Some Matrix Results for Stable m -dimensional Rotations

by

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Abstract

Continuous (real-time) rotation of an object about a line in 3-space can be done using an incremental technique. An equivalent technique in 2-space has a simple modification that makes it stable. The same is shown to be true in m -space.

A general three dimensional rotation of angle θ about a line with direction coordinates $[a,b,c]^T$ through the origin is given by

$$\underline{x}' = A(\theta)\underline{x}$$

where $\underline{x} = [x,y,z]^T$ are the old coordinates, $\underline{x}'$ the new, and $A(\theta)$ is

$$\begin{bmatrix} k+(1-k)a^2 & ab(1-k)-cs & ac(1-k)+bs \\ ab(1-k)+cs & k+(1-k)b^2 & bc(1-k)-as \\ ac(1-k)-bs & bc(1-k)+as & k+(1-k)c^2 \end{bmatrix}$$

where $s = \sin\theta$ and $k = \cos\theta$. (If we wish to rotate about a line not through the origin, translations before and after take care of it. Alternatively we could use homogeneous coordinates.) If it is desired to rotate an object "continuously" in 3-space (say for display purposes), we could choose a small θ and compute

$$\underline{x}_{n+1} = A(\theta)\underline{x}_n$$

thus deriving successive "frames". $A(\theta)$ is fairly complex, and involves $\sin\theta$ and $\cos\theta$. For very small θ , we could neglect $O(\theta^2)$ and replace k by 1 and s by θ , getting

$$A(\theta) \approx A = \begin{bmatrix} 1 & -a\theta & b\theta \\ a\theta & 1 & -a\theta \\ -b\theta & a\theta & 1 \end{bmatrix}$$

Unfortunately, at least one eigenvalue of this approximation is larger than one in magnitude for $\theta \neq 0$, hence $\underline{x}_n = A^n \underline{x}_0$ is unbounded for almost all $\underline{x}_0$. This means that $\underline{x}_n$ will spiral outwards as n increases.

The solution in two dimensions is well known. Instead of

$$\underline{x}_{n+1} = \begin{bmatrix} 1 & -c\theta \\ c\theta & 1 \end{bmatrix} \underline{x}_n$$

that is,

$$x_{n+1} = x_n - c\theta y_n$$

$$y_{n+1} = c\theta x_n + y_n$$

we compute

$$x_{n+1} = x_n - c\theta y_n$$

$$y_{n+1} = c\theta x_{n+1} + y_n$$

That is equivalent to

$$\begin{bmatrix} 1 & 0 \\ -c\theta & 1 \end{bmatrix} \underline{x}_{n+1} = \begin{bmatrix} 1 & -c\theta \\ 0 & 1 \end{bmatrix} \underline{x}_n$$

or

$$\underline{x}_{n+1} = \begin{bmatrix} 1 & 0 \\ -c\theta & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -c\theta \\ 0 & 1 \end{bmatrix} \underline{x}_n = \hat{A} \underline{x}_n$$

Simple computation shows that the eigenvalues of $\hat{A}$ are distinct and on the unit circle if $0 < c^2 \theta^2 < 4$. Hence, for small θ , $\hat{A}^N$ is bounded for all N .

More important, simple calculation shows that $x_n^2 + y_n^2 - c\theta x_n y_n$ is invariant, hence all points lie on the conic $x^2 + y^2 - c\theta xy = K$ where K is determined by the starting point $\underline{x}_0$.

The equivalent three dimensional modification is to write

$$x_{n+1} = x_n - c\theta y_n + b\theta z_n$$

$$y_{n+1} = c\theta x_{n+1} + y_n - a\theta z_n$$

$$z_{n+1} = -b\theta z_{n+1} + a\theta y_{n+1} + z_n$$

or

$$\begin{bmatrix} I & U^T \end{bmatrix} \underline{x}_{n+1} = \begin{bmatrix} I & U \end{bmatrix} \underline{x}_n$$

where

$$U = \begin{bmatrix} 0 & -c\theta & b\theta \\ 0 & 0 & -a\theta \\ 0 & 0 & 0 \end{bmatrix}$$

Hence $\underline{x}_{n+1} = \hat{A} \underline{x}_n$ where $\hat{A} = [I + U^T]^{-1} [I + U]$. $\hat{A}$ has all unit magnitude eigenvalues if U is small, as shown by

Theorem

$$\text{If } \hat{A} = [I + E^T]^{-1} [I + E]$$

for any $m \times m$ matrix E then

- (i) The quadratic form $\underline{x}^T Q \underline{x}$ is invariant where $Q = 2I + E + E^T$
- (ii) The eigenvalues of $\hat{A}$ are on the unit circle and correspond to linear elementary divisors if $\|E\|$ is small enough.

Proof

By simple computation

$$\begin{aligned} \underline{x}_{n+1}^T Q \underline{x}_{n+1} &= \underline{x}_n^T \hat{A}^T Q \hat{A} \underline{x}_n \\ &= \underline{x}_n^T [I + E^T] [I + E]^{-1} [(I + E) + (I + E^T)] [I + E^T]^{-1} [I + E] \underline{x}_n \\ &= \underline{x}_n^T [(I + E) + (I + E^T)] \underline{x}_n \\ &= \underline{x}_n^T Q \underline{x}_n \end{aligned}$$

as claimed in part (i). If $\|E\|$ is small, the eigenvalues of Q can be bounded close to +1 by the Gerschgorin circle theorem. Q is symmetric, hence its eigenvalues are real and positive. Hence Q is positive definite and $\underline{x}^T Q \underline{x} = \|\underline{x}\|_Q$ is a norm. If an eigenvalue of $\hat{A}$ is greater than one in magnitude, or of magnitude one and corresponds to a non-linear elementary divisor, there exists an eigenvector or principal vector $\underline{x}$ such that $\underline{x}_N = \hat{A}^N \underline{x}$ grows unboundedly.

But this is not possible since $||\underline{x}_N||_Q = ||\underline{x}||_Q$ is bounded. Similarly, if there exists an eigenvalue of $\hat{A}$ inside the unit circle, its eigenvector $\underline{x}$ is such that $\underline{x}_N = \hat{A}^N \underline{x} \rightarrow 0$ as $n \rightarrow \infty$. But $||\underline{x}_N||_Q = ||\underline{x}||_Q$ in contradiction. Q.E.D.

Returning to the 3-space problem, we note that $\underline{x}_N$ remains on the conic surface $\underline{x}^T Q \underline{x} = \underline{x}_0^T Q \underline{x}_0$ which is an ellipsoid if $||U||$ is small enough. (To be precise, if $0 < [a^2 + b^2 + c^2 - abc\theta]\theta^2 < 4$.) This is not quite enough, as the "locus" of $\underline{x}_N$ (actually, a set of points) could wander all over the conic surface. However, in this case, one of the eigenvalues of $\hat{A}$ is 1. (Since A is real, at least one of its three eigenvalues is real, that is, ± 1 . When $U = 0$, the eigenvalues are all 1, and the eigenvalues are continuous functions of U , hence one eigenvalue is 1 for all small $||U||$.) Suppose the left eigenvector for the unit eigenvalue is $\underline{l}^T$. Then

$$\underline{l}^T \underline{x}_{n+1} = \underline{l}^T A \underline{x}_n = \underline{l}^T \underline{x}_n = \underline{l}^T \underline{x}_0$$

That is, the plane $\underline{l}^T \underline{x} = K$ is another invariant subspace for the transformation. Hence, the intersection of $\underline{x}^T Q \underline{x} = K$, and $\underline{l}^T \underline{x} = K_2$ is an invariant, and computed points $\underline{x}_N$ move around this ellipse.

Note These invariant planes are perpendicular to the vector $\underline{l}$, so, in a sense, $\underline{l}$ is the axis of rotation. $\underline{l} = [a, b-ac, c]^T$ is a small perturbation of the original axis $[a, b, c]^T$. However, in another sense, the line $\underline{v}$, where $\underline{v}$ is the right unit eigenvector of A , is the axis of rotation, because, if $\underline{x}_n = \underline{v}$, then

$$\underline{x}_{n+1} = A \underline{x}_n = A \underline{v} = \underline{v} = \underline{x}_n$$

so points on $\underline{v}$ are not moved. $\underline{v}$ is $[a, b, c]^T$. Hence the approximate rotation leaves the desired axis unchanged, but skews the planes of rotation a small amount.

In an m -dimensional space, we must first define what we mean by a rotation. A general unitary transformation whose determinant is $+1$ is often called a rotation. This has $m(m-1)/2$ "degrees of freedom". These could be taken to be the angles of rotation in the plane of each axis pair. In three dimensions, any combination of rotations is equivalent to a single rotation in some plane, but that is not true in higher dimensions. This means that we can continuously rotate an object in four or more dimensions, and never get back to the original position. Consequently, for display manipulation purposes, we are interested in the class of rotations in a two dimensional subspace (plane) which leave all points in an $m-2$ dimensional subspace invariant. We will characterize these by the unitary matrix $A(\theta) = MR(\theta)M^T$ where

$$R(\theta) = \left[\begin{array}{cc|cc} \cos\theta & -\sin\theta & & \\ \sin\theta & \cos\theta & & \\ \hline & & & \\ 0 & & & I_{m-2} \end{array} \right]$$

and where M is a unitary matrix of dimension m . To compute successive "frames" for display, we would like to replace $\cos\theta$ by 1 and $\sin\theta$ by θ in $A(\theta)$ and then use the trick used earlier to "stabilize" the method. We will show that the modified method leaves the same points invariant as does the original rotation, so that it is a reasonable approximation. Also, the rotation will be in a plane which is an $O(\theta)$ perturbation to the original plane.

For small θ , let

$$R = \left[\begin{array}{cc|cc} 1 & -\theta & & \\ \theta & 1 & & \\ \hline & & & \\ 0 & & & I_{m-2} \end{array} \right]$$

Note that the eigenvalues of R include $m-2$ equal to one, and that these have $m-2$ distinct right and left eigenvectors $\underline{v}_i$, $i=3, \dots, m$. These are also eigenvalues and eigenvectors of $R(\Theta)$.

Note that the right and left eigenvectors of $A(\Theta)$ and $A = MRM^T$ corresponding to the eigenvalues of one are $M\underline{v}_i$, $i=3, \dots, m$. R can be expressed as $(I + \hat{E} - \hat{E}^T)$ where $\|\hat{E}\| = O(\Theta)$. Then $A = I + M\hat{E}M^T - M\hat{E}^TM^T$. Let $M\hat{E}M^T = D + L + \hat{U}$ where L is strictly lower triangular, D is diagonal and $\hat{U}$ is strictly upper triangular. Define $E = \hat{E} - M^T(D + L + \hat{U})M$. Then we see that $R = I + E - E^T$ where $\|E\| = O(\Theta)$ and $E^T = \hat{U} - L^T = U$, a strictly upper triangular matrix. Hence $A = I + U - U^T$, and the stabilized method is given by

$$\underline{x}_{n+1} = A \underline{x}_n$$

where $\hat{A} = (I + U^T)^{-1} (I + U)$

We will show below that $\hat{A}$ has $m-2$ eigenvalues equal to one with eigenvectors equal to $M\underline{v}_i$ (the same as for A and $A(\Theta)$). This means that $\hat{A}$ has the same invariant subspace as $A(\Theta)$. We will also show that the left eigenvectors of $\hat{A}$ are of the form $\underline{v}_i^T M^T + \underline{d}$ where $\|\underline{d}\| = O(\Theta)$.

Lemma If $\underline{z}$ is a unit eigenvector of $A = (I + U - U^T)$ then it is a unit right eigenvector of $\hat{A} = (I + U^T)^{-1} (I + U)$, $\hat{A}_1 = (I + U)^{-1} (I + U^T)$, $\hat{A}_2 = (I - U^T)^{-1} (I - U)$ and $\hat{A}_3 = (I - U)^{-1} (I - U^T)$.

Proof

$$\underline{z} = A\underline{z} = (I + U - U^T)\underline{z}.$$

Hence

$$(I + U^T)\underline{z} = (I + U)\underline{z},$$

and
$$(I - U)\underline{z} = (I - U^T)\underline{z}.$$

The results follow by multiplying the first equation by $(I + U^T)^{-1}$ or $(I + U)^{-1}$, and the second equation by $(I - U)^{-1}$ or $(I - U^T)^{-1}$. Q.E.D.

Thus $\hat{A}$ has at least $m-2$ unit right eigenvectors when A does and hence leaves all points in the subspace spanned by these invariant.

Lemma If $\underline{z}$ is a unit eigenvector of $A = (I + U - U^T)$, a unit left eigenvector of $\hat{A} = (I + U^T)^{-1} (I + U)$ is $\underline{\ell}^T = \underline{z}^T (I + U^T) = \underline{z}^T (I + U) = \underline{z}^T + \underline{d}^T$ where $||\underline{d}^T|| = O(||U||) = O(\theta)$.

Proof

Note that $\underline{z}^T (I + U - U^T) = \underline{z}^T$. Hence $\underline{z}^T U = \underline{z}^T U^T$. Thus

$$\begin{aligned}\underline{\ell}^T \hat{A} &= \underline{z}^T (I + U^T) (I + U^T)^{-1} (I + U) \\ &= \underline{z}^T (I + U) = \underline{\ell}^T\end{aligned}$$

Obviously $||\underline{\ell}^T - \underline{z}^T|| = ||\underline{z}^T U^T|| = O(\theta)$.

Q.E.D.

Thus, for small θ , $\hat{A}$ has at least $m-2$ distinct right unit eigenvectors $\underline{\ell}_i$ which are a $O(\theta)$ perturbation of the vectors $\underline{v}_i^T M$. The plane $P = \sum_{i=3}^m (\underline{\ell}_i \underline{x} = k_i)$ is invariant, so the conic intersection of this and $\underline{x}^T (2I + U - U^T) = k$ is a locus of the rotation. As $\theta \rightarrow 0$, this tends to a circle in a plane perpendicular to the subspace of invariant points.

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