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CONVERGENCE AND SUMMABILITY OF ORTHOGONAL SERIES

John Killeen

October 6, 1953

Berkeley, California

TABLE OF CONTENTS

INTRODUCTION	4
I. PRELIMINARIES	8
Measure and Integration	8
Space L_2 ; Convergence in the Mean; The Riesz-Fischer Theorem	9
Orthonormal Systems	13
General Fourier Series in L_2	15
The O-o Notation	19
II. CONVERGENCE ALMOST EVERYWHERE	20
Conditions of the First Kind	20
Conditions of the Second Kind	26
Systems of Convergence	27
The Lebesgue Function and Convergence	31
III. SUMMABILITY ALMOST EVERYWHERE	38
Generalities	38
Conditions of the First Kind for Summability (C, 1)	51
Conditions of the Second Kind for Summability (C, 1)	53
Systems of Convergence and Linear Methods of Summation	54
The Lebesgue Function and Summability (C, 1)	57

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ABSTRACT

Consider an orthogonal series

$$\sum_{n=1}^{\infty} a_n \phi_n(x)$$

where $\{\phi_n(x)\}$ is an arbitrary orthonormal system for the interval (a, b) and $\{a_n\}$ is a sequence of real numbers. This report is a collection of theorems which give sufficient conditions for the convergence almost everywhere in (a, b) and summability a. e. in (a, b) of the above series. These theorems were established by Menchoff, Kaczmarz, Orlicz, Rademacher, and Zygmund.

This investigation was performed as part of the qualifying examination for the Ph. D. degree in Mathematics at the University of California.

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October 6, 1953

INTRODUCTION

We shall consider orthogonal series

$$\sum_{n=1}^{\infty} a_n \phi_n(x) \quad (1)$$

where $\{\phi_n(x)\}$ is an arbitrary orthonormal system for the interval (a, b) and $\{a_n\}$ is a sequence of real numbers. The first theorem concerning the convergence of such series is that of Riesz-Fischer, which states that if $\sum_{n=1}^{\infty} a_n^2 < \infty$, then there exists a function $f(x)$ belonging to the class $L_2(a, b)$ to which series (1) converges in the mean, that is

$$\lim_{n \rightarrow \infty} \int_a^b [f(x) - S_n(x)]^2 dx = 0$$

where $S_n(x) = \sum_{k=1}^n a_k \phi_k(x)$. Furthermore the numbers a_n are the Fourier coefficients of $f(x)$ with respect to $\{\phi_n(x)\}$, i. e.

$$a_n = \int_a^b f(x) \phi_n(x) dx.$$

There exists no general theory, independent of the form of the functions $\phi_n(x)$, relating to the convergence at a point, x , of series (1) in analogy with Fourier series. There exists, however, a theory of convergence, almost everywhere in the interval (a, b) . There is then a collection of theorems giving sufficient conditions for the convergence almost everywhere and summability almost everywhere of series (1). These are the theorems that are studied in this report.

The conditions for convergence almost everywhere are of two kinds: the first involves the convergence almost everywhere for a given order of terms; the second does not depend on the order of the terms.

The conditions of the first kind are of the form that if $\sum_{n=1}^{\infty} a_n^2 w(n)$ converges, where $w(n)$ is a function tending to $+\infty$ with n , then the series (1) converges almost everywhere. The first of this type was given by Weyl where $w(n) = n^{1/2}$. Hobson replaced the exponent $1/2$ by the arbitrary positive number α . Plancherel found that if $w(n) = (\log n)^3$ the series (1) converges almost everywhere; and finally Menchoff and Rademacher found that if $\sum_{n=1}^{\infty} a_n^2 (\log n)^2 < \infty$ then series (1) converges almost everywhere. Menchoff showed that this is final, in the sense that $(\log n)^2$ cannot be replaced by any function of n which is $o[(\log n)^2]$, so long as the orthonormal system is not specialized.

In the conditions of the second kind Menchoff established that $\sum_{n=1}^{\infty} |a_n|^{2-\epsilon} < \infty$, $0 < \epsilon < 2$, is a sufficient condition which is a special case of the more general condition $\sum w^2(\log |\log |a_n||) (\log n)^2 a_n^2 < \infty$ where w is an increasing function. Orlicz has replaced the w^2 by w in the above condition and in addition has shown $\sum_{n=1}^{\infty} w(n)(\log n)^2 a_n^2 < \infty$ to be a sufficient condition for the convergence almost everywhere of series (1) for any order of terms.

The existence of orthogonal series everywhere divergent leads to the problem of summability almost everywhere by a given linear method. With respect to summability we can also distinguish two kinds of conditions; however Orlicz proved that conditions of the second kind, which imply summability almost everywhere for any order of terms, also imply convergence almost everywhere for any order of terms.

For the conditions of the first kind for the method $(C, 1)$ Weyl has shown $\sum a_n^2 \log n < \infty$ to be sufficient; Kaczmarz replaced the $\log n$ by $(\log n)^{1/2}$ and finally Menchoff has established the condition $\sum a_n^2 (\log \log n)^2 < \infty$ as the least restrictive.

In addition, Kaczmarz and Zygmund have shown that a series summable almost everywhere by Poisson's method is also summable almost everywhere by every method (C, ϵ) , $\epsilon > 0$.

By a theorem of Menchoff there exists for each linear method of summation an orthogonal series that (for which $\sum a_n^2 < \infty$) is not summable by that method. Menchoff has shown, however, that for any positive function $w(n)$ such that $\lim_{n \rightarrow \infty} w(n) = +\infty$ there exists a linear method of summation for which $\sum_{n=1}^{\infty} a_n^2 w(n) < \infty$ implies the summability almost everywhere of series (1) by that method.

An orthonormal system is a system of convergence when the convergence of the series $\sum_{n=1}^{\infty} a_n^2$ implies the convergence almost everywhere in (a, b) of series (1). The existence of systems of convergence has been shown by Haar and Rademacher. Menchoff shows that each infinite orthonormal system contains a part which is an infinite system of convergence. As application of this theorem he proves a theorem on convergence and several theorems on summability by linear methods. These theorems concern the changing of the order of the functions in an orthonormal system.

The conditions mentioned thus far concern only the coefficients a_n ; the functions $\phi_n(x)$ are all made arbitrary. The question arises: what are the conditions on the coefficients if one makes hypotheses on the system $\{\phi_n(x)\}$? Menchoff and Kolmogoroff have shown that with the hypothesis $|\phi_n(x)| < A$ the condition $\sum a_n^2 w(n) < \infty$ where $w(n) = o(\log n)$ is not sufficient for convergence almost everywhere of series (1). To study this in a general way we define the Lebesgue functions for the system $\{\phi_n(x)\}$:

$$\rho_n(x) = \int_a^b \left| \sum_{k=1}^n \phi_k(x) \phi_k(t) \right| dt.$$

1. For all orthogonal systems

$$\rho_n^2(x) = o(n \log^{1+\epsilon} n) \text{ almost everywhere, } \epsilon > 0$$

in the case where $|\phi_n(x)| < A$

$$\rho_n^2(x) = O(n) \text{ almost everywhere.}$$

2. If $\rho_n(x) \leq F(x)$, $F(x)$ integrable, all n then series (1) converges almost everywhere.
3. If $\rho_n(x) = O[u^2(n)]$ almost everywhere and $\sum_{n=1}^{\infty} a_n^2 u^2(n) < \infty$ then series (1) converges almost everywhere.

The analagous theorems for summability $(C, 1)$ are also established.

I. PRELIMINARIES

Measure and Integration

The integral in the Lebesgue sense will be used throughout this report. The elements of the Lebesgue theory of measure and integration as covered in Titchmarsh's Theory of Functions, Ch. X-XII will be assumed. However, for convenience we shall state a few necessary definitions and theorems.

A function which is bounded and measurable over a set E is integrable (L) over E .

Let $f(x)$ be an unbounded measurable function on a set E and let $f(x) \geq 0$

Define $[f(x)]_n = \begin{cases} f(x) & \text{for } f(x) \leq n \\ n & \text{for } f(x) > n \end{cases}$

Then $[f(x)]_n$ is bounded and measurable, and therefore integrable (L). Now define the integral of $f(x)$ over the set E by

$$\int_E f(x) dx = \lim_{n \rightarrow \infty} \int_E [f(x)]_n dx.$$

We say that $f(x)$ is summable if the above limit exists, that is, is finite. The integral of a negative function may be defined in a similar way. In the general case, let $f(x) \geq 0$ in E_1 , $f(x) < 0$ in E_2 , then

$$\int_E f(x) dx = \int_{E_1} f(x) dx + \int_{E_2} f(x) dx.$$

We denote by $L(E)$ the class of all functions which are summable on E and write $f(x) \in L(E)$.

A set of measure zero is called a null set. Null sets are irrelevant in the theory of integration. If $f(x)$ and $g(x)$ differ only in a null set we say that they are equivalent and write

$$f(x) = g(x) \text{ almost everywhere.}$$

A relation that holds for all points in a set E except for those points in a null set of E is said to hold almost everywhere in E .

We write $m[E]$ for the measure of a set E .

Other classes of functions considered are B, C, C_k , and V -- the classes of bounded functions, continuous functions, functions with k continuous derivatives, and functions of bounded variation.

Lebesgue's Convergence Theorem. If $f_n(x)$ is a sequence of measurable functions such that $|f_n(x)| \leq F(x)$, $F(x) \in L(E)$, for all n, and $f_n(x) \rightarrow f(x)$ almost everywhere then $f(x) \in L(E)$ and

$$\int_E f(x) dx = \lim_{n \rightarrow \infty} \int_E f_n(x) dx.$$

Fatou's Lemma. If $f_n(x) \geq 0$ for all n and $f_n(x) \rightarrow f(x)$ almost everywhere then

$$\int_E f(x) dx \leq \liminf_{n \rightarrow \infty} \int_E f_n(x) dx.$$

Egoroff's Theorem. If a sequence of measurable functions converges to a finite limit almost everywhere on a set E, then, given $\delta > 0$, we can find a set of measure greater than $mE - \delta$ in which the sequence converges uniformly.

Fubini's Theorem. If $f(x, y)$ is integrable (L) then

$$\int dx \int f(x, y) dy = \int dy \int f(x, y) dx = \iint f(x, y) dx dy.$$

Weierstrass's Theorem. If $f(x)$ is continuous in (a, b) , then there is a polynomial $P(x)$ such that

$$|f(x) - P(x)| < \epsilon \text{ for } a \leq x \leq b.$$

Space L_2 ; Convergence in the Mean; the Riesz-Fischer Theorem

Consider real functions $f(x)$ which are measurable on a set E and whose squares $[f(x)]^2$ are summable on E. The class of all such functions will be called $L_2(E)$. From Schwarz's inequality it follows that

$$\left(\int_E [f(x)]_n dx \right)^2 \leq \int_E [f(x)]_n^2 dx \cdot \int_E dx \leq mE \int_E f^2(x) dx.$$

Hence if $f(x) \in L_2(E)$ it also belongs to $L(E)$. Similarly we have

$$\left(\int_E f(x) g(x) dx \right)^2 \leq \int_E f^2(x) dx \cdot \int_E g^2(x) dx.$$

It is convenient to define the inner product of two functions in $L_2(E)$ as

$$(f, g) = \int_E f(x) g(x) dx$$

and the norm of a function $f(x) \in L_2(E)$

$$\|f\| = \sqrt{(f, f)} = \sqrt{\int_E [f(x)]^2 dx}.$$

Then Schwarz's inequality becomes

$$(f, g) \leq \|f\| \cdot \|g\|.$$

From Minkowski's inequality

$$\|f + g\| \leq \|f\| + \|g\|$$

we see that if $f(x)$ and $g(x)$ belong to $L_2(E)$ then $f(x) + g(x)$ also belongs to $L_2(E)$. From the Minkowski inequality one obtains the triangle inequality

$$\|f_1 - f_3\| \leq \|f_1 - f_2\| + \|f_2 - f_3\|.$$

This suggests defining the "distance" between two functions in $L_2(E)$ by

$$\delta(f, g) = \|f - g\|.$$

So we shall consider the space L_2 of "points" $f(x) \in L_2$ with the metric defined above.

Consider a sequence of functions $\{f_n(x)\}$ which belong to $L_2(E)$. We say that the sequence $\{f_n\}$ converges in the mean to $f(x)$ when $\lim_{n \rightarrow \infty} \|f - f_n\| = 0$.

This is also called strong convergence or convergence in L_2 and is written $f_n \rightarrow f(L_2)$.

The most useful theorem concerning this notion is the following analogue of Cauchy's theorem for ordinary convergence.

Theorem 1 (Riesz-Fischer Theorem). In order that f_n should converge in the mean to a function $f \in L_2(E)$, it is necessary and sufficient that

$$\lim_{m, n \rightarrow \infty} \|f_m - f_n\| = 0.$$

There is then a subsequence (n_k) such that $f_{n_k} \rightarrow f$ almost everywhere in E .

Proof: The condition is necessary; in fact, it follows from the triangle inequality that

$$\|f_n - f_m\| \leq \|f - f_n\| + \|f - f_m\| \rightarrow 0.$$

In order to see that it is also sufficient, choose the indices

$$m_1 < m_2 < \dots$$

in such a fashion that for $n > m_k$ we have $\|f_n - f_{m_k}\| < 2^{-k}$. Then, in particular,

$\|f_{m_{k+1}} - f_{m_k}\| < 2^{-k}$ and since from Schwarz's inequality,

$$\int_{E'} |f_{m_{k+1}}(x) - f_{m_k}(x)| dx \leq \sqrt{mE'} \|f_{m_{k+1}} - f_{m_k}\| \leq \sqrt{mE'} 2^{-k}$$

for all subsets $E' \subset E$ with $mE' < \infty$, the series

$$\sum_{k=1}^{\infty} \int_{E'} |f_{m_{k+1}}(x) - f_{m_k}(x)| dx$$

is convergent. By the theorem of Beppo Levi on the integration term by term of a series we have then that the series

$$\sum_{k=1}^{\infty} (f_{m_{k+1}}(x) - f_{m_k}(x))$$

converges absolutely almost everywhere and furthermore the sequence $f_{m_k}(x)$ converges almost everywhere to a limit $f(x)$. From the inequality

$$\begin{aligned} \|f_{m_k}\| &\leq \|f_{m_1}\| + \|f_{m_k} - f_{m_1}\| \\ &\leq \|f_{m_1}\| + 2^{-1} \end{aligned}$$

where it is seen that the integral of $|f_{m_k}|^2$ is bounded, we get then from Fatou's lemma that $f(x) \in L_2(E)$. Furthermore on applying the same lemma to the sequence $\{f_{m_k} - f_n\}$ with n fixed, $n > m_r$, and observing that when $k > r$, we have

$$\|f_{m_k} - f_n\| \leq \|f_{m_k} - f_{m_r}\| + \|f_n - f_{m_r}\| < 2^{-r+1}$$

then

$$\|f - f_n\| \leq 2^{-r+1}.$$

Finally, for $n \rightarrow \infty$, r can increase indefinitely, which gives

$$\|f - f_n\| \rightarrow 0.$$

The limit $f(x)$ is determined to a set of measure 0. If f^* is also a limit in the mean of the same sequence then we have

$$\|f - f^*\| \leq \|f - f_n\| + \|f_n - f^*\| \rightarrow 0$$

$$\therefore \|f - f^*\| = 0.$$

Strong convergence does not imply convergence almost everywhere or convergence at a point nor is the converse true, but it follows from Theorem 1 that if

$$f_n \rightarrow f(L_2) \text{ and } f_n \rightarrow g \text{ (almost everywhere) then } f = g \text{ (almost everywhere).}$$

One of the consequences of strong convergence of f_n to f is the relation

$$(f_n, g) \rightarrow (f, g)$$

where $g(x) \in L_2$. This follows from

$$|(f, g) - (f_n, g)| = |(f - f_n, g)| \leq \|f - f_n\| \|g\| \rightarrow 0.$$

But the reverse is not true, i. e. $(f_n, g) \rightarrow (f, g)$ for all g does not imply strong convergence. This relation holding for all $g \in L_2$ and also the equivalent relation $(g, f_n) \rightarrow (g, f)$ is called weak convergence. One of the essential differences between strong convergence and weak convergence is that the first implies that $\|f_n\| \rightarrow \|f\|$ and this does not follow from weak convergence. That this relation does follow from strong convergence is an immediate consequence of

$$\|f_n\| \leq \|f_n - f\| + \|f\| \text{ and } \|f\| \leq \|f - f_n\| + \|f_n\|.$$

However, weak convergence coupled with the relation $\|f_n\| \rightarrow \|f\|$ implies strong convergence, in fact it follows that

$$\begin{aligned} \|f - f_n\|^2 &= \|f\|^2 - (f, f_n) - (f_n, f) + \|f_n\|^2 \\ &\rightarrow \|f\|^2 - (f, f) - (f, f) + \|f\|^2 = 0. \end{aligned}$$

Another consequence of strong convergence is asymptotic convergence or convergence in measure. The sequence of functions $\{f_n(x)\}$ measurable on a set E is said to converge in measure to a function $f(x)$ when for every $\epsilon > 0$

$$\lim_{n \rightarrow \infty} m_E \left(|f_n(x) - f(x)| > \epsilon \right) = 0$$

A sequence f_n convergent in measure to the function f contains a sub-sequence which converges almost everywhere to f . A sequence of functions which is convergent in the mean to a function is also convergent in measure to the same function.

Orthonormal Systems

A system of functions $\{\phi_n(x)\}$ belonging to $L_2(a, b)$ is said to form an orthonormal (ON) system in (a, b) if

$$(\phi_m, \phi_n) = \int_a^b \phi_m(x) \phi_n(x) dx = 0 \quad m \neq n$$

and

$$(\phi_n, \phi_n) = \|\phi_n\|^2 = \int_a^b \phi_n^2(x) dx = 1.$$

If $f(x) \in L_2(a, b)$ then $C_n = (f, \phi_n)$ are called the Fourier coefficients of $f(x)$ with respect to $\{\phi_n\}$ and the series $\sum_{n=1}^{\infty} C_n \phi_n(x)$ is its Fourier series written $f \sim \sum C_n \phi_n$. Any series $\sum_{n=1}^{\infty} a_n \phi_n(x)$ where the $\{\phi_n\}$ is an ON system and $\{a_n\}$ is a sequence of real numbers is called a series of orthogonal functions or orthogonal series. It will not necessarily be a Fourier series nor will it necessarily be convergent or summable. These are the problems to be considered in this report for any ON system.

A polynomial in the ϕ_n is a finite linear combination

$$P(\phi) = a_1 \phi_1 + \dots + a_n \phi_n$$

If no $P(\phi)$ is null unless all its coefficients vanish, then the ϕ_n are called linearly independent. There is a process of orthogonalization by which we can derive an ON system $\{\phi_n\}$ in (a, b) from any linearly independent system $\{\psi_n\}$ of functions that belong to $L_2(a, b)$. This is as follows: set

$$\phi_1(x) = \frac{\psi_1(x)}{\|\psi_1\|}$$

$$\phi_2(x) = \frac{\psi_2(x) - (\psi_2, \phi_1) \phi_1(x)}{\|\psi_2(x) - (\psi_2, \phi_1) \phi_1(x)\|}$$

$$\phi_n(x) = \frac{\psi_n(x) - \sum_{k=1}^{n-1} (\psi_n, \phi_k) \phi_k(x)}{\|\psi_n(x) - \sum_{k=1}^{n-1} (\psi_n, \phi_k) \phi_k(x)\|}$$

A system of functions $\{\psi_n(x)\}$, not necessarily orthogonal, is said to be complete in $L_2(a, b)$ if there is no nonzero function of $L_2(a, b)$ orthogonal to every ψ_n , i. e. if

$$(f, \psi_n) = 0 \text{ all } n \text{ then } f(x) = 0 \text{ almost everywhere.}$$

We now need the concept of dense sets in a metric space. Consider a metric space M and a subset of points S . The set S is said to be dense in M if given any $p \in M$ and any $\epsilon > 0$ there exists a $q \in S$ such that $\delta(p, q) < \epsilon$. A system of functions $\{\psi_n\}$ belonging to L_2 is said to be closed in L_2 if the system of polynomials $P(\psi)$ is dense in L_2 , i. e. for any $f \in L_2$ and any $\epsilon > 0$

$$\|f - P\| < \epsilon.$$

We already know from Weierstrass's theorem that the system $\{x^n\}$ is closed in C , the space of all continuous functions.

If the system $\{\psi_n\}$ is complete or closed in L_2 then the orthogonalized system $\{\phi_n\}$ is complete or closed in L_2 .

General Fourier Series in L_2

Let $\{\phi_n(x)\}$ be an ON system belonging to $L_2(a, b)$ and let $f(x)$ be any function of $L_2(a, b)$. Furthermore, let $C_n = (f, \phi_n)$ be the Fourier coefficients of f and call

$$f_n = \sum_{k=1}^n C_k \phi_k$$

the Fourier polynomial of degree n of f . Now let $\sum_{k=1}^n \gamma_k \phi_k$ be any polynomial in the $\{\phi_n\}$. We have the following theorem.

Theorem 2. Among all polynomials $\sum_{k=1}^n \gamma_k \phi_k$ of degree n , that

which gives the best mean square approximation to f is the Fourier polynomial f_n .

Proof:

$$\begin{aligned} \left\| f - \sum_{k=1}^n \gamma_k \phi_k \right\|^2 &= \left(f - \sum_{k=1}^n \gamma_k \phi_k, f - \sum_{j=1}^n \gamma_j \phi_j \right) \\ &= (f, f) - \sum_{j=1}^n \gamma_j (f, \phi_j) - \sum_{k=1}^n \gamma_k (\phi_k, f) + \sum_{k=1}^n \gamma_k^2 \\ &= \|f\|^2 - \sum_{k=1}^n (f, \phi_k)^2 + \sum_{k=1}^n [(f, \phi_k) - \gamma_k]^2 \end{aligned}$$

A minimum of the above expression is attained when

$$\gamma_k = (f, \phi_k) = C_k,$$

so the theorem is proved.

On writing this value in the above equation we get Bessel's identity,

$$\left\| f - \sum_{k=1}^n (f, \phi_k) \phi_k \right\|^2 = \|f\|^2 - \sum_{k=1}^n (f, \phi_k)^2.$$

The first term is nonnegative so we get

$$\sum_{k=1}^n (f, \phi_k)^2 \leq \|f\|^2$$

which is called Bessel's inequality. The above inequality holds as n increases indefinitely so we have for infinite ON systems

$$\sum_{k=1}^{\infty} (f, \phi_k)^2 \leq \|f\|^2.$$

Now let $\{\phi_n(x)\}$ be an ON system for (a, b) and let $\{a_n\}$ be any sequence of real numbers. Denote by $f_n(x)$ the function $f_n(x) = \sum_{j=1}^n a_j \phi_j(x)$.

We have the following theorem:

Theorem 3. (Riesz-Fischer Theorem). If $\sum_{n=1}^{\infty} a_n^2$ converges, there exists an $f(x) \in L_2(a, b)$ such that $\{f_n(x)\}$ converges in the mean to $f(x)$. The

a_k are the Fourier coefficients of f and $\|f\|^2 = \sum_{k=1}^{\infty} a_k^2$.

Proof: If $m > n$ we have

$$\|f_m - f_n\|^2 = \sum_{j=n+1}^m a_j^2 \leq \sum_{j=n+1}^{\infty} a_j^2 \rightarrow 0.$$

It follows by Theorem 1 that there is an $f(x)$ in $L_2(a, b)$ to which $\{f_n\}$ converges in the mean.

Also we have for $n \geq k$

$$(f - f_n, \phi_k) = (f, \phi_k) - a_k.$$

But $(f - f_n, \phi_k)^2 \leq \|f - f_n\|^2 \|\phi_k\|^2 = \|f - f_n\|^2$

Hence $|(f, \phi_k) - a_k|^2 \leq \|f - f_n\|^2, n \geq k.$

So if we let $n \rightarrow \infty$ then

$$a_k = (f, \phi_k)$$

that is the a_k are the Fourier coefficients of f . From Bessel's identity we have

$$\|f\|^2 - \sum_{k=1}^n a_k^2 = \|f - f_n\|^2$$

but the second term tends to zero as $n \rightarrow \infty$ so we have the last point:

$$\|f\|^2 = \sum_{k=1}^{\infty} a_k^2.$$

Theorem 4 (Uniqueness of Riesz-Fischer Theorem). Let $\{\phi_n(x)\}$ be a complete ON system on (a, b) and let $\{a_n\}$ be a sequence of real numbers such that $\sum_{n=1}^{\infty} a_n^2$ converges, then there is one and only one $f(x) \in L_2(a, b)$ which has a_n as its Fourier coefficients.

Proof: By theorem 3 there is such an $f(x)$. Suppose $g(x)$ is another. Then $(f - g, \phi_k) = 0$ all k . But $f - g$ belongs to $L_2(a, b)$ and $\{\phi_n(x)\}$ is complete.

Hence $f(x) - g(x) = 0$ almost everywhere in (a, b) .

Theorem 5 (Corollary). If $\{\phi_n(x)\}$ is a complete ON system on (a, b) and $f(x) \in L_2(a, b)$ then the partial sums $f_n(x) = \sum_{k=1}^n (f, \phi_k) \phi_k(x)$ converge in the mean to $f(x)$.

Proof: By Bessel's inequality $\sum_{k=1}^{\infty} a_k^2 < \infty$. It follows from theorem 3 that there is a $g(x)$ to which $f_n(x)$ converges in the mean. By theorem 4

$$f(x) = g(x) \text{ almost everywhere.}$$

Theorem 6 (Parseval's Identity). If $\{\phi_n(x)\}$ is a complete ON system on (a, b) and $f(x)$ and $g(x)$ belong to $L_2(a, b)$ then

$$(f, g) = \sum_{n=1}^{\infty} (f, \phi_n)(g, \phi_n).$$

Proof: Let $f_n = \sum_{k=1}^n (f, \phi_k) \phi_k$ then

$$(f_n, g) = \sum_{m=1}^n (f, \phi_m)(\phi_m, g) = \sum_{m=1}^n (f, \phi_m)(g, \phi_m).$$

But $f_n \rightarrow f(L_2)$ by theorem 5; so $(f_n, g) \rightarrow (f, g)$ by weak convergence. Therefore

$$(f, g) = \sum_{k=1}^{\infty} (f, \phi_k)(g, \phi_k).$$

From Bessel's inequality it follows that the Fourier coefficients of a function belonging to L_2 tend to zero. If the $\{\phi_n\}$ are subject to further conditions; then we can define the Fourier coefficients of a wider class of functions. If the $\{\phi_n\}$ are uniformly bounded then we have the following theorem.

Theorem 7 (Mercer's Theorem). If $\{\phi_n\}$ is an ON system, and $|\phi_n| < H$, then the Fourier coefficients of any $f \in L$ tend to zero.

Proof: The class of functions in L_2 is dense in L so if $g \in L_2$ then

$$\int |f - g| dx < \epsilon$$

Also $|(f, \phi_n)| \leq |(g, \phi_n)| + \int |f - g| |\phi_n| dx \leq |(g, \phi_n)| + H\epsilon.$

But $(g, \phi_n) \rightarrow 0$ since $g \in L_2$ therefore $|(f, \phi_n)| < 2H\epsilon$ for sufficiently large values of n .

As a result of our previous theorems closure and completeness are equivalent in L_2 .

Theorem 8. A system of functions of $L_2(a, b)$ is closed if and only if it is complete.

Proof: 1. If $\{\phi_n\}$ is complete, and $f \in L_2(a, b)$, then $f_n \rightarrow f(L_2)$ by theorem

5. Since f_n is a polynomial in the $\{\phi_n\}$, we have that $\{\phi_n\}$ is closed in L_2 .

2. Suppose that $\{\phi_n\}$ is closed in L_2 , that $f \in L_2$, and that all the Fourier coefficients of f are 0, i. e., that f is orthogonal to every ϕ_n . Since $\{\phi_n\}$ is closed, there is a sequence of polynomials $P_n(\phi)$ for which

$$\|f - P_n\| \rightarrow 0$$

therefore by theorem 2, $\|f - f_n\| \rightarrow 0$. But $f_n = 0$ since $(f, \phi_n) = 0$, so

$$\|f\| = 0.$$

Therefore $f \equiv 0$ almost everywhere and $\{\phi_n\}$ is complete.

The O-o Notation

In the later work we shall frequently use the following notation. Let $\psi(n) > 0$ be defined for $n = 1, 2, 3, \dots$ and let $\phi(n)$ also be a function of n .

Then if

$$\frac{|\phi(n)|}{\psi(n)} < K, \quad K > 0$$

independent of n we say that $\phi(n) = O[\psi(n)]$. If $\lim_{n \rightarrow \infty} \frac{\phi(n)}{\psi(n)} = 0$ then we say that

$\phi(n) = o[\psi(n)]$.

II. CONVERGENCE ALMOST EVERYWHERE

Conditions of the First Kind

We shall consider an ON system $\{\phi_n(x)\}$ in the interval $[0,1]$ and the series

$$\sum_{n=1}^{\infty} a_n \phi_n(x) \quad (1)$$

where

$$\sum_{n=1}^{\infty} a_n^2 < \infty \quad (2)$$

The main theorem on convergence is the following:

Theorem 9 (Menchoff (1) and Rademacher (1)). If the series $\sum_{n=1}^{\infty} a_n^2 (\log n)^2$ is convergent then the series $\sum_{n=1}^{\infty} a_n \phi_n(x)$ converges almost everywhere in $(0, 1)$.

Proof: (due to Chen (1))

Denote $S_n(x) = \sum_{i=1}^n a_i \phi_i(x)$

Let $D(x, \ell, s) = S_{2^m + (s+1)2^\ell}(x) - S_{2^m + s2^\ell}(x)$ where m, s, ℓ are positive integers such that $0 \leq \ell < m$, $0 \leq s < 2^{m-\ell}$.

1. If δ be a prescribed positive number, then there is a positive integer $m_0(x)$ such that

$$m \sum_{\ell, s} [D(x, \ell, s)]^2 < \delta^2 \quad \text{for } m \geq m_0,$$

where $x \in T \subset (0,1)$ and the $m[T] = 1$.

Proof: Put

$$u_m(x) = \int_0^x m \sum_{\ell, s} (D(x, \ell, s))^2 dx,$$

where $0 < x < 1$.

Then

$$\begin{aligned} u_m(x) &\leq m \int_0^1 \sum_{\ell, s} (D(x, \ell, s))^2 dx \\ &= m \sum_{\ell, s} \int_0^1 (D(x, \ell, s))^2 dx \end{aligned}$$

$$\begin{aligned}
 &= m \sum_{\ell, s} \frac{2^{m+(s+1)2^\ell}}{2^{m+s2^\ell} + 1} a_v^2 \\
 &= m \sum_{\ell} \sum_{\frac{2^{m+1}}{2^{m+1}}} \frac{2^{m+1}}{2^{m+1}} a_v^2 \\
 &= m^2 \sum_{\frac{2^{m+1}}{2^{m+1}}} \frac{2^{m+1}}{2^{m+1}} a_v^2 \leq \sum_{\frac{2^{m+1}}{2^{m+1}}} \frac{2^{m+1}}{2^{m+1}} a_v^2 (\log_2 v)^2.
 \end{aligned}$$

Therefore the series $\sum_{m=1}^{\infty} u_m(x)$ converges in $(0, 1)$. We have the following theorem due to Fubini (Riesz-Nagy page 12): Let $\sum_{m=1}^{\infty} u_m(x) = s(x)$ be a convergent series of monotone non-decreasing functions defined on $a \leq x \leq b$, then $\sum_{m=1}^{\infty} u'_m(x) = s'(x)$ almost everywhere, that is the series is differentiable, term by term, almost everywhere in (a, b) . So the series

$$\sum_{m=1}^{\infty} \left[m \sum_{\ell, s} (D(x, \ell, s))^2 \right]$$

converges almost everywhere in $(0, 1)$. Consequently, there exists a set T such that $m[T] = 1$, and

$$\lim_{m \rightarrow \infty} \left[m \sum_{\ell, s} (D(x, \ell, s))^2 \right] = 0 \quad \text{for } x \in T$$

which establishes 1.

2. If $2^m \leq n < 2^{m+1}$, then

$$|S_n(x) - S_{2^m}(x)| < \delta \text{ for } m \geq m_0,$$

where $x \in T$.

Proof: Write

$$n = \delta_0 + \delta_1 2 + \delta_2 2^2 + \dots + \delta_i 2^i + \dots + \delta_m 2^m$$

where $\delta_i = 0$ or 1 and $\delta_m = 1$.

We have

$$\begin{aligned}
 S_n(x) - S_2m(x) &= \delta_0 D(x, 0, \delta_1 \cdot 2 + \dots + \delta_{m-1} \cdot 2^{m-1}) \\
 &+ \dots \\
 &+ \delta_i D(x, i, \delta_{i+1} \cdot 2 + \dots + \delta_{m-1} \cdot 2^{m-i-1}) \\
 &+ \dots \\
 &+ \delta_{m-2} D(x, m-2, \delta_{m-1} \cdot 2) \\
 &+ \delta_{m-1} D(x, m-1, 0) \\
 S_n(x) - S_2m(x) &= \sum_{i=0}^{m-1} \delta_i D(x, i, s_i).
 \end{aligned}$$

Applying Cauchy's inequality:

$$\begin{aligned}
 [S_n(x) - S_2m(x)]^2 &\leq \sum_{i=0}^{m-1} \delta_i^2 \sum_{i=0}^{m-1} (D(x, i, s_i))^2 \\
 &\leq m \sum_{\ell, s} (D(x, \ell, s))^2
 \end{aligned}$$

Hence if $x \in T$

$$[S_n(x) - S_2m(x)]^2 < \delta^2 \quad \text{for } m \geq m_0$$

by 1. So

$$|S_n(x) - S_2m(x)| < \delta \quad \text{for } m \geq m_0$$

and $x \in T$ which proves 2.

3. If $\delta > 0$, then there is an integer $m_1(x)$ such that

$$|S_2 m(x) - S_2 m'(x)| < \delta \text{ for } m, m' \geq m_1$$

where $x \in R \subset (0, 1)$ and $m[R] = 1$.

Proof: Since $\sum_{n=1}^{\infty} a_n^2 < \infty$, there exists a function $f(x) \in L_2(0, 1)$ such that

$$\int_0^1 [f(x)]^2 dx = \sum_{n=1}^{\infty} a_n^2, \quad \int_0^1 f(x) \phi_n(x) dx = a_n$$

Consider the series

$$\sum_{m=1}^{\infty} \int_0^1 [f(x) - S_2 m(x)]^2 dx, \quad 0 < x < 1$$

The term

$$\int_0^x [f - S_2 m]^2 dx \leq \int_0^1 [f - S_2 m]^2 dx = \sum_{2^m+1}^{\infty} a_n^2,$$

and

$$\begin{aligned} \sum_{m=1}^{\infty} \sum_{2^m+1}^{\infty} a_n^2 &= \sum_{m=1}^{\infty} m \sum_{2^m+1}^{2^{m+1}} a_n^2 \leq \sum_{m=1}^{\infty} \sum_{2^m+1}^{2^{m+1}} a_n^2 \log n = \\ &= \sum_{n=3}^{\infty} a_n^2 \log n \end{aligned}$$

which is convergent.

We thus have proved the convergence of the series

$$\sum_{m=1}^{\infty} \int_0^x [f(x) - S_2 m(x)]^2 dx \quad (0 < x < 1).$$

By the same theorem of Fubini used in 1, we have that

$$\sum_{m=1}^{\infty} [f(x) - S_2^m(x)]^2$$

converges a. e. in $(0, 1)$. So we have

$$\lim_{m \rightarrow \infty} [f(x) - S_2^m(x)]^2 = 0$$

for $x \in R$ where $m[R] = 1$. The remaining part of the proof is easily seen from

$$\begin{aligned} |S_2^m(x) - S_2^{m'}(x)| &\leq |f(x) - S_2^m(x)| + |f(x) - S_2^{m'}(x)| \\ &< \frac{\delta}{2} + \frac{\delta}{2} \text{ for } m, m' \geq m_1 \end{aligned}$$

$$|S_2^m(x) - S_2^{m'}(x)| < \delta \text{ for } m, m' \geq m_1$$

4. Let $E = R \cap T$, then $m[E] = 1$.

Take the integer $n_0(x) = \max(2^{m_0(x)}, 2^{m_1(x)})$ for $x \in E$, then for all pairs $n, n' \geq n_0(x)$ we have

$$2^m \leq n < 2^{m+1}, \quad 2^{m'} \leq n' < 2^{m'+1}$$

where $m, m' \geq \max(m_0(x), m_1(x))$

Since $|S_n(x) - S_2^m(x)| < \delta$ by 2°

$$|S_2^m(x) - S_2^{m'}(x)| < \delta \text{ by } 3^\circ$$

$$|S_2^{m'}(x) - S_{n'}(x)| < \delta \text{ by } 2^\circ$$

it follows that

$$|S_n(x) - S_{n'}(x)| < 3\delta \text{ for } n, n' \geq n_0$$

for $x \in E$. Therefore the series

$$\sum_{n=1}^{\infty} a_n \phi_n(x)$$

converges a. e. in $(0, 1)$.

Corollary

If $\sum_{n=1}^{\infty} a_n^2 (\log n)^2 < \infty$ then we have a. e.

$$\sum_{k=1}^n a_k \phi_k(x) = o(\log n)$$

Proof: From Theorem 9 we have that

$$\sum_{n=2}^{\infty} \frac{a_n \phi_n(x)}{\log n} \text{ converges a. e.}$$

by Kronecker's lemma we have a. e.

$$\frac{\sum_{k=1}^n \log k \frac{a_k \phi_k(x)}{\log k}}{\log n} \rightarrow 0 \text{ as } n \rightarrow \infty$$

$$S_n(x) = o(\log n) \text{ a. e.}$$

The question arises whether the coefficient $(\log n)^2$ in the preceding theorem is the least restrictive condition. The answer is given by the following theorem, which says that as long as the ON system remains arbitrary the coefficient $(\log n)^2$ cannot be replaced by a function, $W(n)$, the growth of which is inferior to $(\log n)^2$.

Theorem 10 (Menchoff (1)). If the increasing sequence of numbers $W(n) = o(\log^2 n)$, then there exists an ON system $\{\phi_n(t)\}$ and a sequence of numbers $\{a_n\}$ such that:

- a) $\sum_{n=1}^{\infty} a_n^2 W(n) < \infty$
- b) the series $\sum_{n=1}^{\infty} a_n \phi_n(t)$ is a. e. divergent.

As a consequence of the preceding theorem we obtain immediately
Theorem 11 (Menchoff (1)). There exists an ON system $\{\phi_n(x)\}$ such that $\sum_{n=1}^{\infty} a_n \phi_n(x)$ diverges when $\sum_{n=1}^{\infty} a_n^2$ converges.

Theorems 9 and 10 settle completely the question of conditions of the first kind. We may summarize these results in the following way. Let $W(n)$ be called a "function of Weyl" or a multiplier of convergence if the convergence of $\sum_{n=1}^{\infty} a_n^2 W(n)$ implies the convergence a. e. of $\sum_{n=1}^{\infty} a_n \phi_n(x)$. Then the function $W(n)$ is a "function of Weyl" if $(\log n)^2 = O[W(n)]$; it is not a "function of Weyl" if $W(n) = o[(\log n)^2]$.

Conditions of the Second Kind (Unconditional Convergence)

We pass now to the problem of the conditions of the second kind, that is, conditions implying the convergence a. e. of the series considered for all orders of terms.

Theorem 12 (Orlicz (1)). If (1) $\mu(n)$ is an increasing function tending to ∞ , (2) there exists a constant c and a set of natural numbers $\{n_i\}$ such that

$$\sum_{i=1}^{\infty} \frac{1}{\mu(n_i)} < \infty, \quad \log n_{i+1} < c \log n_i$$

(3) $\sum_{n=1}^{\infty} a_n^2 \mu(n) (\log n)^2 < \infty$ then the series $\sum_{n=1}^{\infty} a_n \phi_n(x)$ is convergent a. e. in (a, b) for any order of its terms.

The proof of the above theorem can be found in Kaczmarz-Steinhaus (1), page 171.

As a consequence of the preceding theorem we have the following theorem of Menchoff.

Theorem 13 (Menchoff (3)). If

$$\sum_{n=1}^{\infty} |a_n|^{2-\epsilon} < \infty, \quad 0 < \epsilon < 2$$

then the series $\sum_{n=1}^{\infty} a_n \phi_n(x)$ converges a. e. in (a, b) for any order of its terms.

For this purpose write the sequence $|a_n|$ in decreasing order $|a_n(s)|$. Since $\sum |a_n(s)|^{2-\epsilon} < \infty$, then

$$s |a_n(s)|^{2-\epsilon} \rightarrow 0$$

and $\sum_{s=1}^{\infty} a_n^2(s) (\log s)^2 (\log \log s)^{1+\epsilon} < \infty$. From theorem 12 the series $\sum a_n \phi_n(x)$ converges a. e. for all orders of terms if we put $\mu(n) = (\log \log n)^{1+\epsilon}$.

Conditions of the second kind, but of a different category from theorems 12 and 13 are given in the next theorem. The multiplier of convergence $\mu(p)$ is given as a function of the coefficients, a_n .

Theorem 14 (Orlicz (1), Menchoff (3)). If (1) $\mu(n)$ is an increasing function tending to ∞ , (2) $\sum_{n=1}^{\infty} \frac{1}{\mu(n)} < \infty$, (3) $\sum_{n=1}^{\infty} \mu(\log |\log a_n|) (\log |a_n|)^2 a_n^2 < \infty$ and $a_n \rightarrow 0$ then $\sum_{n=1}^{\infty} a_n \phi_n(x)$ converges a. e. in (a, b) for any order of terms.

Systems of Convergence

An ON system is a system of convergence when the convergence of the series $\sum_{n=1}^{\infty} C_n^2$ implies the convergence a. e. of the series $\sum_{n=1}^{\infty} C_n \phi_n(x)$. Menchoff (4) has shown the following theorem.

Theorem 15 (Menchoff (4)). Each ON system contains a part

$$\{\phi_{m_k}(x)\} = \{\phi_{m_1}(x), \phi_{m_2}(x), \phi_{m_3}(x), \dots, \phi_{m_k}(x), \dots\}$$

$$m_{k-1} < m_k$$

which is an infinite system of convergence.

We shall consider an application of this theorem that concerns the changing of the order of the functions in the ON system. Let $W(n)$ be a positive function such that $\lim_{n \rightarrow \infty} W(n) = \infty$. We shall say that such a function is a "function of Weyl" for an ON system if $\sum_{n=1}^{\infty} C_n^2 W(n) < \infty$ implies that $\sum_{n=1}^{\infty} C_n \phi_n(x)$ converges a. e. Menchoff shows the following theorem.

Theorem 16 (Menchoff (4)). Given an arbitrary function $W(n)$, $W(n) > 0$, $\lim_{n \rightarrow \infty} W(n) = \infty$, and any ON system $\{\phi_n(x)\}$ in (a, b) , one can effect in this system a change of order of the functions $\phi_n(x)$ in such a fashion that for the system $\{\phi_{\nu_n}(x)\}$, thus obtained, the function $W(n)$ will be a "function of Weyl".

Proof: From the preceding theorem, the system $\{\phi_n(x)\}$ contains a part

$$\{\phi_{m_k}(x)\} = \{\phi_{m_1}(x), \phi_{m_2}(x), \dots\}, m_k < m_{k+1}$$

which is an infinite system of convergence. Taking into account the relation $\lim_{n \rightarrow \infty} W(n)$, we define a sequence of positive integers ρ_ℓ , $\ell = 1, 2, 3, \dots$, which verify the conditions

$$(1) \quad \rho_1 = 1, \rho_{\ell-1} < \rho_\ell, W(\rho_\ell) > 2^\ell \quad (\ell > 1)$$

Determine a sequence of positive integers

$$(2) \quad \nu_1, \nu_2, \nu_3, \dots, \nu_n, \dots$$

in the following manner:

$$\text{When } \rho_\ell < n < \rho_{\ell+1}, \ell = 1, 2, 3, \dots,$$

$$\text{put } \nu_n = m_k \text{ where } k = n - \ell.$$

$$\text{When } n = \rho_\ell, \ell = 1, 2, 3, \dots,$$

$$\text{put } \nu_n = \mu_\ell,$$

where $\mu_1 < \mu_2 < \dots < \mu_\ell < \dots$ are all the positive integers in increasing order which are different from m_k , $k = 1, 2, \dots$.

The numbers ν_n are defined for all positive, integral values of n , and we see immediately that in the sequence (2) each positive integer appears one and only one time, that is the sequence (2) is obtained from the sequence

$$1, 2, 3, \dots, n, \dots$$

by a change of the order of terms.

We are going to show that the function $W(n)$ is a "function of Weyl" for the system $\{\phi_{\nu_n}(x)\}$. Consider a series

$$(3) \quad \sum_{n=1}^{\infty} C_n \phi_{\nu_n}(x)$$

where C_n , $n = 1, 2, 3, \dots$, are arbitrary constants such that $\sum_{n=1}^{\infty} C_n^2 W(n) < \infty$. In order to prove the theorem it suffices to prove that the series (3) converges a. e. in (a, b) .

$$(4) \quad \text{Let } \sum' C_n \phi_{\nu_n}(x)$$

and

$$(5) \quad \sum'' C_n \phi_{\nu_n}(x)$$

be two infinite series, where the sum $\sum'$ is extended for all indices n different from ρ_ℓ , $\ell = 1, 2, 3, \dots$, while the sum $\sum''$ is extended for all indices $n = \rho_\ell$, $\ell = 1, 2, 3, \dots$. Suppose, further that in the series (4) and (5) the indices n are in increasing order. We shall show that each of the series (4) and (5) converges a. e. in (a, b) , which will results in the convergence a. e. of the series (3). We see immediately that

$$(6) \quad \sum' C_n \phi_{\nu_n}(x) = \sum_{k=1}^{\infty} C_k' \phi_{m_k}(x),$$

where $C_k' = C_{k+\ell}$, the number ℓ is defined by the inequality $\rho_\ell < k < \rho_{\ell+1}$. It results from the convergence of the series $\sum_{n=1}^{\infty} C_n^2 W(n)$ and from the relation $\lim_{n \rightarrow \infty} W(n) = \infty$ that the series $\sum_{n=1}^{\infty} C_n^2$ converges, from which we conclude that the series

$$\sum_{k=1}^{\infty} C_k'^2 = \sum_{n=1}^{\infty} C_n^2$$

is also convergent. The ON system $\{\phi_{mk}(x)\}$ is a system of convergence, we see them from the equality (6) that the series (4) converges a. e. in (a, b).

It results from the definition of the series (5) and from the equality $\nu_n = \mu_\ell$ that

$$\sum_{n=1}^{\infty} C_n \phi_{\nu_n}(x) = \sum_{\ell=1}^{\infty} C_{\rho_\ell} \phi_{\mu_\ell}(x)$$

where $\{\phi_{\mu_\ell}(x)\}$ is an ON system. Putting

$$A = \sum_{n=1}^{\infty} C_n^2 W(n)$$

we shall have

$$C_{\rho_\ell}^2 \leq \frac{A}{W(\rho_\ell)}$$

and, consequently, by (1)

$$C_{\rho_\ell}^2 < \frac{A}{2^\ell}$$

from which it results that the series

$$\sum_{\ell=1}^{\infty} (\log \ell)^2 C_{\rho_\ell}^2 \quad \text{converges.}$$

It is seen then from theorem 9 that the series $\sum_{\ell=1}^{\infty} C_{\rho_{\ell}} \phi_{\mu_{\ell}}(x)$, which is identical to series (5), converges a. e. in (a, b). We see, on reviewing, that each of the series (4) and (5), and consequently, series (3) converges a. e. in (a, b), which proves the theorem.

The Lebesgue Function and Convergence

In the preceding sections we have studied the convergence of orthogonal series under hypotheses concerning only the coefficients; the results thus obtained are valuable for all orthogonal systems. The question arises, what are the conditions with respect to the coefficients if one makes hypotheses on the ON system?

For this purpose we write the partial sums in the following manner:

$$S_n(x) = \sum_{k=1}^n \int_0^1 f(t) \phi_k(t) \phi_k(x) dt$$

where the $\{\phi_k(x)\}$ are an ON system on (0, 1), and $f(t) \in L_2(0, 1)$ is a function being expanded in $\{\phi_n(x)\}$. Putting

$$K_n(x, y) = \sum_{k=1}^n \phi_k(x) \phi_k(y)$$

we have

$$S_n(x) = \int_0^1 f(t) K_n(x, t) dt$$

From Rademacher (1) we say that

$$\rho_n(x) = \int_0^1 |K_n(x, y)| dy$$

is the Lebesgue function for the system $\{\phi_n(x)\}$. The expression $K_n(x, y)$ is called the n th kernel of the system $\{\phi_n(x)\}$.

We study at first the order of magnitude of $\rho_n(x)$. Rademacher showed that almost everywhere $\rho_n^2(x) = O(n \log^{3+\epsilon} n)$, $\epsilon > 0$. Kaczmarz gives a better approximation in the following theorem.

Theorem 17 (Kaczmarz (1)). For all orthogonal systems,

$$\rho_n^2(x) = o(n \log^{1+\epsilon} n) \text{ a. e.}$$

Proof: The series

$$\sum_{n=2}^{\infty} \frac{1}{n \log^{1+\epsilon} n} \phi_n^2(x)$$

is convergent a. e. in $(0, 1)$ for from the integration

$$\sum_{n=2}^{\infty} \frac{1}{n \log^{1+\epsilon} n} \int_0^1 \phi_n^2(x) dx = \sum_{n=2}^{\infty} \frac{1}{n \log^{1+\epsilon} n} < \infty$$

then from Kronecker's theorem (Knopp p. 129)

$$\sum_{i=1}^n \phi_i^2(x) = o(n \log^{1+\epsilon} n) \text{ a. e.}$$

Applying Schwarz's inequality to $\rho_n(x)$, we have a. e.

$$\rho_n^2(x) \leq \int_0^1 K_n^2(x, t) dt = \sum_{k=1}^n \phi_k^2(x),$$

which implies a. e. that

$$\rho_n^2(x) = o(n \log^{1+\epsilon} n).$$

Theorem 18 (Kaczmarz (1)). If almost everywhere $|\phi_n(x)| < A$, then a. e. $\rho_n(x) = O(\sqrt{n})$.

Proof: $\rho_n^2(x) \leq \sum_{k=1}^n \phi_k^2(x) \leq A^2 n$

$$\rho_n(x) = O(\sqrt{n}) \text{ a.e.}$$

Note: If the $\rho_n(x)$ are independent of x (Lebesgue constants) then $\rho_n(x) = O(\sqrt{n})$, since

$$\rho_n(x) = \int_0^1 \rho_n(x) dx = \int_0^1 \int_0^1 |K_n(x, t)| dx dt \leq \sqrt{\int_0^1 \int_0^1 K_n^2(x, t) dx dt} = \sqrt{n}$$

$$\rho_n(x) = O(\sqrt{n}) \text{ a.e.}$$

In order to go further we employ the following lemma.

Lemma A (Plessner (1))

If (1) $F(x, y, t)$ is a symmetric function with respect to x and y

(2) $t = \min[\lambda(x), \lambda(y)] = \lambda(x, y)$ then

$$\left| \int_0^1 \int_0^1 F(x, y, \lambda(x, y)) dx dy \right| \leq 2 \int_0^1 dx \int_0^1 |F(x, y, \lambda(x))| dy$$

We are helped in the proof of convergence by the following theorem.

Theorem 19 (Kaczmarz (1)). If

$$1) \quad \rho_n(x) \leq F(x) \mu^2(n), \quad F(x) \in L(0, 1)$$

$$2) \quad \mu(n) \leq \mu(n+1)$$

then the sequence $\frac{S_n(x)}{\mu(n)}$ converges a.e. in $(0, 1)$.

Proof: We are considering a development of a function $f(t) \in L_2(0, 1)$ where

$$S_n(x) = \int_0^1 f(t) K_n(x, t) dt$$

Let
$$V_n(x) = \max_{1 \leq k \leq n} \frac{S_k(x)}{\mu(k)} = \frac{S_p(x)}{\mu(p)}$$

where p is a function $p(x)$ of x and the parameter n . Consider the integrals

$$I_n = \int_0^1 V_n(x) dx$$

We have

$$I_n = \int_0^1 dx \int_0^1 f(t) \frac{K_p(x, t)}{\mu(p)} dt = \int_0^1 f(t) \int_0^1 \frac{K_p(x, t)}{\mu(p)} dx dt$$

Designate by A the constant $\|f\|^2 = \int_0^1 f^2(t) dt$. Applying Schwarz's inequality, we obtain

$$I_n^2 \leq A \int_0^1 \left[\int_0^1 \frac{K_p(x, t)}{\mu(p)} dx \right]^2 dt$$

Putting $q = p(y)$; we will have

$$\int_0^1 \left[\int_0^1 \frac{K_p(x, t)}{\mu(p)} dx \right] \left[\int_0^1 \frac{K_q(y, t)}{\mu(q)} dy \right] dt = \int_0^1 \int_0^1 \int_0^1 \frac{K_p(x, t) K_q(y, t)}{\mu(p) \mu(q)} dx dy dt$$

The orthogonality gives

$$\int_0^1 K_p(x, t) K_q(y, t) dt = K_\lambda(x, y) \text{ where } \lambda = \min(p, q)$$

then

$$I_n^2 \leq A \int_0^1 \int_0^1 \frac{|K_\lambda(x, y)|}{\mu^2(\lambda)} dx dy$$

From lemma A we have

$$I_n^2 \leq 2A \int_0^1 dx \int_0^1 \frac{|K_p(x, y)|}{\mu^2(p)} dy \leq 2A \int_0^1 F(x) dx = K$$

The I_n are then bounded; but the $V_n(x)$ increase, then the sequence $V_n(x)$ is a.e. convergent.

Theorem 20 (Kaczmarz (1)). If

$$\rho_n(x) \leq F(x), \quad F(x) \in L(0, 1)$$

then the development of $f(t) \in L_2(0, 1)$ converges a.e.

Proof: Put $\mu(n) = 1$, by the preceding theorem

$S_n(x)$ converges a.e.

Note: In Haar's system, where $\rho_n(x)$ is bounded, the developments in that system of functions in L_2 are a.e. convergent. We shall now establish a necessary lemma.

Lemma 1 (Rademacher (1)).

If (1) $\mu(n)$ is an increasing function with n ;

$$(2) \sum A_n^2 \mu(n) < \infty$$

$$(3) k \leq \mu(n_k) < k+1$$

then the sequence $S_{n_k}(x)$ converges a.e. in $(0, 1)$, where $S_{n_k}(x) = \sum_{i=1}^{n_k} A_i \phi_i(x)$.

Proof:

$$\int_0^1 [f(x) - S_n(x)]^2 dx = \sum_{k=n+1}^{\infty} A_k^2 = r_n$$

then

$$\int_0^1 [f(x) - S_{n_k}(x)]^2 dx = r_{n_k}$$

The series $\sum_{k=1}^{\infty} r_{n_k}$ is convergent by

$$\sum_k r_{n_k} = \sum_k \left(r_{n_k} - r_{n_{k+1}} \right) k + \lim_{k \rightarrow \infty} k r_{n_{k+1}} \leq \sum_n A_n^2 \mu(n)$$

so the series

$$\sum_k \int_0^1 (f - S_{n_k})^2 dx < \infty.$$

$$\sum_k (f(x) - S_{n_k}(x))^2 \text{ converges a.e.}$$

$$S_{n_k}(x) \rightarrow f(x) \text{ a.e.}$$

Theorem 21 (Kaczmarz (1)). Given an ON system $\{\phi_n(x)\}$ and a positive function $\mu(n)$ tending to $+\infty$; if

$$1) \rho_n(x) = O[\mu^2(n)] \text{ a.e.}$$

$$2) \sum_{n=1}^{\infty} A_n^2 \mu^2(n) < \infty$$

then the series $\sum_{n=1}^{\infty} A_n \phi_n(x)$ converges a.e.

Proof: From hypothesis 2 of the preceding lemma the sequence $S_{n_k}(x)$ converges a.e. if $k \leq \mu^2(n_k) < k+1$. To prove the theorem it suffices to prove that for $n_k < n < n_{k+1}$ we have $S_n(x) - S_{n_k}(x) \rightarrow 0$. For this purpose put $b_n = A_n \mu(n)$; then

$$\begin{aligned} S_n(x) - S_{n_k}(x) &= \sum_{i=n_k+1}^n b_i \phi_i(x) \frac{1}{\mu(i)} \\ &= \sum_{i=n_k+1}^n \left(\sum_{\ell=1}^i b_{\ell} \phi_{\ell}(x) \right) \left(\frac{1}{\mu(i)} - \frac{1}{\mu(i+1)} \right) \\ &\quad - \sum_{i=1}^{n_k} b_i \phi_i(x) \frac{1}{\mu(n_k+1)} + \sum_{i=1}^n b_i \phi_i(x) \frac{1}{\mu(n+1)} \end{aligned}$$

Theorem 19 furnishes

$$\left| \sum_{\ell=1}^i b_{\ell} \phi_{\ell}(x) \right| < \epsilon \mu(i) \text{ for } i > N$$

then on putting $n_k > N$ we have

$$|S_n(x) - S_{n_k}(x)| < \epsilon \mu(n_{k+1}) \left[\frac{1}{\mu(n_k)} - \frac{1}{\mu(n_{k+1})} \right] + 2\epsilon$$

$$|S_n(x) - S_{n_k}| \leq 3\epsilon$$

Note: 1) For the trigonometric series we have $\rho_n(x) = O(\log n)$, then the condition $\sum_{n=1}^{\infty} A_n^2 \log k < \infty$ suffices for the convergence a. e.

2) Theorems 17 and 21 give the following condition for the convergence for any ON system

$$\sum_{n=1}^{\infty} A_n^2 \sqrt{n \log^{1+\epsilon} n} < \infty$$

and theorems 18 and 21 give

$$\sum_{n=1}^{\infty} A_n^2 n^{1/2} < \infty$$

as the condition for $\{\phi_n(x)\}$ for which $|\phi_n(x)| < A$. However, by theorem 9 it can be seen that theorem 21 is not the least restrictive.

Theorem 22. If

$$1) \rho_n(x) = O(1) \quad 2) \sum A_n^2 < \infty$$

then the series $\sum_{n=1}^{\infty} A_n \phi_n(x)$ converges a. e.

Proof: Put $\mu(n) = 1$ in theorem 21.

III. SUMMABILITY ALMOST EVERYWHERE

Generalities

We already know from theorem 11 that there exist orthogonal series $\sum_{n=1}^{\infty} A_n \phi_n(x)$ which are everywhere divergent even though $\sum_{n=1}^{\infty} A_n^2 < \infty$. However, we do know that this condition is sufficient to insure convergence in the mean, that is to say they verify

$$\lim_{n \rightarrow \infty} \int_a^b (f(x) - S_n(x))^2 dx = 0$$

for $f(x) \in L_2(a, b)$. Consequently they are also convergent in measure, that which prevents diverging to $+\infty$ or $-\infty$ on a set of positive measure. It is then possible that there exist methods of summation whose scope includes a broader class of orthogonal series than that of ordinary convergence.

The most important methods of summation are linear. Toeplitz was the first to consider such methods systematically, and they are called T methods. Given a matrix

$$(a_{mn}) ; \quad \begin{array}{l} m = 0, 1, 2, \dots \\ n = 0, 1, 2, \dots \end{array}$$

define the transforms $\{\tau_m\}$ of the sequence of partial sums $\{S_n\}$ by

$$\tau_m = \sum_{n=0}^{\infty} a_{mn} S_n$$

We assume that all these series are convergent. If $\lim_{m \rightarrow \infty} \tau_m = S$, we say that S_n has the T-limit S , or converges (T) to S . Denote this by $S_n \rightarrow S(T)$. We then say that the series considered is summable (T) to the sum S .

If the T method sums every convergent series to the right sum, i. e. if $S_n \rightarrow S$ implies $S_n \rightarrow S(T)$, then the method is called regular. The conditions on the a_{mn} which make a T method regular were found by Toeplitz and are the following:

- 1) $\sum_{n=0}^{\infty} |a_{mn}| < M$ where M is independent of m .
- 2) $\lim_{m \rightarrow \infty} a_{mn} = 0$ all n
- 3) $\lim_{m \rightarrow \infty} \sum_{n=0}^{\infty} a_{mn} = 1$

Menchoff showed that there does not exist a linear method of summation such that the convergence of $\sum_{n=1}^{\infty} A_n^2$ implies the summability of $\sum_{n=1}^{\infty} A_n \phi_n(x)$ by that method. This is expressed in the following theorem.

Theorem 23 (Menchoff (2)). A linear method of summation is given; one can define an ON system $\{\phi_n(x)\}$ and a sequence of constants $\{A_n\}$ with $\sum_{n=1}^{\infty} A_n^2 < \infty$ such that $\sum_{n=1}^{\infty} A_n \phi_n(x)$ is not summable at any point by that method.

We can demand, then, that there be a positive function $W(n)$ for which the convergence of $\sum_{n=1}^{\infty} A_n^2 W(n)$ implies the summation a. e. of the series $\sum_{n=1}^{\infty} A_n \phi_n(x)$ by a given linear method. It is convenient to say that the function $W(n)$ is a "function of Weyl for a linear method" if the convergence of $\sum_{n=1}^{\infty} A_n^2 W(n)$ implies the summability a. e. of $\sum_{n=1}^{\infty} A_n \phi_n(x)$ by that linear method. Menchoff gives the following theorem.

Theorem 24 (Menchoff (2)). For any positive function $W(n)$ such that $\lim_{n \rightarrow \infty} W(n) = \infty$, there exists a linear method of summation for which $W(n)$ is a function of Weyl.

Proof: Let $W(n)$ be an arbitrary function satisfying the conditions of the theorem. Then there exists a positive function $h(n)$ with the properties:

- 1) $h(n)$ is a continuous function, increasing and positive, defined for all positive values of n
- 2) $h(n) < W(n)$ for all n integral, $n > 0$.
- 3) $h(n+1) - h(n) < 1$, $n > 0$
- 4) $\lim_{n \rightarrow \infty} h(n) = +\infty$

In virtue of property (2), it suffices to prove the theorem by putting $h(n)$ in place of $W(n)$.

Designate by $H(\rho)$ the inverse function of $h(n)$, that is $h[H(\rho)] = \rho$ and let $k_\rho = E H(\rho)$ (integer part of $H(\rho)$). From properties 1 and 4 of $h(n)$, it results that $H(\rho)$ is a positive function, uniform and increasing, which tends to ∞ with ρ . We have further from property 3

$$k_{\rho-1} < k_\rho, \quad \rho > 1$$

We shall suppose in the following that ρ is an integer.

Let $\{\phi_n(x)\}$ be an ON system and let $\{A_n\}$ be a sequence of constants such that $\sum_{n=1}^{\infty} A_n^2 h(n) < \infty$.

Now to prove that $S_{k_\rho}(x)$ converges a. e. as $\rho \rightarrow \infty$. Put

$$A_1 = + \sqrt{\sum_{n=1}^{k_1} A_n^2}, \quad A_\rho = + \sqrt{\sum_{n=k_{\rho-1}+1}^{k_\rho} A_n^2}$$

and
$$\bar{\phi}_1(x) = \frac{1}{A_1} S_{k_1}(x)$$

$$\bar{\phi}_\rho(x) = \frac{S_{k_\rho}(x) - S_{k_{\rho-1}}(x)}{A_\rho}$$

The functions $\bar{\phi}_\rho(x)$, $\rho = 1, 2, 3, \dots$ form an ON system and

$$S_{k_\rho}(x) = \sum_{s=1}^{\rho} A_s \bar{\phi}_s(x)$$

From property 3 of $h(n)$ and the equation

$$h[H(\rho)] = \rho$$

we see that $k_{\rho-1} < n$ implies $\rho < ch(n)$ where c is an absolute constant. Then

$$\rho A_\rho^2 \leq c \sum_{n=k_{\rho-1}+1}^{k_\rho} A_n^2 h(n)$$

$\sum_{p=1}^{\infty} \rho A_p^2$ converges since

$$\sum_{n=1}^{\infty} h(n) A_n^2 < \infty.$$

$\sum_{p=1}^{\infty} A_p \Phi_p(x)$ converges a. e.

So the $S_{k_p}(x)$ converges a. e. as $p \rightarrow \infty$.

Putting $a_n(\rho) = \begin{cases} 1 & \text{when } n = k_p \\ 0 & \text{when } n \neq k_p \end{cases}$

we see immediately that the quantities $a_n(\rho)$ have the properties of a regular linear method of summation. Furthermore

$$\sum_{n=1}^{\infty} a_n(\rho) S_n(x) = S_{k_p}(x)$$

so the series $\sum_{n=1}^{\infty} A_n \phi_n(x)$ is summable by the linear method considered since $S_{k_p}(x)$ converge a. e. Therefore $h(n)$, and it follows $W(n)$ is a function of Weyl for this method which establishes the theorem.

As special cases of linear methods we shall now consider the methods of Cesaro and Poisson. Cesaro summability implies Poisson summability. We shall consider the method of Cesaro first.

For this purpose designate by $\{\sigma_n^r(x)\}$ the sequence of Cesaro means of order r of the series $\sum_{n=0}^{\infty} A_n \phi_n(x)$, i. e.

$$\sigma_n^r(x) = \frac{1}{A_n^r} \sum_{k=0}^n A_{n-k}^r A_k \phi_k(x)$$

where $A_n^r = \binom{r+n}{n} = \frac{(r+1)(r+2)\cdots(r+n)}{n!}$

For $r = 1$ we have

$$\sigma_n(x) = \frac{S_0(x) + S_1(x) + \cdots + S_n(x)}{n+1}$$

which is the arithmetic mean of the partial sums, so summation by method (C, 1) is sometimes called summation by first arithmetic means.

Consider an ON system $\{\phi_n(x)\}$ on (a, b) and a sequence of constants $\{A_n\}$ such that $\sum_{n=0}^{\infty} A_n^2 < \infty$. We have the following theorem due to Zygmund.

Theorem 25 (Zygmund (2), (3)). If in a set $E \subset (a, b)$ the series $\sum_{n=0}^{\infty} A_n \phi_n(x)$ is summable (C, 1) to the sum $S(x)$, then a. e. in E

$$\lim_{n \rightarrow \infty} \frac{|S - S_0| + |S - S_1| + \cdots + |S - S_n|}{n+1} = 0$$

or

$$\lim_{n \rightarrow \infty} \frac{(S - S_0)^2 + (S - S_1)^2 + \cdots + (S - S_n)^2}{n+1} = 0$$

(The first relation is obtained from the second by Schwarz's inequality.)

The proof of the theorem follows from

Lemma 2 (Zygmund (2), (3))

A. e. in (a, b)

$$\lim_{n \rightarrow \infty} \frac{(S_0 - \sigma_0)^2 + (S_1 - \sigma_1)^2 + \cdots + (S_n - \sigma_n)^2}{n+1} = 0$$

Proof: (Zygmund (3)). It suffices to prove the convergence a. e. of the series

$$\sum_{n=0}^{\infty} \frac{(S_n - \sigma_n)^2}{n+1} \quad (1)$$

for, from the convergence of a series $\sum_{n=1}^{\infty} C_n$ we know that the

$$\lim_{n \rightarrow \infty} \frac{C_1 + 2C_2 + \cdots + nC_n}{n+1} = 0, \text{ and}$$

this latter expression is equal to the difference between the partial sum and the arithmetic means of the partial sums of $\sum_{n=1}^{\infty} C_n$.

Integrating series (1) term by term, we obtain

$$\sum_{n=0}^{\infty} \frac{1}{n+1} \int_a^b (S_n - \sigma_n)^2 dx = \sum_{n=0}^{\infty} \frac{1}{(n+1)^3} \sum_{k=0}^n k^2 A_k^2$$

$$= \sum_{k=0}^{\infty} k^2 A_k^2 \sum_{n=k}^{\infty} \frac{1}{(n+1)^3} \leq \frac{1}{2} \sum_{k=0}^{\infty} A_k^2 < \infty$$

If series (1) had nonnegative terms divergent on a set of positive measure, the integrated series would be divergent; this is not the case, so the lemma is proved.

To prove the theorem it suffices to use the inequality

$$\frac{\sum_{k=0}^n (S_k - S)^2}{n+1} = \frac{\sum_{k=0}^n [(S_k - \sigma_k) + (\sigma_k - S)]^2}{n+1} \ll$$

$$2 \frac{\sum_{k=0}^n (S_k - \sigma_k)^2}{n+1} + 2 \frac{\sum_{k=0}^n (\sigma_k - S)^2}{n+1} \rightarrow 0$$

The right-hand expression tends to zero by the lemma and the (C, 1) summability of the series.

From the lemma just proved the following important theorem can be deduced.

Theorem 26 (Zygmund (2)). If the series $\sum_{n=0}^{\infty} A_n \phi_n(x)$ is summable in a set $E \subset (a, b)$ by the method of Poisson, it is summable a. e. in E by the method (C, 1).

Note at first that a. e. in (a, b)

$$\frac{(S_0 - \sigma_0) + \dots + (S_n - \sigma_n)}{n+1} = \sigma_n - \frac{\sigma_0 + \sigma_1 + \dots + \sigma_n}{n+1}$$

$$= \sigma_n - \bar{\sigma}_n \rightarrow 0$$

If the series is summable by the method of Poisson, we have

$$\lim_{r \rightarrow 1-0} (1-r) \sum_{n=0}^{\infty} S_n r^n = S$$

where the S_n are the partial sums and r is a continuous parameter which tends to 1 through positive values less than 1. The arithmetic means σ_n also have the Poisson limit S , since

$$(1-r) \sum_{n=0}^{\infty} S_n r^n = (1-r)^2 \sum_{n=0}^{\infty} (S_0 + S_1 + \dots + S_n) r^n \sim S$$

and integrating with respect to r we have

$$\sum_{n=0}^{\infty} \frac{(S_0 + S_1 + \dots + S_n) r^{n+1}}{n+1} \sim S \int_0^r \frac{dr}{(1-r)^2} \sim \frac{S}{(1-r)}$$

$$\lim_{r \rightarrow 1-0} (1-r) \sum_{n=0}^{\infty} \sigma_n r^n = S$$

It can then be assumed that the series $\sum_{n=0}^{\infty} (\sigma_n - \sigma_{n-1})$ is summable by the method of Poisson. In order to deduce the ordinary convergence it suffices to make use of the following relation which is obtained by Abel's transformation: (ref: Knopp p. 313)

$$\frac{\sum_{k=0}^n k (\sigma_k - \sigma_{k-1})}{n+1} = \sigma_n - \bar{\sigma}_n \rightarrow 0$$

We can generalize the above theorem by replacing the $(C, 1)$ summability by (C, ϵ) , $\epsilon > 0$.

To do this it suffices to prove two lemmas. Recalling the definitions of Cesaro means of order r

$$\sigma_n^r = \frac{1}{A_n^r} \sum_{k=0}^n A_{n-k}^r A_k \phi_k(x)$$

where $A_n^r = \frac{(r+1)(r+2)\dots(r+n)}{n!} \sim \frac{n^r}{\Gamma(r+1)}$

we define $\eta_n^r = \frac{\sum_{k=0}^n (\sigma_k^r - \sigma_k^{r-1})^2}{n+1}$

Lemma 3 (Zygmund (2))

A. e. in (a, b)

$$\lim_{n \rightarrow \infty} \eta_n^r = 0 \quad \text{for } r > \frac{1}{2}$$

Proof:

$$\begin{aligned} \sigma_k^r - \sigma_k^{r-1} &= \frac{1}{A_k^r A_k^{r-1}} \sum_{j=0}^k (A_{k-j}^r A_k^{r-1} - A_{k-j}^{r-1} A_k^r) A_k \phi_k \\ &= \frac{1}{A_k^r A_k^{r-1}} \sum_{j=0}^k \left(-\frac{j}{r} A_{k-j}^{r-1} A_k^{r-1} \right) A_k \phi_k \end{aligned}$$

Then

$$\int_a^b (\sigma_k^r - \sigma_k^{r-1})^2 dx = \frac{1}{r^2 (A_k^r)^2} \sum_{j=0}^k j^2 (A_{k-j}^{r-1})^2 A_j^2$$

It results from this equation that

$$\begin{aligned} \int_a^b \eta_{2^n}^r(x) dx &= \frac{1}{r^2 (2^n + 1)} \sum_{k=0}^{2^n} \frac{1}{(A_k^r)^2} \sum_{j=0}^k j^2 (A_{k-j}^{r-1})^2 A_j^2 \\ &= \frac{1}{r^2 (2^n + 1)} \sum_{j=0}^{2^n} j^2 A_j^2 \sum_{k=j}^{2^n} \frac{(A_{k-j}^{r-1})^2}{(A_k^r)^2} \end{aligned}$$

But

$$\begin{aligned} \sum_{k=j}^{2^n} \frac{(A_{k-j}^{r-1})^2}{(A_k^r)^2} &< \sum_{k=j}^{\infty} = \sum_{k=j}^{2j} + \sum_{k=2j+1}^{\infty} \frac{(A_{k-j}^{r-1})^2}{(A_k^r)^2} \\ &< \frac{C_1}{(A_j^r)^2} \sum_{k=0}^j (A_k^{r-1})^2 + C_2 \sum_{k=2j+1}^{\infty} \frac{k^{2r-2}}{k^{2r}} \end{aligned}$$

$$< \frac{C_3}{(A_j^r)^2} \sum_{k=0}^j A_k^{2r-2} + \frac{C_4}{j} < \frac{C_5}{j}$$

Then

$$\begin{aligned} \sum_{n=0}^{\infty} \int_a^b \eta_{2n}^r dx &\leq C_6 \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \sum_{j=0}^{2^n} j A_j^2 = \\ &= C_6 \sum_{j=0}^{\infty} j A_j^2 \sum_{n \geq \log j} \frac{1}{2^{n+1}} \\ &= C_7 \sum_{j=0}^{\infty} A_j^2 < \infty \end{aligned}$$

So the series $\sum_n \eta_{2n}^r$ converges a.e. from which $\lim_{n \rightarrow \infty} \eta_{2n}^r = 0$

or $\lim_{n \rightarrow \infty} \eta_n^r = 0 \quad (r > \frac{1}{2})$

Lemma 4 (Zygmund (2))

Let $\{\tau_n^r\}$ be the sequence of Cesaro means of order r for an arbitrary series $\sum_{n=0}^{\infty} \mu_n$. If

$$\lim_{n \rightarrow \infty} \frac{\sum_{k=0}^n (\tau_k^r)^2}{n+1} = 0 \quad (r > -\frac{1}{2})$$

then $\lim_{n \rightarrow \infty} \tau_n^{r+\frac{1}{2}+\epsilon} = 0 \quad (\epsilon > 0)$

Proof: Put $\tau_n^a = \frac{\tau_n^r}{A_n^a}$

Then $\left| \tau_t^{r+\frac{1}{2}+\epsilon} \right| = \left| \sum_{k=0}^n t_k^r A_{n-k}^{-\frac{1}{2}+\epsilon} \right| =$

$$\begin{aligned}
 &= \left| \sum_{k=0}^n \tau_k^r A_k^r A_{n-k}^{-\frac{1}{2}+\epsilon} \right| \leq \sqrt{\sum_{k=0}^n (\tau_k^r)^2} \sqrt{\sum_{k=0}^n \left(A_k^r A_{n-k}^{-\frac{1}{2}+\epsilon} \right)^2} \\
 &\leq o(\sqrt{n}) C_8 \sqrt{\sum_{k=0}^n A_k^{2r} A_{n-k}^{-1+2\epsilon}} = o(\sqrt{n}) C_8 \sqrt{A_n^{2r+2\epsilon}} \\
 &= o\left(n^{r+\frac{1}{2}+\epsilon}\right)
 \end{aligned}$$

The theorem is now proved in the following manner. If the series is summable (C, 1) to the sum S on a set $E \subset (a, b)$ then from theorem 25

$$\lim_{n \rightarrow \infty} \frac{\sum_{k=0}^n (S_k - S)^2}{n+1} = \lim_{n \rightarrow \infty} \frac{\sum_{k=0}^n (\sigma_k^0 - S)^2}{n+1} = 0$$

a. e. in E. The quantity $\sigma_k^0 - S$ is k-th partial sum of the series

$$(A_0 \phi_0 - S) + A_1 \phi_1 + A_2 \phi_2 + \dots$$

Applying Lemma 4 we obtain

$$\lim_{n \rightarrow \infty} \sigma_n^{\frac{1}{2}+\epsilon} - S = 0 \quad \text{a. e. in E.}$$

Now using Lemma 3 we have, putting $r = \frac{1}{2} + \epsilon$,

$$\begin{aligned}
 \frac{\sum_{k=0}^n \left(\sigma_k^{\frac{1}{2}+\epsilon} - \sigma_k^{-\frac{1}{2}+\epsilon} \right)^2}{n+1} &= \frac{\sum_{k=0}^n \left[\left(\sigma_k^{\frac{1}{2}+\epsilon} - S \right) - \left(\sigma_k^{-\frac{1}{2}+\epsilon} - S \right) \right]^2}{n+1} \rightarrow 0 \\
 \frac{\sum_{k=0}^n \left(\sigma_n^{-\frac{1}{2}+\epsilon} - S \right)^2}{n+1} &\rightarrow 0
 \end{aligned}$$

then from lemma 4, we have

$$\lim_{n \rightarrow \infty} \sigma_n^{2\epsilon} - S = 0$$

So the following theorem has been proved.

Theorem 27 (Zygmund (2)). If the series $\sum A_n \phi_n(x)$ is summable in a set $E \subset (a, b)$ by the method of Poisson, it is summable a. e. in E by the method (C, ϵ) , $\epsilon > 0$.

It suffices then in studying summability by the methods of Poisson and Cesaro to consider only the method $(C, 1)$.

We shall now consider the sequence of Cesaro means of order 1 of the series $\sum_{n=1}^{\infty} A_n \phi_n(x)$. We have

$$\begin{aligned} \sigma_n(x) &= \sum_{k=1}^n \frac{n-k+1}{n} A_k \phi_k(x) \\ &= \frac{nA_1 \phi_1(x) + (n-1)A_2 \phi_2(x) + \cdots + A_n \phi_n(x)}{n} \\ &= \frac{S_1(x) + \cdots + S_n(x)}{n} \end{aligned}$$

We shall now give the following lemma which is necessary to establish the theorems on summability.

Lemma 5 (Kaczmarz (1))

The series $\sum_{n=1}^{\infty} n(\sigma_n - \sigma_{n-1})^2$ converges a. e. in (a, b) , provided $\sum_{n=1}^{\infty} A_n^2 < \infty$.

Proof: The $\sigma_n(x)$ are the arithmetic means of the partial sums $S_n(x)$ of the series $\sum A_n \phi_n(x)$. Consider

$$\begin{aligned} \int_a^b (\sigma_n - \sigma_{n-1})^2 dt &= \int_a^b \left[\sum_{k=1}^{n-1} A_k \phi_k(t) (k-1) \left(\frac{1}{n} - \frac{1}{n-1} \right) + \frac{1}{n} A_n \phi_n \right]^2 dt \\ &= \sum_{k=1}^{n-1} A_k^2 \frac{(k-1)^2}{n^2 (n-1)^2} + \frac{A_n^2}{n^2}, \end{aligned}$$

So

$$n \int_a^b (\sigma_n - \sigma_{n-1})^2 dt \leq \frac{1}{(n-1)^2} \sum_{k=1}^{n-1} k^2 A_k^2 + \frac{A_n^2}{n}$$

$$\sum_{n=2}^{\infty} n \int_a^b (\sigma_n - \sigma_{n-1})^2 dt \leq \sum_{n=2}^{\infty} \frac{1}{(n-1)^3} \sum_{k=1}^{n-1} k^2 A_k^2 + \sum_{n=2}^{\infty} \frac{A_n^2}{n}$$

The last series is convergent and the next to last series is equal to

$$\sum_{k=1}^{\infty} k^2 A_k^2 \left(\frac{1}{k^3} + \frac{1}{(k+1)^3} + \dots \right) < \frac{3}{2} \sum_{k=1}^{\infty} A_k^2 < \infty$$

$$\sum_{n=1}^{\infty} n (\sigma_n - \sigma_{n-1})^2 < \infty \text{ a. e.}$$

Theorem 28 (Kaczmarz (1)). The convergence a. e. of the sequence $\{S_2 n(x)\}$ is the necessary and sufficient condition for the a. e. summability $(C, 1)$ of the series $\sum_{n=1}^{\infty} A_n \phi_n(x)$.

Proof: (A) Put $\mu_k = S_2 k - \sigma_2 k$. We have

$$\int_a^b \mu_k^2 dx = \sum_{i=1}^{2^k} A_i^2 \frac{(i-1)^2}{2^{2k}},$$

then

$$\sum_{k=1}^{\infty} \int_a^b \mu_k^2 dx = \sum_{k=1}^{\infty} \frac{1}{4^k} \sum_{i=1}^{2^k} A_i^2 (i-1)^2$$

$$\leq \sum_{i=1}^{\infty} \left(A_i^2 (i-1)^2 \sum_{n=E \log_2 i}^{\infty} \frac{1}{4^n} \right) \quad \text{where } E \log k \text{ is the integer part of } \log k$$

$$< \sum_{i=1}^{\infty} A_i^2 (i-1)^2 \frac{4}{3} \frac{2}{i}^2 \leq \frac{16}{3} \sum_{i=1}^{\infty} A_i^2 < \infty$$

$$\sum_{n=1}^{\infty} [S_{2^n} - \sigma_{2^n}]^2 < \infty \text{ a.e.}$$

$$\lim_{n \rightarrow \infty} (S_{2^n} - \sigma_{2^n}) = 0 \text{ a.e.}$$

So the convergence of $S_{2^n}(x)$ is a necessary condition.

(B) Suppose that $S_{2^n}(x)$ converges a.e. Then from (A) the sequence $\sigma_{2^n}(x)$ converges also. Consider the difference

$$\sigma_k - \sigma_{2^n} \text{ for } 2^n < k < 2^{n+1}$$

We have

$$\sigma_k - \sigma_{2^n} = \sum_{i=2^n}^{k-1} (\sigma_{i+1} - \sigma_i)$$

Applying Schwartz's inequality we obtain

$$\begin{aligned} (\sigma_k - \sigma_{2^n})^2 &\leq \sum_{i=2^n}^{2^{n+1}} i (\sigma_{i+1} - \sigma_i)^2 \cdot \sum_{i=2^n}^{2^{n+1}} \frac{1}{i} \\ &\leq \frac{3}{2} \sum_{i=2^n}^{2^{n+1}} i (\sigma_{i+1} - \sigma_i)^2 \end{aligned}$$

which tends to zero a.e. by lemma 5.

σ_n converges a.e. in (a, b).

Conditions of the First Kind for Summability (C, 1)

Consider an ON system $\{\phi_n(x)\}$ on (0, 1) and a sequence of constants $\{A_n\}$.

We have a theorem analogous to theorem 9.

Theorem 29 (Menchoff (2))

If the series $\sum_{n=1}^{\infty} A_n^2 (\log \log n)^2 < \infty$ then the series $\sum_{n=1}^{\infty} A_n \phi_n(x)$ is summable (C, 1) a. e. in $(0, 1)$.

Proof: It suffices to show the following:

Lemma 6 (Kaczmarz (1))

If $\sum_{n=1}^{\infty} A_n^2 (\log \log n)^2 < \infty$ then the sequence $S_2 n(x)$ converges a. e. in $(0, 1)$.

Proof: (Due to Chen (1))

Assume

$$\sum_{\nu=2^{m+1}}^{2^{m+1}} A_{\nu}^2 > 0 \quad m = 1, 2, \dots$$

Define

$$\psi_m(x) = \frac{1}{\gamma_m} \sum_{\nu=2^{m+1}}^{2^{m+1}} A_{\nu} \phi_{\nu}(x)$$

where

$$\gamma_m = \sqrt{\sum_{\nu=2^{m+1}}^{2^{m+1}} A_{\nu}^2}$$

we have

$$\int_0^1 \psi_m(x) \psi_n(x) dx = 0 \quad \text{for } m \neq n$$

$$\int_0^1 [\psi_m(x)]^2 dx = 1 \quad m = 1, 2, \dots$$

$\{\psi_m(x)\}$ is an ON system in $(0, 1)$.

Consider
$$\sum_{m=1}^{\infty} \gamma_m \psi_m(x) \quad (1)$$

$$\sum_{m=1}^{\infty} \gamma_m^2 (\log_2 m)^2 = \sum_{m=1}^{\infty} (\log_2 m)^2 \sum_{\nu=2^{m+1}}^{2^{m+1}} A_{\nu}^2 \leq$$

$$\sum_{m=1}^{\infty} \sum_{\nu=2^{m+1}}^{2^{m+1}} (\log_2 \log_2 \nu)^2 A_{\nu}^2 = \sum_{\nu=3}^{\infty} A_{\nu}^2 (\log_2 \log_2 \nu)^2 < \infty$$

So series (1) converges a. e. by theorem 9. That is

$$\sum_{m=1}^{\infty} \gamma_m \psi_m(x) = \sum_{m=1}^{\infty} \left\{ \sum_{\nu=2^{m+1}}^{2^{m+1}} A_{\nu} \phi_{\nu}(x) \right\}$$

converges a. e. in $(0, 1)$

$S_2 n(x)$ converges a. e. in $(0, 1)$. Theorem 29 now follows immediately since by theorem 28 the convergence a. e. of $S_2 n(x)$ is a necessary and sufficient condition for the summability $(C, 1)$ of the series $\sum_{n=1}^{\infty} A_n \phi_n(x)$. The factor $(\log \log n)^2$ is the least restrictive as the following theorem shows.

Theorem 30 (Menchoff (2))

If $V(n)$ is an increasing function which verifies $V(n) = O[(\log \log n)^2]$ then there exists an ON system $\{\phi_n(x)\}$ and a sequence of constants $\{b_n\}$ such that

- 1) $\sum_{n=1}^{\infty} b_n^2 V(n) < \infty$
- 2) $\sum_{n=1}^{\infty} b_n \phi_n(x)$ is not summable $(C, 1)$ in the whole interval $(0, 1)$.

Theorems 29 and 30 give a complete answer to the question of conditions of the first kind for the summability $(C, 1)$.

Conditions of the Second Kind for Summability $(C, 1)$

We shall show that the conditions of the second kind for summability $(C, 1)$ are identical to those of convergence.

Theorem 31 (Orlicz (1)). If the series $\sum_{n=1}^{\infty} A_n \phi_n(x)$ is summable $(C, 1)$ a. e. for any order of terms, this series is a. e. convergent.

Proof: Let the series be a. e. summable $(C, 1)$. From theorem 28 the sequence $S_{2^n}(x)$ converges a. e. Consider the partial series $\sum_{k=1}^{\infty} A_{n_k} \phi_{n_k}(x)$ which converges a. e. if $\sum |A_{n_k}| < \infty$ by theorem 13.

Order the series $\sum A_n \phi_n(x)$ in the following manner: The terms having indices 2^n of the new series will be successively the first, second, etc. terms of $\sum A_n \phi_n(x)$ with the exception of the terms $A_{n_k} \phi_{n_k}(x)$, which will occupy the places having indices $m \approx 2^n$. From the hypothesis this new series is summable $(C, 1)$ a. e. Then the sequence $S_{2^n}^{(1)}(x)$ converges a. e. Consider now a second sequence $S_{2^n}^{(2)}(x)$, by replacing in $S_{2^n}^{(1)}(x)$ the coefficients of terms having indices 2^k by zero. The sequence $S_{2^n}^{(2)}(x)$ converges, since it is the sequence of partial sums of $\sum A_{n_k} \phi_{n_k}(x)$ transformed, then the difference $S_{2^n}^{(1)}(x) - S_{2^n}^{(2)}(x)$ converges also. But this difference is the sequence of partial sums of the series $\sum' A_n \phi_n(x)$, where the prime indicates that terms having indices n_k have been dropped out, at the same time putting the other terms $A_n \phi_n(x)$ at the 2^n th place and zeros at the other places. If we add now to this series the terms $A_{n_k} \phi_{n_k}(x)$ suitably, we shall have our original series, and it is then convergent a. e.

The above theorem holds for any linear method.

Systems of Convergence and Linear Methods of Summation

In section 3, chapter II the so-called systems of convergence are defined and theorem 15 is applied to a theorem on convergence. In this section we give some theorems on the summation of orthogonal series by linear methods. These theorems are also applications of theorem 15.

Recall at first the definition of these methods. Consider the series

$$\sum_{k=1}^{\infty} A_{ik} S_k$$

where S_k is the k^{th} partial sum of the series to be summed. We say that the linear method defined by the matrix (A_{ik}) , sums the series if the series

above converges for all values of i sufficiently large and has a sum that tends to a finite limit as $i \rightarrow \infty$. A linear method is regular if

$$(1) \lim_{i \rightarrow \infty} \sum_{k=1}^{\infty} A_{ik} = 1$$

$$(2) \sum_{k=1}^{\infty} |A_{ik}| < M \quad \text{where } M \text{ is independent of } i$$

$$(3) \lim_{i \rightarrow \infty} A_{ik} = 0 \quad \text{all } k = 1, 2, 3, \dots$$

Let
$$\gamma_i = \max_{1 \leq k < \infty} |A_{ik}|$$

Designate by T' each linear method for which the elements A_{ik} of the matrix (A_{ik}) verify conditions 1, 2 and also the condition

$$(4) \lim_{i \rightarrow \infty} \gamma_i = 0$$

Each method T' is a regular method, but the reverse is not necessarily true. The methods of Cesaro of all positive order are methods T' .

We can consider a method of Cesaro of arbitrary positive order δ as a linear method for which the elements A_{ik} of the corresponding matrix are defined by the relations

$$(A) \quad A_{ik} = \frac{\binom{\delta + i - k - 1}{i - k}}{\binom{\delta + i - 1}{i - 1}} \quad \text{for } 1 \leq k \leq i$$

$$(B) \quad A_{ik} = 0 \quad \text{for } k > i$$

$$\begin{aligned} \text{where } \binom{r}{0} &= 1 \quad \text{and} \quad \binom{r+s}{s} = \frac{(r+1)(r+2)\cdots(r+s)}{s!} \\ &= \frac{\Gamma(r+s+1)}{\Gamma(r+1)(s+1)} \end{aligned}$$

for $s = 1, 2, 3, \dots$

Now to show that the methods of Cesaro are methods T' .

By the properties of $\binom{r+s}{s}$ we have

$$A_{ik} \geq 0, \quad \sum_{k=1}^i A_{ik} = 1,$$

from which it results by (B) that the A_{ik} verify conditions (1) and (2). Taking into account the relations

$$0 < \binom{r+s}{s} = \frac{s^r}{\Gamma(r+1)} (1 + T_s), \quad \lim_{s \rightarrow \infty} T_s = 0 \text{ for } r > -1$$

we find further that

$$C_1 \frac{s^r}{\Gamma(r+1)} < \binom{r+s}{s} < C_2 \frac{s^r}{\Gamma(r+1)}, \quad s = 1, 2, 3, \dots, \quad r > -1$$

where C_1 and C_2 are positive constants.

Then for $\delta > 0$, $k \leq i$, $i > 1$, we have:

$$(1) \quad A_{ik} < \frac{C_2}{C_1} \frac{\Gamma(\delta+1)}{\Gamma(\delta)} \frac{(i-k)^{\delta-1}}{(i-1)^\delta} < L(\delta) \left[\frac{1}{(i-1)^\delta} + \frac{1}{i-1} \right]$$

where $L(\delta)$ depends only on δ , we obtain in the same manner:

$$(2) \quad A_{ii} = \frac{1}{\binom{\delta+i-1}{i-1}} < \frac{1}{C_1} \frac{\Gamma(\delta+1)}{(i-1)^\delta}$$

Comparing (1), (2) and $\gamma_i = \max_{1 \leq k < \infty} |A_{ik}|$ we find

$$\gamma_i < L_1(\delta) \left[\frac{1}{(i-1)^\delta} + \frac{1}{i-1} \right] \quad i > 1$$

where $L_1(\delta)$ depends only on δ . It results then that the A_{ik} verify condition (4), so the Cesaro methods are methods T' .

Menchoff shows the following theorem.

Theorem 32 (Menchoff (4)). Given an arbitrary ON system and a sequence of constants $\{C_n\}$ such that $\sum_{n=1}^{\infty} C_n^2 < \infty$, then for each method T' one can effect in the series $\sum_{n=1}^{\infty} C_n \phi_n(x)$ a change of order of terms in

such a fashion that the new series thus obtained will be summable a. e. in (a, b) by that method T' .

In a later paper he shows a more general theorem.

Theorem 33 (Menchoff (5)). For a given ON system and a method T' one can always change in this system the order of the functions $\phi_n(x)$ in such a fashion that for the new system thus obtained $\{\phi_{\nu_n}(x)\}$ the series $\sum_{n=1}^{\infty} C_n \phi_{\nu_n}(x)$ will be summable a. e. by the method T' if $\sum_{n=1}^{\infty} C_n^2 < \infty$.

In this theorem the changing of the order of the functions in the system depends only on the system and the method T' . We can ask if the theorem remains valid if the change in order does not depend on the method T' . The answer is affirmative as Menchoff shows in the following theorem:

Theorem 34 (Menchoff (6)). Given an ON system and a denumerable sequence $\{T_{\sigma'}\}$ of methods T' , one can always effect in the ON system a change of order of the functions $\phi_n(x)$ in such a fashion that for the new system $\{\phi_{\nu_n}(x)\}$ thus obtained the series $\sum_{n=1}^{\infty} C_n \phi_{\nu_n}(x)$ will be summable a. e. by each method $T_{\sigma'}$ if $\sum_{n=1}^{\infty} C_n^2 < \infty$.

As a consequence of this theorem we have the following.

Theorem 35 (Menchoff (6)). The order of every ON system can be changed in such a fashion that for the new system $\{\phi_{\nu_n}(x)\}$ thus obtained the series $\sum_{n=1}^{\infty} C_n \phi_{\nu_n}(x)$ will be summable a. e. for all methods (C, δ) , $\delta > 0$ if $\sum_{n=1}^{\infty} C_n^2 < \infty$.

The Lebesgue Function and Summability $(C, 1)$

We extend now to $(C, 1)$ summability the theorems of section 4 chapter II concerning the properties of the ON system and their influence on convergence. Consider an ON system $\{\phi_n(x)\}$ on $(0, 1)$ and an arbitrary function $f(x) \in L_2(0, 1)$. Designate by $K_n'(x, t)$ the expression

$$K_n'(x, t) = \sum_{k=1}^n \phi_k(x) \phi_k(t) \left(1 - \frac{k-1}{n}\right)$$

then

$$\sigma_n(x) = \int_0^1 f(t) K_n'(x, t) dt \quad \text{and}$$

$$\rho_n'(x) = \int_0^1 |K_n'(x, t)| dt \quad \text{is the Lebesgue function for } (C, 1)$$

summability.

Theorem 36 (Kaczmarz (1)). If

$$(1) \mu(n) \leq \mu(n+1)$$

$$(2) \rho_n'(x) \leq F(x) \mu^2(n), \quad F(x) \in L(0, 1)$$

then $\lim_{n \rightarrow \infty} \frac{\sigma_n(x)}{\mu(n)}$ exists a. e. in $(0, 1)$.

Proof: Put

$$v_n(x) = \max_{1 \leq k \leq n} \frac{\sigma_k(x)}{\mu(k)} = \frac{\sigma_p(x)}{\mu(p)}$$

then

$$v_n(x) = \int_0^1 f(t) \frac{K_p'(x, t)}{\mu(p)} dt$$

Let

$$I_n = \int_0^1 f(t) \left[\int_0^1 \frac{K_p'(x, t)}{\mu(p)} dx \right] dt$$

then

$$I_n^2 \leq A \int_0^1 \left[\int_0^1 \frac{K_p'(x, t)}{\mu(p)} dx \right]^2 dt \quad \text{by}$$

Schwarz's inequality where $A = \|f\|^2$.

$$I_n^2 \leq A \int_0^1 \int_0^1 \frac{1}{\mu(p)\mu(q)} \sum_{k=1}^{\lambda} \phi_k(x) \phi_k(y) \left(1 - \frac{k-1}{p}\right) \left(1 - \frac{k-1}{q}\right) dx dy$$

where $\lambda = \min(p, q)$

$$\sum_{i=1}^{\lambda} K_i(x, y) \left(\frac{1}{p} + \frac{1}{q} - \frac{2i-1}{pq} \right) =$$

$$\sum_{i=1}^{\lambda} i K_i'(x, y) \frac{2}{pq} + \lambda K_{\lambda}'(x, y) \left(\frac{1}{p} + \frac{1}{q} - \frac{2\lambda+1}{pq} \right)$$

Then from lemma A we have

$$I_n^2 \leq A \int_0^1 \int_0^1 \left[\frac{2}{\lambda^2} \sum_{i=1}^{\lambda} i |K_i'| + |K_{\lambda}'| \right] \frac{dx dy}{\mu^2(\lambda)} \leq$$

$$\leq 4A \int_0^1 \left[\frac{1}{p^2} \sum_{i=1}^p i F(x) \mu^2(p) + F(x) \mu(p) \right] \frac{dx}{\mu^2(p)}$$

$$\leq 8A \int_0^1 F(x) dx = C$$

So $v_n(x)$ converges a. e. in $(0, 1)$

$$\lim_{n \rightarrow \infty} \frac{\sigma_n(x)}{\mu(n)} \text{ exists a. e. in } (0, 1).$$

Theorem 37 (Kaczmarz (1)). If

$$(1) F(x) \in L(0, 1)$$

$$(2) \rho_n'(x) \leq F(x)$$

then the development $\sum_{n=1}^{\infty} A_n \phi_n(x)$ of a $f(x) \in L_2(0, 1)$ is summable $(C, 1)$ a. e. in $(0, 1)$.

Proof: Put $\mu(n) = 1$ in the preceding theorem.

Theorem 38 (Kaczmarz (2)). If

(1) $W(n)$ is a positive, increasing function, tending to ∞

(2) $\sum_{n=1}^{\infty} A_n^2 W^2(n) < \infty$

(3) $\rho'_n(x) \leq W(n)$

then the series $\sum_{n=1}^{\infty} A_n \phi_n(x)$ is summable $(C, 1)$ a. e. in $(0, 1)$.

Proof: From lemma 1 we have $S_{n_k}(t)$ converges a. e. if $k \leq W^2(n_k) < k+1$

Let p be an integer such that $n_k < p < n_{k+1}$, then it is sufficient to prove that

$\sigma_p - S_{n_k} \rightarrow 0$. We have

$$(1) \quad \sigma_p - S_{n_k} = \sum_{i=1}^{n_k} A_i \phi_i(t) (i-1) \frac{1}{p} \\ + \sum_{i=n_k+1}^p A_i \phi_i(t) \left[1 - \frac{i-1}{p} \right]$$

Using Abel's transformation (ref. Knopp p. 313) in the second sum on the right we obtain

$$\frac{1}{W(n_k)} \sum_{i=n_k+1}^p A_i \phi_i(t) W(n_k) \left[1 - \frac{i-1}{p} \right] \\ = \frac{1}{W(n_k)} \left[\sum_{i=n_k+1}^p \bar{S}_i \frac{1}{p} - \bar{S}_{n_k} \left(1 - \frac{n_k}{p} \right) + \bar{S}_p \cdot 0 \right]$$

where $\bar{S}_i$ denotes the partial sums of the series obtained by multiplying the terms with indices between n_j and n_{j+1} of $\sum A_n \phi_n(x)$ by $W(n_j)$. The sequence $\{S_{n_k}\}$ is convergent, thus

$$\bar{S}_{n_k} = o \left[W(n_k) \right] \text{ and further}$$

$$\frac{1}{p} \sum_{i=n_k+1}^p \bar{S}_i = \bar{\sigma}_p - n_k \bar{\sigma}_{n_k} \frac{1}{p}$$

where $\bar{\sigma}_i$ denotes the sequence of arithmetic means of $\{\bar{S}_i\}$. Therefore the second sum in (1) is

$$o \left(\frac{\bar{\sigma}_p - \bar{\sigma}_{n_k}}{W(n_k)} \right).$$

But the expression $\frac{\bar{\sigma}_n}{W(n)}$ converges to zero by theorem 36, hence the second sum is convergent to zero a. e.

The same transformation gives us in the first sum

$$\frac{1}{p} \left[\sum_{i=1}^k n_i \left(\bar{S}_{n_i} - \bar{\sigma}_{n_i} \right) \left(\frac{1}{W(n_i)} - \frac{1}{W(n_{i+1})} \right) + n_k \left(\bar{S}_{n_k} - \bar{\sigma}_{n_k} \right) \frac{1}{W(n_{k+1})} \right]$$

Here we have as before $\bar{S}_{n_k} - \bar{\sigma}_{n_k} = o[W(n_k)]$. It remains to prove that the sum is $o(p)$. It is less than

$$C \sum_{i=1}^k n_i |\bar{S}_{n_i} - \bar{\sigma}_{n_i}| i^{-3/2}$$

on account of $\sqrt{i} < W(n_i)$, $W(n_{i+1}) \leq \sqrt{i+1}$. On the other hand

$$\int_0^1 \frac{|\bar{S}_{n_i} - \bar{\sigma}_{n_i}|}{i^{3/2}} dx \leq i^{-3/2} \sqrt{\sum_{k=1}^{\infty} A_k^2 W^2(k)}$$

$$\sum_{i=1}^{\infty} |\bar{S}_{n_i} - \bar{\sigma}_{n_i}| i^{-3/2}$$

is convergent a. e. From Kronecker's theorem it follows that

$$\sum_{i=1}^k n_i |\bar{S}_{n_i} - \bar{\sigma}_{n_i}| i^{-3/2} = o(p)$$

which proves the theorem.

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1-62