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To: G. A. Garrett

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Subject: The Theory of a Rotating
Cylinder Viscometer in the
Knudsen Pressure Regime

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R. A. Millikan and his co-workers have reported values of Maxwell's reflection parameter f which were obtained from analysis of some results of viscosity measurements via a rotating cylinder arrangement. The theory used to derive these results was given by Millikan on the basis of the so-called "slip" boundary conditions. It is not certain that such slip calculations have anything more than a qualitative significance in any case but Millikan et al. have employed them in a strictly qualitative manner. In this memorandum quantitative results are obtained regarding rotating cylinder viscometers at very low pressures.

Theory

Before taking up the theory of the viscometer proper it will be convenient to solve the following problem:

Molecules are evaporating from a wall which moves with uniform speed, s , along its own plane. To an observer moving with the plane the projections of the directions of emission on a plane perpendicular to the

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... wall and containing the direction of motion conforms to the two-dimensional cosine law. What is the law of emission of the projections with respect to a stationary observer?

The stationary observer chooses the direction of the axis of X along the direction of motion and the axis of Y along the normal to the plane. θ is measured clockwise from the Y axis. The moving observer has axes X' and Y' in similar directions but moving with speed s with respect to the X,Y system. In the moving system let an emitted molecule have X' and Y' components of velocity $v \sin \theta'$, $v \cos \theta'$, respectively, v being the speed of the emitted molecule in the moving frame. In the stationary frame the velocity components are $v \sin \theta' + s$, $v \cos \theta'$. Hence,

$$\tan \theta = \frac{v \sin \theta' + s}{v \cos \theta'} = \tan \theta' + \frac{s}{v} \sec \theta'.$$

We shall only consider the case for $s \ll v$. Hence, we have

$$\theta = \theta' + \frac{s}{v} \cos \theta'$$

or

$$\theta' = \theta - \frac{s}{v} \cos \theta.$$

Let the law of emission apparent to the stationary observer be $P(\theta)$ such that $P(\theta)d\theta$ is the probability of the angle θ being in the range θ to $\theta + d\theta$.

Then

$$\begin{aligned} P(\theta)d\theta &= \frac{1}{2} \cos \theta' d\theta' \\ &= \frac{1}{2} \cos \theta' \left(1 + \frac{s}{v} \sin \theta\right) d\theta \\ &= \frac{1}{2} \left(1 + \frac{s}{v} \sin \theta\right) \cos \theta \left(1 + \frac{s}{v} \sin \theta\right) d\theta \end{aligned}$$

and

$$P(\theta) = \frac{1}{2} \cos \theta \left(1 + 2 \frac{s}{v} \sin \theta \right),$$

which is the required result.

A typical rotating cylinder viscometer consists of an inner cylinder of radius a suspended from a torsion fibre and a concentric cylinder radius b ($> a$) which rotates with uniform angular speed ω . The lengths of the cylinders are long compared to $b-a$ and in the theory we take them infinitely long. The space between the cylinders contains the fluid whose coefficient of viscosity is to be measured and for a perfect fluid with no slipping at the cylinder walls, the torque transmitted to the inner cylinder per unit length is

$$T = \frac{4\pi\eta \omega a^2 b^2}{b^2 - a^2}$$

wherein η is the coefficient of viscosity. We now compute the torque when the fluid is a gas of molecular weight m at such a pressure that the mean free path is very long compared to $b-a$. A molecule emitted from the rotating cylinder (radius b) is emitted from a moving frame of reference which moves with speed ωb relative to the laboratory frame. If its speed with respect to the moving frame is v , its speed in the laboratory frame, correct to the first power in ω will be $v + \omega b \sin \theta$. Its angular momentum about the axis of the cylinders will be $m(v + \omega b \sin \theta)b \sin \theta$. It will hit the inner cylinder if it is emitted in the angular range

$$-\sin^{-1} \frac{a}{b} \leq \theta \leq \sin^{-1} \frac{a}{b}$$

and, hence, if it is emitted according to the distribution function $P(\theta)$, it

will carry, to cylinder a , angular momentum

$$\frac{1}{2} m \int_{-\sin^{-1} \frac{a}{b}}^{\sin^{-1} \frac{a}{b}} (v + \omega b \sin \theta) b \sin \theta \cos \theta \left(1 + \frac{2\omega b}{v} \sin \theta\right) d\theta$$

or dropping terms in ω^2

$$\begin{aligned} \frac{1}{2} m b \int_{-\sin^{-1} \frac{a}{b}}^{\sin^{-1} \frac{a}{b}} (v + 3\omega b \sin \theta) \sin \theta \cos \theta d\theta \\ = m b^2 \omega \frac{a^3}{b^3} = m \omega a^3 / b. \end{aligned}$$

The number of molecules leaving unit area of the walls of b per unit time is (at least correct to the zeroth power of ω , which is all we require) $p(2\pi m k T)^{-1/2}$ wherein p is the pressure and T the absolute temperature. The torque per unit length on the inner cylinder if the molecules are diffusely scattered on impact is thus

$$\frac{p}{\sqrt{2\pi m k T}} \cdot 2\pi b \cdot m \omega \frac{a^3}{b} = p \sqrt{\frac{2\pi m}{k T}} \omega a^3.$$

Consequently, the apparent viscosity η^* is given by

$$\eta^* = \frac{1}{2} \frac{a(b^2 - a^2)}{b^2} m \frac{p}{\sqrt{2\pi m k T}}.$$

We now consider how this expression is modified when the central cylinder diffusely reflects only the fraction f_a of incident molecules and

the outer cylinder diffusely reflects f_b . We assume those not diffusely reflected are perfectly specularly reflected, i.e., only their velocity components normal to the wall change and these only reverse and that this happens independently of the state of motion of the wall. Then molecules leaving the inner fixed cylinder after a diffuse collision and eventually diffusely colliding with the inner cylinder with no intervening diffuse collisions contribute, on the average, nothing to the angular momentum balance of the inner cylinder. A molecule leaving the outer cylinder and headed so that it misses the inner cylinder will of necessity make its next subsequent diffuse collision with the outer wall since if it misses and specularly reflects, it will miss again by exactly the same amount, etc. If it leaves the outer wall toward the inner cylinder and diffusely reflects there it transmits, on the average, angular moment of amount $m\omega a^3/b$. If it specular reflects on the inner cylinder and then diffusely on the outer it does not contribute to the momentum balance but if it specular reflects at the inner cylinder, then at the outer cylinder and then diffusely reflects at the inner cylinder it is easily seen that it contributes $m\omega a^3/b$ to the angular momentum balance. In fact, it contributes this amount if its first diffuse collision after emission is with the inner cylinder regardless of the number of intervening specular reflections. The probability of the next diffuse collision being with cylinder a after emission subsequent to a diffuse collision with b is thus

$$f_a + (1-f_a)(1-f_b)f_a + (1-f_a)^2(1-f_b)^2f_a + \dots = \frac{f_a}{1 - (1-f_a)(1-f_b)}.$$

To get the torque we have only to remark that the rate of diffuse collisions with wall b is

$$\frac{pf_b}{\sqrt{2\pi mkT}}$$

whence

$$T = p \sqrt{\frac{2\pi m}{kT}} \omega a^3 \frac{f_a f_b}{1 - (1 - f_a)(1 - f_b)}$$

$$= p \sqrt{\frac{2\pi m}{kT}} \omega a^3 \frac{1}{\frac{1}{f_a} + \frac{1}{f_b} - 1}$$

If the harmonic mean of the two f 's is f^* where

$$\frac{2}{f^*} = \frac{1}{f_a} + \frac{1}{f_b}$$

then

$$T = p \sqrt{\frac{2\pi m}{kT}} \omega a^3 \frac{f^*}{2 - f^*}$$

which reduces to the previous expression when $f^* = 1$, i.e., both f_a and $f_b = 1$.

To compare our results with those of Millikan and his co-workers, we note the following:

Using Maxwell's theory of slip the equation derived by Millikan⁽¹⁾ reduces as $p \rightarrow 0$ to

$$T = 2\omega \frac{a^3 b^3}{(a^3 + b^3)} \sqrt{\frac{2\pi m}{kT}} \frac{f}{2 - f}$$

⁽¹⁾ R. A. Millikan; Phys. Rev. 21, 217 (1923).

which agrees with our result only if $b-a \ll a$. Blankenstein⁽¹⁾ purports to consider the same situation as this memorandum and obtains an equation equivalent to

$$T = 2\omega \frac{a^2 b^2}{b+a} \sqrt{\frac{2\pi m}{kT}} f$$

which disagrees with our result both as regards the dependence on the radii and in the form in which f enters the equation.

W. C. DeMarcus
W. C. DeMarcus

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⁽¹⁾E. Blankenstein: Phys. Rev. 22, 583 (1923).

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