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MAGNETIC CONFIGURATIONS

BY

E. MAZZUCATO

PLASMA PHYSICS
LABORATORY

MASTER

PRINCETON UNIVERSITY
PRINCETON, NEW JERSEY

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Exact Equilibria of Axisymmetric
Magnetic Configurations

E. Mazzucato

Plasma Physics Laboratory, Princeton University
Princeton, New Jersey 08540

ABSTRACT

Whether for easing the restrictions imposed by equilibrium and stability or for improving the plasma purity with the use of magnetic divertors, many devices of the next generation of tokamaks will have a shaped minor cross section.

In this paper a class of solutions of the ideal MHD equilibrium equations of an axisymmetric magnetic configuration is derived. These solutions, besides including magnetic configurations with a large variety of cross sections and non-uniform current distributions, are exact solutions obtained without making any assumptions on the value of the aspect ratio.

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I. INTRODUCTION

It is a widespread opinion that for improving the performances of tokamak devices it is necessary to shape "ad hoc" their minor cross section, the object being that of easing the restrictions imposed by equilibrium and stability.

The present tokamak devices are plagued with impurities. The spectacular improvements of plasma parameters achieved by the T-3 device during the last decade¹ are not matched by the present high-power level tokamaks, like the T-4² and the S-T³ devices, owing to increased level of impurities. A possible and probably the only solution of the impurities problem is the use of magnetic divertors which inevitably lead to magnetic configurations with non-circular cross sections.

A systematic analysis of the properties of axisymmetric systems with non-circular cross sections is still very incomplete. Only simple shapes, like the ellipse, and/or quasi-uniform current density distributions have been studied.⁴

A class of exact solutions of the ideal MHD equations of equilibrium of an axisymmetric configuration has been recently found.¹⁴ In this paper we derive another class. These solutions, besides including magnetic configurations with a large variety of cross section shapes and non-uniform current distributions, are exact solutions of the equilibrium equations and are therefore obtained without making any assumption on the value of the aspect ratio. This is very important because the next generation of tokamak devices will have large values of the inverse aspect ratio (ϵ) and therefore any solution obtained

expanding in power of ϵ will become doubtful if applied to these devices.

In Section II we derive the general solution. In Section III we give some examples.

II. GENERAL SOLUTION

The MHD equations of equilibrium of an axisymmetric magnetic configuration can be reduced to a single partial differential equation⁵ which in cylindrical coordinates (r, ϕ, z) has the form

$$\frac{\partial^2 \chi}{\partial r^2} + \frac{\partial^2 \chi}{\partial z^2} - \frac{1}{r} \frac{\partial \chi}{\partial r} = -r^2 \frac{dp}{d\chi} - \frac{1}{2} \frac{dI^2}{d\chi}, \quad (1)$$

where χ , I and p are surface quantities. p is the plasma pressure, χ and I are (apart from the factor $(2\pi)^{-1}$) the azimuthal fluxes of the magnetic field and the electric current respectively. In terms of these surface quantities, the magnetic field $\underline{B}$ and the current density $\underline{j}$ are given by the equations

$$\begin{aligned} \underline{B} &= \frac{I}{r} \underline{e}_\phi - \frac{1}{r} \underline{e}_\phi \times \nabla \chi \\ \underline{j} &= j_\phi \underline{e}_\phi - \frac{1}{r} \underline{e}_\phi \times \nabla I \\ j_\phi &= r \frac{dp}{d\chi} + \frac{1}{2r} \frac{dI^2}{d\chi} \end{aligned} \quad (2)$$

where $\underline{e}_\phi$ is the unit vector in the ϕ direction.

We shall assume a particular functional dependence of $p(\chi)$ and $I(\chi)$ that transforms Eq. (1) into a linear partial differential equation:

$$p = P\chi + P_0$$

$$I^2 = B\chi^2 + C\chi + D \quad (3)$$

where P , P_0 , B , C and D are constants and $B \neq 0$.

In general, the functional dependence (3) is not very convenient because it is not possible to have both the plasma pressure and the longitudinal current density equal to zero at the plasma boundary. Increasing the plasma pressure the longitudinal current density distribution tends to become a linear function of the radius. Nevertheless it is possible to find solutions of Eqs. (1) and (3) such that:

(a) The plasma pressure is zero at the plasma boundary,

(b) The longitudinal current density at the plasma boundary is not greater than a few percent of its maximum value inside the plasma column,

(c) Imposing the restriction that the safety factor q (the inverse rotational transform) is greater than one everywhere inside the plasma column, the value of the total plasma β is greater than that achieved in the present tokamak devices by more than an order of magnitude.

Equation (1) with the substitution

$$\chi = \psi - \frac{P}{B} r^2 - \frac{C}{2B} \quad (4)$$

becomes

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{\partial^2 \psi}{\partial z^2} - \frac{1}{r} \frac{\partial \psi}{\partial r} + B\psi = 0 \quad (5)$$

According to the maximum principle⁶, any solution of Eq. (5) with $B \leq 0$ cannot have a maximum or a minimum inside a bounded region without being a constant. Therefore we shall assume $B > 0$.

Let $F(r, p)$ be the one-sided Laplace transform of $\Psi(r, z)$ respect to the variable z

$$F(r, p) = \int_0^{\infty} e^{-pt} \Psi(r, t) dt \quad . \quad (6)$$

From Eq. (5) we get

$$\frac{\partial^2 F}{\partial r^2} - \frac{1}{r} \frac{\partial F}{\partial r} + (p^2 + b^2)F = p \Psi(r, 0) + \frac{\partial \Psi(r, 0)}{\partial z} \quad , \quad (7)$$

where $b = (B)^{1/2}$.

We shall assume symmetry respect to z and therefore we shall put $\partial \Psi(r, 0)/\partial z \equiv 0$ in Eq. (7). With the substitutions

$$\begin{aligned} F(r, p) &= r G(r, p) \\ x &= (p^2 + b^2)^{1/2} r \quad , \end{aligned} \quad (8)$$

Eq. (7) becomes

$$x^2 \frac{\partial^2 G}{\partial x^2} + x \frac{\partial G}{\partial x} + (x^2 - 1)G = \frac{p}{(p^2 + b^2)^{1/2}} x \Psi(r, 0) \quad (9)$$

Let us now assume for a moment

$$\Psi(r, 0) = r^\mu \quad , \quad (10)$$

μ being an unrestricted constant. Then a solution of Eq. (9) is⁷

$$G(r, p) = \frac{p}{(p^2 + b^2)^{(\mu+1)/2}} S_{\mu, 1} \left((p^2 + b^2)^{1/2} r \right) \quad (11)$$

where $S_{\mu, 1}(x)$ is a Lommel function. Let us consider the functions

$$f(z) = r^{\mu-1} {}_3F_2 \left(1, 1 - \frac{\mu}{2}, -\frac{\mu}{2}; \frac{1}{2}, 1; -\left(\frac{z}{r}\right)^2 \right) \quad (12)$$

$$g(p) = \frac{S_{\mu, 1}(pr)}{p^{\mu}} \quad (13)$$

where ${}_3F_2$ is the generalized hypergeometric series

$$\begin{aligned} & {}_3F_2(a_1, a_2, a_3; b_1, b_2; x) \\ &= \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n (a_3)_n}{(b_1)_n (b_2)_n} \frac{x^n}{n!} \end{aligned} \quad (14)$$

with

$$(a)_0 = 1, \quad (a)_n = a(a+1)\dots(a+n-1)$$

When one of the parameters a is zero or equal to a negative integer the series (14) has a finite number of elements. In general the series (14) converges for $|x| < 1$.

The function $g(p)$ is the one-sided Laplace transform of $f(z)$ ⁸.

Therefore the solution of Eq. (5) which satisfies Eq. (10) is⁹

$$\begin{aligned} \Psi(r, z) = & r^{\mu} \left[{}_3F_2 \left(1, 1 - \frac{\mu}{2}, -\frac{\mu}{2}; \frac{1}{2}, 1; -\left(\frac{z}{r}\right)^2 \right) \right. \\ & - (bz) \int_0^z (z^2 - u^2)^{-1/2} J_1(b(z^2 - u^2)^{1/2}) \\ & \left. \times {}_3F_2 \left(1, 1 - \frac{\mu}{2}, -\frac{\mu}{2}; \frac{1}{2}, 1; -\left(\frac{u}{r}\right)^2 \right) du \right] \end{aligned} \quad (15)$$

where $J_1(x)$ is a Bessel function of the first kind.

Any linear combination of terms of this type is a solution of Eq. (5). In this manner it is possible to construct solutions of Eq. (5) which are equal to any power series of r on the equatorial plane. Therefore we have found a very general solution of Eq. (5) in the region $|z| < r$.

III. APPLICATIONS

The hypergeometric series ${}_3F_2$ converges uniformly on every circle contained into its circle of convergence. Therefore the integral of Eq. (15) that contains the series ${}_3F_2$ can be performed term by term in the region $|z| < r$. For doing this we need to calculate integrals of the type

$$h_{n+1}(z) = bz \int_0^z \frac{u^{2n}}{(z^2 - u^2)^{1/2}} J_1(b(z^2 - u^2)^{1/2}) du \quad (16)$$

$$(n = 0, 1, 2, \dots)$$

It is possible^{10,11} to express these integrals in terms of the Bessel functions of fractional orders $J_{n-1/2}$. We get the following recurrence relation:

$$h_{n+1}(z) = z^{2n} - \frac{(2n-1)(2n-3)}{b^2} \left(-h_n(z) + z^2 h_{n-1}(z) \right) \quad (17)$$

$$(n = 2, 3, \dots)$$

and

$$h_1(z) = 1 - \cos(bz) \quad (18)$$

$$h_2(z) = z^2 \left(1 - \frac{\sin(bz)}{bz} \right)$$

Substituting Eq. (17) and (18) into Eq. (15) we get

$$\psi(r, z) = r^\mu \sum_{i=0}^{\infty} A_i^\mu \left(\frac{z}{r} \right)^{2i} g_{i+1}(z) \quad (19)$$

where

$$A_i^\mu = \begin{cases} 1 & \text{for } \mu = i = 0 \\ (-1)^i \frac{\mu - 2i}{i! \mu} \frac{[(-\mu/2)_i]^2}{(1/2)_i} & \text{for } \mu \neq 0 \text{ or } i \neq 0 \end{cases} \quad (20)$$

The functions of $g_{i+1}(z)$ are given by the recurrence relation

$$g_{i+1}(z) = \frac{(2n-1)(2n-3)}{(bz)^2} \left(g_i(z) - g_{i-1}(z) \right) \quad (21)$$

$$(i = 2, 3, \dots)$$

and

$$\begin{aligned} g_1(z) &= \cos(bz) , \\ g_2(z) &= \sin(bz)/(bz) . \end{aligned} \tag{22}$$

The series (19) contains only a finite number of terms, and therefore converges everywhere, when μ is zero or equal to a positive even integer. In fact, in this case the coefficients A_i^μ are zero for $\mu = 0$ and $i > 1$ or for $\mu > 0$ and $i \geq \mu/2$. In Figs. 1 and 2 we give two examples of this subclass of solutions. Figure 1 shows the flux surfaces of the "D" shaped configuration¹² which can be produced in the PDX machine proposed at the PPL. Figure 2 shows the magnetic surfaces of a Doublet type configuration¹³.

IV. CONCLUSION

We have found a class of exact solutions of the ideal MHD equilibrium equations of an axisymmetric configuration in the region $|z| < r$. The subclass of solutions with μ equal to zero or to an even positive integer is large enough to contain various interesting types of geometries. The advantage of this subclass of solutions is that they converge for every value of r and z and moreover they have very simple finite expressions containing only polynomials in r and z and the trigonometric functions $\sin(bz)$ and $\cos(bz)$.

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Fig. 1. Magnetic surfaces of a "D" shaped configuration.

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Fig. 2. Magnetic surfaces of a "Doublet" type configuration.

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