neg

MASTER

511

SCR-98

Sandia Corporation

... MONOGRAPH

GENERALIZED FUNCTIONS OF GREEN
FOR SYSTEMS OF
ORDINARY DIFFERENTIAL EQUATIONS

by

Oswald Wyler

July 1959

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency Thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

DISCLAIMER

Portions of this document may be illegible in electronic image products. Images are produced from the best available original document.

LEGAL NOTICE

This report was prepared as an account of Government sponsored work. Neither the United States, nor the Commission, nor any person acting on behalf of the Commission:

A. Makes any warranty or representation, expressed or implied, with respect to the accuracy, completeness, or usefulness of the information contained in this report, or that the use of any information, apparatus, method, or process disclosed in this report may not infringe privately owned rights; or

B. Assumes any liabilities with respect to the use of, or for damages resulting from the use of any information, apparatus, method, or process disclosed in this report.

As used in the above, "person acting on behalf of the Commission" includes any employee or contractor of the Commission, or employee of such contractor, to the extent that such employee or contractor of the Commission, or employee of such contractor prepares, disseminates, or provides access to, any information pursuant to his employment or contract with the Commission, or his employment with such contractor.

Printed in USA. Price \$ 0.75. Available from the Office of
Technical Services, Department of Commerce,
Washington 25, D.C.

SCR-98
TID-4500 (14th Edition)
Physics and Mathematics

SANDIA CORPORATION MONOGRAPH

GENERALIZED FUNCTIONS OF GREEN FOR SYSTEMS
OF ORDINARY DIFFERENTIAL EQUATIONS

by

Oswald Wyler
Department of Mathematics
University of New Mexico
and
Consultant to Sandia Corporation

July 1959

* *

TABLE OF CONTENTS

	Page
1. Introduction	5
2. Adjoint Matrix Operators	8
3. Adjoint Boundary-Value Problems	6
4. The Algebra of Adjoint Boundary Conditions	8
5. Matrix Operators Associated With a Scalar Operator	9
6. Canonically Associated Operators	11
7. On Solutions of Boundary-Value Problems	13
8. An Auxiliary Kernel Matrix	14
9. The Generalized Function of Green	15
10. The Operator $\mathcal{G}$	17
11. Properties of $G(t, s)$	19
12. Generalized Functions of Green for Adjoint Problems	20
13. On Scalar Boundary-Value Problems	21
14. Special Cases	23
REFERENCES	25

GENERALIZED FUNCTIONS OF GREEN FOR SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

1. Introduction

We study in this paper boundary-value problems for systems of ordinary linear differential equations, and their solution by means of generalized functions of Green, as defined in Reference 1, p. 306 ff.

The adjoint operator, $\mathcal{L}^0 y$, of a matrix operator, $\mathcal{L}x = Px + Q\dot{x}$, is defined in Section 2, and adjoint boundary conditions for adjoint matrix operators are defined in Section 3. We show in Section 5 that a matrix operator, $\mathcal{L}x$, can be associated in various ways with a scalar differential operator, Lx_1 . If we require that two adjoint matrix operators, $\mathcal{L}x$ and $\mathcal{L}^0 y$, are associated with scalar differential operators, Lx_1 and $L^0 y_1$, then $\mathcal{L}$ is uniquely determined by L , and the scalar operators Lx_1 and $L^0 y_1$ are adjoint (Section 6).

Generalized functions of Green are obtained in Section 9, and their use for solving boundary-value problems is discussed in Section 10. Properties which characterize a generalized function of Green are obtained in Section 11, and it is shown in Section 12 that the generalized functions of Green of adjoint boundary-value problems are adjoint kernel matrices. Generalized functions of Green for scalar boundary-value problems are discussed in Section 13, and special cases are taken up in Section 14.

All functions occurring in this paper, except the entries of kernel matrices, are assumed to be functions of a real variable t , defined and continuous for t in an interval $[a, b]$, with real or complex values. We denote by $\mathcal{C}$ the complex vector space of all functional columns (of a given length n) with entries satisfying this condition, and by $\mathcal{D}$ the subspace of $\mathcal{C}$ consisting of all columns in $\mathcal{C}$ with continuously differentiable entries.

We denote by T^* the adjoint or conjugate transpose matrix of a matrix T with complex entries; i.e., $T^* = ||\bar{t}_{ji}||$ if $T = ||t_{ij}||$. Two columns f and g of $\mathcal{C}$ are called orthogonal if $\int_a^b f^* g = 0$.

2. Adjoint Matrix Operators

Throughout the present paper, we are concerned with a matrix operator

$$(2.1) \quad (\mathcal{L}x)(t) = P(t)x(t) + Q(t)\dot{x}(t)$$

which operates on $\mathcal{D}$. We shall assume that P and Q are square matrices of order n , and that Q is a nonsingular matrix with continuously differentiable entries.

The adjoint operator, $\mathcal{L}^0 y$, of the matrix operator, $\mathcal{L}x$, is defined by

$$(2.2) \quad y^*(\mathcal{L}x) - (\mathcal{L}^0 y)^*x = \frac{d}{dt}(y^*Qx),$$

for x and y in $\mathcal{D}$. This equation leads immediately to:

$$(2.3) \quad \mathcal{L}^0 y = (P - \dot{Q})^*y - Q^*\dot{y}.$$

It follows from Equation (2.3) that $\mathcal{L}$ is the adjoint operator of $\mathcal{L}^0$ if $\mathcal{L}^0$ is the adjoint operator of $\mathcal{L}$.

The operator $\mathcal{L}$ is called self-adjoint if $\mathcal{L}^0 = \mathcal{L}$. This is the case if and only if $P^* = P - \dot{Q}$ and $Q^* = -Q$.

The solutions of the homogeneous system $\mathcal{L}x = 0$ form an n -dimensional subspace of $\mathcal{D}$ which is the column space of a nonsingular functional matrix U . Such a matrix is called a fundamental matrix of the operator $\mathcal{L}$. Throughout this paper, we shall denote by U and V fundamental matrices of the operators $\mathcal{L}$ and $\mathcal{L}^0$, respectively. Then $\mathcal{L}U = \mathcal{L}^0V = 0$, and it follows immediately from Equation (2.2) that V^*QU is a nonsingular constant matrix. We shall denote this matrix by E^{-1} . Then we have:

$$(2.4) \quad \mathcal{L}U = 0, \quad \mathcal{L}^0V = 0, \quad Q^{-1} = UEV^*.$$

3. Adjoint Boundary-Value Problems

For a functional matrix, Z , with columns in $\mathcal{D}$, we put $\partial Z = \begin{vmatrix} Z(a) \\ Z(b) \end{vmatrix}$. Then ∂ defines a linear mapping of $\mathcal{D}$ onto the space of columns of length $2n$. If we put

$$(3.1) \quad M = \begin{vmatrix} -Q(a) & 0 \\ 0 & Q(b) \end{vmatrix},$$

then M is a nonsingular matrix of order $2n$, and it follows from Equation (2.2) that

$$(3.2) \quad \int_a^b y^*(Lx) - \int_a^b (L^0 y)^* x = (\partial y)^* M (\partial x),$$

for x and y in $\mathcal{D}$. It follows in particular that we have:

$$(3.3) \quad (\partial V)^* M (\partial U) = 0.$$

If $\mathcal{B}$ is a subspace of $\mathcal{D}$, we denote by $\mathcal{B}^0$ the set of all columns y of $\mathcal{D}$ such that

$$(3.4) \quad \int_a^b y^*(Lx) = \int_a^b (L^0 y)^* x$$

for all columns x in $\mathcal{B}$, and by $\mathcal{B}^+$ the set of all columns x of $\mathcal{D}$ such that Equation (3.4) holds for any column y in $\mathcal{B}^0$. Now Equation (3.4) is equivalent to

$$(3.5) \quad (\partial y)^* M (\partial x) = 0,$$

by Equation (3.2). Thus it depends only on ∂z whether a column z of $\mathcal{D}$ belongs to one of the spaces $\mathcal{B}^0$ or $\mathcal{B}^+$ or not, and these spaces are completely determined by $\partial \mathcal{B}$. Now $v^* M u$ is a nonsingular semibilinear form, and v is in $\partial \mathcal{B}^0$, if and only if $v^* M u = 0$ for all u in $\partial \mathcal{B}$. Thus u is in $\partial \mathcal{B}$, if and only if $v^* M u = 0$ for all v in $\partial \mathcal{B}^0$. It follows that $\mathcal{B}^+$ consists of all columns x of $\mathcal{D}$ such that ∂x is in $\partial \mathcal{B}$.

From now on, we shall assume that $\mathcal{B}^+ = \mathcal{B}$. This is the case if and only if $\mathcal{B}$ contains all columns x of $\mathcal{D}$ such that $\partial x = 0$. Then the problems

$$(3.6) \quad Lx = f, \quad x \text{ in } \mathcal{B},$$

$$(3.7) \quad L^0 y = g, \quad y \text{ in } \mathcal{B}^0,$$

are boundary-value problems. We call these problems adjoint.

There exist matrices A and B such that a column x of $\mathcal{D}$ is in $\mathcal{B}$ if and only if $A(\partial x) = 0$, and that a column y of $\mathcal{D}$ is in $\mathcal{B}^0$ if and only if $B(\partial y) = 0$. If A is given, then $\mathcal{B}$ and hence $\mathcal{B}^0$ are determined, and thus B is determined up to row-equivalence. The following theorem describes B .

THEOREM 1. -- If $\mathcal{B}$ consists of all columns x of $\mathcal{D}$ such that $A(\partial x) = 0$, and if m is the rank of A , then $\mathcal{B}^0$ consists of all columns y of $\mathcal{D}$ such

that $(\partial y)B = 0$ if and only if B is a matrix of rank $2n - m$ which satisfies the condition $AM^{-1}B^* = 0$.

PROOF. -- If B is a matrix of rank $2n - m$ such that $AM^{-1}B^* = 0$, then $C = M^{-1}B^*$ is a matrix of rank $2n - m$ such that $AC = 0$, and thus $\partial\mathcal{B}$ is the column space of C . Then a column y of $\mathcal{D}$ is in $\mathcal{B}^0$ if and only if $(\partial y)^*MC = 0$. But this is equivalent to $B(\partial y) = 0$. Now AM^{-1} is a matrix of rank m with $2n$ columns, and thus B^* is determined up to column-equivalence, and hence B up to row-equivalence, by the given conditions. It follows that these conditions are necessary as well as sufficient.

The boundary-value problem, Equation (3.6), is called self-adjoint if $\mathcal{L} = \mathcal{L}^0$ and $\mathcal{B} = \mathcal{B}^0$. The conditions for $\mathcal{L}$ to be self-adjoint have been given in Section 2. If $\mathcal{B}$ consists of all columns x of $\mathcal{D}$ such that $A(\partial x) = 0$, then it follows from Theorem 1 that $\mathcal{B} = \mathcal{B}^0$ if and only if A is of rank n and satisfies $AM^{-1}A^* = 0$.

4. The Algebra of Adjoint Boundary Conditions

We assume from now on that A is a matrix of rank m such that $\mathcal{B}$ consists of all columns x of $\mathcal{D}$ such that $A(\partial x) = 0$, and that B is a matrix of rank $2n - m$ such that $\mathcal{B}^0$ consists of all columns y of $\mathcal{D}$ such that $B(\partial y) = 0$. Then

$$(4.1) \quad AM^{-1}B^* = 0.$$

Put $R = A(\partial U)$, $S = B(\partial V)$, and let H be a matrix such that $RHR = R$. Such a matrix H can always be found.

These equations determine A and B , and hence R and S , up to row-equivalence. HR and RH are idempotent, and $I_n - HR$ is a right annihilator of R of maximal rank $n - r(R)$, where $r(R)$ is the rank of R . If A and R have p rows, then $I_p - RH$ is a left annihilator of R of maximal rank $p - r(R)$.

(4.2). $(I_p - RH)A \begin{vmatrix} 0 \\ U(b) \end{vmatrix} E$ is a left annihilator of S^* of maximal rank $n - r(s)$, where $r(s)$ is the rank of S .

PROOF. -- It follows immediately from Equations (3.1) and (2.4) that

$$(4.3) \quad M^{-1} = \begin{vmatrix} 0 \\ U(b) \end{vmatrix} E(\partial V)^* - (\partial U)E \begin{vmatrix} V^*(a) \\ 0 \end{vmatrix}.$$

Using Equations (4.1) and (4.3), we obtain

$$(4.4) \quad A \begin{vmatrix} 0 \\ U(b) \end{vmatrix} ES^* = RE \begin{vmatrix} V^*(a) & 0 \end{vmatrix} B^*.$$

It follows immediately that the matrix of Lemma (4.2) is a left annihilator of S^* .

Conversely, let $\gamma S^* = \gamma (\partial V)^* B^* = 0$. Because of Equation (4.1), there is a row δ such that $\gamma (\partial V)^* = \delta A M^{-1}$. Then $\delta R = \gamma (\partial V)^* M \Theta U$, and thus $\delta R = 0$ by Equation (3.3). Using this and Equation (4.3), we obtain

$$\delta A M^{-1} = \delta A \begin{vmatrix} 0 \\ U(b) \end{vmatrix} E (\partial V)^*.$$

As the rows of $(\partial V)^*$ are independent, it follows that $\gamma = \delta A \begin{vmatrix} 0 \\ U(b) \end{vmatrix} E$. Since $\delta R = 0$, δ is in the row space of $I_p - RH$, and hence γ in the row space of $(I_p - RH) A \begin{vmatrix} 0 \\ U(b) \end{vmatrix} E$. Thus this matrix is a left annihilator of S^* of maximal rank, which proves Lemma (4.2).

It follows immediately from Lemma (4.2) that $n - r(S) \leq p - r(R)$. Since (4.2) does not depend on any particular choice of A , this is true for any integer $p \geq m$, and thus $n - r(S) \leq m - r(R)$. A similar argument shows that $n - r(R) < 2n - m - r(S)$. Combining the two inequalities, we obtain

$$(4.5) \quad r(S) - r(R) = n - m.$$

The results of Sections 3 and 4 of this paper, except (4.2), are essentially the results of Reference 2, Ch. 11, in a somewhat broader context, and with more direct proofs.

5. Matrix Operators Associated With a Scalar Operator

If a scalar function $x_1(t)$ is n times continuously differentiable for t in $[a, b]$, we denote by $\hat{x}_1$ the column $\|x_1\|$ of $\mathcal{D}$ given by $x_i = x_1^{(i-1)}$ for $1 \leq i \leq n$. Similarly, given a row $\xi = \|x_{1k}\|$ with n times continuously differentiable entries, we denote by $\hat{\xi}$ the matrix $\|x_{ik}\|$ defined by $x_{ik} = x_1^{(i-1)}$ for $1 \leq i \leq n$. For any continuous scalar function f_1 , we denote by $\hat{f}_1$ the column $\|f_1\|$ with the given scalar function as first entry, and with $f_i = 0$ for $2 \leq i \leq n$. Similarly, given a row $\phi = \|f_k\|$, we denote by $\tilde{\phi}$ the matrix $\|f_{ik}\|$ with $f_{1k} = f_k$ and $f_{ik} = 0$ for $2 \leq i \leq n$.

Let $Lx_1 = \sum_{j=0}^n r_j x_1^{(j)}$ be a linear differential operator of order n . We

assume that $r_j(t)$ is of class C_j for $0 \leq j \leq n$ and t in $[a, b]$, and that $r_n(t) \neq 0$ for t in $[a, b]$. We say that a matrix operator, $\mathcal{L}x$, is associated with the scalar operator, Lx_1 , if the system of differential equations:

$$(5.1) \quad Lx_1 = \sum_{j=0}^n r_j x_1^{(j)} = f_1,$$

$$\dot{x}_{k-1} = x_k \quad \text{for } 2 \leq k \leq n,$$

is equivalent to the matrix equation

$$(5.2) \quad \mathcal{L}x - Px + Q\dot{x} = \tilde{f}_1.$$

If $x_k = \dot{x}_{k-1}$ for $2 \leq k \leq n$, then the entries of the left-hand side of (5.2) become:

$$p_{i1} x_1 + \sum_{k=2}^n (p_{ik} + q_{i,k-1}) x_1^{(k-1)} + q_{in} x_1^{(n)}.$$

In order to obtain Lx_1 for $i = 1$ and 0 for $i > 1$, we must have:

$$(5.3) \quad \begin{aligned} p_{11} &= r_0, & p_{1k} &= r_{k-1} - q_{1,k-1}, & q_{1n} &= r_n, \\ p_{i1} &= 0, & p_{ik} &= -q_{i,k-1}, & q_{in} &= 0, \end{aligned}$$

for $2 \leq i \leq n$, $2 \leq k \leq n$.

Conversely, line i of (5.2), for $2 \leq i \leq n$, becomes:

$$\sum_{k=2}^n q_{i,k-1} (\dot{x}_{k-1} - x_k) = 0,$$

if these relations hold. The matrix $||q_{i,k-1}||$, $2 \leq i \leq n$, $2 \leq k \leq n$, is non-singular if Q is nonsingular and $q_{in} = 0$ for $2 \leq i \leq n$. Thus these equations imply $\dot{x}_{k-1} = x_k$, for $2 \leq k \leq n$. If these relations and the first line of Equations (5.3) hold, then the first line of (5.2) becomes $Lx_1 = f_1$. Thus we have:

THEOREM 2. -- A matrix operator $\mathcal{L}x = Px + Qx$ is associated with a scalar differential operator $Lx_1 = \sum_{j=0}^n r_j x_1^{(j)}$ if and only if the Equations (5.3) are satisfied.

COROLLARY. -- If a matrix operator, $\mathcal{L}x$, is associated with a scalar differential operator, Lx_1 , then $\mathcal{L}\hat{x}_1$ is of the form $\tilde{f}_1$, and the matrix equation $\mathcal{L}\hat{x}_1 = \tilde{f}_1$ is equivalent to the scalar equation $Lx_1 = f_1$, for any scalar function x_1 which is n times continuously differentiable.

This follows immediately from Equations (5.3).

6. Canonically Associated Operators

We say that a matrix operator, $\mathcal{L}x$, is canonically associated with a scalar differential operator, Lx_1 , if $\mathcal{L}x$ is associated with Lx_1 , and the adjoint operator $\mathcal{L}^0 y$ of $\mathcal{L}x$ is associated with a scalar differential operator $L_1 y_1 = \sum_{j=0}^n s_j y_1^{(j)}$. This means that we must have Equations (5.3) and:

$$(6.1) \quad \begin{aligned} p_{11} - \dot{q}_{11} &= \bar{s}_0, & p_{i1} - \dot{q}_{i1} &= \bar{s}_{i-1} + q_{i-1,1}, \\ -q_{n1} &= \bar{s}_n, & q_{nk} &= 0, \end{aligned}$$

$$p_{1k} - \dot{q}_{1k} = 0, \quad p_{ik} - \dot{q}_{ik} = q_{i-1,k},$$

for $2 \leq i \leq n$ and $2 \leq k \leq n$.

From (5.3) and the last line of (6.1), we obtain

$$(6.2) \quad \begin{aligned} q_{1,k-1} + \dot{q}_{1k} &= r_{k-1}, & q_{1n} &= r_n, \end{aligned}$$

$$q_{i,k-1} + q_{i-1,k} + \dot{q}_{ik} = 0, \quad q_{in} = 0,$$

for $2 \leq i \leq n$ and $2 \leq k \leq n$. Using again (5.3), we obtain from this:

$$(6.3) \quad \begin{aligned} p_{11} &= r_0, & p_{1k} &= \dot{q}_{1k}, \\ p_{i1} &= 0, & p_{ik} &= -q_{i,k-1}, \end{aligned}$$

for $2 \leq i \leq n$ and $2 \leq k \leq n$. The recursion formulas (6.2) determine Q completely if the functions r_j are given. Using them, we obtain:

$$(6.4) \quad q_{ik} = \begin{cases} \sum_{h=k+i-1}^n (-1)^{h-k} \binom{h-k}{i-1} r_h^{(h-i-k+1)} & \text{if } i+k \leq n+1, \\ 0 & \text{if } i+k > n+1. \end{cases}$$

In order to prove (6.4), it is sufficient to verify that the terms q_{ik} given by (6.4) satisfy the recursion formulas (6.2). We omit the easy calculations.

If Q and P are given by (6.4) and (6.3), then (6.2) is valid, and (5.3) and the last line of (6.1) follow. $q_{nk} = 0$ for $2 \leq k \leq n$ follows from (6.4), and the remaining equations of (6.1) determine the functions s_j . We obtain:

$$(6.5) \quad s_j = \sum_{h=j}^n (-1)^h \binom{h}{j} \bar{r}_h^{(h-j)},$$

for $0 \leq j \leq n$, by easy calculations. Thus L_1 is the adjoint operator

$$(6.6) \quad L^0 y_1 = \sum_{j=0}^n s_j y_1^{(j)} = \sum_{h=0}^n (-1)^h (\bar{r}_h y_1)^{(h)}$$

of Lx_1 , in the usual sense (cf. Reference 2, p. 84). The second part of Equation (6.6) is an immediate consequence of Equation (6.5). We have proved:

THEOREM 3. -- Let $Lx_1 = \sum_{j=0}^n r_j x_1^{(j)}$ be a scalar differential operator,

with r_j of class C_j for $0 \leq j \leq n$, and $r_n(t) \neq 0$ for t in $[a, b]$. There is exactly one matrix operator $\mathcal{L}x = Px + Qx$ which is canonically associated with the given operator Lx_1 . This operator $\mathcal{L}x$ is given by (6.4) and (6.3), and its adjoint operator $\mathcal{L}^0 y$ is canonically associated with the adjoint operator $L^0 y_1$ of the given scalar operator Lx_1 .

The last part of this theorem can be considered as a definition of adjoint scalar operators in terms of adjoint matrix operators.

If $\mathcal{L}x$ is canonically associated with Lx_1 , then $\hat{y}_1^* (\mathcal{L} \hat{x}_1)$ and $(\mathcal{L}^0 \hat{y}_1)^* \hat{x}_1$ become $\bar{y}_1 (Lx_1)$ and $(L^0 y_1) x_1$ respectively, by the Corollary of

Theorem 2. Thus (2.2) leads to:

$$(6.7) \quad \bar{y}_1 (Lx_1) - \overline{(L^0 y_1)} x_1 = \frac{d}{dt} (\hat{y}_1 * Q \hat{x}_1),$$

where Q is given by (6.4).

7. On Solutions of Boundary-Value Problems

Let x be a solution of problem (3.6), and assume that y is a solution of the homogeneous adjoint problem:

$$(7.1) \quad \mathcal{L}^0 y = 0, \quad B\theta y = 0.$$

Then (3.4) becomes: $\int_a^b y^* f = 0$. Thus (3.6) has a solution only if f is orthogonal to every solution of (7.1).

If $\mathcal{L}^0 y = 0$, then $y = V\gamma$ and $B\theta y = S\gamma$, for a constant column γ . Thus f is orthogonal to all solutions of (7.1) if and only if $\gamma^* \int_a^b v^* f = 0$ for all columns γ such that $S\gamma = 0$. By (4.2), this is the case if and only if

$$(7.2) \quad (I_p - RH)A \left\| \begin{matrix} 0 \\ U(b) \end{matrix} \right\| E \int_a^b V^* f = 0.$$

The general solution of the system $\mathcal{L}x = f$ is

$$(7.3) \quad x(t) = U(t) \left[c + E \int_a^t V^*(s) f(s) ds \right],$$

where c is an arbitrary constant column. This leads to:

$$A\theta x = Rc + A \left\| \begin{matrix} 0 \\ U(b) \end{matrix} \right\| E \int_a^b V^* f.$$

Putting $c = -HA \left\| \begin{matrix} 0 \\ U(b) \end{matrix} \right\| E \int_a^b V^* f$, we obtain

$$(7.4) \quad A\theta x = (I_p - RH)A \left\| \begin{matrix} 0 \\ U(b) \end{matrix} \right\| E \int_a^b V^* f,$$

and hence a solution of (3.6) if f satisfies (7.2). Thus we have:

THEOREM 4. -- The boundary-value problem $\mathcal{L}x = f$, $A\theta x = 0$, has a solution if and only if $\int_a^b y^* f = 0$ for every solution y of the homogeneous adjoint problem $\mathcal{L}^0 y = 0$, $B\theta y = 0$.

Similarly, the adjoint problem (3.7) of (3.6) has a solution if and only if $\int_a^b x^* g = 0$ for any solution x of its homogeneous adjoint problem

$$(7.5) \quad \mathcal{L}x = 0, \quad A(\partial x) = 0,$$

and hence, if and only if,

$$(7.6) \quad (I_n - HR)^* \int_a^b U^* g = 0.$$

We obtain (7.6) in the same way as (7.2).

8. An Auxiliary Kernel Matrix

If we put

$$(8.1) \quad \Gamma(t, s) = -U(t) HA \begin{vmatrix} 0 \\ U(b) \end{vmatrix} E V^*(s) + \begin{cases} U(t) E V^*(s), & s < t, \\ 0 & s > t, \end{cases}$$

for t and s in $[a, b]$ and $s \neq t$, and

$$(8.2) \quad (\mathcal{G}_1 f)(t) = \int_a^b \Gamma(t, s) f(s) ds,$$

for f in $\mathcal{C}$ and t in $[a, b]$, then (7.3), with $c = -HA \begin{vmatrix} 0 \\ U(b) \end{vmatrix} E \int_a^b V^* f$, becomes $x = \mathcal{G}_1 f$. Thus we have

$$(8.3) \quad \mathcal{L}\mathcal{G}_1 f = f$$

for any column f in $\mathcal{C}$, and $\mathcal{G}_1 f$ is a solution of (3.6) if this problem can be solved, i.e., if f satisfies (7.2).

$\Gamma(t, s)$ is discontinuous for $t = s$. $\Gamma(t, t-)$ and $\Gamma(t, t+)$ are defined for t in (a, b) , and we have:

$$(8.4) \quad \Gamma(t, t-) - \Gamma(t, t+) = Q^{-1}(t)$$

for t in (a, b) . We also have, for s in (a, b) :

$$(8.5) \quad A \begin{vmatrix} \Gamma(a, s) \\ \Gamma(b, s) \end{vmatrix} = (I_p - RH) A \begin{vmatrix} 0 \\ U(b) \end{vmatrix} E V^*(s),$$

and thus $A(\partial(\mathcal{G}_1 f))$ is given by (7.4) for f in $\mathcal{C}$.

The adjoint operator $\mathcal{G}_1^0$ of $\mathcal{G}_1$ is defined by:

$$(8.6) \quad (\mathcal{G}_1^0 g)(t) = \int_a^b \Gamma^*(s, t) g(s) ds,$$

for t in $[a, b]$ and g in $\mathcal{C}$. Thus

$$(\mathcal{G}_1^0 g)(t) = V(t) E^* \left[\int_t^b U^* g - \begin{vmatrix} 0 & U^*(b) \end{vmatrix} A^* H^* \int_a^b U^* g \right],$$

for t in $[a, b]$ and g in $\mathcal{C}$. It follows immediately that

$$(8.7) \quad \mathcal{L}^0 \mathcal{G}_1^0 g = g$$

for g in $\mathcal{C}$. Using (8.1), we have

$$B \begin{vmatrix} \Gamma^*(s, a) \\ \Gamma^*(s, b) \end{vmatrix} = -S \left[H A \begin{vmatrix} 0 \\ U(b) \end{vmatrix} E \right]^* U^*(s) + B \begin{vmatrix} V(a) \\ 0 \end{vmatrix} E^* U^*(s),$$

for s in (a, b) , and hence, using (4.4):

$$(8.8) \quad B \begin{vmatrix} \Gamma^*(s, a) \\ \Gamma^*(s, b) \end{vmatrix} = B \begin{vmatrix} V(a) \\ 0 \end{vmatrix} E^* (I_n - HR)^* U^*(s),$$

for s in (a, b) . Thus

$$B \mathcal{G}(\mathcal{G}_1^0 g) = B \begin{vmatrix} V(a) \\ 0 \end{vmatrix} E^* (I_n - HR)^* \int_a^b U^* g,$$

for g in $\mathcal{C}$. It follows that $\mathcal{G}_1^0 g$ is a solution of (3.7) if this problem can be solved, i.e., if g satisfies (7.6).

9. The Generalized Function of Green

It is sometimes convenient to introduce matrices with zero rows or columns. This can be done as follows. We introduce a zero matrix, Z_{mn} , with m rows and n columns of zeros for any two natural numbers m and n . If $m = 0$ or $n = 0$, then Z_{mn} is the only (m, n) -matrix. Products of any (m, n) -matrix A with a zero matrix are defined by the equations $Z_{rm} A = Z_{rn}$ and $A Z_{ns} = Z_{ms}$. Thus $Z_{mn} Z_{ns} = Z_{ms}$ for any natural numbers m, n, s . The matrix Z_{00} has the properties of a unit matrix of order zero, and thus we put $I_0 = Z_{00}$. Sums are defined by $A + Z_{mn} = Z_{mn} + A = A$, for any (m, n) -matrix A , adjoints by $Z_{mn}^* = Z_{nm}$, integrals by $\int_a^b Z_{mn} = Z_{mn}$, and derivatives by $\dot{Z}_{mn} = Z_{mn}$.

With these definitions, the basic laws of matrix algebra, and many theorems, remain valid for the extended algebra.

By using this extended algebra, we can derive all results of Sections 9-13 of this paper without setting apart the special cases which would otherwise arise if (7.1) has only the trivial solution $y = 0$, or if (7.5) has only the trivial solution $x = 0$. We shall discuss these special cases briefly in Section 14.

The solutions of (7.1) are the columns $V\gamma$, where γ is a constant column such that $S\gamma = 0$. These solutions form a vector space of dimension $n - r(S)$, where $r(S)$ is the rank of S . Put $n - r(S) = v$, and let Y be a functional (n, v) -matrix the columns of which form a basis of the space of solutions of (7.1). Then $\int_a^b Y^* Y$ is a nonsingular square matrix of order v . Let Y_1 be any functional (n, v) -matrix such that $\int_a^b Y^* Y_1 = I_v$. A possible choice of Y_1 is $Y_1 = Y T^{-1}$ for $T = \int_a^b Y^* Y$. This is one of the most useful choices for Y_1 . Other useful choices are discussed in Sections 13 and 14.

Put $n - r(R) = \mu$, where $r(R)$ is the rank of R . Then we have:

$$(9.1) \quad \mu - v = n - m,$$

by (4.5), and the solutions of (7.5) form a vector space of dimension μ . Let X and X_1 be functional (n, μ) -matrices such that the columns of X form a basis of the space of solutions of (7.5), and that $\int_a^b X^* X_1 = I_\mu$.

Since $(I_p - RH) A \begin{vmatrix} 0 \\ U(b) \end{vmatrix} E$ is a left annihilator of S^* , it follows from (8.5) that there is a matrix R_1 such that

$$A \begin{vmatrix} \Gamma(a, s) \\ \Gamma(b, s) \end{vmatrix} = R_1 Y^*(s)$$

for s in (a, b) . Then $R_1 = (I_p - RH) A \begin{vmatrix} 0 \\ U(b) \end{vmatrix} E \int_a^b V^* Y_1$ by (8.5). If we put

$$(9.2) \quad U_1(t) = - \int_a^b \Gamma(t, s) Y_1(s) ds,$$

then $A @ U_1 = -R_1$, by (8.5) and (7.4). Thus we have

$$(9.3) \quad A \begin{vmatrix} G(a, s) \\ G(b, s) \end{vmatrix} = 0,$$

for s in (a, b) , if we put:

$$G(t, s) = \Gamma(t, s) + U_1(t) Y^*(s) + X(t) W^*(s),$$

for any choice of a functional (n, μ) -matrix W . We obtain:

$$(9.4) \quad \int_a^b X_1^*(t) G(t, s) dt = 0,$$

for s in $[a, b]$, if we put:

$$W^*(s) = - \int_a^b X_1^*(t) [\Gamma(t, s) + U_1(t) Y^*(s)] dt.$$

Let now

$$(9.5) \quad V_1(t) = - \int_a^b \Gamma^*(s, t) X_1(s) ds,$$

for t in $[a, b]$, and

$$(9.6) \quad T = \int_a^b \int_a^b X_1^*(t) \Gamma(t, s) Y_1(s) ds dt.$$

Then $\int_a^b X_1^*(t) \Gamma(t, s) dt = -V_1^*(s)$ and $\int_a^b X_1^* U_1 = -T$. Thus we put:

$$(9.7) \quad G(t, s) = \Gamma(t, s) + U_1(t) Y^*(s) + X(t) V_1^*(s) + X(t) T Y^*(s),$$

for s, t in $[a, b]$ and $s \neq t$. The kernel matrix thus defined is called a generalized function of Green for the boundary-value problem (3.6).

10. The Operator $\mathcal{G}$

We associate with a generalized function of Green $G(t, s)$ the operator

$$(10.1) \quad (\mathcal{G}f)(t) = \int_a^b G(t, s) f(s) ds,$$

for f in $\mathcal{C}$ and t in $[a, b]$. We study in this section the relations between the operators $\mathcal{L}$ and $\mathcal{G}$.

For x in $\mathcal{B}$, we have $\int_a^b Y^*(\mathcal{L}x) = \int_a^b (\mathcal{L}^0 Y)^* x = 0$ by (3.4).

Using this, and (9.3) and (9.4), we obtain:

(10.2) For f in $\mathcal{C}$, $\mathcal{G}f$ is in $\mathcal{B}$ and $\int_a^b X_1^* (\mathcal{G}f) = 0$. For x in $\mathcal{B}$, $\mathcal{L}x$ is in $\mathcal{C}$, and $\int_a^b Y^* (\mathcal{L}x) = 0$.

For f in $\mathcal{C}$, we have $\mathcal{L}\mathcal{G}f = \mathcal{L}\mathcal{G}_1 f + (\mathcal{L}U_1) \int_a^b Y^* f$, using (9.7). Using (8.3) and (9.2), we obtain:

$$(10.3) \quad \mathcal{L}\mathcal{G}f = f - Y_1 \int_a^b Y^* f,$$

for f in $\mathcal{C}$. Replacing f with $\mathcal{L}x$ in this, and using (10.2), we get:

$$(10.4) \quad \mathcal{L}\mathcal{G}\mathcal{L}x = \mathcal{L}x,$$

for x in $\mathcal{B}$.

For x in $\mathcal{B}$, $x - \mathcal{G}\mathcal{L}x$ is in $\mathcal{B}$, and $\mathcal{L}(x - \mathcal{G}\mathcal{L}x) = 0$ by (10.4). There is thus a constant column α such that $x - \mathcal{G}\mathcal{L}x = X\alpha$. Then $\alpha = \int_a^b X_1^* (x - \mathcal{G}\mathcal{L}x) = \int_a^b X_1^* x$, using (10.2). Thus we have:

$$(10.5) \quad \mathcal{G}\mathcal{L}x = x - X \int_a^b X_1^* x,$$

for x in $\mathcal{B}$. Replacing x with $\mathcal{G}f$ in this, and using (10.2), we get:

$$(10.6) \quad \mathcal{G}\mathcal{L}\mathcal{G}f = \mathcal{G}f,$$

for f in $\mathcal{C}$.

These relations show that $\mathcal{G}$ maps $\mathcal{C}$ onto the space of all columns x in $\mathcal{B}$ such that $\int_a^b X_1^* x = 0$, that $\mathcal{L}$ maps $\mathcal{B}$ onto the space of all columns f in $\mathcal{C}$ such that $\int_a^b Y^* f = 0$, and that $\mathcal{L}$, restricted to the image space of $\mathcal{G}$, and $\mathcal{G}$, restricted to the image space of $\mathcal{L}$, become inverses of each other. Thus $\mathcal{G}f$ is a solution of (3.6) for any column f in $\mathcal{C}$ for which this problem can be solved, i.e., for any column f in $\mathcal{C}$ such that $\int_a^b Y^* f = 0$.

Replacing f by Y_1 in (10.3), we get $\mathcal{L}\mathcal{G}Y_1 = 0$ as $\int_a^b Y^* Y_1 = I_v$, and then $\mathcal{G}Y_1$ by (10.6). Conversely, $f = Y_1 \int_a^b Y^* f$ if $\mathcal{G}f = 0$, by (10.3). Recalling the definition of X , or using (10.5) and (10.4) in the same way, we obtain:

(10.7) $\mathcal{G}f = 0$, for f in $\mathcal{C}$, if and only if f is in the column space of Y_1 , and $\mathcal{L}x = 0$, for x in $\mathcal{B}$, if and only if x is in the column space of X .

11. Properties of $G(t, s)$

THEOREM 5. -- The generalized function of Green $G(t, s)$ of a boundary-value problem $\mathcal{L}x = f$, x in $\mathcal{B}$, has the following properties by which it is completely determined:

(1) For fixed s in $[a, b]$, t in $[a, b]$ and $t \neq s$, $G(t, s)$ is continuously differentiable with respect to t , and $\mathcal{L}G(t, s) = -Y_1(t)Y^*(s)$.

(2) For t in (a, b) , $G(t, t-)$ and $G(t, t+)$ are defined and satisfy the equation $G(t, t-) - G(t, t+) = Q^{-1}(t)$.

(3) $A \begin{vmatrix} G(a, s) \\ G(b, s) \end{vmatrix} = 0$, for s in (a, b) .

(4) $\int_a^b X_1^*(t) G(t, s) ds = 0$, for s in $[a, b]$,

PROOF. -- Properties (3) and (4) are (9.3) and (9.4). Since $\Gamma(t, s)$ is the only term of $G(t, s)$ with discontinuities, (2) follows directly from (8.4). For fixed s and $t \neq s$, we have $\mathcal{L}\Gamma(t, s) = 0$ by (8.1), $\mathcal{L}X = 0$ by definition of X , and $\mathcal{L}U_1 = -Y_1$ by (8.3) and (9.2). Using this in (9.7), we see that $G(t, s)$ satisfies Property (1).

Let now $G_1(t, s)$ be any kernel matrix which has the four given properties, and let $G_2(t, s) = G_1(t, s) - G(t, s)$. Since $G(t, s)$ and $G_1(t, s)$ have the same discontinuities, we can make $G_2(t, s)$ continuous for $t = s$. Then we have $\mathcal{L}G_2(t, s) = 0$ for fixed s in $[a, b]$ and any t in $[a, b]$. Since $G_2(t, s)$ satisfies (3), there is a matrix $W^*(s)$ such that $G_2(t, s) = X(t)W^*(s)$. But then $W^*(s) = \int_a^b X_1^*(t) G_2(t, s) dt = 0$, as $G_2(t, s)$ satisfies (4), and hence $G_2(t, s) = 0$. This shows that the given properties determine $G(t, s)$ uniquely.

As defined by (9.7), $G(t, s)$ depends on the matrices X , X_1 , Y , Y_1 . We want to show that $G(t, s)$ is already uniquely determined if only the column spaces of the matrices X_1 and Y_1 are prescribed.

$G(t, s)$ is determined by the conditions of Theorem 5. Condition (3) is satisfied if and only if $\int_a^b g^*(t) G(t, s) dt = 0$ for any column g in the column space of X_1 and any s in $[a, b]$. Thus $G(t, s)$ does not depend on any particular choice of X and X_1 if the column space of X_1 is prescribed.

If we replace Y and Y_1 by (n, ν) -matrices Y' and Y'_1 with the same column spaces, then there are nonsingular, constant matrices C and C_1 of order ν such that $Y' = YC$ and $Y'_1 = Y_1C_1$. Then $\int_a^b Y'^* Y'_1 = C^* C_1$.

In order to obtain $\int_a^b Y' * Y_1' = I_v$, we must have $C^* = C_1^{-1}$. But then we have $Y_1'(t) Y'^*(s) = Y_1(t) Y^*(s)$ for any s and t in $[a, b]$. Thus $G(t, s)$ depends only on the column space of Y_1 , but not otherwise on Y and Y_1 .

Since we require $\int_a^b Y * Y_1 = I_v$, the columns of Y_1 must be a basis of a subspace $\mathcal{J}$ of $\mathcal{C}$ of dimension v such that no nonzero column of $\mathcal{J}$ is orthogonal to the solution space of (7.1). Then $\int_a^b Y * Y_1$ is always nonsingular if the columns of Y form a basis of the solution space of (7.1), and the columns of Y_1 a basis of $\mathcal{J}$, and if one of the two matrices has been chosen, the other one is uniquely determined by its column space and the relation $\int_a^b Y * Y_1 = I_v$. We have exactly analogous relations for X and X_1 .

12. Generalized Functions of Green for Adjoint Problems

THEOREM 6. -- If $G(t, s)$ is a generalized function of Green for a boundary-value problem $\mathcal{L}x = f$, x in $\mathcal{B}$, then the adjoint kernel matrix $G^0(t, s) = G^*(s, t)$ of $G(t, s)$ is a generalized function of Green for the adjoint boundary-value problem $\mathcal{L}^0 y = g$, y in $\mathcal{B}^0$.

PROOF. -- We show that $G^*(s, t)$ has the properties of Theorem 5 for a generalized function of Green for (3.7), with the rôles of X and Y and of X_1 and Y_1 interchanged.

We have $\mathcal{L}^0 \Gamma^*(s, t) = 0$ for fixed s and $t \neq s$ by (8.1). $\mathcal{L}^0 Y = 0$ by definition of Y , and $\mathcal{L}^0 V_1 = -X_1$ by (9.5) and (8.7). Thus we have $\mathcal{L}^0 G^*(s, t) = -X_1(t) X^*(s)$ for fixed s and $t \neq s$.

For t in (a, b) , we have $G^0(t, t-) = G^*(t-, t) = G^*(t, t+)$, and similarly $G^0(t, t+) = G^*(t, t-)$. Thus $G^0(t, t-) - G^0(t, t+) = - (Q^*(t))^{-1}$, which is the matrix corresponding to $Q^{-1}(t)$ for the adjoint problem.

It follows from (8.8) that $B \begin{vmatrix} \Gamma^*(s, a) \\ \Gamma^*(s, b) \end{vmatrix} = S_1 X^*(s)$ for some constant matrix S_1 , and then it follows from (9.5) that $S_1 = -B(\partial V_1)$. Since $B(\partial Y) = 0$, we obtain $B \begin{vmatrix} G^*(s, a) \\ G^*(s, b) \end{vmatrix} = 0$.

Finally, we have $\int_a^b G(t, s) Y_1(s) ds = 0$ by (10.7), and we obtain immediately $\int_a^b Y_1^*(t) G^*(s, t) dt = 0$, by taking adjoints and changing variables. This completes the proof of the Theorem.

The adjoint operator $\mathcal{G}^0$ of the integral operator $\mathcal{G}$ is defined by:

$$(12.1) \quad (\mathcal{G}^0 g)(t) = \int_a^b G^*(s, t) g(s) ds,$$

for g in $\mathcal{C}$ and t in $[a, b]$. By Theorem 6, the relations between $\mathcal{L}^0$ and $\mathcal{G}^0$ are analogous to those between $\mathcal{L}$ and $\mathcal{G}$ which we discussed in Section 10. In particular, $\mathcal{G}^0 g$ is a solution of (3.7) if this problem can be solved, i.e., if g is orthogonal to the solution space of (7.5), and $\mathcal{G}^0 g = 0$, for g in $\mathcal{C}$, if and only if g is in the column space of X_1 . Thus the column spaces of X_1 and of Y_1 , which determine $G(t, s)$, are also determined by it.

It follows immediately from Equations (10.1) and (12.1) that:

$$(12.2) \quad \int_a^b (\mathcal{G}^0 g)^* f = \int_a^b g^* (\mathcal{G} f) = \int_a^b \int_a^b g^*(t) G(t, s) f(s) ds dt,$$

for any columns f and g in $\mathcal{C}$.

If the problem (3.6) is self-adjoint, then both $G(t, s)$ and $G^*(s, t)$ are generalized functions of Green for (3.6). In this case, we may put $X = Y$ and $X_1 = Y_1$. Then the conditions of Theorem 5 for $G(t, s)$ and for $G^*(s, t)$ coincide, and thus we have $G^*(s, t) = G(t, s)$, by Theorem 5. In other words:

(12.3). A self-adjoint boundary-value problem $\mathcal{L}x = f$, x in $\mathcal{B}$, has a generalized function of Green, $G(t, s)$, such that $G^*(s, t) = G(t, s)$, for s and t in $[a, b]$ and $s \neq t$, and hence such that $\mathcal{G}^0 = \mathcal{G}$.

13. On Scalar Boundary-Value Problems

We assume in this section that the operator $\mathcal{L}$ is canonically associated with a scalar differential operator L . We denote by ϕ the first row of U , and by η the first row of Y . Then $U = \hat{\phi}$, $Y = \hat{\eta}$, the solutions of $Lx_1 = 0$ are the functions $\phi\gamma$, for any constant column γ , and the solutions of the adjoint homogeneous scalar boundary-value problem:

$$(13.1) \quad L^0 y_1 = 0, \quad B(\partial \hat{y}_1) = 0,$$

are the functions $y_1 = \eta\beta$, for any constant column β . The entries of the row η are linearly independent, and thus $\int_a^b \eta^* \eta$ is a nonsingular matrix of order v . It follows that there is a row η_1 of continuous functions such that

$\int_a^b \eta * \eta_1 = I_v$. If we put $Y_1 = \tilde{\eta}_1$, then $Y * Y_1 = \eta * \eta_1$, and thus we may use Y_1 in the construction of a generalized function of Green.

We denote by ψ the first row of V and by ξ the first row of X . Then $V = \hat{\psi}$, $X = \hat{\xi}$, and the properties of ξ and ψ are analogous to those of η and ϕ . We put $X_1 = \tilde{\xi}_1$, where ξ_1 is a row of continuous functions such that $\int_a^b \xi * \xi_1 = I_\mu$.

With such a choice of X_1 and Y_1 , we have $\mathcal{L}U_1 = -\tilde{\eta}_1$, by Equations (9.1) and (8.3), and hence $U_1 = \hat{\phi}_1$, and $L\phi_1 = -\eta_1$, where ϕ_1 is the first row of U_1 . Similarly, $V_1 = \hat{\psi}_1$, and $L\psi_1 = -\xi_1$, where ψ_1 is the first row of V_1 .

Now let $G(t, s) = ||g_{ik}(t, s)||$ and $g(t, s) = g_{11}(t, s)$. From (8.1) and the formulas of Section 9, we obtain formulas for $g(t, s)$ in terms of ξ, η, ξ_1, η_1 , and it is readily seen that then

$$g_{ik}(t, s) = \frac{\partial^{i-1}}{\partial t^{i-1}} \frac{\partial^{k-1}}{\partial s^{k-1}} g(t, s),$$

for $1 \leq i, k \leq n$. Using these formulas in Theorem 5, we see that $g(t, s)$ has the properties of a generalized function of Green for the scalar problem

$$(13.2) \quad Lx_1 = f_1, \quad \Lambda(\partial \hat{x}_1) = 0.$$

We have $\mathcal{L}\mathcal{G}\tilde{f}_1 = \tilde{f}_1 - \tilde{\eta}_1 \int_a^b \eta * f_1$ by Equation (10.3), and thus $\mathcal{G}\tilde{f}_1$ is of the form $\hat{x}_1$. We define the scalar integral operator with the kernel $g(t, s)$ by:

$$(13.3) \quad (\mathcal{G}_s f_1)(t) = \int_a^b g(t, s) f_1(s) ds,$$

for t in $[a, b]$. Then $\mathcal{G}_s f_1$ is the first entry of $\mathcal{G}\tilde{f}_1$, and thus we have $\mathcal{G}\tilde{f}_1 = \hat{x}_1$ if $\mathcal{G}_s f_1 = x_1$, and $\mathcal{L}\hat{x}_1 = \tilde{h}_1$ if $h_1 = Lx_1$. Thus (10.3) and (10.5) become:

$$(13.4) \quad L\mathcal{G}_s f_1 = f_1 - \eta_1 \int_a^b \eta * f_1,$$

$$(13.5) \quad \mathcal{G}_s Lx_1 = x_1 - \xi \int_a^b \xi_1 * x_1,$$

for $\tilde{f}_1$ in $\mathcal{C}$ and x_1 in $\mathcal{B}$. These formulas show that $\mathcal{G}_s f_1$ is a solution of (13.2) if this problem can be solved, i.e., if f_1 is orthogonal to all solutions of the homogeneous adjoint problem (13.1).

It follows immediately from Theorem 6 and the construction of $g(t, s)$ that $\overline{g(s, t)}$ is a generalized function of Green of the adjoint scalar boundary-value problem $L^0 y_1 = g_1$, $B(\partial \hat{y}_1) = 0$, of (13.2).

14. Special Cases

If problem (3.6) has a solution x , then $x + X\alpha$, for a constant column α , is the general solution of (3.6), with a given column f in $\mathcal{C}$. This solution is unique if and only if $\mu = 0$. If this is the case, then X , X_1 , and V_1 are $(0, n)$ -matrices, and thus all terms involving these matrices, in formulas of Sections 9 through 13, are zero and can be omitted.

By Theorem 4, problem (3.6) can be solved for any column f in $\mathcal{C}$, if and only if, $\nu = 0$. Then Y , Y_1 , and U_1 are $(0, n)$ -matrices, and all terms involving these matrices, in formulas of Sections 9 through 13, are zero and can be omitted.

$\mu = \nu$ if and only if $m = n$, by (9.1). In this case, we have the so-called Alternative: Either (3.6) has a unique solution for any column f in $\mathcal{C}$, or the homogeneous adjoint problem (7.1) has a nontrivial solution. In the first case, $\mu = \nu = 0$, and hence $G(t, s) = \Gamma(t, s)$. Such a problem is called regular, and $G(t, s)$ is called its function of Green. If (3.6) is regular, and if $\mathcal{L}$ is canonically associated with a scalar operator L , then $g(t, s)$ becomes the function of Green of the regular scalar problem (13.2).

We have seen that $G(t, s)$ is completely determined by the column spaces of the matrices X_1 and Y_1 . These column spaces are usually chosen in one of the two following ways.

For any problem (3.6), we can choose as column spaces of X_1 and Y_1 the solution spaces of the homogeneous problems (7.5) and (7.1). If we do this, and if we choose as columns of X and Y orthonormal bases of these solution spaces, then $X_1 = X$ and $Y_1 = Y$.

If $\mathcal{L}$ is canonically associated with a scalar operator L , we can choose as column spaces of X_1 and Y_1 the column spaces of the matrices $\tilde{\xi}$ and $\tilde{\eta}$ of

Section 13. If we do this, and if we choose as entries of ξ and η orthonormal bases of the solution spaces of the problems $Lx_1 = 0$, $A(\partial x_1) = 0$, and (13.1) respectively, then $X_1 = \tilde{\xi}$ and $Y_1 = \tilde{\eta}$.

In either case, $G(t, s)$ is uniquely determined by (3.6), and conversely. We show the converse as follows. The column spaces of X_1 and of Y_1 are determined by $G(t, s)$, by (10.7) and the corresponding property of $\mathcal{G}^0$, and these column spaces determine the column spaces of X and of Y . By (10.5), $\mathcal{B}$ is the sum of the image space of $\mathcal{G}$ and the column space of X . We have $\mathcal{L}x = 0$ for x in the column space of X , and $\mathcal{L}x$ is given by (10.3) for x in the image space of $\mathcal{G}$. This determines $\mathcal{L}x$ for any x in $\mathcal{B}$.

We obtain $\mathcal{B}^0$, and $\mathcal{L}^0 y$ for y in $\mathcal{B}^0$, in the same way from $\mathcal{G}^0$. If $G^*(s, t) = G(t, s)$ for s, t in $[a, b]$ and $s \neq t$, then $\mathcal{G}^0 = \mathcal{G}$, and thus $\mathcal{B}^0 = \mathcal{B}$, and $\mathcal{L}^0 x = \mathcal{L}x$ for x in $\mathcal{B}$. In this case, (3.7) is the same problem as (3.6), and thus (3.6) is self-adjoint. Thus we may strengthen (12.3) as follows.

THEOREM 7. -- If the matrices X_1 and Y_1 are determined in one of the two ways described above, then the boundary-value problem $\mathcal{L}x = f$, x in $\mathcal{B}$, is self-adjoint if and only if $G^*(s, t) = G(t, s)$ for all t, s in $[a, b]$ such that $t \neq s$.

If (3.6) is derived from a scalar boundary-value problem (13.2), and if the second way of determining X_1 and Y_1 is chosen, the following corollary becomes an immediate consequence of the discussion of Section 13:

COROLLARY. -- Under the given hypotheses, the problem (13.2) is self-adjoint if and only if $\overline{g(s, t)} = g(t, s)$ for all t, s in $[a, b]$ such that $t \neq s$.

REFERENCES

1. Courant, R., and Hilbert, D., Methoden der mathematischen Physik, Vol. I, Berlin, 1931.
2. Coddington, E. A., and Levinson, N., Theory of Ordinary Differential Equations, New York, 1955.