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by

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**DIRECT USE OF YOUNG TABLEAU ALGEBRA
TO GENERATE THE CLEBSCH-GORDAN COEFFICIENTS OF SU(2)**

by

Jack Nachamkin

ABSTRACT

Although it is well known that the irreducible representations of the SU(N) groups may be generated by using Young tableau algebra, this technique seems to have found little use for the derivation of closed algebraic expressions for the Clebsch-Gordan coefficients of these groups. A frontal attack on the derivation of these coefficients using tableau symmetrizers is described. For example, the SU(2) group illustrates the fundamental ideas behind the process.

I. INTRODUCTION

This report describes how the Young tableau algebra technique is applied to find explicit algebraic formulas for the Clebsch-Gordan coefficients (CGC), with attention focused on the SU(2) group. Physicists are familiar with this group because it is isomorphic to the three-dimensional rotation group, R_3 .

In the SU(2), or R_3 , case, spin operators are used to define an "internal" symmetry; Young tableau symmetrizers are applied directly to generate operators, R_{JJ} , which transform the state

$$|j_1, j_2; J' = j_1 + j_2, M' = J'\rangle$$

into the state $|j_1, j_2; J, M = J\rangle$. If the angular momentum vectors $\vec{J}_1$ and $\vec{J}_2$ operate on particles with angular momentum j_1 and j_2 , respectively, the R_{JJ} appear as

$$R_{JJ} = \sum_{\gamma=0}^n g_{j\gamma} J_{1-}^{\gamma} J_{2-}^{n-\gamma}, \quad (1)$$

where the $g_{j\gamma}$ depend only on j_1, j_2, J , and γ , and the $J_{1\pm}$'s are components of J_1 , with

$$n = j_1 + j_2 - J. \quad (2)$$

In Sec. II we review some of the properties of states made up of N spin- $\frac{1}{2}$ particles coupled to a total spin S . The key property is that all states with a given S transform among themselves under the permutation on particle numbers, thus forming a basis for the $(\frac{1}{2}N + S, \frac{1}{2}N - S)$ irreducible representation of the symmetric group of N indices.

In Sec. III the Young tableau symmetrizers are defined and used to generate R_{JJ} from an unsymmetrized product of spin operators, identifying sums of spin operators as J_{1-} and J_{2-} . After obtaining R_{JJ} , the operator

$$R_{JM} \propto (J_{1-} + J_{2-})^{J-M} R_{JJ} \quad (3)$$

is used to find an expression for CGC.

$$C_{m_1 m_2 M}^{j_1 j_2 J} = \langle j_1 m_1, j_2 m_2 | j_1 j_2 JM \rangle.$$

II. SOME PROPERTIES OF SPIN STATES AND OPERATORS

When using a set of N coupled spin- $\frac{1}{2}$ particles, it is well known that the states with total spin S

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have simple properties under the group of particle-number permutations. This may be seen immediately by considering the operator

$$Q = \sum_{i,j=1}^N \left\{ (s_{i+} s_{j-} + s_{i-} s_{j+}) / 2 + s_{i0} s_{j0} \right\} \quad (4)$$

where the spin operators s_{i+} , s_{i0} , and s_{i-} operate on the spin states ϕ_{i+} "spin-up" and ϕ_{i-} "spin-down":

$$s_{i+} \phi_{i+} = 0 \quad (5a)$$

$$s_{i+} \phi_{i-} = \phi_{i+} \quad (5b)$$

and

$$s_{i0} \phi_{i\pm} = \pm \frac{1}{2} \phi_{i\pm} \quad (5c)$$

These operators not only obey the usual angular momentum commutation rules

$$[s_{i+}, s_{i-}] = 2s_{i0} \quad (6a)$$

and

$$[s_{i0}, s_{i\pm}] = s_{i\pm} \quad (6b)$$

but from Eqs. (5a) and (5b)

$$(s_{i\pm})^2 = 0 \quad (7)$$

Returning to Eq. (4), note that Q is invariant under any permutation of particle numbers. Defining the components of the total-spin vector $\vec{S}$

$$S_{\zeta} = \sum_{i=1}^N s_{i\zeta}, \quad \zeta = +, 0, - \quad (8)$$

It is also evident that

$$Q = \vec{S} \cdot \vec{S} \quad (9)$$

From the above, we may infer that eigenstates of Q are those for which the total spin S is a good quantum number, and thus form the bases for the irreducible representations of the symmetric group on N particles.^{1,2} The irreducible representation of the symmetric group corresponding to spin S is de-

noted by the two-rowed Young tableau with n_1 boxes in the first row and n_2 boxes in the second row,¹ where

$$n_1 = \frac{1}{2} N + S \quad (10a)$$

and

$$n_2 = \frac{1}{2} N - S \quad (10b)$$

Graphically this looks like

$$\begin{array}{ccc} \begin{array}{|c|c|c|} \hline & & \\ \hline \end{array} & \dots & \begin{array}{|c|c|c|} \hline & & \\ \hline \end{array} \end{array} \quad \begin{array}{l} n_1 \text{ boxes} \\ n_2 \text{ boxes} \end{array} \quad (11)$$

Such diagrams are referred to by the partition of N into n_1 and n_2 , $[n_1, n_2]$. The state with the highest spin, $S = \frac{1}{2} N$, corresponds to the partition $[N, 0]$, usually abbreviated to $[N]$.

In matrix notation the operators and states may be replaced by

$$s_{i+} \rightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}_i, \quad s_{i-} \rightarrow \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}_i, \quad s_{i0} \rightarrow \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_i \quad (12)$$

$$\phi_{i+} \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}_i, \quad \text{and} \quad \phi_{i-} \rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}_i \quad (13)$$

From Eqs. (7) and (8), it can be shown by induction that

$$(S_{\pm})^r = r! \sum_{i_1 < i_2 < \dots < i_r \leq N} s_{i_1 \pm} s_{i_2 \pm} \dots s_{i_r \pm} \quad (14)$$

Because the sum in Eq. (9) contains $\binom{N}{r}$ terms, where

$$\binom{N}{r} = \frac{N!}{r!(N-r)!} \quad (15)$$

and because $(S_{\pm})^r$ applied to a state with definite S changes only the magnetic quantum number, applying $(S_-)^r$ to the state $|S = \frac{1}{2} N, M_S = \frac{1}{2} N\rangle$, which is made up of only one term, yields

$$\begin{aligned} (S_-)^r |S = \frac{1}{2} N, M = \frac{1}{2} N\rangle \\ = r! \sqrt{\frac{N!}{r!}} |S = \frac{1}{2} N, M = \frac{1}{2} N - r\rangle \quad (16a) \end{aligned}$$

This may be generalized immediately to the usual angular momentum stepping operator J_- , i.e.,

$$(J_-)^r |J, M = J\rangle = r! \sqrt{\frac{(2J)!}{r!}} |J, M = J - r\rangle, \quad (16b)$$

where

$$J_- |J, M\rangle = \sqrt{(J+M)(J-M+1)} |J, M-1\rangle. \quad (17)$$

In effect, the operators S_ζ may be regarded in the same way as the angular momentum operators J_ζ , because they have the same commutation properties. The only difference is that the S_ζ 's have an internal structure that is not quite evident in the J_ζ 's but is used in Sec. III.

Before proceeding, we break the S_ζ 's into sums, each containing $2j_1$ terms

$$S_\zeta = \sum s_{1\zeta}, \quad (18a)$$

where

$$s_{1\zeta} = \sum_{k=1}^{2j_1} s_{k\zeta}, \quad (18b)$$

$$s_{2\zeta} = \sum_{k=2j_1+1}^{2j_1+2j_2} s_{k\zeta}, \text{ etc.} \quad (18c)$$

Note that now

$$(s_{1\pm})^{2j_1+1} = 0. \quad (18d)$$

These sums will now represent angular momentum operators.

III. TABLEAU AND ANGULAR MOMENTUM SHIFTING OPERATORS

Given the sums

$$S = s_{1\zeta} \left(\sum_{k=1}^{2j_1} s_{k\zeta} \right) + s_{2\zeta} \left(\sum_{k=2j_1+1}^{2j_1+2j_2=N} s_{k\zeta} \right), \quad (19a)$$

and drawing the analogy to

$$\vec{J} = \vec{J}_1 + \vec{J}_2, \quad (19b)$$

we wish to generate a state with a total spin

$$J = \frac{1}{2} N - n = j_1 + j_2 - n, \quad (20)$$

while preserving the property that the first $2j_1$ spin states are coupled to spin j_1 while the last $2j_2$ spin states are coupled to j_2 . This state would then correspond to one of two coupled angular momenta,

$$|j_1, j_2; JM\rangle. \quad (21)$$

Once again, expressed in terms of spins, there are "internal" symmetries that are not immediately evident in Eq. (21).

The desired state corresponds to the Young tableau $[N-n, n]$. There is a very simple way for generating such states.^{2,4} Fill in the boxes of the Young tableau with the numbers 1, 2, ..., $N = 2j_1 + 2j_2$ in any order. Next form the operators that symmetrize the terms belonging to the particle numbers found in separate rows, and then for the operators that antisymmetrize to the particle numbers in each column.

For example, consider the partition $[F] = [3, 2]$ with the numbers 1, ..., 5 written in order:

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & \\ \hline \end{array}. \quad (22)$$

The operator that symmetrizes the subscripts 1, 2, and 3 is denoted by $[[1, 2, 3]]$, and the operator symmetrizing 4 and 5 is denoted by $[[4, 5]]$. Antisymmetrizers are enclosed by two curly brackets, such as $\{1, 4\}$ and $\{2, 5\}$. Symmetrizers $[[a_1, a_2, \dots, a_r]]$ are merely the sum of $r!$ permutation operators defined on the indices $a_1, a_2, \dots, a_r$

$$[[a_1, a_2, \dots, a_r]] = \sum_{\gamma=1}^{r!} P_\gamma. \quad (22a)$$

Similarly,

$$\{i, j\} = 1 - P_{ij}, \quad (22b)$$

where P_{ij} permutes the subscripts i and j .

A tableau symmetrizer for SU(2) may be defined as a product of the two $[[\]]$ operators and the $\{ \}$ operators, taken in any order. The most convenient symmetrizer is usually determined by the specific problem, but we will adhere to the convention that all the $\{ \}$'s will be to the left of the $[[\]]$'s. Thus, Eq. (22) corresponds to

$$Y_{[F]} = \{1,4\}\{2,5\}[[1,2,3]] [[4,5]] \quad (22c)$$

If, in addition, we want to symmetrize the subscripts 1, 2 and 3, 4, 5, we would need the operator

$$Y'_{[F]} = [[12]] [[345]] Y_{[F]} \quad (22d)$$

We now follow the convention that the numbers 1, 2, $\dots, 2j_1 + 2j_2 = N$ will be written in order, that is, the first row will hold the numbers 1, 2, $\dots, N-n$ while the second row reads $N - n + 1, N - n + 2, \dots, N$. If

$$r_1 = N - n + 1 \quad (23)$$

then the first row reads 1, 2, $\dots, r_0$ and the second $r_1, r_2, \dots, r_n = N$. The Young tableau symmetrizer, which corresponds to Eq. (21), is

$$Y_{[N-n,n]} = \{[1, \dots, 2j_1]\} \{[2j_1 + 1, \dots, N]\} \{1, r_1\} \dots \{n, r_n\} \times [[1, \dots, r_0]] [[r_1 \dots r_n]] \quad (24)$$

In this way subscripts belonging to the same J cannot occur in the same column.

Applying the operator in Eq. (24) to an unsymmetrized product or sum of products of n operators of the form s_{i-} will yield either zero or an angular momentum shifting operator. In schematic form then

$$\rho R_{JJ}(J_1 J_2) = Y'_{[N-n,n]} \times (\text{product of } n s_{i-}'s) \quad (25a)$$

and

$$R_{JJ}(J_1 J_2) |j_1 j_2; j_1 + j_2, j_1 + j_2\rangle = \rho |j_1 j_2, J = j_1 + j_2 - n, M = J\rangle \quad (25b)$$

where ρ is a normalization constant. The most convenient product of $n s_{i-}$'s seems to be

$$P = s_{r_1-} s_{r_2-} \dots s_{r_n-} \quad (26)$$

Because

$$[[r_1 \dots r_n]] P = n! P \quad (27)$$

and

$$[[1 \dots r_0]] P = r_0! P \quad (28)$$

$Y'_{[N-n,n]} P$ reduces to

$$\rho R_{JJ}(J_1 J_2) = Y'_{[N-n,n]} P = n! r! \{[1, \dots, 2j_1]\} \times \{[2j_1 + 1, \dots, N]\} \times \{1r_1\} \dots \{nr_n\} P \quad (29)$$

Straightforward simple manipulation, as shown in Appendix A, yields

$$\rho R_{JJ}(J_1 J_2) = r_0! (2j_1)! (2j_2)! \times \sum_{q=0}^n \frac{(-1)^q \binom{n}{q}}{\binom{2j_1}{q} \binom{2j_2}{n-q}} J_1^q J_2^{n-q} \quad (30)$$

Applying $\rho P_{jj}(J_1 J_2)$ to $|j_1 j_2\rangle$ and using Eqs. (15), (16a), (20), and (23), we obtain

$$\begin{aligned} \rho |j_1 j_2, J = j_1 + j_2 - n, M = J\rangle &= (j_1 + j_2 + J)! (2j_1)! (2j_2)! (j_1 + j_2 - J)! \\ &\times \sum_{q=0}^n \frac{(-1)^q \binom{n}{q}}{\sqrt{\binom{2j_1}{q} \binom{2j_2}{n-q}}} |j_1, j_1 - q\rangle |j_2, j_2 - n + q\rangle \end{aligned} \quad (31)$$

To evaluate ρ , Eq. (15) yields

$$\begin{aligned} \frac{\binom{n}{q}}{\sqrt{\binom{2j_1}{q} \binom{2j_2}{n-q}}} &= \frac{n!}{\sqrt{(2j_1)! (2j_2)!}} \\ &\times \sqrt{\frac{(2j_1 - q)! (2j_2 - n + q)!}{q! (n - q)!}} \end{aligned} \quad (32)$$

Making the identifications

$$m_1 = j_1 - q, m_2 = j_2 - n + q \quad (33)$$

Eq. (31) now reads as

$$\rho |j_1 j_2; J, M = J\rangle = (j_1 + j_2 + J)! \sqrt{(2j_1)!(2j_2)!} \\ \times [(j_1 + j_2 - J)!]^2 \times \sum_{m_1+m_2=J} (-1)^{j_1-m_1} \\ \times \sqrt{\frac{(j_1+m_1)!(j_2+m_2)!}{(j_1-m_1)!(j_2-m_2)!}} |j_1^{m_1}\rangle |j_2^{m_2}\rangle \quad (34)$$

and

$$\rho^2 = (j_1 + j_2 + J)! \sqrt{(2j_1)!(2j_2)!} \\ \times [(j_1 + j_2 - J)!]^2 \\ \times \sum_{m_1+m_2=M} \frac{(j_1+m_1)!(j_2+m_2)!}{(j_1-m_1)!(j_2-m_2)!} \quad (35)$$

Hamermesh,² pp. 372 and 373, shows that

$$\sum_{m_1+m_2=M} \frac{(j_1+m_1)!(j_2+m_2)!}{(j_1-m_1)!(j_2-m_2)!} = (j_1 - j_2 + J)! \\ \times (j_2 - j_1 + J)! \binom{J+j_1+j_2+1}{j_1+j_2-J} \quad (36)$$

Thus,

$$R_{JJ} = \sqrt{\frac{\binom{2j_1}{n} \binom{2j_2}{n}}{(j_1+j_2+J+1) \binom{2J+1}{n}}} \sum_q (-1)^q \binom{n}{q} \\ \times \frac{j_1^q j_2^{n-q}}{q!(n-q)! \binom{2j_1}{q} \binom{2j_2}{n-q}} \quad (37)$$

Multiplying R_{JJ} by

$$J_-^{J-M} = (J_{1-} + J_{2-})^{J-M} = \sum_p \binom{J-M}{p} J_{1-}^p J_{2-}^{J-M-p} \quad (38)$$

and using Eq. (16a), we obtain after some manipulation,

$$|j_1 j_2^{JM}\rangle = \sqrt{\frac{\binom{2j_1}{n} \binom{2j_2}{n}}{(j_1+j_2+J+1) \binom{2J+1}{n}}} \times \sum_{m_1+m_2=M} \\ \sqrt{\frac{\binom{2j_1}{j_1+m_1} \binom{2j_2}{j_2+m_2}}{\binom{2J}{J+M}}} \left[\sum_q (-1)^q \frac{\binom{n}{q} \binom{j_1-m_1}{q} \binom{j_2-m_2}{q}}{\binom{2j_1}{q} \binom{2j_2}{n-q}} \right] \\ \times |j_1^{m_1}\rangle |j_2^{m_2}\rangle \quad (39)$$

Within a phase, then,

$$\langle j_1^{m_1} | j_2^{m_2} j_1 j_2^J = j_1 + j_2 - n, M \rangle$$

$$= \sqrt{\frac{\binom{2j_1}{n} \binom{2j_2}{n} \binom{2j_1}{j_1+m_1} \binom{2j_2}{j_2+m_2}}{(j_1+j_2+J+1) \binom{2J}{J+M}}} \\ \times \sum_q (-1)^q \frac{\binom{n}{q} \binom{j_1-m_1}{q} \binom{j_2-m_2}{q}}{\binom{2j_1}{q} \binom{2j_2}{n-q}} \quad (40a)$$

which is shown in Appendix B to be

$$= \sqrt{\frac{\binom{2j_1}{j_1-j_2+J} \binom{2j_2}{j_2-j_1+J}}{\binom{2j_1}{j_1+m_1} \binom{2j_2}{j_2+m_2} \binom{2J}{J+M} \binom{2J+1}{j_1+j_2+J+1}}} \\ \times \sum_q (-1)^q \binom{j_1+j_2-J}{q} \binom{j_2-j_1+J}{j_2+m_2-q} \binom{j_1-j_2+J}{j_1-m_1-q} \quad (40b)$$

The above formulas are expressed in terms of binomial coefficients instead of factorials for ease of eventual computation.

IV. DISCUSSION

The analysis in this report was aimed primarily at obtaining the CGC of R_3 , which is isomorphic to $SU(2)$. From the group-theoretic viewpoint, the most important relations are expressed by Eqs. (5), (6), and (7), yielding the group's structure. Taking advantage of the fact that irreducible representations (IRs) of $SU(2)$ may be represented by two-rowed Young tableaux and that the highest weight state is totally symmetric leads to a straightforward prescription for generating all desired states.

Therefore, the generalization to higher groups is almost immediate. The IRs of the $SU(N)$ group may

be represented by $(N + L)$ -rowed Young tableaux. Because the highest weight state of a product representation is nondegenerate, we find the correct symmetrizing operators that will project onto the desired state, i.e., the correct symmetry-breaking shift operators analogous to R_{JJ} of Eq. (1).

Subsequent work is being directed toward finding such operators for higher groups, mainly for obtaining CGC by computer. Future reports will deal with higher groups and the "strategies" imposed by nonlinear Young diagrams and multiple reducibility.

A somewhat amusing aspect is shown in Appendix C.

APPENDIX A

SYMMETRY OPERATORS

The symmetrizer

$$[[a_r]] \equiv [[a_1, \dots, a_r]] = \sum_{\gamma=1}^{r!} P_{\gamma} \quad (A-1)$$

contains the $r!$ possible permutations defined on $a_1, \dots, a_r$. We are operating with $[[a_r]]$ on products and sums of products of the s_{i-} , remembering that $(s_{1-})^2 = 0$, and that i 's belong only to the set

$$\{a_1, a_2, \dots, a_r\}.$$

(It is trivial to extend this to the general case.)

Thus, consider

$$P_n = s_{i_1-} s_{i_2-} \dots s_{i_n-} \quad (A-2)$$

and

$$i_1 < i_2 < \dots < i_n. \quad (A-2a)$$

There are $\binom{r}{n}$ different ways of choosing a P_n , and the operator $[[a_r]]$ applied to a P_n produces a sum containing all of the possible P_n 's the same number of times. Operating with $[[a_r]]$ on the already symmetric sum of all the P_n 's reproduces that sum times the factor $r!$. That means

$$[[a_r]] P_n = \left[r! / \binom{r}{n} \right] \times (\text{sum of all } P_n \text{'s}). \quad (A-3)$$

If we define

$$J_- = \sum_{i=1}^r s_{a_{i-}},$$

then, from Eq. (14)

$$[[a_r]] P_n = \left\{ \frac{r!}{n!} / \binom{r}{n} \right\} J_-^n. \quad (A-4)$$

To evaluate the right-hand side of Eq. (29), we define the quantities $P^{i,j,\dots}$, where

$$P^{i,j,\dots} = \prod_{\substack{k=1 \\ k \neq i,j,\dots}}^n s_{k-}. \quad (A-5)$$

The action of $\{i, r_1\}$ on P is

$$\{i, r_1\} P = P - s_{i-} P^i,$$

so that

$$\begin{aligned} \{i, r_1\} \{2, r_2\} \dots \{n, r_n\} P &= P - \sum_{i=1}^n s_{i-} P^i \\ &+ \sum_{j>1} s_{i-} s_{j-} P^{ij} \pm \dots + (-1)^n s_{i-} s_{2-} \dots s_{n-}. \end{aligned} \quad (A-6)$$

Using Eq. (A-3) applied to the P's

$$\begin{aligned} & \{[2j_1 + 1, \dots, 2j_1 + 2j_2]\} \{1, r_1\} \dots \{n, r_n\} \\ &= (2j_2)! \sum_q \left\{ (-1)^q j_{2-}^{n-q} \hat{j}_{1-}^q / \left[(n-q)! q! \binom{n}{n-q} \right] \right\} \end{aligned} \quad (A-7)$$

where j_{1-} is defined by Eqs. (19a) and (19b) and

$$\hat{j}_{1-} = \sum_{i=1}^n s_{i-} \quad (A-8)$$

The last step is to apply Eq. (A-3) to the $\hat{j}_{1-}^q$, each of which contain $\binom{n}{q}$ terms so that

$$\begin{aligned} & \{[1, \dots, 2j_1]\} \{[2j_1, 1, \dots, 2j_1 + 2j_2]\} \{1, r_1\} \\ & \{2, r_2\} \dots \{n, r_n\} P = \frac{(2j_1)! (2j_2)!}{n!} \\ & \times \sum_q (-1)^q \frac{\binom{n}{q}^2}{\binom{2j_1}{q} \binom{2j_2}{n-q}} j_{1-}^{n-q} \end{aligned} \quad (A-9)$$

APPENDIX B

A COMBINATORIAL IDENTITY

In Eq. (40) the sum on q can be recast by using Eq. (15) into the form

$$\begin{aligned} & \sum_q (-1)^q \frac{\binom{n}{q} \binom{2j_1 - q}{j_1 + m_1} \binom{2j_2 - n + q}{j_2 + m_2}}{\binom{2j_1}{j_1 + m_1} \binom{2j_2}{j_2 + m_2}} \\ &= K \sum_q (-1)^q \binom{n}{q} \binom{a-q}{b} \binom{c+q}{d} \end{aligned} \quad (B-1)$$

We may transform to this sum by the following artifice. First expand

$$(Y - X)^n = \sum_q (-1)^q \binom{n}{q} X^q Y^{n-q} \quad (B-2)$$

now

$$X^c Y^{a-n} (Y - X)^n = \sum_q (-1)^q \binom{n}{q} X^{c+q} Y^{a-q} \quad (B-3)$$

Let

$$X = T + U, \quad Y = T + V \quad (B-4)$$

so that

$$X^c Y^{a-n} (Y - X)^n = \sum_{q,b,d} (-1)^q \binom{n}{q} \binom{a-q}{b} \binom{c+q}{d}$$

$$\times U^d V^b T^{a+c-d-b} \quad (B-5)$$

However, it is also true that

$$\begin{aligned} & X^c Y^{a-n} (Y - X)^n = \sum_{\mu, \omega, \eta} (-1)^\mu \binom{n}{\mu} \binom{c}{\omega} \binom{a-n}{\eta} U^{\mu+\omega} \\ & \times V^{n-\mu+\eta} T^{a+c-\eta-n-\omega} \end{aligned} \quad (B-5b)$$

Identifying

$$q = \mu, \quad d = \mu + \omega, \quad b = n - \mu + \eta, \quad (B-6)$$

it follows that

$$\begin{aligned} & \sum_q (-1)^q \binom{n}{q} \binom{a-q}{b} \binom{c+q}{d} \\ &= \sum_q (-1)^q \binom{n}{q} \binom{c}{d-q} \binom{a-n}{b+q-n} \end{aligned} \quad (B-7)$$

Equation (40b) now follows immediately from Eq. (40a) by combining Eqs. (B-1) and (B-7).

An interesting consequence of the analysis in this Appendix is the derivation of a CGC generating function. By comparing Eqs. (B-5b) and (40b), we obtain

$$(T+U)^{j_1-j_2+j_3} (T+V)^{-j_1+j_2+j_3} (V-U)^{j_1+j_2-j_3}$$

$$= \sqrt{\frac{(j_1+j_2-j_3)!(-j_1+j_2+j_3)!(j_1+j_2-j_3)!(j_1+j_2+j_3+1)!}{2j_3+1}}$$

$$\times \sqrt{\frac{\binom{j_1+j_2+j_3+1}{j_1+j_2-j_3}}{\binom{2j_1}{j_1-j_2+j_3}\binom{2j_2}{j_1+j_2+j_3}}}$$

$$\times \sum \frac{U^{j_2+m_2} V^{j_1+m_1} T^{j_3-m_3}}{\sqrt{(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!(j_3+m_3)!(j_3-m_3)!}}$$

$$\times \sum_{m_1 m_2} \sqrt{\binom{2j_1}{j_1+m_1}\binom{2j_2}{j_2+m_2}\binom{2j_3}{j_3+m_3}} c_{m_1 m_2 m_3}^{j_1 j_2 j_3}$$

$$\times c_{m_1 m_2 m_3}^{j_1 j_2 j_3} \quad (B-8)$$

$$\times U^{j_2+m_2} V^{j_1+m_1} T^{j_3-m_3}$$

APPENDIX C

SOME SPECIAL CGC

Using Eq. (40a) or (40b), some CGC may be put into very simple form, i.e.,

Using the relation

$$c_{m_1 m_2 M}^{j_1 j_2 J=j_1+j_2} = \sqrt{\frac{\binom{2j_1}{j_1+m_1}\binom{2j_2}{j_2+m_2}}{\binom{2J}{J+M}}} \quad (C-1)$$

$$c_{m_1 m_2 M}^{j_1 j_2 J} = (-1)^{j_1-m_1} \sqrt{\frac{2J+1}{2j_2+1}} c_{m_1-M-m_2}^{j_1 j_2 J} \quad (C-3)$$

and

Eqs. (C-1) and (C-2) may serve as the basis of tables for CGC where $j_2 = 1/2, 1$.

$$c_{m_1 m_2 M}^{j_1 j_2 J=j_1+j_2-1} = \sqrt{\frac{j_1 j_2}{2(j_1+j_2)}} \left(\frac{m_1}{j_1} - \frac{m_2}{j_2} \right)$$

An amusing aspect of Eq. (C-2) is that some of the "accidentally zero" coefficients can be generated. Equation (C-1) is quoted in several places^{2,5} and was the original basis for this work. It, too, has an amusing point. The square of the CGC in Eq. (C-1) is the probability that a draw of $J \pm M$ balls from an urn containing $2j_1$ red balls and $2j_2$ green balls will contain $j_1 \pm m_1$ red balls and $j_2 \pm m_2$ green balls.

$$\times \sqrt{\frac{\binom{2j_1}{j_1+m_1}\binom{2j_2}{j_2+m_2}}{\binom{2J}{J+M}}} \quad (C-2)$$

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