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THERMAL NEUTRON EXTRAPOLATION DISTANCE
FROM ABSORBING MEDIUM INTO BLACK CONTROL ELEMENT

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Thermal Neutron Extrapolation Distance
from Absorbing Medium into Black Control Element

Raymond L. Murray

1. Introduction.

The use in diffusion theory of an extrapolation distance d at the boundary between a reactor medium and a void or black material has the following basis: Diffusion Theory, by its basic assumptions, is not valid in the vicinity of such boundaries. In order to assign a proper boundary condition, information is borrowed from transport theory as to the nature of the true flux shape. Details on flux distributions are replaced by a statement as to the ratio of flux to its slope at a few mean free paths from the boundary of the form $\phi/\phi' = d$.

In this report, the importance of the effect of finite absorption in the core is assessed, particularly under the conditions in which the thermal source is finite up to the rod surface. Estimates are made of the proper extrapolation distances to use with plane or large cylindrical control elements.

2. Basic Theory.

The simplest version of the extrapolation distance is that of diffusion theory, which in spite of its fundamental weakness gives approximately the correct value of d . A one-dimensional (plane) geometry is presumed.

One specifies zero return current from the void or black region. The neutron current density in the $+Z$ direction by

strict diffusion theory is

$$j_+ = \frac{\phi}{4} - \frac{\lambda_s}{6} \frac{d\phi}{dz} \quad \dots\dots(1)$$

As indicated in Figure 1, $j_+ = 0$. Thus at $Z = 0$,

Figure 1. Geometry of Extrapolation Distance.

The extrapolation distance is

$$|d| = \frac{\phi}{\frac{d\phi}{dz}} = \frac{2}{3} \lambda_s \quad \dots\dots(2)$$

If the transport correction of replacing λ_s by λ_t is made, this of course becomes $|d| = 2/3 \lambda_t$.

The next more sophisticated approach is that of the first order transport theory, P_1 approximation, with zero sources and absorption in the medium. Description of the procedure will help in understanding higher order methods to follow. The first two equations of the spherical harmonics approach* to transport

*

Murray, Nuclear Reactor Physics, Section 9.5.

theory (the P_1 approximation) are

$$\frac{dN_1}{dZ} + \Sigma_a N_0 = \frac{S_T}{v} \quad \dots\dots(3)$$

$$\frac{dN_0}{dZ} + 3\Sigma_T N_1 = 0 \quad \dots\dots(4)$$

where N_0 and N_1 are functions of Z that correspond respectively to the neutron density and $1/v$ times the net current. Also

$$\Sigma_T = \Sigma_a + \Sigma_t \quad \dots\dots(5)$$

where Σ_t is the transport cross section $\Sigma_s(1 - \overline{\cos\theta})$. S_T is the total number of neutrons arising in a unit volume per second. The distribution of flux in angle θ relative to the positive Z axis is, in first order,

$$N(Z, \mu) = \frac{1}{4\pi} [N_0(Z) + 3\mu N_1(Z)] \quad \dots\dots(6)$$

where

$$\mu = \cos\theta \quad \dots\dots(7)$$

A boundary condition that approximates the true statement (that $N(Z, \mu)$ is zero at $Z = 0$ for all angles θ between 0 and $\pi/2$) is

$$\int_0^1 \mu N_1(0, \mu) d\mu \quad \dots\dots(8)$$

This is the same as saying that the return current is zero, on the average. Substituting equation (6) into equation (8)

and integrating yields the relation

$$N_0(0) = 2N_1(0) \quad \dots\dots(9)$$

Zero absorption case. Now assume that Σ_a and S_T are both zero, implying that the medium is a pure scatterer and that neutron sources are far from the boundaries. Then Equation (3) becomes

$$\frac{dN_1}{dz} = 0 \quad \dots\dots(11)$$

Differentiating Equation (4) and inserting Equation (11), one obtains

$$\frac{d^2N_0}{dz^2} = 0 \quad \dots\dots(12)$$

Its solution is clearly

$$N_0(z) = Az + B \quad \dots\dots(13)$$

and invoking Equations (4) and (5) with $\Sigma_a = 0$

$$N_1(z) = -\frac{1}{3\Sigma_t} A \quad \dots\dots(14)$$

Now, apply the boundary condition of Equation (9). This yields

$$B = \frac{2}{3\Sigma_t} A \quad \dots\dots(15)$$

and thus

$$N_0(z) = A \left[z + \frac{2}{3\Sigma_t} \right] \quad \dots\dots(16)$$

The extrapolation distance is therefore,

$$|d| = \frac{N_0(0)}{N_0'(0)} = \frac{2}{3 \Sigma_t} = \frac{2}{3} \lambda_t \quad \dots\dots(17)$$

The result is the same as for transport-modified diffusion theory, but is presented since it is the start of the sequence of improved calculations by spherical harmonics. For isotropic scattering, $\Sigma_t = \Sigma_s$ becomes

$$d = \frac{2}{3 \Sigma_s}$$

and

$$d = 0.6667 \lambda_s$$

The derivation of the extrapolation length by P_1 approximation for a plane source distant from the boundary is not reproduced here. The result is, however, $|d| = 2/3 \lambda_t$ again.

These results may be compared with that quoted for rigorous transport theory under the same conditions of zero absorption and no volume sources, i.e. the familiar

$$|d| = 0.7104 \lambda_t \quad \dots\dots(17)$$

and for isotropic scattering

$$d = 0.7104 \lambda_s$$

3. Data from Literature on Extrapolation Distance with Absorption.

Reference may be made to two articles on this subject. Mark* lists values of the extrapolation distance (in units of the scattering mean free path $\lambda_s = 1/\Sigma_s$) for isotropic scattering

*

J. Carson Mark, "The Spherical Harmonics Method II" CRT-338,

May 31, 1945.

and absorption ratios Σ_a/Σ varying from 0 to 1.0. These are listed below:

$\frac{\Sigma_a}{\Sigma}$	$\frac{d}{\lambda_S}$	$\frac{\Sigma_a}{\Sigma}$	$\frac{d}{\lambda_S}$
0	0.7104	.6	0.7300
.1	0.7106	.7	0.7486
.2	0.7112	.8	0.7843
.3	0.7126	.9	0.8590
.4	0.7154	1.0	1.0000
.5	0.7204		

The Reactor Handbook* quotes values also, but extends the results to include anisotropic scattering. From each of these references the conclusion would be drawn that for weak but finite absorption, one should replace λ_t in the standard formula $d = .7104 \lambda_t$ by λ_S . This view is also taken by Weinberg.† Of note is the fact that the asymptotic distribution is expressed as a form that implies that the extrapolation distance is not $N_0(0)/N'_0(0)$, but the point at which $N_0(Z)$ goes to zero, i.e. in our notation

$$N_0(Z) = \frac{\Sigma_S}{\Sigma \nu} \sinh \nu(Z + Z'_0) \quad \dots (18)$$

Here ν is the same as k/Σ , with k given by the transcendental transport equation

$$\frac{k}{\Sigma} = \tanh \frac{k}{\Sigma_S} \quad \dots (19)$$

*

Reactor Handbook, Volume 1, Physics. page 392. Original source, Davison.

†

Weinberg and Noderer, "Theory of Neutron Chain Reaction", AECD-3471. See page I-47.

and $Z_0^?$ is the "extrapolation distance" in units of the total mean free path (not the scattering length as erroneously indicated in RH-1*). This functional form of $N_0(Z)$ will be discussed later in connection with the distributed source problem.

4. Effect of the Thermal Neutron Source.

The transport solutions of Weinberg, of Davison and of Mark are presumably all of the type in which the source of neutrons is located at some distance from the boundary. For example, Weinberg considers the black sphere (page II-29 ff) with this condition. The question may well be raised as to the applicability of such methods to reactors, in which the thermal source extends completely up to the boundary. Qualitatively the extrapolation distance should be somewhat larger, as can be seen from Figure 2.

Figure 2. Comparison of Extrapolation Distances for Distant and Uniform Sources.

*

A check of an approximate derivation by Feynman, "A Theorem and its Application to Finite Tamperers". LA-608 reveals that this unit of measure is correct.

One would expect that the existence of the supply near $Z = 0$ would tend to "hold the flux level up", as sketched.

Finite Absorption and Sources. In order to establish how the extrapolation distance is changed with the reestablishment of a uniform source and with absorption included, the diffusion problem was first solved again, using the original equations (3) and (4). Differentiating Equation (4) and inserting in Equation (3) gives

$$-\frac{1}{3\Sigma_T} \frac{d^2 N_0}{dz^2} + \Sigma_a N_0 = \frac{S_T}{v} \quad \dots\dots(20)$$

Its solution is straightforward, being

$$N_0(Z) = A e^{-\kappa Z} + \frac{S}{v\Sigma_a} \quad \dots\dots(21)$$

where

$$\kappa = \sqrt{3\Sigma_a\Sigma_T} \quad \dots\dots(22)$$

The positive exponential has been omitted to preserve finite flux at $Z = \infty$.

Also from Equation (5)

$$N_1(Z) = + \frac{\kappa A}{3\Sigma_T} e^{-\kappa Z} \quad \dots\dots(23)$$

To determine A, invoke Equation (9) again. This yields

$$A = \frac{\frac{S_T}{v\Sigma_a}}{1 + \frac{2\kappa}{3\Sigma_T}} \quad \dots\dots(24)$$

and the complete solution

$$N_0(z) = \frac{S_T}{v \Sigma_T} \left[1 - \frac{e^{-\kappa z}}{1 + 2\kappa/3\Sigma_T} \right]$$

The extrapolation distance becomes

$$|d| = \frac{N_0(0)}{N'_0(0)} = \frac{2}{3\Sigma_T} \quad \dots\dots(25)$$

This would suggest that Σ_T should merely replace Σ_t in the standard extrapolation distance formula. In the special case of isotropic scattering, $\Sigma_t = \Sigma_s$. This reduces to $|d| = \frac{2}{3\Sigma}$

Higher order approximations. Since diffusion theory is inadmissible in the region near the boundary, it cannot be relied on to give accurate results. We turn to the P_3 spherical harmonics method, treated in direct analogy to the first order (P_1) just completed. The appropriate boundary conditions on neutron density are taken to be that integrals of the Legendre polynomials $P_1(\mu)$ and $P_3(\mu)$ over half of space are zero:

$$\int_0^1 \mu N(0, \mu) d\mu = 0 \quad \dots\dots(26)$$

$$\int_0^1 \frac{1}{2}(5\mu^3 - 3\mu)N(0, \mu) d\mu = 0 \quad \dots\dots(27)$$

Now in the P_3 method

$$N(z, \mu) = \frac{1}{4\pi} \left[N_0(z) + 3\mu N_1(z) + \frac{5}{2}(3\mu^2 - 1)N_2(z) + \frac{7}{2}(5\mu^3 - 3\mu)N_3(z) \right] \quad \dots\dots(28)$$

Substituting Equation (28) into Equations (26) and (27) and performing the necessary integrations gives new boundary conditions at $Z = 0$.

$$\frac{N_0(0)}{2} + N_1(0) + \frac{5}{8} N_2(0) = 0 \quad \dots\dots(29)$$

and

$$-\frac{N_0(0)}{4} + \frac{5}{4} N_2(0) + 2N_3(0) = 0 \quad \dots\dots(30)$$

Source at a distance-- P_3 calculation. A procedure outlined by Weinberg, but applicable to a plane source in an infinite medium was extended to describe the plane source in a finite medium. The method consists of devising a finite region near the origin, symmetrically located between the plane boundaries of a slab, in which sources exist. A two-region P_3 solution is performed, after which the width of the source region is allowed to go to zero in such a way as to maintain constant neutron emission.

A set of eight functions involved are:

$$N_0(Z) = A_1 \cosh \kappa_1 Z + A_3 \cosh \kappa_3 Z + \frac{S}{\Sigma_a}$$

$$N_1(Z) = r_{11} A_1 \sinh \kappa_1 Z + r_{13} A_3 \sinh \kappa_3 Z$$

$$N_2(Z) = r_{21} A_1 \cosh \kappa_1 Z + r_{23} A_3 \cosh \kappa_3 Z$$

$$N_3(Z) = r_{31} A_1 \sinh \kappa_1 Z + r_{33} A_3 \sinh \kappa_3 Z$$

$$\bar{N}_0(Z) = B_1 e^{\kappa_1(Z-b)} + B_2 e^{-\kappa_1(Z-b)} + C_1 e^{\kappa_3(Z-b)} + C_2 e^{-\kappa_3(Z-b)}$$

$$\bar{N}_1(Z) = r_{11} B_1 e^{\kappa_1(Z-b)} - r_{11} B_2 e^{-\kappa_1(Z-b)} + r_{13} C_1 e^{\kappa_3(Z-b)} - r_{13} C_2 e^{-\kappa_3(Z-b)}$$

$$\bar{N}_2(Z) = r_{21} B_1 e^{\kappa_1(Z-b)} + r_{21} B_2 e^{-\kappa_1(Z-b)} + r_{23} C_1 e^{\kappa_3(Z-b)} + r_{23} C_2 e^{-\kappa_3(Z-b)}$$

$$\bar{N}_3(Z) = r_{31} B_1 e^{\kappa_1(Z-b)} - r_{31} B_2 e^{-\kappa_1(Z-b)} + r_{33} C_1 e^{\kappa_3(Z-b)} - r_{33} C_2 e^{-\kappa_3(Z-b)}$$

Boundary conditions are: continuity of each $N_i(Z)$ at the interface $Z = b$, and the relations expressed by Equations (29) and (30) at $Z = S$. Six simultaneous equations in the arbitrary coefficients $A_1, A_3, B_1, B_2, C_1, C_2$ are thus developed. Of these B_1 and B_2 are needed to determine d . The final result, after considerable more manipulation even than is suggested by Weinberg's comment on the infinite medium problem "the computation of B_1is straightforward but lengthy", is

$$d = \frac{u + v}{\chi(u - v)} \quad \text{.....(32)}$$

where

$$\begin{aligned} u &= \delta_2 \gamma_1 - \beta_2 \epsilon_1 \\ v &= \beta_1 \epsilon_1 - \delta_1 \gamma_1 \\ \delta_2 &= -\frac{1}{4} + 5r_{21} + 2r_{31} \\ \gamma_1 &= \frac{1}{2} - r_{13} + \frac{5}{8} r_{23} \\ \beta_2 &= \frac{1}{2} + r_{11} + \frac{5}{8} r_{21} \\ \epsilon_1 &= -\frac{1}{4} + \frac{5}{4} r_{23} - 2r_{33} \\ \beta_1 &= \frac{1}{2} - r_{11} + \frac{5}{8} r_{21} \\ \delta_1 &= -\frac{1}{4} + \frac{5}{4} r_{21} - 2r_{31} \end{aligned}$$

and the r_{ij} are listed elsewhere.

A numerical calculation was performed using Equation (32) for the case of isotropic scattering, $\Sigma_t = \Sigma_s$ and for weak absorption $\Sigma_a = 0.1\Sigma$, with $\Sigma = \Sigma_a = \Sigma_s$. The results

≠

Murray, NRP, page 253.

are listed below:

$$\begin{aligned}
 k_1 &= 0.525560 \\
 k_3 &= 2.055185 \\
 r_{11} &= -.190273 \\
 r_{13} &= -.0486574 \\
 r_{21} &= 0.0430583 \\
 r_{23} &= -.464487 \\
 r_{31} &= -.00969845 \\
 r_{33} &= 0.409117 \\
 \beta_1 &= 0.717185 \\
 \beta_2 &= 0.336638 \\
 \delta_1 &= -.176780 \\
 \delta_2 &= -.215574 \\
 \gamma_1 &= 0.258353 \\
 \epsilon_1 &= -1.64884
 \end{aligned}$$

$$d = \frac{0.74132}{\Sigma} = \frac{0.6672}{\Sigma_S}$$

Two conclusions are deduced from this calculation. First that the P_3 method is an improvement on diffusion theory. Consider the following tabulation for this case of $\Sigma_a/\Sigma = 1$.

	d/λ	d/λ_S
DIFFUSION (P_1)	0.6667	0.6000
P_3	0.7413	0.6672
TRANSPORT	0.7896	0.7106

The second is that the transport formula of RH-1 clearly refers to a source at a distance, not a distributed source. A separate calculation was performed by the method for the limiting case of zero absorption, obtaining $d = 0.7051/\Sigma_s$, which is the same as the value obtained by Weinberg.*

Uniform source---P₃ calculation. This physical situation is somewhat more manageable than the previous case, since only a one-region problem is involved. Assumed were uniform source, with finite flux, current, and higher harmonics at infinity, and with boundary conditions (29) and (30) at the surface.

The equations to solve are:

$$\begin{aligned} N_0(Z) &= B e^{-k_1 Z} + D e^{-k_3 Z} + \frac{S_T}{v \Sigma_a} \\ N_1(Z) &= -r_{11} B e^{-k_1 Z} - r_{13} D e^{-k_3 Z} \\ N_2(Z) &= r_{21} B e^{-k_1 Z} + r_{23} D e^{-k_3 Z} \\ N_3(Z) &= -r_{31} B e^{-k_1 Z} - r_{33} D e^{-k_3 Z} \end{aligned} \quad \dots\dots(33)$$

In the same notation as before, the asymptotic extrapolation distance is given by

$$d = \frac{1 + \frac{4(\delta_1 \gamma_1 - \beta_1 \epsilon_1)}{2\epsilon_1 + \gamma_1}}{k_1} \quad \dots\dots(34)$$

* Weinberg and Noderer AECD-3471 Theory of Neutron Chain Reactions.

Again the test case $\Sigma_s = \Sigma_t$, $\Sigma_a/\Sigma = 0.1$ was computed, using Equation (34), with the result

$$d = \frac{0.9441}{\Sigma} = \frac{0.8497}{\Sigma_s}$$

A distinct increase in extrapolation distance is found over the value $d = \frac{.74132}{\Sigma} = \frac{.6672}{\Sigma_s}$ for the distant source. Thus a verification of the expectation that the extrapolation is actually larger as discussed earlier is achieved.

Extension to more general transport theory. One may make a deduction as to the nature of the ultimate form of the extrapolation distance, by the use of an analogy. The flux distribution according to diffusion theory for a distant source may be written in the form

$$N_0(Z) \sim \sinh \kappa(Z - Z_0) \quad \text{.....(35)}$$

where Z_0 is the point at which $N_0(Z)$ is zero, namely

$$\tanh^{-1} \left(- \frac{2 \Sigma_a}{\kappa} \right)$$

In contrast, the solution with a uniform source equation (21) is $N_0(Z) \sim 1 - b e^{-\kappa(Z - Z_0)}$ where b is a function of the material constants.

By analogy, with the result of Davison for a distant source, RH-1, page 392, the asymptotic form for uniform source would be expected to be

$$N_0(Z) \sim 1 - b'e^{-\nu(Z + Z_0')} \quad \text{.....(36)}$$

After these conclusions were reached, an obscure Canadian report not cited in RH-1 was found, in which a discussion of the uniform source with isotropic scattering was given. The asymptotic flux is indeed of the form of the above equation with

$$b' = \sqrt{\frac{2 \frac{\Sigma_a}{\Sigma} (1 - \nu^2)}{\nu^2 - \frac{\Sigma_a}{\Sigma}}} \quad \dots\dots(37)$$

where

$$\frac{\Sigma_s}{\Sigma} = \frac{\nu}{\tanh^{-1} \nu} \quad \dots\dots(38)$$

A formula for the distance x , in units of λ , at which the flux goes to zero was derived as follows:

$$x = \frac{\ln b'}{\nu} - Z_0' \quad \dots\dots(39)$$

where

$$Z_0' \approx \frac{0.7104 \Sigma}{\Sigma_a} \quad \dots\dots(40)$$

A series of calculations were made using Equations (37), (38), (39) and (40) for absorption values in the range $\frac{\Sigma_a}{\Sigma} = 0$ to 0.2, which covers most normal situations in enriched reactors. In Figure 3, the d in units of the total mean free path is given for all cases computed in this report, with distant and uniform sources by the P_1 approximation, the P_3 approximation, and isotropic scattering transport theory. Figure 4 shows the working results expressed in a convenient form. Let d_0 be the extrapolation distance as usually calculated from the rod radius a and λ_t (where $\lambda_t = \lambda_s / (1 - \overline{\cos \theta})$, not the general transport cross section including absorption.) This d_0 is normally found from

the cylindrical rod graph of Davison and Kushneriuk, a copy of which, for large rods, is included in Figure 5. From the value of d/d_0 the correct extrapolation distance d is determined. This approach implies that the source effect is superposed on the effect of curvature. To illustrate, calculate d for a rod of radius $a = 4.0$ cm in a medium with $\lambda_t = 0.5$ cm and $\Sigma_a/\Sigma = 0.18$. Thus $a/\lambda_t = 8.0$. From Figure 5, $d_0/\lambda_t = 0.7432$, or $d_0 = 0.3716$ cm. Entering Figure 4 at $\Sigma_a/\Sigma = 0.18$ $d/d_0 = 1.322$, and finally $d = 0.4912$ cm.

FIGURE 3

EXTRAPOLATION DISTANCE AS A
FUNCTION OF
ABSORPTION

λ = TOTAL MEAN FREE PATH

FIGURE 4

RATIO OF EXTRAPOLATION DISTANCE
TO
 $0.7104 \lambda_c$ WITH UNIFORM
SOURCE

FIGURE 5
CYLINDRICAL EXTRAPOLATION DISTANCE

a = ROD RADIUS

d_0 = EXTRAPOLATION DISTANCE

$$\lambda_c = \frac{\lambda_s}{1 - \cos \theta}$$

d_0/λ_c