Q_o r}{\eta} \frac{H}{(dH/dr)} \quad (2.16)$$

The distribution of the magnetic field may be determined from Eq. (2.13), which reduces for this case to

$$\frac{16\pi^2 Q_o kT}{H_o^2 \eta} r dr = \frac{dH}{H} - \frac{1}{H_o^2} H dH \quad (2.17)$$

Integrating both sides of this equation,

$$\frac{8\pi^2 Q_o kT r_o^2}{H_o^2 \eta} \frac{r^2}{r_o^2} = \frac{1}{2} \log \left(\frac{H}{H_o} \right)^2 - \frac{1}{2} \left(\frac{H}{H_o} \right)^2 + \frac{C_o^2}{2} \quad (2.18)$$

where C_o is a constant of integration. Solving for $(r/r_o)^2$,

$$\left(\frac{r}{r_o} \right)^2 = \frac{H_o^2 \eta}{16\pi^2 Q_o kT r_o^2} \left[\log \left(\frac{H}{H_o} \right)^2 - \left(\frac{H}{H_o} \right)^2 + C_o^2 \right] \quad (2.19)$$

The constant C_o must now be so chosen that the boundary conditions are satisfied. At $r = r_o$, $H = H_o$, so that

$$C_o^2 = 1 + \frac{16\pi^2 Q_o k T r_o^2}{H_o^2 \eta} \quad (2.20)$$

It is customary to denote the ratio of gas pressure to vacuum magnetic field pressure by

$$\beta = \frac{nkT}{(H_o^2/8\pi)} \quad (2.21)$$

From Eqs. (2.6) and (2.8) this quantity may be written

$$\beta = 1 - \left(\frac{H}{H_o}\right)^2 \quad (2.22)$$

The value of β at the axis, which will be written β_o , corresponds to the minimum value of H , written H_m , and these extreme values occur at $r = 0$, a fact made evident by inspection of Eq. (2.16), which shows that $\frac{dH}{dr} = 0$ for $r = 0$. Thus β_o may be written

$$\beta_o = 1 - \left(\frac{H_m}{H_o}\right)^2 \quad (2.23)$$

and setting $r = 0$ in Eq. (2.19), the constant of integration may be written

$$C_o^2 = -\log \left(\frac{H_m}{H_o}\right)^2 + \left(\frac{H_m}{H_o}\right)^2 \quad (2.24)$$

The constant C_o can now be expressed in terms of β_o by using Eqs. (2.23) and (2.24). The expression is

$$C_o^2 = (1 - \beta_o) - \log (1 - \beta_o) \quad (2.25)$$

Using Eq. (2.20), there results finally the important relation

$$-\beta_0 - \log(1 - \beta_0) = \frac{16\pi^2 Q_0 k T r_0^2}{H_0^2 \eta} \quad (2.26)$$

The maximum value of β is now related to all of the significant parameters appearing in the problem, and the result is illustrated in Fig. 2.1. The electrical conductivity, injection rate, and temperature all have the same effect, namely, that increasing them will increase β_0 -- rapidly for small values of β_0 , and only small amounts for large values of β_0 . The square of the radius of the cylinder has the same effect, while the square of the vacuum magnetic field has the inverse effect.*

All of the relevant variables may now be written as functions of β_0 and H/H_0 . Letting

$$\frac{H}{H_0} = h \quad (2.27)$$

in order to simplify the expressions, Eq. (2.19) becomes

$$\frac{r}{r_0} = \left[\frac{\log h^2 - h^2 + 1 - \beta_0 - \log(1 - \beta_0)}{-\beta_0 - \log(1 - \beta_0)} \right]^{1/2} \quad (2.28)$$

The distribution of the magnetic field represented by Eq. (2.28) is plotted in Fig. 2.2 with β_0 as a parameter. As β_0 approaches unity, more and more of the field is excluded from the inside of the cylinder. However, even for $\beta_0 = 0.98$, the value of the field at the center is relatively large, being 14% of the vacuum magnetic field.

* For $-1 < \beta_0 < 1$,

$$-\beta_0 - \log(1 - \beta_0) = \frac{1}{2} \beta_0^2 + \frac{1}{3} \beta_0^3 + \frac{1}{4} \beta_0^4 + \dots,$$

and consequently for values of β_0 between zero and 0.2 one may easily compute β_0 using the approximation

$$\beta_0^2 \approx \frac{32\pi^2 Q_0 k T r_0^2}{H_0^2 \eta}$$

Fig. 2.1 Curve showing how the value of β at the cylinder axis varies with the uniform injection rate Q_0 and other parameters.

Fig. 2.2 Distribution of the magnetic field inside the cylinder for the case of uniform injection. The magnetic field vector always points along the z axis.

The current density is found by differentiation to be

$$\begin{aligned}
 J &= -\frac{1}{4\pi} \frac{dH}{dr} \\
 &= -\frac{H_o}{4\pi r_o} \left(\frac{h}{1-h^2} \right) \left\{ \left[-\beta_o - \log(1-\beta_o) \right] \times \right. \\
 &\quad \left. \left[\log h^2 - h^2 + 1 - \beta_o - \log(1-\beta_o) \right] \right\}^{1/2} \quad (2.29)
 \end{aligned}$$

and the curves representing the distribution of current density are shown in Fig. 2.3. Note that, for a given value of β_o , the current density is proportional to the vacuum magnetic field, and inversely proportional to the radius of the cylinder. Also note that the current density at a particular value of radius has a maximum with respect to variations in β_o . In fact, as $\beta_o \rightarrow 1$, $J \rightarrow 0$, except near the boundary.

The macroscopic velocity is found to be

$$\begin{aligned}
 v &= \frac{\eta}{4\pi} \frac{1}{H} \frac{dH}{dr} \\
 &= \frac{\eta}{4\pi r_o} \left(\frac{1}{1-h^2} \right) \left\{ \left[-\beta_o - \log(1-\beta_o) \right] \times \right. \\
 &\quad \left. \left[\log h^2 - h^2 + 1 - \beta_o - \log(1-\beta_o) \right] \right\}^{1/2} \quad (2.30)
 \end{aligned}$$

and this result is shown in Fig. 2.4. For a fixed value of β_o , the macroscopic velocity is proportional to the resistivity of the plasma.

The particle density, given by Eq. (2.9), is plotted in Fig. 2.5. The density is of course maximum where the magnetic field is minimum, and approaches zero at the boundary. Actually, the density cannot be zero at the boundary, and this point will be discussed further in connection with the electrostatic field.

To summarize the results obtained so far, it has been shown that uniform injection causes the density of particles to approach a particular equilibrium distribution. At this equilibrium the gas pressure forces plasma outward across the magnetic field lines at the same rate at which the new material is being supplied. Motion across the magnetic field lines generates an azimuthal current in the plasma which has two effects: it diminishes the magnetic field inside the cylinder, and it reacts with the existing mag-

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Fig. 2.3 The current density under uniform injection. The current density vector points opposite to the azimuthal (θ) direction, and is therefore shown negative.

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Fig. 2.4 The macroscopic velocity, which points in the radial direction, under uniform injection.

netic field to provide the electromagnetic pressure that inhibits the flow of plasma outward. The restraining force is viscous in the sense that it is proportional to the macroscopic velocity. The pressure and density inside the cylinder can be increased by increasing the injection rate, but the pressure can never exceed $H_0^2/8\pi$, the energy density of the vacuum magnetic field. If the injection rate is increased indefinitely, the macroscopic velocity will also increase indefinitely. However, the viscous forces associated with $\vec{J} \times \vec{H}$ do not increase indefinitely, because the magnetic field itself diminishes. The generated current, also sometimes called the Hall current, is responsible for the depression of the magnetic field, the effects of this current being opposite in sign to that of the current source producing the field H_0 . This is the reason for saying that a confined plasma exhibits diamagnetic effects. A plasma confined by material walls does not exhibit such effects in the presence of a magnetic field.

The radial component of electric field E given by Eq. (2.2) is not required in determining the other variables in the problem, but once the pressure and density are known it can be calculated directly. Using Eq. (2.2) and the results given in Eq. (2.8) and (2.9), the electric field, which is of course a static field, may be written

$$E = \frac{2ckT}{er_0} \frac{h^2}{(1-h^2)^2} \left\{ \left[-\beta_0 - \log(1-\beta_0) \right] x \left[\log h^2 - h^2 + 1 - \beta_0 - \log(1-\beta_0) \right] \right\}^{1/2} \quad (2.31)$$

Curves showing the distribution of the electrostatic field are drawn in Fig. 2.6. The electric field vector points outward from the axis of the cylinder, so that the electrostatic potential at the axis is positive with respect to the cylinder wall.

The electrostatic potential U can be calculated by integrating Eq. (2.2). Letting

$$E = -\frac{dU}{dr} \quad (2.32)$$

and using

$$E = -\frac{c}{en} \frac{dp}{dr}$$

Fig. 2.5 Distribution of the particle density inside the cylinder for the case of uniform injection.

Fig. 2.6 The electrostatic field under uniform injection. The electric field vector points in the radial direction.

from Eq. (2.2), the potential at any radius r , relative to the wall, becomes

$$U(r) = \frac{ckT}{e} \int_{r_0}^r \frac{1}{n} \left(\frac{dn}{dr} \right) dr = \frac{ckT}{e} \int_{n_0}^n \frac{dn}{n} = \frac{ckT}{e} \log \frac{n}{n_0} \quad (2.33)$$

where n is the particle density at the radius r , and n_0 is the particle density at the wall. The density at the wall requires some explanation. In all of the previous analysis the forces associated with acceleration of the macroscopic mass density have been neglected, and this leads to negligible error for the steady-state problem except very near the wall. At the wall v and J were infinite, and n was zero. It has been shown in Reference 2 that if terms which represent inertial forces are retained, the velocity attained by the escaping particles is not actually infinite but instead is equal to the thermal velocity of the particles. The particle density is therefore not zero but some small finite value. The same number of electrons and ions pass to the wall each second, but they have different velocities and therefore slightly different densities, and this fact accounts for the positive potential developed in the interior. It is shown in Appendix B that the density near the wall is given by

$$n_0 = \frac{Q_0 r_0}{2} \frac{m_i}{3kT} \quad (2.34)$$

for the case of a gas with one type of ion having mass m_i . Substituting this value of density into Eq. (2.33), the electrostatic potential becomes

$$U(r) = \frac{ckT}{e} \log \frac{2n}{Q_0 r_0} \left(\frac{3kT}{m_i} \right)^{1/2} \quad (2.35)$$

In particular, the potential at the axis of the cylinder, $U(0)$, is obtained by substituting for n the density of particles at the axis.

2.3 Decay Time

Note that if the supply of plasma were suddenly stopped the distribution of density would not change abruptly, but $\partial n / \partial t$ initially would be uni-

formly negative over the whole region, with the value $-Q_0$. Perhaps a good estimate of the decay time τ for the system, obtainable without complete analysis of the transient problem, is

$$\tau = \frac{n(0)}{2Q_0} \quad (2.36)$$

This is the time that would be required for the density at the center to reach 50% of the original density assuming the density there to decrease at the initial rate. Using Eqs. (2.21) and (2.36).

$$\tau = \frac{H_0^2 \beta_0}{16\pi Q_0 kT} \quad (2.37)$$

For values of β_0 less than 0.2 the formula given in the footnote, page 11, can be used, and the decay time in that case can be written

$$\tau = \frac{2\pi r_0^2}{\beta_0 \eta} \quad (2.38)$$

This expression is analogous to the time constant L/R for an inductive electric circuit. Here $2\pi r_0^2/\beta_0$ is analogous to L , while the resistivity η is analogous to the resistance R . Some values of $\beta_0 \tau$ are given in Table 2.1 for the case of hydrogen (or deuterium) plasma using Eq. (2.38) and the expression

$$\eta = 2 \times 6.53 \times 10^{12} \frac{\log \Lambda}{T^{3/2}} \quad (2.39)$$

given by Spitzer.^{1c} The quantity $\log \Lambda$ is assumed to have the nominal value 20, which is a good approximation over a wide range of temperature and density.

2.4. Injection along the Axis.

In the previous section it was found that the gas pressure is always less than the vacuum magnetic field pressure regardless of how large the

TABLE 2.1 Decay Time

$T, ^\circ K$	$\beta_o \tau, \text{ seconds}$	
	$r_o = 10 \text{ cm.}$	$r_o = 100 \text{ cm.}$
10^4	2.4×10^{-6}	0.24×10^{-3}
10^5	$75. \times 10^{-6}$	7.50×10^{-3}
10^6	2.4×10^{-3}	0.24
10^7	$75. \times 10^{-3}$	7.5
10^8	2.4	240

(uniform) injection rate may be. In fact, to change the value of β_o from 40% to 80% requires increasing the injection rate by a factor of eight. It may be argued that uniform injection represents a great waste of plasma, because that plasma injected near the cylinder wall does not have to move through a large space containing magnetic field before it is lost. To investigate this question requires the analysis of distributions $Q(r)$ other than a constant. Consider the extreme case in which all of the plasma is injected at the axis of the cylinder. We shall find that in this case β_o is always unity, the magnetic field is zero at the axis, and another parameter α is more appropriate for characterizing the state of the system for different values of the injection rate. This parameter determines the extent to which the magnetic field is excluded from the interior of the cylinder in much the same way as β_o does for the case of uniform injection. If the plasma is all injected along the axis, then at any radius r , the value of Q is zero, and Eq. (2.10) takes the form

$$2\pi r n v = q_o \quad (2.40)$$

where q_o is the number of particles injected along the axis per centimeter of length per second.

Following the same procedure as that used before, Eq. (2.13) becomes

$$\frac{16\pi q_o kT}{H_o^2 \eta} \frac{dr}{r} = \frac{dH}{H} - \frac{1}{H_o^2} H dH \quad (2.41)$$

and this equation can be integrated to yield

$$a^2 \log \left(\frac{r}{r_0} \right) = \log h^2 - h^2 + 1 \quad (2.42)$$

where the constant of integration unity is added to the right side in order to satisfy the boundary condition that $H = H_0$ for $r = r_0$, and where

$$\frac{1}{2} a^2 = \frac{16\pi q_0 kT}{H_0^2 \eta} \quad (2.43)$$

Curves can now be drawn which show the distribution of the magnetic field for the case of injection along the axis of the cylinder. A different curve will be obtained for each distinct value of a . Instead of choosing the values of a arbitrarily, it is convenient to choose a set of values that is related to the values of β_0 used in the plots of the previous section. Note that injection of the same total number of particles into the cylinder as in the previous section would require that

$$q_0 = \pi r_0^2 Q_0 \quad (2.44)$$

By satisfying this equation we can obtain a direct comparison of uniform injection with axial injection. Substituting this value of q_0 into Eq. (2.43), the values of a can be obtained from the expression

$$\frac{1}{2} a^2 = \frac{16\pi^2 Q_0 kT r_0^2}{H_0^2 \eta} = -\beta_0 - \log(1 - \beta_0) \quad (2.45)$$

which holds only for the assumption implied in Eq. (2.44). The values of a corresponding to $\beta_0 = 0.25, 0.50, 0.75$, and 0.98 are $a = 0.272, 0.622, 1.122$, and 2.38 , respectively. Curves showing the distribution of the magnetic field are shown in Fig. 2.7 for these values of a . These curves may be compared directly with the curves of Fig. 2.2. Note that the magnetic field is completely excluded at the axis of the cylinder.

The distribution of particle density, shown in Fig. 2.8, should be compared with that of Fig. 2.5. The density is always larger when the particles are injected along the axis; it is not much larger near the walls, but at the center the ratio of densities equals the reciprocal of β_0 .

Fig. 2.7 Distribution of the magnetic field inside the cylinder for the case of injection along the cylinder axis.

Fig. 2.8 Particle density inside the cylinder for injection along the axis.

3. CONFINEMENT BETWEEN PARALLEL PLANES

Plasma confinement between two parallel planes is even further from physical realizability than confinement within a cylinder of infinite length, but it is of some theoretical importance, because in plasma confinement problems involving time variations, temperature variations, or other complications, one may find it expedient to carry out an analysis for a configuration in which the equations take the simplest form. The purpose of this chapter is to compare the results obtained for the cylindrical configuration and the results one gets in rectangular coordinates. For the case of uniform injection, the solutions are found to be identical except for a numerical factor (not involving π) in the normalization; the solutions for concentrated injection differ only in minor detail.

3.1. Equations in Rectangular Coordinates

Let the planes be defined by the equations $x = \pm L$, and let the magnetic field again be H_0 , pointing along the z axis, which is taken to be perpendicular to the plane of the paper. The macroscopic velocity will now have only one component, v_x , which will be written simply v . Likewise, the only components required for the other vectors will be E_x , J_y , and H_z , and subscripts will not be required to identify them.

The equations in rectangular coordinates are identical with the equations in cylindrical coordinates, if r is replaced by x , except for the equation of continuity (2.4). In rectangular coordinates, that equation is written

$$0 = -\frac{d}{dx} (n\vec{v}) + Q(x). \quad (3.1)$$

3.2. Uniform Injection

For $Q(x) = Q_1$, a constant, the integral of Eq. (3.1) becomes

$$n = \frac{Q_1 x}{v} \quad (3.2)$$

which is identical with Eq. (2.15) except for a factor of two. Since all of the other equations are the same, all of the solutions obtained in Sec. 2.2, for the case of uniform injection within a cylinder, apply to the present case,

provided we replace Q_0 by $2Q_1$, r by x , and r_0 by L . In other words, under uniform injection the cylinder and plane problems have identical solutions if the number of particles per cubic centimeter per second injected between the two planes is half of the number injected per cubic centimeter per second within the cylinder. The distribution of some of the variables is illustrated in Fig. 3.1 for the particular case $\beta_0 = 50\%$. These curves are the same numerically as the curves presented previously for $\beta_0 = 0.50$, but they show in addition which functions are odd and which are even for the problem in rectangular coordinates.

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Fig. 3.1 Curves showing the distribution of some of the macroscopic variables in the space between the parallel planes assuming uniform injection, all drawn for the particular value $\beta_0 = 0.50$.

3.3. Injection in the Median Plane

The median plane is defined by $x = 0$, and the analog of axial injection is injection in the median plane. For this case,

$$nv = \frac{1}{2} q_1 \quad (3.3)$$

where q_1 is the number of particles per square centimeter per second appearing at the plane $x = 0$. The factor $1/2$ means that half of the particles go to the left and half go to the right. Comparing Eq. (3.3) with Eq. (2.40), it is evident that the solutions will be different for this injection distribution.

Instead of Eq. (2.10) it is necessary to write for rectangular coordinates

$$nv = \int_0^x Q(x) dx + q_1 \quad (3.4)$$

and Eq. (2.13) becomes

$$\frac{32\pi^2 kT}{H_o^2 \eta} \int_0^x [Q(x) dx + q_1] dx = \frac{dH}{H} - \frac{1}{H_o^2} \quad (3.5)$$

and, for $Q = 0$, this equation can be integrated to yield

$$\frac{64\pi^2 q_1 kTL}{H_o^2 \eta} \left(\frac{x}{L} \right) = \log h^2 - h^2 + C_1^2 \quad (3.6)$$

where C_1 is a constant of integration. At $x = L$, $h = 1$, so that

$$C_1^2 = 1 + \frac{64\pi^2 q_1 kTL}{H_o^2 \eta} \quad (3.7)$$

The maximum value of β occurs at $x = 0$, and satisfies the equation

$$-\beta_o - \log(1 - \beta_o) = \frac{64\pi^2 q_1 kTL}{H_o^2 \eta} \quad (3.8)$$

Since the function on the left side of this equation is the same as that in Eq. (2.26), the curve of Fig. 2.1 can again be used to determine β_0 , provided the abscissa is replaced by the quantity appearing on the right side of Eq. (3.8). Figs. (3.2) and (3.3) illustrate how the magnetic field and particle density, respectively, are distributed for injection at the median plane.

Injection of the same total number of particles between the two planes as in the previous section requires that

$$q_1 = 2Q_1 L, \quad (3.9)$$

For uniform injection

$$-\beta_0 - \log(1 - \beta_0) = \frac{32\pi^2 Q_1 kT}{H_0^2 \eta} \quad (3.10)$$

By inspection of Eqs. (3.8), (3.9) and (3.10) it is evident that injection at the median plane and uniform injection will produce identical values of β_0 , provided four times as many particles are injected under the scheme of uniform injection. Thus, although the curves of this section are drawn for the nominal values of β_0 , 0.25, 0.50, etc., it should be remembered that only one fourth as many particles per second are required to produce these results as would be required under uniform injection.

APPENDIX A

THE CONTINUITY EQUATION

The equation of continuity for the ions in a plasma having one type of positive ion is

$$\frac{\partial n_i}{\partial t} = -\frac{1}{m_i} \vec{\nabla} \cdot (\rho \vec{v}) + Q_i(\vec{r}) \quad (A.1)$$

Fig. 3.2 Distribution of the magnetic field between the two planes under injection at the median plane.

Fig. 3.3 Distribution of the particle density between the two planes under injection at the median plane.

and for electrons

$$\frac{\partial n_e}{\partial t} = -\frac{z}{m_i} \vec{\nabla} \cdot (\rho \vec{v}) + \frac{c}{e} \vec{\nabla} \cdot \vec{J} + Q_e(\vec{r}) \quad (\text{A.2})$$

where the subscripts e and i refer to electrons and ions, respectively,

and where

$$\rho = m_i n_i + m_e n_e \quad (\text{A.3})$$

Let

$$p_i = n_i k T_i \quad (\text{A.4})$$

and

$$p_e = n_e k T_e \quad (\text{A.5})$$

Also let

$$n_e = z n_i \quad (\text{A.6})$$

so that

$$n = n_e + n_i = z n_i + n_i = (1+z) n_i \quad (\text{A.7})$$

If

$$T_i = T_e = T \quad (\text{A.8})$$

it follows that

$$\begin{aligned} p &= p_e + p_i = (1+z) n_i k T \\ &= \frac{n}{(1+z)} (1+z) k T = n k T \quad (\text{A.9}) \end{aligned}$$

Adding Eqs. (A.1) and (A.2),

$$\frac{\partial n}{\partial t} = -\frac{1+z}{m_i} \vec{\nabla} \cdot (\rho \vec{v}) + \frac{c}{e} \vec{\nabla} \cdot \vec{J} + Q(\vec{r}) \quad (\text{A.10})$$

where

$$Q(\vec{r}) = Q_e(\vec{r}) + Q_i(\vec{r}) \quad (\text{A.11})$$

Now neglecting $m_e n_e$ compared with $m_i n_i$ in Eq. (A.3), ρ may be replaced in Eq. (A.10) by $m_i n_i$ and that equation then reads

$$\frac{\partial n}{\partial t} = -(1+z) \vec{\nabla} \cdot (n_i \vec{v}) + \frac{c}{e} \vec{\nabla} \cdot \vec{J} + Q(\vec{r}) \quad (\text{A.12})$$

and using Eq. (A.7),

$$\frac{\partial n}{\partial t} = -\vec{\nabla} \cdot (n \vec{v}) + \frac{c}{e} \vec{\nabla} \cdot \vec{J} + Q(\vec{r}) \quad (\text{A.13})$$

In the steady-state problems discussed in this paper, $\vec{\nabla} \cdot \vec{J} = 0$, $\partial n / \partial t = 0$, and Eq. (A.13) reduces to Eq. (1.4).

APPENDIX B

PARTICLE DENSITY AT THE BOUNDARY

For a nonrelativistic gas escaping from the boundary at thermal velocity, as discussed in Reference 3,

$$\frac{1}{2} m_e v_e^2 = \frac{3}{2} kT \quad (B.1)$$

and

$$\frac{1}{2} m_i v_i^2 = \frac{3}{2} kT \quad (B.2)$$

where the same temperature is assumed for both types of particles. Now near the walls we must have

$$2\pi r_o n_o v = \pi r_o^2 Q_o \quad (B.3)$$

for the case of uniform injection within a cylinder (the right-hand side of this equation is simply q_o for the case of injection along the axis). Since by definition the macroscopic velocity is given by

$$\vec{v} = \frac{n_i m_i \vec{v}_i + n_e m_e \vec{v}_e}{n_i m_i + n_e m_e} \quad (B.4)$$

we may solve for n_o , and we find

$$n_o = \frac{Q_o r_o}{2v} = \frac{Q_o r_o}{2} \frac{(n_i m_i + n_e m_e)}{(n_i m_i v_i + n_e m_e v_e)} \quad (B.5)$$

All of the velocity vectors in question are assumed to be radial. Substituting the values of v_e and v_i implied in Eqs. (B.1) and (B.2), we may write

$$n_o = \frac{Q_o r_o}{2\sqrt{3kT}} \frac{n_i m_i + n_e m_e}{n_i \sqrt{m_i} + n_e \sqrt{m_e}} \quad (B.6)$$

and, neglecting $n_e \sqrt{m_e}$ compared with $n_i \sqrt{m_i}$ we find

$$n_o = \frac{Q_o r_o}{2} \sqrt{\frac{m_i}{3kT}} \quad (B.7)$$

LIST OF SYMBOLS

- c Velocity of light, 2.9979×10^{10} cm/sec.
- C_o, C_1 Constants
- e Charge of proton, 4.803×10^{-10} esu.
- $\vec{E}$ Electric field strength, in emu, equal to 10^8 times field strength in volts/cm; $E_r = E$, radial component of electric field.
- h Ratio of actual magnetic field intensity H to vacuum magnetic field intensity H_o .
- $\vec{H}$ Magnetic field intensity in oersted, equal to the magnetic field strength in gauss; H_o , magnetic field strength in the absence of plasma, or the vacuum magnetic field; $H_z = H$, actual magnetic field pointing along the z axis.
- $\vec{J}$ Current density, in emu, equal to 1/10 times the current in amperes; $J_\theta = J$, azimuthal (θ) component of current density.
- k Boltzmann constant, 1.380×10^{-16} erg/degree.
- L Distance; $2L$, distance between parallel planes.
- m Particle mass in grams; m_e and m_i , electron and ion mass.
- n Particle density, per cm^3 ; n_e and n_i , density of electrons and positive ions; n_o , particle density at the boundary.
- p Pressure; p_e, p_i , pressure of electrons and positive ions
- q_o Rate of supply of plasma along the axis of the cylinder, particles/cm sec.
- q_1 Rate of supply of plasma at the median plane, particles/cm² sec.
- Q Rate of supply of plasma at an arbitrary point, particles/cm³ sec.
- Q_o Uniform rate of supply of plasma inside the cylinder, particles/cm³ sec.
- Q_1 Uniform rate of supply of plasma between two parallel planes, particles/cm³ sec.

- $\vec{r}$ Position vector
- r Cylindrical coordinate
- r_0 Radius of cylinder.
- t Time
- T temperature; T_e and T_i , electron and positive ion temperature.
- U Electrostatic potential, in emu, equal to 10^8 times the potential in volts.
- $\vec{v}$ Macroscopic velocity, equation (B.4); $v_r = v$, radial component of $\vec{v}$;
 $v_x = v$, component of $\vec{v}$
along x .
- x, y, z Coordinate axes.
- Z Particle charge, in units of the proton charge.
- α Constant
- β Ratio of gas pressure to vacuum (unperturbed) magnetic field pressure; β_0 , value of β at cylinder axis and at median plane.
- η Resistivity, in emu, equal to 10^9 times resistivity in ohm-cm.
- θ Angle in cylindrical coordinates
- Λ Approximately 20 (See Spitzer, loc. cit. p 84)
- ρ Mass density, in grams per cm^3 .
- τ Decay time for confined plasma.
- $\vec{\nabla}$ Gradient

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