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General Derivation of Bäcklund Transformations
from Inverse Scattering Problems*

by

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ABSTRACT

A general way to derive Bäcklund transformations for non-linear partial differential equations that are solvable by inverse scattering method in the scheme of Ablowitz, Kaup, Newell and Segur is demonstrated.

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Recently, Ablowitz et. al.¹ discovered a general scheme of finding the set of nonlinear partial differential equations that are solvable by inverse scattering method. We will show in this paper how one can derive the Bäcklund transformation from the auxiliary equations for the inverse problem.² This derivation provides the basis of unifying the two different approaches of solving these non-linear equations.

1. Equations solvable by inverse method.

Ablowitz et. al.¹ have found that the integrability conditions for the systems of linear partial differential equations

$$\begin{cases} v_{1x} + i\zeta v_1 = qv_2 \\ v_{2x} - i\zeta v_2 = rv_1 \end{cases} \quad (1)$$

and

$$\begin{cases} v_{1t} = Av_1 + Bv_2 \\ v_{2t} = Cv_1 - Av_2 \end{cases} \quad (2)$$

are exactly those equations which allow soliton solutions solvable by inverse scattering method. The integrability conditions are:

$$\begin{cases} A_x = qC - rB \\ B_x + 2i\zeta B = q_t - 2Aq \\ C_x - 2i\zeta C = r_t + 2Ar \end{cases} \quad (3)$$

Finite expansions of A, B and C in terms of ζ reduces it to specific equations of interest. For example, KdV², mKdV³, sine-Gordon⁴ and nonlinear Schrödinger equation.⁵

2. Bäcklund Transformations.

Equations that are solvable by inverse scattering problem are found to be also solvable by Bäcklund transformations.^{6,7} But till now only two such transformations have been found.^{6,7} They are derived independently from tedious ad hoc elimination procedure (or simply from a guess). We present in this section a unified way of finding them from the inverse problem, and therefore, provides a base for the statement that corresponding to each inverse problem there exists a Bäcklund transformation.

From Eq. (1), (2) and (3), we can easily get a system of equations for the quantity $u \equiv v_1/v_2$:

$$\begin{cases} u_x = 2i\zeta u - ru^2 + q \\ u_t = 2Au - Cu^2 + B \end{cases} \quad (4)$$

This equation is very important. We will demonstrate in the following by showing examples that all Bäcklund transformations can indeed be reduced (or derived) to (from) this set of Riccati equations. A specific identification of u as a functional of q and q' will provide a Bäcklund transformation to a particular differential equation. These Bäcklund transformations can be divided into different classes.

Class I: $r = \text{constant} = -2$

Equation (4) then becomes, $i\zeta = k$

$$\begin{cases} u_x = -2ku + 2u^2 + q \\ u_t = 2Au - Cu^2 + B \end{cases} \quad (5)$$

The simplest example in this class is the Korteweg-de Vries equation,⁷ $q_t + 12qq_x + q_{xxx} = 0$. Following Ablowitz et. al., identifying A, B and C to be

$$\begin{aligned} A &= 4k^3 + 4kq - 2q_x \\ B &= -4k^2q + 2kq_x - q_{xx} - 4q^2 \\ C &= 8k^2 + 8q \end{aligned} \quad (6)$$

If we eliminate q in Eq. (5) we get an equation for u

$$u_t - 24u^2u_x + 24kuu_x + u_{xxx} = 0 \quad (7)$$

This is a mixed KdV - mKdV equation. Equation (5) provides a Bäcklund transformation between solutions of KdV equation and Equation (7). In particular, if $k = 0$, Equation (7) reduces to the pure mKdV equation and Equation (5) becomes the famous Miura transformation. Now, we can see that if (u, k) satisfies Equation (7), then $(-u, -k)$ also satisfies Equation (7). This gauge-like invariance of Equation (7) tells us immediately that a second solution q' exists for the KdV equation, such that

$$\begin{cases} -u_x = -2ku + 2u^2 + q' & (8a) \\ -u_t = -2A(q', -k)u - C(q', -k)u^2 + B(q', -k) & (8b) \end{cases}$$

From Equation (5) and Equation (8), we get

$$\begin{cases} 2u_x = q - q' & (9a) \\ q + q' = 4ku - 4u^2 & (9b) \end{cases}$$

and

$$\begin{cases} 2u_t = 2u(A+A') - u^2(C-C') + (B-B') & (10a) \\ 0 = 2u(A-A') - u^2(C+C') + (B+B') & (10b) \end{cases}$$

Two different forms of Bäcklund transformations can be derived from Equation (9)

$$(i) \quad \text{let } q = w_x,$$

then $u = \frac{w - w' + k}{2}$, we get the Bäcklund transformation between w and w' .

$$\begin{cases} (w + w')_x = k^2 - (w' - w)^2 & (11a) \end{cases}$$

$$\begin{cases} (w - w')_t = 2A(w - w' + k) - \frac{C}{2}(w - w' + k)^2 + 2B & (11b) \end{cases}$$

where w satisfies the equation

$$w_t + 6(w_x^2) + w_{xxx} = 0$$

$$(ii) \quad \text{solve } u \text{ from Equation (9b)}$$

we get

$$u = \frac{2k \pm \sqrt{4k^2 - 4(q+q')}}{4}$$

Because Equation (9b) is a local equation, we can choose a constant x_0 such that q would be continuous soliton solution and then

$$u = \frac{k \pm \sqrt{k^2 - (q+q')} H(x-x_0 - 4k^2 t)}{2} \quad (12)$$

where H is the Heaviside step function. Substitute it into Equation (9a) we get the Bäcklund transformation between q and q' directly

$$\begin{cases} (q+q')_x = \mp 2(q-q') \sqrt{k^2 - (q+q')} H(x-x_0 - 4k^2 t) \\ u_t = 2Au - Cu^2 + B \end{cases} \quad (13)$$

We make two remarks here. First, Equation (10b) is equivalent to Equation (9a) plus Equation (9b), so that either Equation (10a) or Equation (8b) with the appropriate u can serve to be the time part of the Bäcklund transformation. Second, the two forms of Bäcklund transformation, Eqs. (11) and (13) are actually equivalent. We do not wish the reader to get the wrong impression that Bäcklund transformation is not unique.

Class II: $r, -q, i\zeta = \frac{k}{2}$

In this class, we can also get two equivalent forms of Bäcklund transformation. Let us consider two examples.

a) Modified KdV equation, $q_t + 6q^2 q_x + q_{xxx} = 0$.

$$\begin{aligned} A &= \frac{k^3}{2} + kq^2 \\ B &= -k^2 q + kq_x - q_{xx} - 2q^3 \\ C &= k^2 q + kq_x + q_{xx} + 2q^3 \end{aligned} \quad (14)$$

Now, eliminate q , we get the equation for u as the following,

$$\begin{cases} u = \operatorname{tg} \frac{v}{2} \\ v_t + v_{xxx} + \frac{v^3}{2} + \frac{3k^2}{2} v_x \sin^2 v = 0 \end{cases} \quad (15)$$

This equation also possesses the invariance $(u, k) \rightarrow (\pm u, \pm k)$.

It results in the following self-Bäcklund transformation for mKdV equation

(i) $v = w \mp w'$

$$\begin{cases} (w \pm w')_x = k \sin(w \mp w') \end{cases} \quad (10a)$$

$$\begin{aligned} \begin{cases} (w \pm w')_t = -2w_x^2 k \sin(w \mp w') \mp 2w_{xx} k \cos(w - w') \\ \mp k^3 \sin(w \mp w') \pm 2k^2 w_x \end{cases} \end{aligned} \quad (10b)$$

$$(ii) \quad u = \frac{k \pm \sqrt{k^2 - (q \mp q')^2}}{(q \mp q')} H(x - x_0 - k^2 t) \quad (17)$$

$$\begin{cases} (q \mp q')_x = \mp (q \pm q') \sqrt{k^2 - (q \mp q')^2} H(x - x_0 - k^2 t) \\ u_t = 2Au - Cu^2 + B \end{cases} \quad (18)$$

b) sine-Gordon equation, $2w_{xt} = \sin 2w$

$$A = -\cos 2w/2k$$

$$B = C = q_t/k \quad (19)$$

Equation satisfied by u is

$$\begin{cases} u = \operatorname{tg} \frac{v}{2} \\ v_{xt} = \sqrt{1 - k^2 v_t^2} \sin v \end{cases} \quad (20)$$

It is also invariant under the transform $(u, k) \rightarrow (\pm u, \pm k)$. The Bäcklund transformations are then

$$(i) \quad v = w \mp w'$$

$$\begin{cases} (w \pm w')_x = k \sin (w \mp w') \\ (w \mp w')_t = \frac{1}{k} \sin (w \pm w') \end{cases} \quad (21)$$

(ii) same as Equation (17) and Equation (18) but with appropriate A, B, C and H

Class III: $r = -q^*$, $\zeta = \xi + i\eta$

In this class

$$\begin{cases} u_x = -2i\zeta u + q^* u^2 + q \\ u_t = 2Au - Cu^2 + B \end{cases} \quad (22)$$

The simplest example is the nonlinear Schrödinger equation

$$iq_t + q_{xx} + 2q^2 q^* = 0. \quad \text{For this equation, we have}$$

$$\begin{aligned} A &= -2i\zeta^2 + iq q^* \\ B &= +2q\zeta + iq_x \\ C &= -2q^* \zeta + iq_x^* \end{aligned} \quad (23)$$

After elimination of q and q^* in Equation (22), we get a nonlinear partial differential equation for u and u^* . It is straightforward to show that this equation for u is invariant under the gauge like transformation

$$(u, \zeta) \rightarrow (\pm u, \zeta^*)$$

Therefore, we have

$$\begin{cases} \pm u_x = \mp 2i\zeta^* u + q'^* u^2 + q' & (24a) \\ \pm u_t = \pm 2A(q', \zeta^*)u - C(q', \zeta^*)u^2 + B(q', \zeta^*) & (24b) \end{cases}$$

From Equation (23) and Equation (25) we have

$$\begin{cases} 0 = 4\eta u + (q \mp q')^* u^2 + (q \mp q') & (25a) \end{cases}$$

$$\begin{cases} 2u_x = -4i\xi u + (q \pm q')^* u^2 + (q \pm q') & (25b) \end{cases}$$

Equation (25b) is not integrable as in the case of KdV and mKdV by transformation $q = w_x$. We get therefore only one form of Bäcklund transformation for nonlinear Schrödinger equation itself.

$$u = \frac{-2\eta \pm \sqrt{4\eta^2 - |q \mp q'|^2}}{(q \mp q')^*} H(x_0 - x_0 + 4\xi t) \quad (26)$$

$$\begin{cases} (q \pm q')_x = -2i\xi (q \pm q') + (q \mp q') \sqrt{4\eta^2 - |q \pm q'|^2} H(x-x_0+4\xi t) \\ (q \pm q')_t = -2i\xi (q \pm q')_x + i(q \mp q')_x \sqrt{4\eta^2 - |q \pm q'|^2} H(x-x_0+4\xi t) \\ \quad + i(q \pm q')(qq^* + q'q'^*) \end{cases} \quad (27a)$$

$$(27b)$$

We remark here that Class II is only a special subclass of Class III. That is, when $\xi = \frac{-ik}{2}$ purely imaginary and $q = q^*$ purely real; Backlund transformations (27a) would reduce to Equation (18a).

A complex form of mKdV equation, $q_t + 6qq^*q_x + q_{xxx} = 0$, exists as a member in Class III and shares the same spatial part of Bäcklund transformation as the nonlinear Schrödinger equation, so does the Hirota equation⁸ $q_t + 6\alpha qq^*q_x + \alpha q_{xxx} + i\beta q_{xx} + 2i\beta q^2 q^* = 0$.

3. Conclusions.

It is demonstrated above that from the inverse scattering problem defined by Equation (1), (2) and (3), Bäcklund transformations between different class of nonlinear partial differential equations can be found. The existence of a gauge-like invariance for the equation satisfied by (u, ξ) makes it possible to find a self-Bäcklund transformation for q and q' . These Bäcklund transformations can be divided into classes. Equations in the same class have an identical spatial part Bäcklund transformation. Their solutions therefore satisfy the same superposition formula. For example the mKdV and sine-Gordon equations have the same superposition formula

$$\operatorname{tg} \left(\frac{w_3 - w_0}{2} \right) = \frac{k_1 + k_2}{k_1 - k_2} \operatorname{tg} \left(\frac{w_1 - w_2}{2} \right).$$

This formula renders it possible to construct N-soliton solutions by algebraic manipulations only.

Further generalizations to higher order inverse problems of what we have done above is possible. We will report some examples in another paper.

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