10

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NEUTRON SCATTERING ANGULAR DISTRIBUTION

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TABLE OF CONTENTS

Page

ABSTRACT	3
SECTION I: EXPERIMENTAL APPARATUS	5
Introduction	5
Historical	5
Apparatus	5
SECTION II: ANALYSIS OF THE DATA	11
Introduction	11
Calculation of the Legendre Polynomial Coefficients	11
(Attenuation Correction)	12
Termination of the Legendre Series	18
Multiple Scattering Corrections	21
SECTION III: SINGLE AND DOUBLE SCATTERING IN PLATES	36
A. Introduction	36
B. Notation and Sign Conventions	36
C. Idealized Singly Scattered Intensity	38
D. Emergent Singly Scattered Intensity	39
E. Integrated Emergent Single Scattering	40
F. Emergent Doubly Scattered Intensity	42
G. The Integrated Doubly Scattered Flux	48
SECTION IV: PLANS FOR FUTURE ANGULAR DISTRIBUTION DATA (The Angular Resolution Correction)	50
SECTION V: ACKNOWLEDGEMENTS	52
SECTION VI: THE EXPERIMENTAL DATA	53
APPENDIX A. A GENERALIZED EXPONENTIAL INTEGRAL	146
APPENDIX B. THE FUNCTIONS $F_j(\tau, \alpha)$, $F_j(\tau, \alpha, s)$ and $M_0(\tau, \alpha)$	155
APPENDIX C. THE INTEGRAL $I_{s,j+1}^{(1)}(h, \alpha)$	158
APPENDIX D. THE INTEGRAL $I_{s,j+1}^{(2)}(h, \alpha)$	161

- 3 First paragraph should begin:
"The accompanying 41 graphs on pp. 57-137. . ."
- 9 Last paragraph: After third sentence insert:
(For further discussion on mean scattering angles see pp. 50 and 54.)
- 21 Equation (13). Insert α before the ratio $\sigma(u)/\sigma_t$ in the case of reflection.
- 24-28 Titles should have Q_{lk} where Q_{kj} appears.
- 29 Third line from bottom, should read Eq. (21) instead of Eq. (20).
- 34 Line 6, should read $\sigma_t Z$ instead of σ_{st} .
- 47 Equation 58 should read δ_{kl} instead of δ_{kl} in first bracket term.
- 47 Line 4. Replace lower by less.
- 48 Equation 69. Subscripts on I and J should have comma between s and j + 1.
- 53 The second sentence should read:
"The large graphs which follow on pp. 57-137. . ."
- 54 Add: For further general information see also notes in Figure titles on pp. 60, 62, etc., and comments on p. 50 (paragraph 3) and p. 9 (paragraph 4).
- 58 Delete last sentence.
- 149 Equation (A20). Add primes to x inside integrals, where absent.
- 151 Between Eq. (A31) and (A32), add subscript n to read $\phi_n(z, x)$
- 154 Eq. (A48). The large bracket is the binomial coefficient symbol and should be $\binom{n-1}{k}$ without any horizontal line inside.
- 154 Footnote 12. Thanks are due to Richard S. Preston for finding this reference.
- 163 The reference is to Eq. (D2) instead of to (C2).
- 163 Eq. (D8) and preceding equation. A bracket] should follow so one has: $h M_0 \left(2h, \frac{\alpha-1}{2} \right)]$.

NEUTRON SCATTERING ANGULAR DISTRIBUTION

A. Langsdorf, Jr., R. O. Lane, J. Monahan

ABSTRACT

The accompanying 41 graphs present the results of a survey of the angular distribution of neutrons scattered from 36 elements, two compounds, and one alloy. The data are presented in the form of an expansion in a series of Legendre polynomials,

$$\sigma(\theta) = \sum_{l=0}^4 \frac{4\pi}{l} \omega_l P_l(\cos \theta),$$

in the laboratory coordinate system. The graphs give σ_t , ω_1 , ω_2 , ω_3 , and ω_4 as a function of neutron energy.

Since the detector system used BF₃ proportional counters immersed in an oil moderator so arranged as to have constant (within 10%) counting efficiency for neutrons of all energies from 100 kev to 1500 kev, $\sigma(\theta)$ is the total differential scattering cross section.

The actual scattering detectors were positioned at five scattering angles: 22°, 54°, 92° (on the transmission side of the plate scatterer), 112° and 144° (on the reflection side). Therefore $\sigma(\theta)$, as given by the above expansion, will be correct at these angles even if the value thus extrapolated to other angles is slightly erroneous.

The angular distribution is measured usually at about 80-kev intervals for each element; the resolution was ~100 kev at the lower energies, decreasing to ~60 kev at the higher incident energies. For the transmission angles the angular resolution is ~5°, while for the reflection angles it is ~3°.

A statistical analysis of part of the data, as well as an observation of the general consistency, indicated standard deviations of ~±0.05 for ω_1 and ~±0.10 for ω_4 , with the accuracy of ω_2 and ω_3 being intermediate between these values. These deviations are roughly independent of the values of the ω_l . In most cases the total cross section, σ_t , is probably accurate to better than 10% statistically.

It was necessary to "can" some eight materials in containers of 0.005-in. shim stock steel. For these samples the background counting rates - measured with an empty can in place of the scatterer - were

higher than in other cases, leading to slightly less accurate results statistically and also to slightly larger systematic errors.

All total cross sections are evaluated in terms of the total cross section of a given sample of carbon, the value of which was taken from BNL-325 and unpublished data.

On the graphs open circles are used for σ_t and ω_4 , open triangles (Δ) for ω_1 , open squares for ω_2 , and open triangles (∇) for ω_3 . There is no graph for ω_0 , which was assumed to be equal to unity; this is equivalent to neglecting all reactions other than neutron scattering.

The data points with open symbols are not corrected for plural scattering. The corresponding solid symbols represent points corrected for plural scattering as described in this report. Such corrections were made on part of the data for twelve of the elements. For these twelve samples the solid lines were drawn following the shape of the uncorrected data but displaced by an amount indicated by the corrected points. Thus the solid lines are the best indication of the true behavior of ω over the entire energy region covered. On the other graphs the solid lines were drawn by "guessimation" - a combination of quantitative estimation plus some guessing - by analogy with the twelve cases for which corrections had been made. The solid lines on these graphs are a reasonable approximation to the correct σ_t and ω , but are less accurate than in the cases where the actual corrections were made. No line is drawn for ω_3 and/or ω_4 when either is zero within the accuracy of the data.

In several graphs data is given for two or more thicknesses of scattering sample at the same neutron energy. In some of these cases (^{16}C , ^{11}Na , ^{12}Mg , ^{47}Ag , and ^{23}U) several thicknesses were corrected for plural scattering.

The remainder of this report describes the experiment and the analytical methods which were used to obtain corrected cross sections in the representation shown in the graphs. Section I contains a detailed description of the apparatus and of the experimental techniques which were used. In Section II the analytical methods of obtaining a microscopic differential cross section from the raw data are described. Section III contains a discussion of the plural scattering corrections for this geometry (a collimated beam of particles incident on a thin flat scatterer). Section IV discusses some modifications, both in experimental procedure and analytical treatment of the data, which are planned for future experiments of this type. Section V contains a list of those people who have contributed to the various phases of this work. Finally, in Section VI the symbolism of the graphical presentation of the differential cross section data is explained.

EXPERIMENTAL APPARATUSGeneral Purpose; Introduction

The determination of the angular distribution of scattered mono-energetic neutrons is of interest and importance from various points of view. The lack of agreement between experiments and relevant theories may well be resolved in the light of more extensive experimental information. These discrepancies, particularly in theories¹ which average over a band of energies including many resonances, are of current interest. In addition, angular distribution information in the neighborhood of individual resolvable resonances (in the lighter elements, particularly) is a powerful aid in the assignment of correct angular momenta associated with such resonances. Finally, such data have a wide range of practical uses in the solution of transport problems including the design of reactors and their shielding.

Historical

Some experimental work on this problem, using the van de Graaff generator at the Argonne National Laboratory as a neutron source, was reported early in 1952.² This work showed the feasibility of the method being employed but indicated that refinements in technique were required for extensive or accurate work in this field. The apparatus and results described in this report stem from the development initiated at that time.

Apparatus - (See illustrations, p. 6 & 7)

The primary neutron source was a target bombarded by a beam of ions from the van de Graaff generator; in the work reported here neutrons were obtained from the $\text{Li}^7(p,n)\text{Be}^7$ reaction.

The neutron-producing source was placed in the cavity in a tank filled with boric acid water. This tank had a collimating tube from the cavity to the exterior of the tank, permitting a beam of neutrons to emerge from the source target so as to impinge on a thin flat plate of scattering material.

Five neutron detectors were placed in a circular array around the scattering material. Thus scattering data were determined simultaneously at five angles.

¹See, for example, Feshbach, Porter, and Weisskopf, Phys. Rev. 96, 448 (1954).

²ANL-4798, pp. 5-18 and earlier reports.

A view of apparatus in an early stage of assembly. Neutron source shield tank appears in the upper left. Each of the five detector tanks is aimed at scatterer position in center.

Completed apparatus shown with proton beam line entering source shield tank from lower right. Plate scatterer in open frame can be seen in center.

Central region of experiment showing plate scatterer and the faces of the source shield and detector tanks.

View of BF_3 counters suspended in moderating oil.

Each neutron detector contained ten BF₃ proportional counters, 1.5 in. diameter, 20 in. active length. These were immersed in a neutron moderator of mineral oil in an aluminum tank 20 in. diameter by 4 ft high. Each aluminum tank was placed inside a steel tank containing borated water. A collimator hole permitted neutrons from the scatterer to reach the counter assembly.

The dimensions of the entire array were such that the designed distance from the neutron source to the center of the scatterer was 5 ft; scatterer-to-detector distance, 7 ft; illuminated area of scatterer measured normal to the axis of the incident beam, 6 in. wide by 16 in. high; effective detector area, also 6 in. wide by 16 in. high. The water tanks were in the form of truncated pie-shaped wedges (right-isosceles trapezoidal cylinders) which fit together side by side. The five detector shield tanks each subtended 30 degrees in the horizontal plane at the center of the scatterer and the source shield tank subtended 40 degrees. The water shielding thus filled all the available space. Its adequacy was tested by plugging the collimator hole of one unit. Then the counting rate with 1-Mev neutrons being produced was less than 1/2000 of that with the collimator open. This shielding was entirely adequate at 1 Mev and probably will be adequate up to or above 3 Mev.

The detector shield tanks each contained about 700 gallons of (borated) water; the source shield tank contained about 1000 gallons. The source shield tank had a removable lid, permitting reasonably simple access for servicing the lithium target.

Since BF₃ counters in a moderator detect neutrons of all energies, the apparatus measured total scattering rather than just elastic scattering. It was therefore desirable that the efficiency of the detectors be independent of incident neutron energy so that if inelastic scattering were present, the measured cross section would not be a complicated function of the amount and type of inelastic scattering. Various arrays of the ten counters in each assembly were tried, until an array was obtained such that the response was within 10% of that of a "long counter" (which was presumed to have a "flat" response) from 0.1 Mev to 1.5 Mev. It is considered that the present degree of uniformity of response is adequate for these initial experiments.

By tests with neutron sources (photo-sources of Sb-Be, Na-Be, Na-D₂O, and mixed Ra (α)-Be) the five counters were found to have equal efficiencies within about 2 percent. (The data were of course corrected to uniform efficiency.) The efficiency for counting the neutrons actually impinging on the 6 in. by 16 in. face of each detector was between 12 percent and 15 percent; this was about three times the minimum estimate design figure used in planning the construction.

The counting rates were such that with a lithium target of 10-kilovolt stopping power as a neutron source (which has a mean yield of about 10^6 neutrons per steradian at 0° per microcoulomb of protons), about 300 counts per minute per detector were obtained with a 15-microampere beam.

In most neutron scattering apparatus, adequately precise monitoring of the flux incident on the scatterer is very difficult. By measuring five angles at once, monitoring becomes somewhat less important. However, quite satisfactory monitoring is in fact obtained by locating two BF₃ counters in the source shield tank assembly, close to the collimator hole. In this geometry, even with very thick scatterers, backscattering perturbs the monitor a negligible amount (less than 1% for a scatterer over a mean free path thick).

The background counting rate (the counting rate in the absence of a scattering sample) in this apparatus is predominantly due to air. Each detector, in addition to "seeing" the scattering sample, also "sees" a column of air about 2 ft long which is "illuminated" by the primary neutron beam. This amount of air scatters about 1% of the neutrons. Thus background rates were predicted to be, and indeed were found to be, on the average, about 10% of the rate due to the scatterer. For some angles with some samples, the background was as great as 30%. The air scattering could be reduced by building a large vacuum tank; however, such a step is not believed to be important at the present time.

The scattering measurements reported here were made with the five neutron detectors placed at fixed positions. The scatter-angles between the axis of the incident beam and the axes of the five counters were: $22^\circ 18'$, $54^\circ 4'$, $91^\circ 40'$, $112^\circ 25'$, and $143^\circ 54'$. For these nominal angles, but with the extended sizes of scatterers and detectors employed, the mean angles of scatter and root mean square deviations from the mean have been calculated. These mean angles, again in order of increasing angle, were: $26.5^\circ \pm 5.4^\circ$, $55.4^\circ \pm 5.0^\circ$, $91.2^\circ \pm 5.6^\circ$, $112.8^\circ \pm 2.4^\circ$ and $143.7^\circ \pm 2.4^\circ$. These calculations were made for the geometrical conditions of the experiment, including the fact that the scattering samples were mounted vertically, with the incident beam at an angle of 45° to the normal to the plate surface. The first three detectors were placed to observe neutrons scattered from the side of the plate opposite that from which the beam was incident (that is, in a transmission-side geometry). The last two detectors (those at 112° and 144°) observed neutrons from the same side from which the beam was incident on the plate (that is, in a reflection-side geometry). It will be noted that the transmission geometry had twice as large a root-mean-square spread in angle as the reflection geometry; this is a characteristic geometric difference between the two cases. It should also be pointed out that this oblique orientation of the scatterer plate was a practical compromise chosen to minimize the difficulties in correcting for plural

scattering. It is clear that nearly all neutrons will be plurally scattered away from any direction which is nearly in the plane of the plate. With the orientation chosen, no data were taken at an angle of emergence from the plate greater than 47° , measured from the normal to the plate surface. Thus we have been able to control the magnitude of the plural scattering perturbations so that in the most extreme cases that have been calculated, the corrections to any initial (uncorrected) differential scattering cross sections were never over about 35% and were usually a good deal less than this.

Most of the factors in the geometry of the system necessary to make highly refined analyses of the angular data can be determined by auxiliary experiments. Many of these will be quite practical in the near future when the five detectors, which have been fixed in location, will be mounted on a track so that they pivot about the location of the scattering sample. A few of the geometrical factors have been partially studied already. In particular, the collimation of the emergent beam from the source collimator seems to be excellent, at least up to 1.8 Mev. A good deal of the sharpness of definition of the beam may be ascribed to the use of an accurately rectangular, pyramidally tapered source collimator, with the source spot of neutrons ($1/4$ in. diameter spot) at the apex. The actual shielding material started 6 in. from the source spot and extended a distance of 28 in.

Another factor which was studied was the extent of background neutron counting which might be ascribed to neutrons already scattered by a scattering sample, which were then rescattered by air or shielding materials visible to a given detector. This effect was observed with the aid of a sodium-beryllium photoneutron source placed at or near the usual scatterer position, with and without a shadow-cone shield interposed between the source and given detector. In this manner an upper limit of 2% was placed on the amount of such scattering relative to the direct flux from the source. Of this 2%, roughly half was produced by air scattering and half by other materials (located as an oppositely located shield tank would be). No corrections for this phenomenon have been made in the data as presented in this report. In a few cases, this effect could be a correction of some importance, but adequate information is not yet at hand to know just how to make the correction.

ANALYSIS OF THE DATAIntroduction

In order to present the data in a practical and useful form, it was decided to represent the differential cross section in the laboratory coordinate system by an expansion in a series of Legendre polynomials. It is then possible to obtain a compact and easily visualized description of the angular distribution over a range of incident neutron energies by graphing these series coefficients as a function of incident neutron energy.

In brief, the scheme of calculation involves reducing the actual neutron counting information to a representation in terms of Legendre polynomials. The coefficients in this representation are then used to carry through a correction of the observed counting information for plural scattering by an analytic theory of double scattering in plates. Then by iteration a new set of Legendre polynomial coefficients may be obtained. Reiteration may be employed until the answer has converged as closely as may be desired to a stable answer. A reiteration is necessary in very few cases.

In this scheme, one may see that the Legendre polynomials are required in the Laboratory coordinate system in order to complete the calculations. This representation is also useful for all practical problems in neutron transport. For use in nuclear theory, a representation in the center-of-mass coordinate system would be more desirable. It was decided not to delay issuing the data by trying to carry through the transformation, which would be very time consuming for the amount of data presented in this report.

In this analysis it is convenient to make several assumptions.

These are:

- (1) Since the resolution used in obtaining these data is approximately 100 kev, the quantity measured is very nearly

$$\frac{\sigma(E, \theta)}{\Delta} = \frac{1}{\Delta} \int_{\Delta} I(E) \sigma(E, \theta) dE,$$

where $I(E)$ is the spectrum of incident neutrons and Δ is their resolution width. Thus, all sharp resonance variation in the cross section is averaged out. It is assumed that all effects, such as "beam hardening," energy loss on scattering, etc., which cause a variation in $I(E)$ or Δ can be neglected in the interpretation of $\sigma(E, \theta)$. In the following we shall omit the bar in designating this cross section.

(2) All polarization effects are neglected, i.e., $\sigma(\theta)$ is not considered to be a function of the azimuthal angle, ϕ .

(3) In the experiment a plane parallel scattering sample of finite thickness, h , is used (in practice $0 < h < 0.5$ and usually $0.05 < h < 0.10$; in units of mean free path). In order to correct the data for multiple scattering, it is assumed that the scatterer is infinite in the plane axis direction. It is further assumed that the scattering sample is "illuminated" by a parallel beam of neutrons and that the scattered neutrons are detected by a counter system which subtends a small solid angle at the scattering plate. The actual geometry, of course, cannot quite match these assumptions.

(4) As a consequence of assumptions (1) and (2), we do not distinguish elastic from inelastic scattering. We assume that the neutron detector measures all scattered neutrons with equal efficiency, a situation which is very nearly achieved in practice.

(5) Neutron capture is assumed to be negligible relative to neutron scattering; thus

$$\sigma_{st}/\sigma_t \approx 1,$$

where σ_{st} denotes the total scattering cross section and σ_t the total cross section. For almost every material measured in this survey

$$0.9 < \sigma_{st}/\sigma_t \leq 1.0$$

within the energy range in which measurements were made. In this discussion all charged particle reactions such as (n, p) , etc., are to be lumped with neutron capture.

(6) Finally, all multiple neutron producing reactions such as $(n, 2n)$, $(n, \text{fission})$, etc., are assumed to be negligible.

Calculation of the Legendre Polynomial Coefficients

We define the following symbols:

Φ_0 , the flux incident on the scattering sample, a flat plate. In the following discussion we use "flux" in the sense of neutrons per unit area integrated over a specified time.

A , the area, measured perpendicular to the direction of Φ_0 , of the sample "illuminated" by the incident flux.

N_Z , the thickness, in atoms/cm², of the scattering sample, Z , measured in the direction normal to the plane of the plate.

θ_0 , the angle included by the directions of Φ_0 and the normal to the scattering plate.

$\sigma_t Z$, the total neutron cross section in barns for the sample Z .

$\sigma(\mu^m)$, the differential neutron scattering cross section, in barns per steradian, for the sample Z evaluated at the scattering angle whose cosine is μ^m .

Thus, the total flux incident on the scattering sample is $A\Phi_0$. The fraction of this which is transmitted is

$$A\Phi_0 e^{-\alpha_N \sigma_t Z},$$

where $\alpha_N Z \equiv N/\cos \theta_0$ is the thickness of the scatterer in the direction of the incident beam. The fraction of the incident flux which is not transmitted, and which, therefore, has interacted, is

$$A\Phi_0 (1 - e^{-\alpha_N \sigma_t Z}).$$

Of this interacting flux, the fraction $\sigma(\mu^m)/\sigma_t Z$ is scattered into a unit solid angle at the angle whose cosine is μ^m . Thus the flux scattered into a detector which subtends the solid angle $\Delta\Omega^m$ is

$$\Delta\Omega^m \frac{\sigma(\mu^m)}{\sigma_t Z} A\Phi_0 (1 - e^{-\alpha_N \sigma_t Z}).$$

If the efficiency of this detector is ϵ^m , the number of counts it records, corrected for background, is

$$\epsilon^m \Delta\Omega^m A\Phi_0 (1 - e^{-\alpha_N \sigma_t Z}) \frac{\sigma(\mu^m)}{\sigma_t Z}.$$

The counts are measured with several separate detectors set up at the various angles identified by the index m . These counters have slightly differing values of ϵ^m and $\Delta\Omega^m$. By a separate measurement

14

using a rotating neutron source (e.g., Sb-Be, Na-Be, and Na-D₂O photo-neutron sources and a Ra-Be source), it is possible to determine the relative efficiency of each counter. Therefore, a set of correction factors, ρ_m , can be determined such that the product $\rho_m \Delta \Omega_m$ is the same for all angles measured. The counting rate, corrected to uniform efficiency, is then

$$\rho_m \epsilon_m \Delta \Omega_m A \Phi_0 (1 - e^{-\alpha_N Z_0^t Z_0^{\mu}}) = k (1 - e^{-\alpha_N Z_0^t Z_0^{\mu}}) \frac{Z_0^t Z_0^{\mu}}{Z_0^{\mu}} \quad (1)$$

where

$$k = \rho_m \epsilon_m \Delta \Omega_m A \Phi_0$$

is a constant independent of m .

It is clear that the flux, $A \Phi_0$, is the same for all the angles measured simultaneously. However, it is further assumed that a constant value of this incident flux can be maintained by normalizing all counts to a standard count as measured by a monitor whose count is proportional to the forward neutron output of the Li(p, n) source.

In brief, for the counter at the scattering angle $\cos^{-1} \mu_m$:

$g(\mu_m)$ is the observed gross counts for a particular scatterer.

m is the monitor count observed simultaneously.

$G(\mu_m) = \frac{g(\mu_m)}{m}$ is then the gross count normalized to a unit monitor reading.

$b(\mu_m)$ is the observed background count.

m is the simultaneously observed monitor reading.

$B(\mu_m) = \frac{b(\mu_m)}{m}$ is then the normalized background count.

Thus

$$G(\mu^m) - B(\mu^m)$$

is the normalized net count at $\cos^{-1} \mu^m$ due to the scattering sample. Defining the corrected and normalized count to be

$$C^Z(\mu^m) = \rho^m [G(\mu^m) - B(\mu^m)] ,$$

for the scatterer Z, we obtain approximately the quantity defined by Eq. (1) so that as a first approximation

$$C^Z(\mu^m) = k (1 - e^{-\alpha N_Z \sigma^Z(\mu^m) t_Z}) \frac{\sigma^Z(\mu^m)}{\sigma^Z(\mu^m)} , \quad (2)$$

where k is a constant which is yet to be determined.

Next the differential cross section is expanded in a series of Legendre polynomials as

$$\sigma(\mu^m) = \sum_{l=0}^{N-1} a_l P_l(\mu^m) , \quad m = 1, \dots, N \quad (3)$$

where we have omitted the subscript, Z, which identifies a particular scattering sample. It is assumed that N is sufficiently large that the coefficients a_l are negligibly small for all $l \geq N$. A discussion of this assumption is given in the following section of this report.

The evaluation of such an expansion presupposes, of course, that measurements of the scattering are made for at least N different values of the scattering angle. Regarding the polynomials $P_l(\mu^m)$ as the elements of a square matrix, it is possible to introduce a reciprocal matrix whose elements, P_{mn} , are defined by

$$\sum_{m=1}^N P_l(\mu^m) P_{mn} = \delta_{ln} . \quad (4)$$

Then

$$\sum_{m=1}^N \sigma(\mu^m) P_{mn} = \sum_{m=1}^N \sum_{n=1}^N a_{mn} P_{mn} = a_n,$$

or

$$a_{\ell} = \sum_{m=1}^N \sigma(\mu^m) P_{m\ell}. \quad (5)$$

The total scattering cross section, σ^{stZ} , for the element Z is defined by the integral

$$\sigma^{stZ} = \int_0^{\infty} \sigma^Z(\mu) d\mu = \int_0^{\infty} \int_{-1}^1 \phi d\mu d\Omega = 4\pi a_0,$$

where we have used Eq. (3) and the orthogonality of the polynomials $P_{\ell}(\mu)$. Combining this result with Eq. (5), we have

$$\sigma^{stZ} = \sum_{m=1}^N \sigma^Z(\mu^m) P_{m0}. \quad (6)$$

From Eq. (2) we now obtain

$$\sum_{m=1}^N C^Z(\mu^m) P_{m0} = k \sum_{m=1}^N \frac{\sigma^Z(\mu^m)}{N \sigma^{stZ}} P_{m0} = k_{m0} = \sum_{m=1}^N \sigma^Z(\mu^m) P_{m0} = \sigma^{stZ}.$$

$$k_{m0} = \frac{4\pi \sigma^{stZ}}{N \sigma^{stZ}} (1 - e^{-\alpha_N \sigma^Z(\mu^m) tZ}). \quad (7)$$

Equation (7) can be used to evaluate the constant k by assuming that measurements have been made with a target for which σ^{stZ} is known and $\sigma^{stZ} = \sigma^Z$. For neutron energies below 2 Mev, the best material for this purpose is carbon; it exhibits no inelastic scattering, no resonance structure, essentially no capture, and its differential cross section is not very far from isotropic in this energy region. Replacing the symbol Z by C to denote a carbon scatterer and considering the value of $\sigma^{stC} = \sigma^C$ as known from independent transmission measurements, it is possible to calculate the value of the constant

is of the order of 0.1. Only in exceptional cases will g exceed a maximum of 0.3. For $\delta = 0.1$ and $g = 0.3$, the error in using Eq. (10), instead of Eq. (9), is 1.7%. Normally $g \approx 0.1$; then if $\delta = 0.1$ the error is 0.7%. Since $\delta > 0.1$ in most cases, the error in calculating the total scattering cross section from Eq. (10) is negligibly small. All of the calculations which we have made are based on Eq. (10).

$$g = \frac{4\pi}{N} \sum_{m=1}^N C^Z(\mu_m) p_{mo}$$

As long as the scattering sample is not too thick, the total scattering cross section can be obtained from Eq. (10) quite accurately even for $\delta \approx 0.1$. In these measurements we choose scattering samples of such thickness that the quantity

$$\sigma^{stZ} = \frac{1}{N} \sum_{m=1}^N C^Z(\mu_m) p_{mo} \left\{ 1 - \frac{4\pi}{N} \sum_{m=1}^N C^Z(\mu_m) p_{mo} \right\} \quad (10)$$

For scattering elements, Z , whose capture cross section is zero, $\delta = 0$, and Eq. (9) becomes

$$\sigma^{stZ} = \frac{1}{N} \sum_{m=1}^N C^Z(\mu_m) p_{mo} \left\{ 1 - \frac{4\pi}{N} \sum_{m=1}^N C^Z(\mu_m) p_{mo} \right\} \quad (9)$$

and solve these equations simultaneously for σ^{stZ} , we obtain

$$\sigma^{stZ} = (1 + \delta) \sigma^{stZ} \quad (\delta > 1),$$

If we define a quantity δ by

$$\frac{\sigma^{stZ}}{\sigma^{stZ}} = \frac{4\pi}{N} \sum_{m=1}^N C^Z(\mu_m) p_{mo} \left\{ 1 - \frac{4\pi}{N} \sum_{m=1}^N C^Z(\mu_m) p_{mo} \right\}$$

Once k is determined, we use Eq. (7) to obtain for any scattering sample other than carbon

$$k = \frac{4\pi}{N} \sum_{m=1}^N C^C(\mu_m) p_{mo} \left\{ 1 - \frac{4\pi}{N} \sum_{m=1}^N C^C(\mu_m) p_{mo} \right\} \quad (8)$$

Combining Eqs. (2) and (7) we obtain

$$C_{\sigma}^Z(\mu^m) = \frac{4\pi \sum_{m=1}^N C_{\sigma}^Z(\mu^m) P_{m\sigma}}{N} \quad (11)$$

Once the N quantities $C_{\sigma}^Z(\mu^m)$ are determined by Eqs. (10) and (11), the coefficients a_{σ} can be determined from Eqs. (5) and (6).

The representation of the results of this survey is given by the expansion

$$C_{\sigma}^Z(\mu) = \frac{4\pi}{N-1} \sum_{\sigma=0}^{\ell} \omega_{\sigma} a_{\sigma} P_{\sigma}(\mu),$$

where $\omega_0 = 1$. There are two reasons why such a representation was chosen: 1) In the correction for multiple scattering the coefficients ω_{σ} enter naturally. 2) The coefficients ω_{σ} are independent of any normalization to a known total cross section: This may be shown by combining Eqs. (5), (6) and (11) to obtain

$$\omega_{\sigma} = \frac{\sum_{m=1}^N C_{\sigma}^Z(\mu^m) P_{m\sigma}}{\sum_{m=1}^N C_{\sigma}^Z(\mu^m) P_{m\sigma}}.$$

From Eq. (5) it may be noted that the quantities $P_{m\sigma}$ serve to weight the individual measurements, $C_{\sigma}^Z(\mu^m)$, in the determination of a_{σ} . It is, therefore, of some practical interest to choose the angles at which measurements are made in such a manner that these matrix elements, $P_{m\sigma}$, are all of comparable magnitude.

Termination of the Legendre Series

As was mentioned in the previous section, the maximum value of N in the series,

$$C_{\sigma}^Z(\mu) = \sum_{\sigma=0}^{N-1} a_{\sigma} P_{\sigma}(\mu), \quad (12)$$

is determined by the number of angles at which measurements are made. In order that this series be a valid representation of $\sigma(n)$, at angles other than the measured $\cos^{-1} n$, it is necessary that δ vanish for $\delta \gg N$.

There are, of course, good theoretical grounds for believing that only a finite number of terms contribute appreciably to the series (12). The termination of this series corresponds to neglecting the contribution of all partial waves whose orbital angular momentum is greater than $(N+1)/2$. For this reason, we have consistently made measurements at an odd number of angles.

Furthermore, there are several methods of checking the validity of the expansion (12). These include:

(1) The values of the total scattering cross sections, σ^{stZ} , calculated from Eq. (10) agree with known values of σ^{tZ} , the total cross section, determined by previous transmission measurements. For those elements where absorption is known to be negligible, the calculated σ^{stZ} usually has been well within 10% of the independently measured σ^{tZ} even though we have made measurements at only five scattering angles thus far. At the present stage of the experiments this is as good agreement as could be expected for a number of reasons.

(2) The values of a_3 and a_4 remain small compared with a_1 and a_2 for lower neutron energies and lighter scattering elements where this is to be expected.

(3) The results for scattering by hydrogen, where the angular distribution is known very well, came out as predicted.

(4) The scattering by elements previously measured elsewhere agrees with our results not only at the angles we measured, but also at interpolated angles using the series (12).

(5) There exists a general smoothness and self-consistency of the results for most materials over the energy range which was measured.

(6) At low enough energy, the scattering is expected to be isotropic in the center-of-mass system. For the lightest elements this leads to a considerable anisotropy in the laboratory system which should be exhibited by a non-zero value for a_1 and much smaller values of a_2 , a_3 and a_4 . Finding this predicted behavior is a further check on the reliability of this representation of the data.

It is, of course, possible to obtain expressions for the effect of higher order non-vanishing coefficients on the coefficients, a_l , in the series (12) with $N = 5$. Let the true differential cross section be given as

$$\sigma(\mu) = \sum_{l=0}^{\infty} A_l P_l(\mu),$$

and compare the coefficients, a_l , as calculated using an expansion

$$\sigma(\mu) = \sum_{l=0}^4 a_l P_l(\mu),$$

with the true coefficients, A_l . The results are

$$\begin{aligned} a_0 &= A_0 - .020 A_5 - .016 A_6 + \dots \\ a_1 &= A_1 - .075 A_5 - .005 A_6 + \dots \\ a_2 &= A_2 + .261 A_5 + .218 A_6 + \dots \\ a_3 &= A_3 - .115 A_5 - .060 A_6 + \dots \\ a_4 &= A_4 + .267 A_5 - .483 A_6 + \dots \end{aligned}$$

It is evident from these expressions that for any reasonable behaviour of the higher A_l , the total cross section, $4\pi a_0$, and the integral

$$\int \sigma(\mu) (1 - \mu) d\Omega = 4\pi (A_0 - \frac{1}{3} A_1)$$

which occurs in the definition of the transport cross section are essentially independent of the higher order coefficients.

For the heavier elements at high neutron energy, there remains some question about the adequacy of the expansion (12) with $N = 5$. In the future the above scheme of calculation will be generalized to give a most probable set of coefficients, a_l , in the sense of a least squares analysis. We plan to extend these measurements to probably nine angles, spaced at about equal increments in degrees between the minimum practical angle ($\sim 20^\circ$) and the maximum (145°). This will allow a meaningful least squares analysis for all coefficients up to a_7 . One advantage of this type of analysis is that it provides a measure of the probable error for each a_l depending, essentially, on how accurately the calculated curve, $\sigma_Z(\mu)$, can be fitted to the individual measured values $\sigma_Z(\mu_m)$.

In the previous sections, we have shown how a representation of angular distribution data is obtained in terms of a series of Legendre polynomials. We consider now the problem of correcting the approximate answers so obtained for the effects of multiple scattering.

We introduce the following notation:

$a = \alpha^{-1} = \cos \theta_0$; $\phi_0 = 0$ defines the direction of the incident neutron beam.
 $c = \gamma^{-1} = \cos \theta_2$; ϕ_2 defines the direction of the emergent beam.

θ is the scattering angle: $\mu = \cos \theta = ac + \sqrt{(1 - a^2)(1 - c^2)} \cos \phi_2$.

$h Z = N Z_0 t Z$ is the normal thickness of the scattering sample, Z , in units of mean free path.

$\Phi_1(h, a, c)$ is the emergent singly scattered flux per unit solid angle, (per unit incident flux) in the direction (c, ϕ_2) .

$\Phi_2(h, a, c)$ is the corresponding emergent doubly scattered flux.

$\Phi_1(h, a)$ is the total (integrated) emergent singly scattered flux.

$\Phi_2(h, a)$ is the total emergent doubly scattered flux.

In a separate section on the theory of double scattering in thin plates, it is shown that

$$\Phi_1(h, a, c) = \begin{cases} \alpha \frac{\sigma(h)}{\sigma} e^{-\alpha h} e^{-\gamma h} & \text{transmission } (\gamma \gg 0) \\ \frac{\sigma(h)}{\sigma} e^{-(\lambda - \alpha)h} e^{-\lambda} & \text{reflection } (\gamma \ll 0) \end{cases}$$

transmission ($\gamma \gg 0$)

reflection ($\gamma \ll 0$)

(13)

$$\Phi_1(h, a) = \sum_{\ell=0}^{N-1} \omega_{\ell} T_{\ell}(h, a) : \omega_{\ell} \equiv \frac{a_{\ell}}{a_0}, \quad (14)$$

where the parameters $T_{\ell}(h, a)$ are given in graphical form (Fig. 1) for $\ell = 0, 1, \dots, 4$; $a = \cos 45^\circ$ and $0 < h \leq .20$. Also

$$\Phi_2(h, a, c) = \sum_{\ell=0}^{N-1} \omega_{\ell} \omega_k Q_{\ell k}(h, a, c), \quad (15)$$

where the parameters $Q_{\ell k}(h, a, c)$ for $k = 0, 1, \dots, 4$; $a = \cos 45^\circ$; $0 < h \leq .20$; and $(\theta_2, \phi_2) = (9^\circ 10', 180^\circ)$, $(23^\circ, 0^\circ)$, $(46^\circ 30', 180^\circ)$, $(157^\circ 30', 0^\circ)$ and $(170^\circ 50', 180^\circ)$ are given in Figs. 2, 3, 4, 5 and 6.

Finally

$$\Phi_2(h, a) = \sum_{\ell=0}^{N-1} \omega_{\ell} \sum_{k=0}^{N-1} \omega_k V_{\ell k}(h, a). \quad (16)$$

In the initial calculation, it was assumed that the number of counts observed by the counter at $\cos^{-1} \mu_m = \theta_m$ is given by Eq. (2)

$$C_Z(\mu_m) = k(1 - e^{-\alpha h Z}) \frac{\sigma_Z(\mu_m)}{\sigma_Z} \quad (2)$$

Actually, however, the observed count is given by

$$k \left\{ \sum_{j=1}^{\infty} \Phi_j(h, a, c_m) \right\}. \quad (17)$$

If we assume that the angular distribution of the sum of emergent particles scattered a number of times greater than two is (approximately) the same as that given by $\Phi_2(h, a, c)$, then the actual number of counts $S_Z(\mu_m)$ recorded by the counter at θ_m is approximately

$$S_Z(\mu_m) = k \left\{ \Phi_1(h, a, c_m) + \eta \Phi_2(h, a, c_m) \right\}. \quad (18)$$

where

$$\eta = \frac{1 - e^{-\alpha h Z} - \Phi_1(h, a)}{\Phi_2(h, a)}. \quad (19)$$

Fig. 1. The coefficients, $T_0(h,a)$, for $a = \cos 45^\circ$ in the expansion for total singly-scattered emergent flux. $T_j(-)$ denotes the coefficient as negative.

1×10^{-2} 1×10^{-3} 1×10^{-4} 1×10^{-5} 1×10^{-6} 1×10^{-7} 1×10^{-8} Q_{jk} $(H) = 22^\circ$ $(H) = 22^\circ$

○ points had computed from approximation for this layer.
 + points Avidas computed by exact formulas.

↓ located value for Q_{jk} from approximate equation for this layer.

↑ indicates sign reversal of Q is at an angle, if it reverses sign at all.

Landsdorf & Monahan
 3-12-55

Fig. 2. The coefficients, $Q_{jk}(h, a, c)$, for $a = \cos 45^\circ$ for the doubly-scattered flux emergent in the direction $(c, \phi_2) = (\cos 90^\circ, 180^\circ)$ k, j (-) denotes the coefficient Q_{kj} as negative.

Fig. 3. The coefficients, $Q_{kj}(h,a,c)$, for $a = \cos 45^\circ$ for the doubly-scattered flux emergent in the direction $(c, \phi_2) = (\cos 23^\circ; 0^\circ)$ $k,j (-)$ denotes the coefficient Q_{kj} as negative.

Fig. 4. The coefficients, $Q_{jk}(h, a, c)$, for $a = \cos 45^\circ$ for the doubly-scattered flux emergent in the direction $(c, \phi_2) = (\cos 46^\circ 30', 180^\circ)$ $k, j (-)$ denotes the coefficient Q_{kj} as negative.

Fig. 6. The coefficients, $Q_{jk}(h, a, c)$, for $a = \cos 45^\circ$ for the doubly-scattered flux emergent in the direction $(\bar{c}, \phi_2) = (\cos 170^\circ 50', 180^\circ)$ k, j (-) denotes the coefficient Q_{kj} as negative.

The labor in calculating the functions $\Delta \lambda^k$ was found to be so great that it was not carried through. Therefore we did not employ Eq. (19) to calculate η . Instead a somewhat less exact η was calculated by assuming that

$$\frac{\Phi_{m+1}(h, a)}{\Phi_m(h, a)} = \text{a constant independent of } n,$$

i.e., we assume that the ratio of the integrated flux due to successive orders of scattering is independent of the order of scatter. It can then be shown that

$$\eta = \frac{1 - e^{-\alpha h} \Phi_2(h, a)}{1 - e^{-\alpha h} \Phi_1(h, a)} \approx \frac{\Phi_2(h, a)}{\Phi_1(h, a)} \quad (20)$$

We may now calculate, using Eq. (18) and Eq. (20),

$$C_Z(\mu_m) = S_Z(\mu_m) \left\{ \frac{\sigma^t Z [\Phi_1(hZ, a, c_m) + \eta \Phi_2(hZ, a, c_m)]}{\sigma^t Z(\mu_m) (1 - e^{-\alpha h Z})} \right\} \quad (21)$$

where

$$S_Z(\mu_m) = \rho_m [G_1(\mu_m) - B_j(\mu_m)] ,$$

and $C_Z(\mu_m)$, the observed counts corrected for multiple scattering, as defined by Eq. (20), is the quantity described by Eq. (2).

The bracketed quantity on the right hand side of Eq. (21) is, of course, evaluated using the first approximate, uncorrected values of $\sigma^t Z(\mu_m)$, hZ , and a . The corrected values, $C_Z(\mu_m)$, etc., so obtained can then be resubstituted into Eqs. (5), (10), and (11) of the previous sections to obtain corrected values for the coefficients a , the total scattering cross section $\sigma^t Z$ (and the equivalent corrected hZ), and the differential cross section $\sigma^t Z(\mu_m)$. It may be worth while to point out that this correction must be carried out simultaneously for the carbon standard. This means that the constant, k , of Eq. (8) must be recomputed using for $C(\mu_m)$ the counting rate for carbon as corrected by Eq. (20). This process is repeated until a consistent set of values of the a 's is obtained.

The calculations of the uncorrected σ_{stZ} and ω were carried through in accordance with the system formulated in the sections above for all the data as shown on the graphs in this report. The corrections made by the system here described were carried through on only a fraction of the data. The detailed explanations for reading the corrected data are given on page 53, and in the figure titles of the graphs.

To determine the consistency achieved after correcting results by this system of calculation, a few materials were measured with a series of samples of various thicknesses. The results of these calculations are shown on Fig. (7) and Fig. (8), with detailed explanations in the figure titles. It is clear from these figures (as a little thought about the geometrical conditions of the experimental arrangements will also make obvious) that as one increases the scattering sample thickness, the increase in plural scattering at the angles we measured is usually to be expected to be greater than the decrease in singly scattered neutrons reaching the detector. Therefore the uncorrected values of σ_{stZ} and $\sigma_{stZ}(\mu^m)$ are usually rising functions of thickness. Indeed, in over a hundred correction calculations, only two or three cases were found where the plural scattering corrected counts were greater than the uncorrected, and then only at the 22° scattering angle. In fact, the only reason why corrected σ_{stZ} and $\sigma_{stZ}(\mu^m)$ etc., are ever increased in making corrections is that their values are relative to carbon, so that when the correction (downward) in carbon counts is greater than the correction (downward) in any other material, the relative corrections are such as to lead to an increase in σ_{stZ} , etc.

It is of interest to point out that the functions $\Omega_k(h, a, c)$ are very rapidly varying functions of h , varying about as h^n where $2 < n < 3$ usually. As a result, $\Phi_2(h, a, c)$ in Eq. (15) is far more sensitive to the value of h , therefore to σ_{stZ} , than to the values of the ω . In consequence, it was found that most often iteration of our results to obtain corrected values was necessary almost exclusively in order to reach a suitably corrected σ_{stZ} . Indeed, the uncorrected σ_{stZ} was often greater than the true σ_{stZ} (total cross section). Thus it has become clear that in future one should whenever possible measure the total cross section at nearly the same time as and for the same neutron spectrum as used for, the differential cross section. The measured σ_{stZ} will then give an h which is closer to a correct value to use in calculating $\Phi_2(h, a, c)$ than when h is calculated from the uncorrected σ_{stZ} . Furthermore, such an (independent) measure of σ_{stZ} will give a good internal consistency check in the experiment.

Fig. 7. The differential cross section of graphite at 394 Kev as a function of sample thickness. The open circles indicate the same measurements corrected for plural scattering.

Fig. 8a. The differential cross section of ^{92}U at 394 Kev as a function of sample thickness

Fig. 8b. The differential cross section of ^{92}U at 394 Kev as a function of sample thickness

Fig. 9. The emergent singly scattered flux, $\Phi_1(\alpha, \gamma)$ per unit incident flux, for flat plates, versus thickness, h , at the five scattering angles, θ , indicated. The incident flux is at an obliquity of $45^\circ = \sec^{-1} \alpha$.

The reason why literature values of σ_t^Z (such as are to be found in BNL-325) were not used, was that in general these were obtained for a different neutron spectrum, different sample thicknesses, and with unknown inaccuracies. Therefore such literature values would often have been more grossly in error than our own uncorrected σ_{st}^Z . In some cases, there is no value of σ_{st}^Z available in the literature at present.

Discussion on the problem of an adequate determination of σ_t by transmission measurements for use with the calculations of scattering data:

The scattering data is best determined using thin sheets, in the geometry of these experiments. We consider that the most desirable thickness is for $0.05 \leq h \leq 0.1$ where $h = N_t \sigma_t$. Then double scattering will not require a very large correction and triple and higher order scattering will be practically negligible.

In the experiments we have performed the flat plates of scatterer have been placed at an obliquity of 45° to the neutron beam so that the transmission, T , will be

$$T = e^{-\sqrt{2} N_t \sigma_t}$$

Then we obtain

$$\frac{d\sigma_t}{d\ln T} = -\frac{\sigma_t}{\sqrt{2} T}$$

Therefore a fractional error $\frac{\Delta T}{T}$ is magnified by an amount $1/\sqrt{2}h$ in determining the fractional error $\Delta\sigma_t/\sigma_t$. In the above range of h , the magnification is of the order of 7 to 14. For example, if a million counts are obtained for "sample in" and "sample out" measurements to obtain I and I_0 respectively, so that I and I_0 are each statistically accurate to 0.1%, the accuracy of $T = I/I_0$ will be about 0.14%. Then the per cent accuracy of σ_t in the range of $0.05 \leq h \leq 0.10$ will be 1% to 2%.

Unless the counting system used for transmission is well adapted to counting of the order of a million counts in a reasonable time, and adequate flux for such an intensity is also available, some other expedient should be used. One possibility is to rotate

the plate to such an oblique angle that the transmission is about $T = 0.5$. The most serious difficulty with this scheme is that if the material being measured has resonances, whether resolved or not, beam hardening effects in such a thick layer may cause the measured σ_t to be as much as 5% lower than its correct value for the scattering geometry.

41

SECTION III.

SINGLE AND DOUBLE SCATTERING IN PLATES

A. Introduction

Our measurements of the angular distribution of neutrons scattered from flat plates indicate that the effect of multiple scattering can be appreciable (greater than 10% of single scattering) even for thicknesses less than 0.1 mean free path. Although there exists an extensive literature on the subject of multiple scattering, little of it has been found to be applicable to the present situation in which a beam of neutrons is incident at an oblique direction on a plane-parallel scattering sample, which has rather strongly anisotropic scattering properties.

The work to be described in this section gives exact expressions for the emergent doubly scattered flux under the conditions described above. The assumptions used in the derivation of these equations are those referred to in Section II. Although the assumption of thin plates does not enter formally in these derivations, this discussion is limited for all practical purposes to thicknesses for which the contribution of third and higher order processes to the total scattering is not very large.

B. Notation and Conventions

Consider a coordinate system oriented with respect to a

scattering target as shown in the figure. The thickness of the target, measured along the Z-direction, i.e., normal to the target face, is denoted by h . This, as well as all other distances, is measured in units of mean free path. The lateral (X and Y) dimensions of the scattering sample are taken as essentially infinite compared with h .

The incident flux in the direction ($a = a^{-1} = \cos \theta_0$, ϕ_0) is taken as unity. Thus the singly scattered flux, $\Phi_1(b, \phi_1)$, emergent in the direction ($b = b^{-1} = \cos \theta_1$, ϕ_1) and the doubly scattered flux, $\Phi_2(c, \phi_2)$, emergent in the direction ($c = c^{-1} = \cos \theta_2$, ϕ_2) are to be interpreted as flux per unit solid angle per unit incident flux.

The scattering material is characterized by a phase function: $p(a, b)$, defined by

$$p(a, b) = p(\mu) = \frac{4\pi\sigma(\mu)}{\sigma_t} \quad (22)$$

where $\mu = \cos \theta$ is the cosine of the scattering angle as ordinarily defined,

$$\mu = ab + \sqrt{(1-a^2)(1-b^2)} \cos(\phi_0 - \phi_1) \quad (23)$$

$\sigma(\mu)$ is the differential cross section for scattering from the direction (a, ϕ_0) to the direction (b, ϕ_1) ; σ_t is the corresponding total scattering cross section. The notation, p_{ab} , for the phase function implies the corresponding azimuthal angles ϕ_0 and ϕ_1 .

In order to fix the coordinate system uniquely with respect to directions which are physically well defined, we choose the

$\phi_0 = 0$ plane as being defined by the direction of the Z-axis and the direction of the incident beam (i.e., the Y-Z plane).

In the following treatment of double scattering it is necessary to adopt a specific convention with regard to sign, particularly for

the trigonometric functions. The positive Z-direction will be defined as the direction pointing toward the scattering plate on the side from which the incident flux impinges. The cosines a , b , and c together with the associated secants α , β , and γ will be considered as intrinsically positive. When a negative magnitude is required, it will be shown explicitly, either by writing, e.g., $-\alpha$ or $-\alpha$. The symbols α , etc., will only be used to indicate the direction of a ray which has a component in the negative Z-direction. The symbols α , etc., as arguments of a function shall imply that the functional relation is valid for both positive and negative directions of a ray. Length variables, τ , λ , κ and h are taken as positive magnitudes.

Since the two cases are treated separately, we distinguish between transmitted and reflected flux by writing $\Phi(+)$ = $\Phi(t)$ for the flux emergent on the transmission side of the scattering plate and $\Phi(-)$ = $\Phi(s)$ for the flux emergent on the reflection side.

The expansion

$$\sigma(\mu) = \frac{4\pi}{t} \sum_{N=0}^{\infty} \omega_N P_N(\mu) \quad (24)$$

has been discussed in the previous section of this report. Thus

- (1) The incident flux, $\Phi_0 = 1$ is incident at an angle θ_0 such that $\sec \theta_0 = \alpha$
- (2) The probability that an incident particle will penetrate to a depth κ without having a previous interaction is just $e^{-\alpha\kappa}$
- (3) The probability that the particle will interact in the element $d\kappa$ at κ is $\alpha d\kappa$

In this part we treat the idealized situation in which one considers that a neutron incident on a plate of scattering material interacts with this material just once, at most, after which no further interaction of any sort is allowed. Then we consider:

In this part we will introduce the technique of constructing the equations which we wish to discuss by using the concept of probabilities, so that in the succeeding more complex cases which follow, the developments will be more easily comprehended.

C. Idealized Single Scattered Intensity

The total emergent flux which has scattered j times will be denoted by Φ_j ; the arguments h and α are not (usually) explicitly written.

because of the symmetry properties of the $P_m^j(x)$.

$$P_{ab}^- = P_{ab}^+; \quad P_{ab} = P_{ab}^- \quad (27)$$

Note that

$$P_{ab}^- = \sum_{\ell=0}^N \omega_{\ell} P_{\ell}^0(h) = \sum_{\ell=0}^N \omega_{\ell} \frac{(\ell-m)!}{(\ell+m)!} (2-\delta_{0m}) \cos m(\phi_1 - \phi_0) P_m^0(a) P_m^0(b). \quad (26)$$

where we have made use of the addition theorem for Legendre polynomials. These equations are generally applicable for any angle of scatter, hence for a and b either positive or negative. In restricting the symbols a, b , etc., to denote positive magnitudes, we simply show a, b , etc., explicitly to identify a negative direction. Thus, e.g., the phase function for a ray scattered from a positive direction (a, ϕ_0) to a direction (b, ϕ_1) in the reflection side of the scatterer will be written.

$$P_{ab}^+ = \sum_{\ell=0}^N \omega_{\ell} P_{\ell}^0(h) = \sum_{\ell=0}^N \omega_{\ell} \frac{(\ell-m)!}{(\ell+m)!} (2-\delta_{0m}) \cos m(\phi_1 - \phi_0) P_m^0(a) P_m^0(b), \quad (25)$$

(4) Having interacted in dk , the probability that the particle will be found to be scattered at an angle θ is $\sigma(\theta)/\sigma_t$ measured as a probability per unit steradian at θ . In an element of solid angle $d\Omega$ at θ , therefore, the probability of finding the particle is

$$(28) \quad \frac{\sigma(\theta) d\Omega}{\sigma_t} = \frac{p(\theta) d\Omega}{4\pi}$$

where $p(\theta)$ is the phase function which may be considered to be defined by this equation.

The details of notation, coordinates, and conventions have already been given in Part B of this section.

Now an element of the idealized scattered flux will be given by multiplying the probabilities in items (2), (3), and (4) just listed. This gives

$$(29) \quad \Phi_{II}(\theta) d\Omega = e^{-\alpha_k} \alpha_k p(\theta) d\Omega / 4\pi.$$

Integrating k over the thickness of the plate, from zero to h , we obtain:

$$(30) \quad \Phi_{II} = (1 - e^{-\alpha_h}) p(\theta) / 4\pi.$$

This result is in agreement with the simpler development in Section II of this report. For, if the scatterer is irradiated over an area A , and the scattering of intensity $A\Phi_{II}$ per steradian falls on a detector subtending $\Delta\Omega$ at the scatterer, we can immediately write the flux on the detector as

$$(31) \quad A\Phi_{II} \Delta\Omega = A\Delta\Omega (1 - e^{-\alpha_h}) p(\theta) / 4\pi.$$

Since $h = N_t \alpha_t$ this is exactly the same expression derived in Section II by another method.

D. Emergent Singly Scattered Intensity

The flux scattered in dk at κ as described in Part C, is actually exponentially attenuated by further interaction before it emerges from the plate. On the reflection side, the probability of the scattered particle emerging is $e^{-\beta\kappa}$ where $\beta = |\sec \theta_1|$ and θ_1 is the altitude angle of the emergent direction with respect to the normal to the plate. On the transmission side of the plate, however, the probability is different. It is $e^{-\beta(h-\kappa)}$.

On the reflection side, an element of emergent flux will be given by multiplying the probabilities in items (2), (3), and (4) of Part C by $e^{-\beta\kappa}$. Thus

$$d\Phi_1^{(s)} d\Omega = e^{-\alpha\kappa} \alpha d\kappa p(\Theta) (d\Omega/4\pi) e^{-\beta\kappa} \quad (32)$$

which when integrated over κ from 0 to h gives

$$\Phi_1^{(s)}(b) = \Phi_0 p(\Theta) \alpha \frac{4\pi(\alpha+\beta)}{1 - e^{-h(\alpha+\beta)}} \quad (33)$$

On the transmission side of the plate a similar treatment but with $e^{-\beta(h-\kappa)}$ substituted for $e^{-\beta\kappa}$ leads to

$$\Phi_1^{(t)}(b) = \Phi_0 p(\Theta) \alpha e^{-h\beta} \frac{4\pi(\alpha-\beta)}{1 - e^{-h(\alpha-\beta)}} \quad (34)$$

As in the previous Part C of this section, the notation, sign, and coordinate conventions are those detailed in Part B.

Also in accordance with the conventions of Part B, we may replace $p(\Theta)$ by p_{ab} in the transmission case and by p_{ab} in the reflection case to obtain

$$\Phi_1^{(t)}(b) = \frac{\Phi_0 p_{ab} \alpha e^{-h\alpha}}{1 - e^{-h(\beta-\alpha)}} \quad (35)$$

$$\Phi_1^{(s)}(b) = \frac{\Phi_0 p_{ab} \alpha}{1 - e^{-h(\beta+\alpha)}} \quad (36)$$

from which follow the useful relations

$$\begin{aligned} \Phi_1^{(s)}(s) \alpha &= -e^{-h\alpha} \Phi_1^{(t)}(t) (-\alpha) \\ \Phi_1^{(t)}(t) \alpha &= -e^{-h\alpha} \Phi_1^{(s)}(s) (-\alpha) \end{aligned} \quad (37)$$

because $p_{ab}^- = p_{ab}$, as noted in Part B.

E. Integrated Emergent Single Scattering

The total single scattered flux which emerges is found by integrating the emergent flux over all outgoing directions on both sides of the plate. We find

$$\Phi_1 = \Phi_1^{(t)} + \Phi_1^{(s)} \quad (38)$$

where

$$\Phi_1^{(t)} = \int_{-2\pi}^0 d\phi_1 \int_1^\infty db \Phi_1^{(t)}(b) \quad (39)$$

$$\Phi_1^{(s)} = \int_{-2\pi}^0 d\phi_1 \int_1^\infty db \Phi_1^{(s)}(b) \quad (40)$$

The addition theorem for expansion of the phase function,

Eq. (25), is used in equations (35) and (36) for $\Phi_1^{(t)}(b)$ and $\Phi_1^{(s)}(b)$. The integration over ϕ_1 is simple; it gives

$$\int_{-2\pi}^0 d\phi_1 p(a, b) = 2\pi \sum_{j=0}^N \omega_j P_j(a) P_j(b) \quad (41)$$

where we have taken $\phi_0 = 0$.

We have then

$$\Phi_1^{(t)} = \frac{\alpha e^{-h\alpha}}{2} \sum_{j=0}^N \omega_j P_j(a) \int_1^\infty db P_j(b) \frac{1 - e^{-h(\beta - \alpha)}}{\beta - \alpha} \quad (42)$$

$$\Phi_1^{(s)} = \frac{\alpha}{2} \sum_{j=0}^N \omega_j P_j(a) \int_1^\infty db P_j(b) \frac{1 - e^{-h(\beta + \alpha)}}{\beta + \alpha} \quad (43)$$

Now expand $P_j(b)$ in terms of the cosine so that

$$P_j(b) = \sum_{\pm=0}^j q_{\pm j}^{(b)} = \sum_{\pm=0}^j q_{\pm j}^{(b)} \quad (44)$$

and we have

$$\int_1^\infty db P_j(b) = \frac{1 - e^{-h(\beta - \alpha)}}{\beta - \alpha} = \sum_{\pm=0}^j q_{\pm j}^{(b)} \int_1^\infty db \frac{1 - e^{-h(\beta - \alpha)}}{\beta - \alpha} \quad (45)$$

The functions $F_j(h, \alpha)$ are discussed by Chandrasekhar* and others and are further discussed in Appendix B of this report. The coefficients $q_{\pm j}$ are

readily obtained from the equations for $P_l(b)$ as in Jahnke-Emde. They are given explicitly up to $l = j = 4$ in the accompanying table.

TABLE OF $q_{l,j}$

$l \backslash j$	0	1	2	3	4
0	1				
1		1			
2	-1/2		3/2		
3		-3/2		5/2	
4	3/8		-30/8		35/8

In terms of the F functions the solution for Φ_1 is, then,

$$\Phi_1 = \frac{2}{\alpha} \sum_{l=0}^N \omega_l P_l(a) \sum_{j=0}^l q_{lj} \left\{ e^{-h\alpha} F_{j+2}(h, \alpha) + (-)^j F_j(h, -\alpha) \right\} \quad (46)$$

In this equation the second F function has a negative sign for the argument $\bar{a}$ in accordance with the notation of Chandrasekhar, in view of the sign of the terms in $\bar{a}$ in our Eq. 36 for $\Phi_1^{(s)}(b)$.

F. Emergent Doubly Scattered Intensity

The first scattering occurs in an element dk at k , the second in $d\tau$ at τ . On both the reflection side of the plate and the transmission side, the integrations must be broken up into two components, one for $k \gg \tau$ and one for $k \ll \tau$, so that four integrals in all are to be evaluated. The following tabulation itemizes limits of integration, probabilities of each differential process, etc., which are involved in constructing each of the four integrals in the same manner that was introduced in Part C above.

	Transmission Side		Reflection Side	
	Part I	Part II	Part I	Part II
	$\Phi_0 = 1$ $\Phi_{2I}^{(t)}$ $0, 2\pi$	$\Phi_{2II}^{(t)}$ $*$	$\Phi_{2I}^{(s)}$ $\Phi_{2II}^{(s)}$	$\Phi_{2I}^{(s)}$ $\Phi_{2II}^{(s)}$
(1) Incident flux				
(2) Symbol for the emergent component flux				
(3) Limits of integration on ϕ_1				
(4) Limits of integration on b	0, 1			
(5) Limits of integration of τ	0, h			
(6) Limits of integration of κ	0, τ	τ, h	0, τ	τ, h
(7) Conditions on κ and τ	$h \gg \tau \gg \kappa \gg 0$	$h \gg \kappa \gg \tau \gg 0$	$h \gg \tau \gg \kappa \gg 0$	$h \gg \kappa \gg \tau \gg 0$
Probability components of the processes:				
(8) Attenuation of incident beam arriving at κ	$e^{-\alpha k}$			
(9) Scattering in $d\kappa$ at κ	αdk			
(10) Scattering into solid angle $d\phi_1$ db	$P_{ab} d\phi_1 db / 4\pi$	$P_{ab}^- d\phi_1 db / 4\pi$	$P_{ab} d\phi_1 db / 4\pi$	$P_{ab}^- d\phi_1 db / 4\pi$
(11) Attenuation on path from κ to τ	$e^{-\beta(\tau - \kappa)}$	$e^{-\beta(\kappa - \tau)}$	$e^{-\beta(\tau - \kappa)}$	$e^{-\beta(\kappa - \tau)}$
(12) Scattering in $d\tau$ at τ	$\beta d\tau$			
(13) Scattering into solid angle $d\phi_2 dc$	$P_{bc} d\phi_2 dc / 4\pi$	$P_{bc}^- d\phi_2 dc / 4\pi$	$P_{bc}^- d\phi_2 dc / 4\pi$	$P_{bc}^- d\phi_2 dc / 4\pi$
(14) Attenuation on path from τ to surface of emergence.	$e^{-\gamma(h - \tau)}$	$e^{-\gamma(h - \tau)}$	$e^{-\gamma\tau}$	$e^{-\gamma\tau}$

*Items not repeated are the same in all four columns.

The four integrals are constructed by multiplying together the seven probability items, numbers (8) - (14) and applying the limits of integration, items (3) - (6).

The complete integral for $\Phi_{2I}^{(t)}$ is, then

$$\Phi_{2I}^{(t)} \text{dc d}\phi_2 = \frac{\Phi_{0\alpha} e^{-h\gamma}}{(4\pi)^2} \int_{-2\pi}^{2\pi} d\phi_1 \int_0^1 d\beta P^{ab} P^{bc} \int_h^T d\tau e^{-T(\alpha-\beta)} \int_T^{\infty} d\kappa e^{\kappa(\beta-\alpha)} \quad (47)$$

which may be integrated over κ and T to give

$$\Phi_{2I}^{(t)}(a, c) = \frac{\Phi_{0\alpha}}{(4\pi)^2} \int_{-2\pi}^{2\pi} d\phi_1 \int_0^1 d\beta P^{ab} P^{bc} \left[\frac{1}{\beta - \alpha} \left[\frac{e^{-h\alpha} - e^{-h\gamma}}{\gamma - \alpha} - \frac{e^{-h\beta} - e^{-h\gamma}}{\gamma - \beta} \right] \right] \quad (48)$$

Similar solutions may be given for the remaining three components,

$\Phi_{2I}^{(t)}$, $\Phi_{2I}^{(s)}$, and $\Phi_{2II}^{(s)}$. If one writes them out, simple substitution will

validate the correctness of the following formalism:

$$\Phi_{2I}^{(t)}(a, c) = \left(e^{-h\alpha} \right) \left(e^{-h\gamma} \right) \Phi_{2I}^{(t)}(-\alpha, -\gamma)$$

$$\Phi_{2I}^{(s)}(a, c) = \left(e^{-h\gamma} \right) \Phi_{2I}^{(t)}(\alpha, -\gamma)$$

$$\Phi_{2II}^{(s)}(a, c) = \left(-e^{-h\alpha} \right) \Phi_{2I}^{(t)}(-\alpha, \gamma)$$

The only additional fact needed in corroborating these formal rules is the symmetry properties of the phase function as given in Eq. 27.

We now return to Eq. 47 and integrate first over ϕ_1 , next over b . Expanding $P^{ab} P^{bc}$ by the expansion theorem for Legendre polynomials,

$$P^{ab} P^{bc} = \left[\sum_N^k \omega_k \sum_{m=0}^k (2 - \delta_{0m}) \frac{(k-m)!}{(k+m)!} P_m^k(a) P_m^k(b) \cos m \phi_1 \right]$$

$$\left[\sum_N^x \omega_x \sum_{\ell=0}^x (2 - \delta_{0\ell}) \frac{(\ell-n)!}{(\ell+n)!} P_n^\ell(b) P_n^\ell(c) \cos n(\phi_2 - \phi_1) \right] \quad (50)$$

and making use of the fact that,

$$\int_{-2\pi}^{2\pi} d\phi_1 \cos m \phi_1 \cos n(\phi_2 - \phi_1) = \begin{cases} (2 - \delta_{0n})^{-1} & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases} \quad (51)$$

we obtain

$$\int_{2\pi}^0 P^{ab} P^{bc} d\phi_1 = 2\pi \sum_{N,N} \sum_{k=0}^{\infty} \omega_k \sum_{n \leq \ell} \frac{(k-n)! (\ell-n)!}{(k+n)! (\ell+n)!} P_n^k(a) P_n^k(b) P_n^k(c) (2 - \delta_{0n}) \cos n\phi_2 \quad (52)$$

We now introduce the coefficients $g_{\ell,k;j}$ in the expansion of the product of $P_n^k(b)$ and $P_n^k(b)$ by the defining equation:

$$P_n^k(b) P_n^k(b) = \sum_{\ell+k}^j b^j g_{\ell,k;j} = \sum_{\ell+k}^j g_{\ell,k;j} / \beta^j. \quad (53)$$

The values of the $g_{\ell,k;j}$ up to $\ell = k = 4$ are given in a table appended. A useful check on any tabulation of these coefficients is the condition that

$$\sum_{\ell+k}^j g_{\ell,k;j} = \delta_{n,0}. \quad (53a)$$

By replacing db by its equivalent, $-d\beta/\beta^2$, we can now write, by aid of Eqs. 47, 52, and 53:

$$\Phi_{21}^{(t)}(a, c) = \sum_{N,N} \sum_{k=0}^{\infty} \frac{8\pi}{\alpha} \omega_k \sum_n (2 - \delta_{0n}) \cos n\phi_2 P_n^k(a) P_n^k(c) \quad k=0$$

$$x \sum_{\ell+k}^j g_{\ell,k;j} e^{-h\gamma} \int_0^{\infty} d\tau e^{-\tau\gamma} \int_0^{\infty} d\kappa e^{-\kappa\alpha} \int_{-\infty}^{\infty} d\beta e^{-\beta(\tau-k)} \frac{1}{\beta^{j+1}}. \quad (54)$$

The remaining integrals in this equation may now be considered solved in terms of the exponential integral functions $E_n(t)$, and repeated integration over these integral functions as defined by:

$$E_n(t) = \int_1^{\infty} e^{-tu} du / u^n, \quad (55)$$

$$F_n^{\ell}(t, r) = \int_0^T e^{tr} E_n(t) dt, \quad (56)$$

$$F_n^{\ell}(h, s, r) = \int_0^{\infty} e^{Ts} F_n^{\ell}(T, r) dt. \quad (57)$$

These are further discussed in Appendix A and Appendix B.

$$R_n^k, k; j \text{ for } k \leq 4$$

k	n	$j = 0$	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$	$j = 7$	$j = 8$
4	4	0	0	0	1					
4	4	1	0	0						
4	4	1	1	1	1/4					
4	4	2	0	0	-1/2					
4	4	2	1	0	1/4					
4	4	2	2	2	1/64					
4	4	3	0	0	-3/2					
4	4	3	1	0	0					
4	4	3	1	1	-1/16					
4	4	3	2	0	0					
4	4	3	2	1	0					
4	4	3	2	2	0					
4	4	3	3	0	0					
4	4	3	3	1	1/64					
4	4	3	3	2	0					
4	4	3	3	3	1/2304					
4	4	4	0	0	3/8					
4	4	4	1	0	0					
4	4	4	1	1	0					
4	4	4	2	0	-3/16					
4	4	4	2	1	0					
4	4	4	2	2	-1/384					
4	4	4	2	3	0					
4	4	4	3	0	0					
4	4	4	3	1	0					
4	4	4	3	2	0					
4	4	4	3	3	0					
4	4	4	3	4	0					
4	4	4	4	0	0					
4	4	4	4	1	0					
4	4	4	4	2	0					
4	4	4	4	3	0					
4	4	4	4	4	0					
4	4	4	4	5	0					
4	4	4	4	6	0					
4	4	4	4	7	0					
4	4	4	4	8	0					
4	4	4	4	9	0					
4	4	4	4	10	0					
4	4	4	4	11	0					
4	4	4	4	12	0					
4	4	4	4	13	0					
4	4	4	4	14	0					
4	4	4	4	15	0					
4	4	4	4	16	0					
4	4	4	4	17	0					
4	4	4	4	18	0					
4	4	4	4	19	0					
4	4	4	4	20	0					
4	4	4	4	21	0					
4	4	4	4	22	0					
4	4	4	4	23	0					
4	4	4	4	24	0					
4	4	4	4	25	0					
4	4	4	4	26	0					
4	4	4	4	27	0					
4	4	4	4	28	0					
4	4	4	4	29	0					
4	4	4	4	30	0					
4	4	4	4	31	0					
4	4	4	4	32	0					
4	4	4	4	33	0					
4	4	4	4	34	0					
4	4	4	4	35	0					
4	4	4	4	36	0					
4	4	4	4	37	0					
4	4	4	4	38	0					
4	4	4	4	39	0					
4	4	4	4	40	0					
4	4	4	4	41	0					
4	4	4	4	42	0					
4	4	4	4	43	0					
4	4	4	4	44	0					
4	4	4	4	45	0					
4	4	4	4	46	0					
4	4	4	4	47	0					
4	4	4	4	48	0					
4	4	4	4	49	0					
4	4	4	4	50	0					
4	4	4	4	51	0					
4	4	4	4	52	0					
4	4	4	4	53	0					
4	4	4	4	54	0					
4	4	4	4	55	0					
4	4	4	4	56	0					
4	4	4	4	57	0					
4	4	4	4	58	0					
4	4	4	4	59	0					
4	4	4	4	60	0					
4	4	4	4	61	0					
4	4	4	4	62	0					
4	4	4	4	63	0					
4	4	4	4	64	0					
4	4	4	4	65	0					
4	4	4	4	66	0					
4	4	4	4	67	0					
4	4	4	4	68	0					
4	4	4	4	69	0					
4	4	4	4	70	0					
4	4	4	4	71	0					
4	4	4	4	72	0					
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4	4	4	4	74	0					
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4	4	4	4	76	0					
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4	4	4	4	78	0					
4	4	4	4	79	0					
4	4	4	4	80	0					
4	4	4	4	81	0					
4	4	4	4	82	0					
4	4	4	4	83	0					
4	4	4	4	84	0					
4	4	4	4	85	0					
4	4	4	4	86	0					
4	4	4	4	87	0					
4	4	4	4	88	0					
4	4	4	4	89	0					
4	4	4	4	90	0					
4	4	4	4	91	0					
4	4	4	4	92	0					
4	4	4	4	93	0					
4	4	4	4	94	0					
4	4	4	4	95	0					
4	4	4	4	96	0					
4	4	4	4	97	0					
4	4	4	4	98	0					
4	4	4	4	99	0					
4	4	4	4	100	0					
4	4	4	4	101	0					
4	4	4	4	102	0					
4	4	4	4	103	0					
4	4	4	4	104	0					
4	4	4	4	105	0					
4	4	4	4	106	0					
4	4	4	4	107	0					
4	4	4	4	108	0					
4	4	4	4	109	0					
4	4	4	4	110	0					
4	4	4	4	111	0					
4	4	4	4	112	0					
4	4	4	4	113	0					
4	4	4	4	114	0					
4	4	4	4	115	0					
4	4	4	4	116	0					
4	4	4	4	117	0					
4	4	4	4	118	0					
4	4	4	4	119	0					
4	4	4	4	120	0					
4	4	4	4	121	0					
4	4	4	4	122	0					
4	4	4	4	123	0					
4	4	4	4	124	0					
4	4	4	4	125	0					
4	4	4	4	126	0					
4	4	4	4	127	0					
4	4	4	4	128	0					
4	4	4	4	129	0					
4	4	4	4	130	0					
4	4	4	4	131	0					
4	4	4	4	132	0					
4	4	4	4	133	0					
4	4	4	4	134	0					
4	4	4	4	135	0					
4	4	4	4	136	0					
4	4	4	4	137	0					
4	4	4	4	138	0					
4	4	4	4	139	0					
4	4	4	4	140	0					
4	4	4	4	141	0					
4	4	4	4	142	0					
4	4	4	4	143	0					
4	4	4	4	144	0					
4	4	4	4	145	0					
4	4	4	4	146	0					
4	4	4	4	147	0					
4	4	4	4	148	0					
4	4	4	4	149	0					
4	4	4	4	150	0					
4	4	4	4	151	0					
4	4	4	4	152	0					
4	4	4	4	153	0					
4	4	4	4	154	0					
4	4	4	4	155	0					
4	4	4	4	156	0					
4	4	4	4	157	0					
4	4	4	4	158	0					
4	4	4	4	159	0					
4	4	4	4	160	0					
4	4	4	4	161	0					
4	4	4	4	162	0					
4	4	4	4	163	0					

We also introduce the notation

$$P_n^k \ell(a, c) = (1 - \frac{1}{2} \delta^k \ell) [P_n^k(a) P_n^k(c) + P_n^k(a) P_n^k(c)] \quad (58)$$

to reduce the summation over all values of k and ℓ to an ordered summation of lower terms. Thus, finally, we have

$$\Phi_{21}^{(t)}(a, c) = \frac{8\pi}{\alpha} \sum_{\ell=0}^{\infty} \sum_{k \geq \ell} \omega_{\ell k} (2 - \delta_{0n}) \cos n \phi_2 P_n^k \ell(a, c) \sum_{j=0}^{\infty} \varepsilon_{\ell k, j} e^{-h \gamma F_{j+1}^j} (h, \gamma - \alpha, \alpha) \quad (59)$$

It is easily shown by an integration by parts that

$$F_n(h, s, r) = e^{hs} F_n(h, -s, s + r) \quad (60)$$

It is also readily shown, using the symmetry properties of the Legendre polynomials, that

$$P_n^k \ell(a, c) = (-1)^{k+\ell} P_n^k \ell(a, c)$$

(61)

$$P_n^k \ell(a, c) = (-1)^{k+\ell} P_n^k \ell(a, c) \cdot$$

One final relation that aids in simplifying the final solution is that the $\varepsilon_{\ell k, j}^j$ are zero if $(\ell + k)$ is odd unless j is also odd, and vice versa, when $(\ell + k)$ is even they are zero unless j is even. Therefore the term $(-1)^{k+\ell}$ inside a sum over ℓ, k , and j may be replaced by $(-1)^j$. Using this fact and the relations (49), (59), (60) and (61), we may now write the final answers

$$\Phi_{21}^{(t)}(a, c) = \Phi_{21}^{(t)} + \sum_{\ell=0}^{\infty} \sum_{k \geq \ell} \omega_{\ell k} \Phi_{21}^{(t)}(h, \alpha, \gamma, \phi_2) \quad (62)$$

where

$$\Phi_{21}^{(t)}(h, \alpha, \gamma, \phi_2) = \frac{8\pi}{\alpha} \sum_{k=0}^{\infty} (2 - \delta_{0n}) \cos n \phi_2 P_n^k \ell(a, c) \times$$

$$\sum_{\ell=0}^{\infty} \varepsilon_{\ell k, j} e^{-h \alpha} \left[F_{j+1}^{j+1}(h, \alpha - \gamma, \gamma) + (-1)^j F_{j+1}^j(h, \alpha - \gamma, -\alpha) \right] \quad (63)$$

and

$$\Phi_{21}^{(s)}(a, c) = \Phi_{21}^{(s)} + \sum_{\ell=0}^{\infty} \sum_{k \geq \ell} \omega_{\ell k} \Phi_{21}^{(s)}(h, \alpha, \gamma, \phi_2) \quad (64)$$

where

$$Q_{\ell k}^{(s)}(h, \alpha, \gamma, \phi_2) = \frac{\alpha}{8\pi} \sum_{k=0}^n (2 - \delta_{0k}) \cos n\phi_2 P_n^k(a, \bar{c}) x$$

$$\left[F_{j+1}^{j+1}(h, -\alpha-\gamma, \alpha) + (-)^j F_{j+1}^{j+1}(h, -\alpha-\gamma, \gamma) \right] \sum_{k=0}^j g_{\ell, k; j}^n$$

G. The Integrated Doubly Scattered Flux

Next consider the total emergent doubly scattered flux

$$\Phi_2 = \int_{2\pi} d\phi_2 \left\{ \int_1 d\phi_2^{(t)}(c, \phi_2) + \int_0^1 d\phi_2^{(s)}(c, \phi_2) \right\} \quad (66)$$

A crude approximation for this integral can be obtained in the following manner: Vineyard* has shown that the ratio Φ_n^{n-1} is approximately independent of the order, n , of scattering, at least for the "quasi-isotropic" case which he considers. This assumption permits the evaluation of the sum of emergent scattered flux for all orders of scatter as

$$\Phi_n = \sum_{i=1}^n \Phi_i = \Phi_2^1 / (\Phi_1 - \Phi_2) \quad (67)$$

Since, in the cases considered here, this sum must also converge to $1 - e^{-\alpha h}$, we have

$$\Phi_2 \approx \Phi_1 \left[1 - \frac{1 - e^{-\alpha h}}{\Phi_1} \right] \quad (68)$$

The following exact evaluation of this integral is somewhat tedious. The integration over the azimuthal angle is trivial giving

$$\Phi_2 = \frac{\alpha}{4} \sum_{k=0}^N \sum_{\ell=0}^k \sum_{j=0}^k \frac{g_{\ell, k; j}^n}{2 - \delta_{\ell k}} \left\{ P_{\ell}(a) \sum_{k=0}^s q_{ks}(I_{sj+1} + J_{sj+1}) + P_k(a) \sum_{k=0}^s q_{ks}(I_{sj+1} + J_{sj+1}) \right\} \quad (69)$$

where we have made use of the definition (Eq. 58) of $P_{\ell}^k(a, c)$ and have expanded $P_k(c)$ in powers of c as before. The integrals I and J are defined by

*George H. Vineyard, Phys. Rev. 96, 93 (1954)

$$(70) \quad I_{s,j+1}^{(h)}(h, \alpha) = (-)^j I_{s,j+1}^{(2)}(h, \alpha) + (-)^j I_{s,j+1}^{(1)}(h, \alpha),$$

and

$$(71) \quad J_{s,j+1}^{(h)}(h, \alpha) = (-)^{s+1} \left\{ J_{s,j+1}^{(1)}(h, \alpha) + (-)^j J_{s,j+1}^{(2)}(h, \alpha) \right\},$$

in terms of the definite integrals

$$(72) \quad I_{s,j+1}^{(1)}(h, \alpha) = \int_h^\infty d\tau \int_1^\infty \frac{d\gamma}{\gamma^{s+2}} e^{-(\gamma-\alpha)\tau} e^{-\gamma h} F_{j+1}^{(1)}(\tau, \alpha),$$

$$(73) \quad I_{s,j+1}^{(2)}(h, \alpha) = \int_h^\infty d\tau \int_1^\infty \frac{d\gamma}{\gamma^{s+2}} e^{(\gamma-\alpha)\tau} e^{-\gamma h} F_{j+1}^{(2)}(h-\tau, -\alpha),$$

$$(74) \quad J_{s,j+1}^{(1)}(h, \alpha) = \int_h^\infty d\tau \int_1^\infty \frac{d\gamma}{\gamma^{s+2}} e^{-(\gamma+\alpha)\tau} F_{j+1}^{(1)}(\tau, \alpha),$$

$$(75) \quad J_{s,j+1}^{(2)}(h, \alpha) = \int_h^\infty d\tau \int_1^\infty \frac{d\gamma}{\gamma^{s+2}} e^{-(\gamma+\alpha)\tau} F_{j+1}^{(2)}(h-\tau, -\alpha).$$

A simple change of variables shows that

$$(76) \quad J_{s,j+1}^{(1)}(h, \alpha) = e^{-\alpha h} I_{s,j+1}^{(2)}(h, -\alpha)$$

and

$$(77) \quad J_{s,j+1}^{(2)}(h, \alpha) = e^{-\alpha h} I_{s,j+1}^{(1)}(h, -\alpha).$$

Thus

$$(78) \quad \Phi_z = \sum_{k=0}^N \sum_{\ell=k}^N \omega_\ell \omega_k A_{\ell k}$$

where

$$A_{\ell k}(h, \alpha) = \frac{4}{\alpha} \frac{2^{-\delta \ell k}}{2N} \sum_{j=0}^{2N} \sum_{s=0}^N g_{0,k,j}^{(s)} [q_{ks} P(a) + q_{\ell s} P(a)]$$

$$(79) \quad \left\{ I_{s,j+1}^{(h, \alpha)} + (-)^{s+j+1} e^{-\alpha h} I_{s,j+1}^{(h, -\alpha)} \right\} \cdot$$

The evaluation of the integral $I_{s,j+1}^{(h, \alpha)}$ is discussed in Appendices C and D.

PLANS FOR FUTURE ANGULAR DISTRIBUTION DATA

It will soon be possible to obtain scattering data, using this equipment, with the detectors mounted on a circular track, at angles other than the five fixed angles at which these results were obtained. These additional data will permit a somewhat more critical analysis than has been possible previously. It is believed that in this way the major uncertainties remaining in the present results can be eliminated.

Briefly the planned modifications of the present analysis are:

- 1) Correction for the finite angular resolution of the detecting apparatus.
- 2) A least square analysis for the coefficients in a series of Legendre polynomials containing seven terms (more if warranted) for the differential cross section, and 3) An internally coded iteration process for the multiple scattering corrections.

The quantity actually measured in these experiments is the cross section averaged not only over the energy spectrum of the incident beam but also over the solid angle of the detector, i.e.,

$$\sigma M(u_j) = \int \Delta \Omega_j \epsilon_j(u) \chi_j(u) \sigma(u) / \int \Delta \Omega_j \epsilon_j(u) \chi_j(u),$$

where $\sigma M(u_j)$ is the measured cross section evaluated at the mean scattering angle of the j th counter; $\epsilon_j(u)$ is the efficiency of this counter, and $\chi_j(u)$ is a function describing the geometry of the experiment. Thus Eq. (12) should be written

$$\sigma M(u_j) = \sum_a \left\{ \int \Delta \Omega_j \epsilon_j(u) \chi_j(u) P_a(u) / \int \Delta \Omega_j \epsilon_j(u) \chi_j(u) \right\}$$

assuming that ϵ_j is not a function of u (i.e., that the counter system is equally efficient over its entire face) and using the $\chi_j(u)$ appropriate to this geometry, we have evaluated these integrals numerically for the five scattering angles at which measurements were made. These corrections were made independently for that portion of the data for which the plural scattering corrections were made. The results are shown in the cross section graphs.

A least square analysis carried out for the differential cross section of Pd at 578 kev gives

$$\sigma(u) = .558 \pm .013 + (.542 \pm .019) P_1(u) + (.324 \pm .020) P_2(u).$$

A comparison of the experimental (statistical) uncertainties with the least square errors is given in tabular form below:

μ	$\sigma(\mu)$ - measured	$\sigma(\mu)$ - least square
.92	$1.35 \pm .02$	$1.33 \pm .04$
.58	$0.86 \pm .02$	$0.87 \pm .03$
-.03	$0.38 \pm .01$	$0.37 \pm .02$
-.38	$0.27 \pm .01$	$0.25 \pm .02$
-.81	$0.27 \pm .01$	$0.28 \pm .03$

This agreement is a measure of the adequacy of a three-term expansion of this differential cross section.

The iteration process which we have used to evaluate the plural scattering corrections would converge more rapidly (at most three iterations for $h = N_0 \approx 0.5$) if the total cross section could be obtained independently. Because of this some consideration is being given to the problem of obtaining transmission data simultaneously with the angular distribution data. Also this would make unnecessary the normalization of all total cross sections to that of a given sample of graphite.

The mobility of the counters will permit also a direct intercalibration of the efficiency of the counters for a given neutron energy. In order to obtain the energy dependence of the counter efficiency more accurately we intend to make use of the known angular distribution and energy degradation effect of a hydrogen scatterer.

Several measurements of the angular distribution of neutrons scattered from a single resonance have been carried out with good resolution. Unfortunately the correction techniques which are described in this report are no longer applicable for this type of data because of the rapid variation with energy of the measured cross section. We are presently considering a Monte Carlo calculation which will include possibly the geometrical as well as the multiple scattering corrections for these data. This should provide also a worthwhile check on the "poor resolution" analysis as used in this report.

Finally we believe that the flexibility of this equipment, together with its high counting efficiencies, will permit both single and double scattering polarization experiments.

SECTION V

ACKNOWLEDGEMENTS

We wish to thank Merle Burgy for his part in work with one of us (A. Langsdorf) which led to development of the type of equipment used in the present work.

Acknowledgement is also made for the vital aid of the staff of the van de Graaff machine, including particularly Jack R. Wallace, William F. Evans, Ronald L. Amrein, and Vincent A. Montelpasse. They helped in the setting up of the equipment, both mechanical and electronic, in preparing scattering samples and in performing the actual experiments.

Thanks are due to Donald A. Flanders, Herbert L. Gray, and others of the mathematics section for cooperation in machine computation of results of these experiments.

Thanks are due to George M. Lobell, Edward J. Lach and others of the machine shop staff for their part in construction of the equipment. Thanks are due to Thomas Brill and the staff of the Electronics group for constructing most of the electronic equipment.

We are indebted to Professor S. Chandrasekhar for discussing with us the nature of the scattering corrections and the possible application of his general theory of scattering to our problem.

THE EXPERIMENTAL DATA

Table I is an example of typical data. The large graphs which follow give the complete results of the survey. The titles give detailed information for each graph.

Table I

TYPICAL DATA

Date 2-16-55, $E_n = 477$ kev

Run No.	Element	N_t	Monitor Counts	Counts at				
254	C	.05297	27762	20319	17653	14070	13304	12448
255	C	.02575	56172	22326	18372	14765	13364	12761
256	No sample		56045	3473	1413	753	846	1425
275	Fe	.02387	28348	13812	9691	6753	6197	6676
480	Ra - Be source for 5 minutes		3757	3814	3856	3777	3872	
				22°	54°	92°	112°	144°

Some general information on the graphs and their figure titles is as follows:

The "Series" number identifies the three groups of experiments carried through at three different times. Series 1 data was obtained in October 1954, Series 2 in February 1955, and Series 3 in August 1955.

The N_t value is the thickness of a scattering sample measured in (atoms per cm^2) $\times 10^{-24}$. The thickness in mean free path units is, then, $h = N_t \sigma_t$, where we consider that the total scattering cross section, σ_{st} equals the total cross section σ_t to a sufficient accuracy for our purposes. The curves for total cross section, marked σ_t on the graphs, should be considered as representing more closely the total scattering cross section than the total cross section; we should probably have used some other symbol than σ_t on the graphs to make this clear.

The neutron energies at which the data points are shown are average energies. At each voltage setting of the van de Graaff machine, the maximum neutron energy was calculated from the voltage setting as standardized by the $\text{Li}^7(p,n)$ reaction threshold voltage of 1.882 Mev. Using the lithium layer thickness as determined by the rise curve above threshold, the energy loss of the protons in the lithium was computed using data on

stopping power such as that given by Allison and Warshaw.* Then the minimum neutron energy was calculated and the average of this and the maximum was taken. There is considerable uncertainty as to whether the points plotted at 30 kev (from Series 2 data only) and at 60 kev in particular should properly be plotted at these positions. Better energy resolution must be used to resolve this question.

Figures 10 through 13 are photographs of cut-outs of the actual graphs, racked in order of increasing atomic number for each ω . A photograph from both the right and left end of each assembly is shown. In the former cases, the actual graph paper rulings show; in the latter, the plain cardboard backings were edged in black to emphasize the structure. We apologize for the crudeness of these figures, but they seem interesting enough to include nevertheless, since preparation of better figures would have delayed publication.

GRAPHS AND FIGURES

19

$C_{1/2}H$ Polyethylene Sheet

62

The results are calculated per average molecule of $C_{1/2}H$.

σ_t	ω_1	ω_2	ω_3	ω_4	N_t	SERIES
.00187	2
.00383	2
.00612	2
.01402	2

Although a Legendre polynomial expansion with terms only up to ω_4 is not a very good description of the angular distribution of hydrogen, the values of σ_t are only about one barn below the literature values (for σ_t of H plus $1/2$ of σ_t of C).

Note that ω_4 is negative and that the scale for ω_4 is displaced from that used for ω_1 , ω_2 , and ω_3 . The zero point of the scale for ω_4 is placed differently on this graph than on all following graphs, which (except for carbon) are standardized.

⁴Be Beryllium Metal Plates

σ†	ω1	ω2	ω3	ω4	N†	SERIES
○	△	□	▽	○	.03277	1
○	△	□	▽	○	.01857	2
○	△	□	▽	○	.03872	2
○	△	□	▽	○	.07907	2

The data points for ω_3 and ω_4 are displaced two (small) divisions to the right and left, respectively, from the correct E_n value, to decrease the overlap of symbols.

Points for ω_3 and ω_4 are displaced two small divisions to the right and left, respectively, to prevent overlap.

64

66

⁵B Elemental Boron, Canned

σ_t	w1	w2	w3	w4	N†	SERIES
○	△	□	▽	○	.02038	3

"Canned" samples are in containers with .005 inch thick steel walls, so that statistical errors, and possibly systematic errors, are greater than for bare samples.

67

SERIES	N_t	w_4	w_3	w_2	w_1	σ_t
1	.02575
2	.02575
3	.02575

This sample of graphite, about 1/8 inch thick, is the standard against which all total cross sections are determined. The σ_t values are input data from BNL 325 and unpublished data.

The solid symbols are data points corrected for plural scattering. Only w_1 points are indicated because all others are essentially the same as the uncorrected points.

Note that the scale of w_i is twice as large as for all other materials but carbon.

More data was obtained on carbon than on other materials so that the degree of agreement between w_i points for the three series of runs is indicative of the degree of consistency of the results attainable over a period of time. Series 1 data was obtained in October 1954, Series 2 in February 1955, Series 3 in August 1955.

The large open symbols are for points obtained from the above plural scattering corrected results, after re-correcting from the geometrical scattering angles to the weighted mean scattering angles. (See Section IV.) When these symbols would coincide with the solid symbols (and make the graphs illegible by superposing too many symbols) they have been omitted.

The curves for ω_1 and ω_2 shown on this graph are duplicated from the preceding graph for this same carbon sample, but with the ω_l scale reduced to that used for all other graphs besides those for carbon.

$${}^6C \quad N_t = .02575$$

70

C^6

72

6C Carbon, Graphite Plates

σ_t	w_1	w_2	w_3	w_4	N_t	SERIES
					.0530	1
					.0530	2
					.0530	3

This sample of graphite is about 1/4 inch thick, about twice as thick as the other sample of carbon.

The solid data points for w_1 and σ_t are corrected for plural scattering. Corrected points for the other w_j are not shown because they are nearly the same as the uncorrected points.

The curves are transcribed from the standard, 1/8 inch thick, graphite curves (for which $N_t = .02575$). The deviations between the solid points and the curves are indicative, in the case of σ_t , of the over-correction of the data in making the corrections without iteration.

The degree of disagreement between the corrected points and the curves is indicative of the uncertainties due to systematic errors, plus statistical errors, after making corrections as described in the text.

73

74

$C_{1/2F}$ "Teflon" Sheets, Polyfluoroethylene

σ_1	w_1	w_2	w_3	w_4	N_1	SERIES
.00771	2
.01680	2

All values here are calculated per effective molecule described as $C_{1/2F}$. The values are close to those for fluorine, but somewhat modified by the presence of the carbon in the compound.

Some points for w_3 and w_4 are displaced two small divisions to prevent overlap of the symbols.

¹¹Na Sodium Metal, Canned

σ_1	w_1	w_2	w_3	w_4	N_1	SERIES
.0136	2
.0287	2

The solid data points are corrected for plural scattering. Corrected points are not shown for w_3 and w_4 because they differ inappreciably from the uncorrected points.

"Canned" samples such as sodium are in containers with .005 inch thick steel walls. The background (obtained with an empty can in place) is higher than for other materials, so that the statistical accuracy is less and the possibility of systematic errors is greater for canned than bare samples.

Points for w_3 and w_4 are displaced two small divisions to the right and left, respectively, to prevent overlap. Corrected points are displaced horizontally enough to make them distinguishable from other points.

The small open symbols are data points calculated using the geometrical (central-point) scattering angles.

The small solid symbols are the data points obtained for these geometrical angles after correcting for plural scattering. (See Section III.)

The large open symbols are for points obtained from the above plural scattering corrected results, after re-correcting from the geometrical scattering angles to the weighted mean scattering angles. (See Section IV.) When these symbols would coincide with the solid symbols (and make the graphs illegible by superposing too many symbols) they have been omitted.

σ_1	w_1	w_2	w_3	w_4	N_1	SERIES
.02624	2
.01375	2
.00686	2

The structure in the w_3 data seems to be large enough to be significant.

The points for w_3 and w_4 are displaced two small divisions to the right and left, respectively, to decrease overlap of the symbols.

Corrected points are displaced (horizontally) enough to make them distinguishable from other points.

The small open symbols are data points calculated using the geometrical (central-point) scattering angles.

The small solid symbols are the data points obtained for these geometrical angles after correcting for plural scattering. (See Section.III.)

The large open symbols are for points obtained from the above plural scattering corrected results, after re-correcting from the geometrical scattering angles to the weighted mean scattering angles. (See Section IV.) When these symbols would coincide with the solid symbols (and make the graphs illegible by superposing too many symbols) they have been omitted.

825 075

80

Aluminum Metal Sheets

σ_t	w_1	w_2	w_3	w_4	N_t	SERIES
○	△	□	▽	○	.01963	1
○	△	□	▽	○	.01465	2
○	△	□	▽	○	.01963	2
○	△	□	▽	○	.03691	2

The points for w_3 and w_4 are displaced two small divisions to the right and left, respectively, to decrease overlap of the symbols.

18

528 22J

82

¹⁵P Red Phosphorus Powder, Canned

σ_1	w_1	w_2	w_3	w_4	N_1	SERIES
○	△	□	△	○	.02734	3

w_1 and w_2 are unusually large at low energies. There is a possibility that the phosphorus contains adsorbed water. Although a freshly opened can of material was used, no analysis was made for water content, and it is known that red phosphorus is somewhat hygroscopic.

83

820 520 025 078

165

Powdered Yellow Sulfur, Canned

84

σ_t	w_1	w_2	w_3	w_4	N_t	SERIES
○	△	□	△	○	.02936	3

The small open symbols are data points calculated using the geometrical (central-point) scattering angles.

The small solid symbols are the data points obtained for these geometrical angles after correcting for plural scattering. (See Section III.)

The large open symbols are for points obtained from the above plural scattering corrected results, after re-correcting from the geometrical scattering angles to the weighted mean scattering angles. (See Section IV.) When these symbols would coincide with the solid symbols (and make the graphs illegible by superposing too many symbols) they have been omitted.

58

86

¹⁹K Potassium Metal, Canned

σ_1	w_1	w_2	w_3	w_4	N_1	SERIES
.01163	2
.02393	2

The points for w_3 and w_4 are displaced two small divisions to the right and left, respectively, from their correct position to decrease overlap of the symbols.

^{22}Ti

Titanium Metal Sheets

σ_t	w_1	w_2	w_3	w_4	N_t	SERIES
.03727	1
.01870	1
.00992	1

Points for w_3 and w_4 are displaced two small divisions to the right and left, respectively, to prevent overlap.

Points for σ_t for the two lowest energies are off scale and omitted.

^{23}V Vanadium Metal Sheets

○	σ†	
△	w1	
□	w2	
▽	w3	
○	w4	
	N†	.01367
	SERIES	3

90

²⁴Cr

Chromium Metal Powder, Canned

92

σ_t	w1	w2	w3	w4	N†	SERIES
○	△	□	△	○	.02325	3

"Canned" samples are in containers with .005 inch thick steel walls, so that the statistical errors and possibly systematic errors are greater than for bare samples.

The small open symbols are data points calculated using the geometrical (central-point) scattering angles.

The small solid symbols are the data points obtained for these geometrical angles after correcting for plural scattering. (See Section III.)

The large open symbols are for points obtained from the above plural scattering corrected results, after re-correcting from the geometrical scattering angles to the weighted mean scattering angles. (See Section IV.) When these symbols would coincide with the solid symbols (and make the graphs illegible by superposing too many symbols) they have been omitted.

93

94

σ_t	w1	w2	w3	w4	N†	SERIES
○	△	□	▽	○	.02387	1
○	▽	□	△	○	.02387	2
○	△	□	▽	○	.02387	3

Note that the same sample was measured in all three series of experiments and that there is quite good agreement between the runs at the same energy.

The small open symbols are data points calculated using the geometrical (central-point) scattering angles.

The small solid symbols are the data points obtained for these geometrical angles after correcting for plural scattering. (See Section III).

The large open symbols are for points obtained from the above plural scattering corrected results, after re-correcting from the geometrical scattering angles to the weighted mean scattering angles (See Section IV). When these symbols would coincide with the solid symbols (and make the graphs illegible by superposing too many symbols) they have been omitted.

95

060 529

96

²⁷Co Cobalt Metal Sheets

σ_1	w_1	w_2	w_3	w_4	N_1	SERIES
					.01279	2
					.02847	2

The cobalt metal was produced by electroplating, and vacuum degassed to remove hydrogen. The final plates were somewhat rough and were not well flattened, so that some systematic error, in σ_1 especially, is possible from these causes.

The points for w_3 and w_4 are displaced two small divisions to the right and left, respectively, to avoid overlap of the points.

²⁸Ni Nickel Metal Sheets

98

σ_t	w_1	w_2	w_3	w_4	N_t	SERIES
					.01426	1
					.02834	1

The points for w_3 and w_4 are displaced two small divisions to the right and left, respectively, to avoid overlap of the symbols.

²⁹Cu Copper Metal Sheets

100

σ_1	w_1	w_2	w_3	w_4	N_1	SERIES
					.01331	1
					.02562	1

The points for w_3 and w_4 are displaced two small divisions to the right and left, respectively, to avoid overlap of the symbols.

101

825 097

102

^{30}Zn

Zinc Metal Sheet

σ_t	Δ	$\square$	∇	$\bigcirc$	
w1					
w2					
w3					
w4					
N†					
SERIES					

.02076

1

103

825 098

104

^{34}Se Selenium Powder, Canned

σ_t	w1	w2	w3	w4	N†	SERIES
○	△	□	△	○	.02325	3

"Canned" samples are in containers with .005 inch thick steel walls, so that the statistical errors and possibly systematic errors are greater than for bare samples.

025 100

105

Zirconium Metal Sheets

⁴⁰Zr

106

σ†	w1	w2	w3	w4	N†	SERIES
○	△	□	▽	○	.01274	1
○	△	□	▽	○	.00670	1

108

⁴¹ Nb		Niobium (Columbium) Metal Sheet				
σ _t	ω ₁	ω ₂	ω ₃	ω ₄	N _t	SERIES
○	△	□	▽	○	.01066	3

601

110
⁴²Mo Molybdenum Metal Sheets

σ_t	w_1	w_2	w_3	w_4	N_t	SERIES
					.00533	2
					.01040	2
					.01573	2
					.02049	2

The points for w_3 and w_4 are displaced two small divisions to the right and left, respectively, to avoid overlap of the symbols.

⁴⁶Pd
Palladium Metal Sheet

σ†	○
w1	△
w2	□
w3	▽
w4	○
N†	.00860
SERIES	3

112

⁴⁷Ag

Silver Metal Sheets

114

σ_1	w_1	w_2	w_3	w_4	N_1	SERIES
					.01866	1
					.00475	2
					.00944	2
					.01419	2
					.01866	2

The data points for w_3 and w_4 are displaced two small divisions to the right and left, respectively, to avoid overlap of the symbols.

The small open symbols are data points calculated using the geometrical (central-point) scattering angles.

The small solid symbols are the data points obtained for these geometrical angles after correcting for plural scattering. (See Section III.)

The large open symbols are for points obtained from the above plural scattering corrected results, after re-correcting from the geometrical scattering angles to the weighted mean scattering angles. (See Section IV.) When these symbols would coincide with the solid symbols (and make the graphs illegible by superposing too many symbols) they have been omitted.

⁴⁸Cd Cadmium Metal Sheets

116

σ_1	w_1	w_2	w_3	w_4	N_1	SERIES
					.007434	2
					.01007	2
					.01235	2
					.01481	2

The points for w_3 and w_4 are displaced two small divisions to the right and left, respectively, to avoid overlap of the symbols.

⁴⁹In

Indium Metal Sheets

σ_1	w_1	w_2	w_3	w_4	$N \dagger$	SERIES
○	△	□	△	○	.01158	3

The small open symbols are data points calculated using the geometrical (central-point) scattering angles.

The small solid symbols are the data points obtained for these geometrical angles after correcting for plural scattering. (See Section III.)

The large open symbols are for points obtained from the above plural scattering corrected results, after re-correcting from the geometrical scattering angles to the weighted mean scattering angles. (See Section IV.) When these symbols would coincide with the solid symbols (and make the graphs illegible by superposing too many symbols) they have been omitted.

119

σ†	w1	w2	w3	w4	N†	SERIES
Q	△	□	▽	Q	.01187	1
Q	△	□	▽	Q	.02375	1

Tin Metal Sheets

⁵⁰Sn

120

121

122

⁵²Te Tellurium Powder, Canned

σ	w_1	w_2	w_3	w_4	$N \dagger$	SERIES
○	△	□	△	○	.01025	3

The small open symbols are data points calculated using the geometrical (central-point) scattering angles.

The small solid symbols are the data points obtained for these geometrical angles after correcting for plural scattering. (See Section III.)

The large open symbols are for points obtained from the above plural scattering corrected results, after re-correcting from the geometrical scattering angles to the weighted mean scattering angles. (See Section IV.) When these symbols would coincide with the solid symbols (and make the graphs illegible by superposing too many symbols) they have been omitted.

123

⁷³Ta Tantalum Metal Sheets

124

σ_t	w_1	w_2	w_3	w_4	N†	SERIES
○	△	□	▽	○	.00434	2
○	▽	□	△	○	.00818	2
○	△	□	▽	○	.01735	2

The small open symbols are data points calculated using the geometrical (central-point) scattering angles.

The small solid symbols are the data points obtained for these geometrical angles after correcting for plural scattering. (See Section III.)

The large open symbols are for points obtained from the above plural scattering corrected results, after re-correcting from the geometrical scattering angles to the weighted mean scattering angles (See Section IV.) When these symbols would coincide with the solid symbols (and make the graphs illegible by superposing too many symbols) they have been omitted.

125

025 120

Tungsten (Wolfram) Metal Sheets ⁷⁴W

126

σ_t	w1	w2	w3	w4	N†	SERIES
○	△	□	▽	○	.00512	2
○	△	□	▽	○	.00998	2
○	△	□	▽	○	.02008	2

127

σ†	w1	w2	w3	w4	N†	SERIES
○	△	□	▽	○	.008516	2

⁷⁸Pt Platinum Metal Sheet

128

129

825 125

130

79Au		Gold Metal Sheets				
σ†	w1	w2	w3	w4	N†	SERIES
○	△	□	▽	○	.006012	2
○	△	□	▽	○	.01215	2

Thallium Metal Sheet

σ_t	Δ	$\square$	Δ	$\circ$	
w1		w2	w3	w4	N†
<hr/>					
					3
					.00548
SERIES					

132

133

825 08

628
129
621

⁸²Pb Lead Metal Sheets

134

σ _t	w ₁	w ₂	w ₃	w ₄	N _†	SERIES
○	△	□	▽	○	.0114	1
○	△	□	▽	○	.0114	2
○	△	□	▽	○	.00558	2

The small open symbols are data points calculated using the geometrical (central-point) scattering angles.

The small solid symbols are the data points obtained for these geometrical angles after correcting for plural scattering. (See Section III.)

The large open symbols are for points obtained from the above plural scattering corrected results, after re-correcting from the geometrical scattering angles to the weighted mean scattering angles (See Section IV.) When these symbols would coincide with the solid symbols (and make the graphs illegible by superposing too many symbols) they have been omitted.

135

Bismuth Metal Plates

⁸³Bi

σ†	○
w1	△
w2	□
w3	▽
w4	○
N†	.00780
SERIES	2

136

137

SERIES	N†	W4	W3	W2	W1	σ†
2	.00564	○	▽	□	△	○
2	.00991	○	▽	□	△	○
2	.01781	○	▽	□	△	○

138

⁹⁰Th Thorium Metal Sheets

139

140

Uranium Metal Sheets

⁹²U

σ_1 w_1 w_2 w_3 w_4 N_1 SERIES

.00394	2
.00777	2
.01536	2
.03020	2

The small open symbols are data points calculated using the geometrical (central-point) scattering angles.

The small solid symbols are the data points obtained for these geometrical angles after correcting for plural scattering. (See Section III.)

The large open symbols are for points obtained from the above plural scattering corrected results, after re-correcting from the geometrical scattering angles to the weighted mean scattering angles. (See Section IV.) When these symbols would coincide with the solid symbols (and make the graphs illegible by superposing too many symbols) they have been omitted.

141

142

Stainless Steel, Type 347

The calculated σ_t and w are per average atom of the mixture.

σ_t	w_1	w_2	w_3	w_4	N†	SERIES
○	△	□	△	○	.01438	2
○	△	□	△	○	.02752	2

The results are rather similar to those for ^{26}Fe (ordinary steel) since this is the major component of the mixture.

"Stainless Steel"

143

Fig. 10a.

144

Fig. 10b.

A profile showing the behavior of the coefficient, ω_1 , as a function of energy, E_n , for the range of atomic numbers, Z , over which measurements were made. The arrows point in the direction of increasing ω_1 , Z , and E_n respectively.

Fig. 11a.

146

Fig. 11b.

A profile showing the behavior of the coefficient, ω_2 , as a function of energy, E_n , for the range of atomic numbers, Z , over which measurements were made. The arrows point in the direction of increasing ω_2 , Z , and E_n respectively.

Fig. 12a.

148

149

Fig. 12b.

A profile showing the behavior of the coefficient, ω_3 , as a function of energy, E_n , for the range of atomic numbers, Z , over which measurements were made. The arrows point in the direction of increasing ω_3 , Z , and E_n respectively. These coefficients which are within statistics of zero over the entire energy range are not shown.

825 144

Fig. 13a.

150

Fig. 13b.

A profile showing the behavior of the coefficient, ω_4 , as a function of energy, E_n , for the range of atomic numbers, Z , over which measurements were made. The arrows point in the direction of increasing ω_4 , Z , and E_n respectively.

APPENDIX A

Some Generalizations of the Exponential Integrals

Placzek¹ defines the functions $E_n(x)$ by the integral

$$E_n(x) = \int_1^\infty e^{-xu} du/u^n. \quad (A1)$$

Many properties of these functions, and integrals whose solutions involve these functions, are given in the tabulations by Placzek,¹ and by Le Caine.²

Chandrasekhar³ and van de Hulst⁴ discuss the function they call $E_1^{(2)}(x)$ which is defined by the integral

$$E_1^{(2)}(x) = \int_x^\infty E_1(x) dx/x. \quad (A2)$$

This same function is also to be found incorporated in Le Caine and in the much earlier work by Nielsen.⁵

Kourganoff⁶ shows that the function $E_1^{(2)}(x)$ may be written as the definite integral

$$E_1^{(2)}(x) = \int_1^\infty e^{-xu} \log u du/u. \quad (A3)$$

¹National Research Council of Canada, Division of Atomic Energy, Chalk River, Ontario, Report N.R.C. No. 1547, MT-1, by G. Placzek: The Functions $E_n(x)$.

²National Research Council of Canada Atomic Energy Project, Chalk River, Ontario, Report N.R.C. No. 1553, MT-131, by J. LeCaine: A Table of Integrals Involving the Functions $E_n(x)$.

³S. Chandrasekhar. Radiative Transfer. Oxford Clarendon Press 1950.

⁴H. C. van de Hulst, Astro. phys. J. 107, 220 (1948).

⁵Nielsen. Theorie des Integral logarithmus, Leipzig, 1906.

⁶V. Kourganoff, Basic Methods in Transfer Problems, Oxford, 1952, page 258 et seq.

None of these authors discuss a generalization of these functions to a larger class or for negative arguments. Only for $E_1(x)$ is the problem considered; in this case Chandrasekhar and others make use of the well-known functions written as $E_1(x)$ by Jahnke-Emde.⁷

The function which we will define, for positive arguments, and positive integer m and n ,

$$E_m^n(x) = \int_0^1 e^{-xu} (\log u)^{m-1} du / u^n \quad (A4)$$

includes the entire class of functions defined by Placzek and the other authors referred to above. We will also show that the functions $E_m^n(x)$ can be defined for negative as well as positive arguments and that infinite series expressions and recursion relations can be derived for the entire class.

The defining equation (A4) for $E_m^n(x)$ may be integrated by parts to give, after a slight rearrangement,

$$E_{m+1}^n(x) = \frac{m E_m^n(x) - x E_{m+1}^{n-1}(x)}{n-1}, \quad n > 1, \quad m > 0 \quad (A5)$$

If $m = 0$, the above recurrence relation is replaced by the one given by Placzek,

$$E_1^n(x) = \frac{e^{-x} - x E_1^{n-1}(x)}{n-1}, \quad n > 1 \quad (A6)$$

The above recurrence relations make it clear that if $E_m^1(x)$ are known for all m up to some specified m , and all $E_1^n(x)$ are known, then the application of Eq. (A5) in principle permits the evaluation of all $E_m^n(x)$ up to the specified m . For positive arguments, many $E_1^n(x)$ are available in the tabulation by Placzek.

For $n = 1$ but general m , an integration by parts yields

$$E_m^1(x) = \int_0^1 e^{-xu} (\log u)^{m-1} du / u$$

$$= \frac{m}{x} \int_0^1 e^{-xu} (\log u)^m du = \frac{m}{x} E_{m+1}^0(x) \quad (A7)$$

⁷E. Jahnke and F. Emde, Third Edition (page 2), Tables of Functions, G. E. Stechert and Co., New York, (Teubner, Leipzig, 1938).

We may now write

$$E_m^I(x) = \frac{m}{x} \int_1^0 e^{-xu} (\log u)^m du - \frac{m}{x} \int_1^0 e^{-xu} (\log u)^m du. \quad (A8)$$

The final integral above may be written

$$W_m^I(x) = \frac{m}{x} \int_1^0 e^{-xu} (\log u)^m du = \int_1^0 \frac{u}{e^{-xu} - 1} (\log u)^{m-1} du \quad (A9)$$

so that

$$W_{m+1}^I(x) = -m \int_x^0 W_m^I(x') dx'/x' \quad (A10)$$

An expansion of the expression for $W_1^I(x)$ gives

$$W_1^I(x) = \sum_{n=1}^{\infty} \frac{n \cdot n!}{(-x)^n} \quad (A11)$$

and by continuation one finds

$$W_m^I(x) = (-)^{m-1} (m-1)! \sum_{n=1}^{\infty} \frac{n \cdot n!}{(-x)^n} \quad (A12)$$

The value of the remaining integral in Eq. (A8) for $E_m^I(x)$ can be determined as follows:

Le Caine uses the relation for the derivatives of the gamma function:

$$\left[\frac{d \Gamma(z)}{dz} \right]_{Z=1} = \int_0^{\infty} e^{-t} (\log t)^n dt \quad (A13)$$

to evaluate

$$\int_0^{\infty} e^{-xu} (\log u)^m du \quad (A14)$$

up to $m = 4$.

It is clear results such as Le Caine gives may be extended, with sufficient labor, to as large an $\bar{m}$ as may be desired. Therefore one can construct, in principle, tables of numerical values of $E_m^I(x)$,

at least for positive argument, to any extent desired. Of course, for a very large argument, x , it would be necessary to find asymptotic expressions because the functions $W_m^I(x)$ become very slowly convergent for large x .

For the functions $E_m^n(x)$ the integral and derivative recurrence relations

$$E_m^{n+1}(x) = \int_{-\infty}^x E_m^n(x') dx' \quad (A15)$$

$$E_{m+1}^n(x) = -E_m^n(x) \quad (A16)$$

follow by simple integration and differentiation.

In the case of Placzek's $E_1^n(x)$, one may consider that

$$E_1^0(x) = \frac{x}{e^{-x}} \quad (A17)$$

is a kernel from which all $E_1^n(x)$ can be derived. Similarly for a general $\overline{m}$, we observe that

$$E_m^1(x) = \int_{-\infty}^1 e^{-xu} (\log u)^{m-1} du / u = \int_{-\infty}^1 e^{-xu} d(\log u)^m / m$$

$$= \frac{m}{x} \int_{-\infty}^1 e^{-xu} (\log u)^m du = \frac{m}{x} E_{m+1}^0(x). \quad (A18)$$

This relation leads to the concept that for each order $\overline{m+1}$, the kernel may be considered to be $E_{m+1}^0(x)$ with

$$E_{m+1}^0(x) = \frac{x}{E_m^1(x)} \quad (A19)$$

so that, for example, we may write

$$E_{m+1}^1(x) = \int_{-\infty}^x E_{m+1}^0(x) dx = m \int_{-\infty}^x \frac{x}{E_m^1(x')} dx' \quad (A20)$$

which is the direct connection between orders, $\overline{m}$, of the functions $E_m^n(x)$.

It will turn out that all the integral, derivative, and recursion relations above apply equally well when the functions have been generalized to negative arguments. Only the defining integrals so far employed have to be abandoned in defining the $E_m^n(x)$ for x negative.

We will now construct a complex generating function for the series expansions of all $E_m^n(x)$. We find that the complex function

$$\mathcal{E}_n(z, x) = (-)^{n-1} \frac{(x)^{n-z} \Gamma(z)}{(2-z)(3-z) \dots (n-z)} + \sum_{r=0}^{\infty} \frac{r! (r+z-n)}{(-x)^r (r+1-n)} \quad (A21)$$

has the property

$$d \mathcal{E}_n(z, x) / dx = - \mathcal{E}_{n-1}(z, x) \quad (A22)$$

consistent with the derivative relation previously derived for the functions $E_m^n(x)$, as in Eq. (A16).

This function also has the property that

$$\left[d \mathcal{E}_m^n(z, x) / dz \right]_{z=1} = E_m^n(x) \quad (A23)$$

That this is so follows from use of the expressions for the incomplete gamma function, as given, for example, in Jahneke-Emde.⁸

From the expression there given, which may be written

$$(x, y)! / (x, \infty)! = \frac{1}{\Gamma(x)} \int_y^\infty e^{-t} t^x dt = \frac{y^{x+1}}{\Gamma(x+1)} \sum_{r=0}^{\infty} \frac{r! (x+r-1)}{(-y)^r (x+1)} \quad (A24)$$

using also $\Gamma(x+1) = x!$,

and the definition

$$\Gamma(x+1, y) = (x, y)!$$

we rewrite

$$\int_y^\infty e^{-t} t^z dt = \Gamma(z) - \Gamma(z, y) \quad (A25)$$

where

$$\Gamma(z, y) = y^z \sum_{r=0}^{\infty} \frac{r! (r+z)}{(-y)^r} \quad (A26)$$

Substituting $t = -xv$ and $y = -x$, one obtains

$$(-x)^z \int_0^1 e^{xv} v^{z-1} dv = \Gamma(z) - \Gamma(z, -x) \quad (A27)$$

and finally,

$$-x \int_0^1 e^{xv} v^{z-1} dv = (-x)^{1-z} \Gamma(z) + \sum_{r=1}^{\infty} \frac{x^r}{r!} (r+z-1) \quad (A28)$$

This equation leads to the expression for $\mathcal{E}_n(z, x)$ of Eq. (A21) by repeated integrations over x from x to infinity.

The actual evaluation of expansions for $E_n^H(x)$ by this generating function method involves some interesting properties related to the technique of differentiation, having to do with the use of logarithmic derivative to evaluate the derivatives of the product term involving the gamma function. We define

$$\mathcal{P}_n(z, x) = \frac{x^{1-z} \Gamma(z)}{x e^{-z} \log x \Gamma(z)} = \frac{(2-z) \dots (n-z)}{(2-z) \dots (n-z)} \quad (A29)$$

so that

$$\mathcal{E}_n(z, x) = (-x)^{n-1} \mathcal{P}_n(z, x) + \sum_{r=0}^{\infty} \frac{x^r}{(-x)^r (r+1-n)} \quad (A30)$$

Then

$$\log \mathcal{P}_n(z, x) = -z \log x + \log \Gamma(z) - \sum_{s=2}^{\infty} \log(s-z) + \log x \quad (A31)$$

We also define the function $\phi_n(z, x)$ to be the logarithmic derivative of $\mathcal{P}_n(z, x)$,

$$\phi_n(z, x) = \frac{d \log \mathcal{P}_n}{dz} = -\log x + \psi(z) + \sum_{s=2}^{\infty} \frac{s-z}{1} \quad (A32)$$

where $\psi(z)$ is the logarithmic derivative of the gamma function.

Now

$$\mathcal{P}_n(1, x) = \frac{1}{(n-1)!} \quad (A33)$$

so that

$$\phi_n(l, x) = -\log x - \gamma + \sum_{s=1}^{n-1} \frac{1}{s} \quad (A34)$$

$$\left[d \phi_n(z) \frac{dz}{z} \right]_{z=1} = \phi_n(l, x) \phi_n(l, x) = \frac{1}{l} (n-1)! \left(-\log x - \gamma + \sum_{s=1}^{n-1} \frac{1}{s} \right) \quad (A35)$$

and, finally

$$\left[d \mathcal{E}_n(z, x) \frac{dz}{z} \right]_{z=1} = \sum_{r=0}^{\infty} \frac{r! (n-1-r)!}{(-x)^r} - \frac{(n-1)!}{(-x)^{n-1}} \left(\gamma + \log x - \sum_{s=1}^{n-1} \frac{1}{s} \right) \quad (A36)$$

which in fact agrees exactly with the equation given for $E_n(x)$ by Placzek.⁹

In the same manner, one may derive the series for $E_2^l(x)$ given by van de Hulst¹⁰ and, in general, for any $E_n^l(x)$ that may be desired. To carry through these more complex cases, the material which follows after the next paragraph is useful.

For negative arguments we use a Cauchy principle value to obtain a real value of $\mathcal{E}_n(z, -x)$. We take

$$\mathcal{E}_n^+(z, -x) = \mathcal{E}_n(z, e^{+i\pi} x) \quad (A37)$$

$$\mathcal{E}_n^-(z, -x) = \mathcal{E}_n(z, e^{-i\pi} x) \quad (A38)$$

and then define

$$\mathcal{E}_n(z, -x) = 1/2 (\mathcal{E}_n^+(z, -x) + \mathcal{E}_n^-(z, -x)) \quad (A39)$$

The value of $\mathcal{E}_n(z, -x)$ so defined may be shown to yield the same answers for $E_n^l(x)$ as are given in its series expansion by always reading each logarithm term as the logarithm of the magnitude, e.g., $\log x$ is read as $\log |x|$.

Some additional material on relations of the derivatives of the gamma functions, useful in evaluating series expansions for exponential integrals, follows:

Following Whittaker and Watson, 11

$$\Gamma(z) = \int_0^\infty e^{-t} e^{(z-1)\log t} dt \quad (A40)$$

Thus

$$\Gamma_n = \Gamma_n(1) = \left[\frac{d^n \Gamma(z)}{dz^n} \right]_{z=1} = \int_0^\infty e^{-t} (\log t)^n dt \quad (A41)$$

We wish to evaluate the Γ_n defined as above.

Now

$$\psi(z) = \frac{d \ln \Gamma(z)}{dz} = -\gamma - \frac{z}{1} + \sum_{\nu=1}^\infty \left(\frac{1}{1} - \frac{z}{z+\nu} \right) \quad (A42)$$

$$\frac{d \psi(z)}{dz} = \sum_{\nu=0}^\infty \frac{1}{(z+\nu)^2} \quad (A43)$$

$$\psi_n^u(z) = \frac{d^n \psi(z)}{dz^n} = (-1)^{n+1} n! \sum_{\nu=0}^\infty \frac{1}{(z+\nu)^{n+1}} \quad (A44)$$

We will abbreviate $\psi_n(1) = \psi_n$ and obtain

$$\psi_n = (-1)^{n+1} n! \zeta(n+1), \quad n \geq 1 \quad (A45)$$

where ζ is the Riemann zeta function, except that we define

$$\zeta(1) = \gamma \quad (A46)$$

where γ is the Euler - Macheroni number so that we can relate this special $\zeta(1)$ to

$$\psi(1) = \psi_0 = \psi_0(1) = -\gamma \text{ and apply Eq. (A45) to the case } n = 0 \text{ also.}$$

Now

$$\Gamma_1(z) = \frac{d \Gamma(z)}{dz} = \psi(z) \Gamma(z) \quad (A47)$$

from which

$$\Gamma_n = \sum_{k=0}^{n-1} \binom{n-1}{k} \psi_k \Gamma_{n-1-k} \quad (A48)$$

where we have defined $\Gamma_0 = \Gamma(1) = 1$.

This equation can also be written

$$\Gamma_n = \sum_{k=0}^{n-1} \frac{(n-1)!}{(n-1-k)!} \Gamma_{n-1-k} (-)^{k+1} \zeta(k+1) \quad (A49)$$

If we substitute

$$\Gamma_n = \frac{n!}{\Gamma_n} \quad (A50)$$

$$N_k = \frac{\psi_{k-1}}{\zeta(k)} = (-)^k \frac{k!}{\zeta(k)} \quad (A51)$$

the equation for Γ_n leads to

$$M_n = \sum_{k=1}^n \frac{k}{n} N_k M_{n-k} \quad (A52)$$

from which it may be shown that

$$M_n = M_0 \sum_{i=1}^n \frac{N_{a_1}!}{N_{a_2}!} \cdot \frac{N_{a_2}!}{N_{a_3}!} \cdots \frac{N_{a_n}!}{N_{a_n}!}$$

$$= M_0 \sum_{i=1}^n \frac{N_{a_i}!}{n!} \quad (A53)$$

where the sum $\sum^*$ extends over all possible products with integers a_i consistent with the condition that

$$n = \sum_{i=1}^n a_i \quad (A54)$$

This is a problem in partitions, such as is discussed in Dickson.¹² This type of solution is valid for more general problems than the particular one of evaluation of gamma function derivatives. It may be used, for example, to obtain detailed solutions for the function $\phi_n(1, x)$ in Eq. (A29).

APPENDIX B

The Functions $F_j(t, \alpha)$, $F_j(t, \alpha, s)$, and $M_0(t, \alpha)$

The Functions $F_j(t, \alpha)$ and $F_j(t, \alpha, s)$

The defining equations for the function $F_j(t, \alpha)$ are

$$F_j(t, -\alpha) = \int_0^t \frac{d\beta}{\beta} [1 - e^{-\beta(\beta+\alpha)}] = \int_0^t e^{-\alpha\beta} F_j(\beta) d\beta. \quad (B1)$$

The recursion relation,

$$F_j(t, -\alpha) = \frac{1}{1} \left\{ \frac{1}{1} - e^{-\alpha t} F_{j-1}(t, -\alpha) \right\}, \quad (B2)$$

follows directly from the definition since

$$F_j(t, -\alpha) = \frac{1}{1} \int_0^t d\beta [\beta^{-j} - \beta^{j-1}(\beta+\alpha)^{-1}] [1 - e^{-\beta(\beta+\alpha)}].$$

This, together with the expression*

$$F_1(t, \alpha) = \alpha^{-1} [e^{\alpha t} E_1(t) - E_1(t - \alpha)] - \ln |\alpha - 1|, \quad (B3)$$

permits, at least in principle, the evaluation of $F_j(t, \alpha)$.

The integral

$$F_j(t, \alpha, s) = \int_0^t e^{sx} F_j(x, \alpha) dx \quad (B4)$$

occurs frequently in the treatment of double scattering. It can be evaluated by partial integration as

$$F_j(t, \alpha, s) = \int_0^t dx \frac{d}{dx} \left(e^{sx} \right) \left(F_j(x, \alpha) - \frac{1}{s} e^{sx} F_j(t, \alpha) - F_j(t, s + \alpha) \right). \quad (B5)$$

From this equation it is immediately obvious that

$$e^{sh} F_j(h, s + \alpha, -s) = F_j(h, \alpha, s). \quad (B6)$$

The Functions $M_0(t, \lambda)$

By analogy with Kourganoff's $M_0(t)$ -function, we define a two parameter family of integrals

$$M_0(t, \lambda) = \int_0^t \frac{d\mu}{\mu} \ln(\mu + \lambda) e^{-\mu t}. \quad (B7)$$

*V. Kourganoff, Basic Methods in Transfer Problems, Oxford (1952), Appendix I., page 253 et. seq.

The Taylor's expansion of $M_0(\tau, \lambda)$ around $\lambda = \lambda_0$ is

$$M_0(\tau, \lambda) = M_0(\tau, \lambda_0) + \sum_{n=1}^{\infty} \frac{(\lambda - \lambda_0)^n}{n!} (-)^{n-1} \int_0^1 \frac{d\mu}{d\mu} h(\mu + \lambda_0) e^{-T\mu} \quad (B8)$$

Since for $\lambda_0 \gg 0$,

$$\int_0^1 \frac{d\mu}{d\mu} h(\mu + \lambda_0) e^{-T\mu} \ll \frac{1}{\lambda_0^{n+1}} = E_{n+1}(\tau) \ll \frac{1}{\lambda_0^n},$$

we see that the Taylor's series is absolutely convergent for $\lambda - \lambda_0 \ll 1$. For $\lambda_0 = 0$, Eq. (B8) gives

$$M_0(\tau, \lambda) = E_1(2)(\tau) - \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} (-)^n E_{n+1}(\tau), \quad (B9)$$

while for $\lambda_0 = 1$, we obtain

$$M_0(\tau, \lambda) = M_0(\tau) + \sum_{n=1}^{\infty} (-)^n \frac{(\lambda - 1)^n}{n!} \left[e^{-T} \sum_{i=0}^n 2^{-i} E_{i+1}(2\tau) - E_1(\tau) \right]. \quad (B10)$$

In this last expression we have used the expansion

$$\int_0^1 \frac{d\mu}{d\mu} h(\mu + \lambda_0) e^{-T\mu} = \frac{1}{\lambda_0^n} \left[E_1(\tau) - \sum_{i=0}^{n-1} \frac{\lambda_0^i}{i!} \int_0^1 \frac{d\mu}{d\mu} h(\mu + \lambda_0) e^{-T\mu} \right].$$

That the function $M_0(\tau, \lambda)$ is related to the integral

$$\int_0^1 \frac{d\beta}{d\beta} E_1[h(1+\beta)] = h \int_0^1 \frac{d\mu}{d\mu} E_1[\mu + h(1-\alpha)], \quad (B11)$$

can be shown in the following manner. Consider the function

$$W_1(hx) = hx \int_0^1 e^{-hx\mu} h\mu d\mu = x \int_0^1 h dy e^{-xy} hny + (e^{-xh} - 1) hnh, \quad (B12)$$

then

$$x \int_0^1 dy e^{-xy} hny = W_1(bx) - W_1(ax) - (e^{-xb} - 1) hnb + (e^{-xa} - 1) hna. \quad (B13)$$

A further change of variables gives

$$\int_0^{b-c} \frac{d\mu}{d\mu} e^{-x\mu} h\mu d\mu = \frac{x}{e^{xc}} \left\{ W_1(bx) - W_1(ax) - (e^{-xb} - 1) hnb + (e^{-xa} - 1) hna \right\}.$$

Integrating both sides of this expression over x from x to z , we obtain

$$\int_{b-c}^{a-c} d\mu \ln(\mu+c) \left\{ \frac{\mu}{e^{-x\mu} - e^{-z\mu}} \right\} = \int_z^x dy \frac{e^{-cy}}{y} \left\{ W_1(by) - W_1(ay) - (e^{-by} - 1)\ln b + (e^{-ay} - 1)\ln a \right\}.$$

In the limit as $b \rightarrow \infty$, this becomes

$$\int_0^\infty d\mu e^{-x\mu} \ln(\mu+c) = \int_0^\infty dy \frac{e^{-cy}}{y} \left\{ E_1[(\delta+c)y] + e^{-(\delta+c)y} \ln(\delta+c) \right\}, \quad (B14)$$

where $\delta = a - c$, and we have made use of the relation

$$W_1(x) = -\gamma - \ln|x| - E_1(x).$$

Equation (B14) can be rewritten

$$\int_0^\infty d\mu e^{-x\mu} \ln(\mu+c) = E_1(\delta x) \ln|\delta+c| + \int_0^\infty dy \frac{e^{-cy}}{y} E_1[(\delta+c)y], \quad (B15)$$

where

$$\int_0^\infty dy \frac{e^{-cy}}{y} E_1[(\delta+c)y] = \int_0^1 d\beta \int_0^\infty \frac{\beta}{dy} e^{[(\delta+c)\beta - c]y} = \int_0^1 d\beta E_1[(\delta+c)x\beta - cx]. \quad (B16)$$

With $\delta = 2$, $x = h$, and $c = \alpha - 1$, Eq (B15) becomes

$$\int_0^\infty d\mu e^{-h\mu} \ln(\mu+\alpha-1) = E_1(2h) \ln|\alpha+1| + \int_0^\infty d\beta E_1(\beta+h(1-\alpha))$$

so that the integral of Eq. (B11) is given by

$$\int_0^\infty d\mu E_1[\mu+h(1-\alpha)] = E_1(2h) \ln \frac{|\alpha+1|}{2} + M_0\left(2h, \frac{2}{\alpha-1}\right). \quad (B17)$$

We shall use this relation in the evaluation of the integral $I_s^{(2)}(h, \alpha)$.

APPENDIX C

The Integral $I_{s,j+1}^{(i)}(h, \alpha)$

In order to make the notation less cumbersome, consider the

integral

$$I_{\ell m}^{(i)}(h, \alpha) = \int_0^1 \frac{d\gamma}{\gamma} e^{-\gamma h} \int_h^\infty d\tau e^{(\gamma - \alpha)\tau} F_m(\tau, \alpha). \quad (C1)$$

Although this involves a change in the meaning of the indices, no confusion should result. A recursion relation for this integral can be obtained by partial integration:

$$I_{\ell m}^{(i)}(h, \alpha) = \int_h^\infty d\tau E_{\ell}(h - \tau) \left\{ \frac{d}{d\tau} \frac{F_m(\tau, 0) - e^{-\alpha\tau} F_m(\tau, \alpha)}{\alpha} \right\}$$

$$= \frac{E_{\ell}(0)}{\alpha} \left\{ F_m(h, 0) - e^{-\alpha h} F_m(h, \alpha) \right\}$$

$$+ \frac{1}{\alpha} \left\{ I_{\ell-1, m}^{(i)}(h, \alpha) - I_{\ell-1, m}^{(i)}(h, 0) \right\}, \quad (\ell > 1),$$

(C2)

where

$$I_{\ell m}^{(i)}(h, 0) = E_{m+1}(0) \left[E_{\ell+1}(0) - E_{\ell+1}(h) \right] - G_{\ell, m+1}^{(i)}(h),$$

$$(\ell, m > 0). \quad (C3)$$

The integral

$$G_{\ell m}^{(i)}(\tau) = \int_0^\tau dt E_m(t) E_{\ell}(\tau - t) = G_{\ell m}^{(i)}(\tau) \quad (C4)$$

is discussed by Kourganoff and Chandrasekhar (op. cit.). Making use of the defining integrals for the $E_{\ell}(\tau)$ and $F_j(\tau, \alpha)$ - functions, Eq. (C1) can be rewritten

$$I_{\ell m}^{(i)}(h, \alpha) = e^{-\alpha h} \int_h^\infty d\tau e^{\alpha\tau} G_{\ell m}^{(i)}(\tau). \quad (C5)$$

From Eq. (C4) it is obvious that

$$I_{\ell m}^{(i)}(h, \alpha) = I_{\ell m}^{(i)}(h, \alpha). \quad (C6)$$

Thus it is necessary to consider only the integral

$$I_{1,1}^{(1)}(h, \alpha) = e^{-\alpha h} \int_0^\infty e^{\alpha T} dT G_{1,1}(T). \quad (C7)$$

However it is convenient to treat the integral

$$I_{1,1}^{(1)}(h, \epsilon, \alpha) = e^{-\alpha h} \int_0^\infty e^{\alpha T} dT e^{-\epsilon T} G_{1,1}(T),$$

and obtain Eq. (C7) by taking the limit as $\epsilon \rightarrow 0$. Making use of Eq. (K 37.30)*

$$I_{1,1}^{(1)}(h, \epsilon, \alpha) = e^{-\alpha h} \int_0^\infty e^{\alpha T} dT \int_0^\infty e^{-\frac{\alpha}{z} T} \frac{d}{dx} \alpha(x - z)$$

$$= -e^{-\alpha h} \int_0^\infty \frac{dz}{dz} e^{h(\alpha - z)} \frac{e^{\alpha(\alpha - z)} - e^{\alpha - z}}{\alpha - z} \ln(z - 1). \quad (C8)$$

Expanding the integrand in partial fractions and making use of Eq. (K 37.27) we obtain

$$I_{1,1}^{(1)}(h, \epsilon, \alpha) = -\frac{\alpha}{2} N_1(h) + \frac{\alpha}{2} e^{\alpha(\epsilon - h)} N_1(\epsilon) - \frac{\alpha}{2} N_0(h) + \frac{\alpha}{2} e^{\alpha(\epsilon - h)} N_0(\epsilon) - \frac{\alpha}{2} \int_0^\infty e^{\alpha T} dT \int_0^\infty e^{-\frac{\alpha}{z} T} \ln(z - 1) dz. \quad (C9)$$

This last integral can be written

$$\int_h^\infty e^{\alpha T} dT \int_0^\infty e^{-\frac{\alpha}{z} T} \ln(z - 1) dz = \int_h^\infty e^{\alpha T} dT \int_0^\infty e^{-(\alpha - 1)T} dy \ln y e^{-Ty}; \quad (C10)$$

*In the remainder of this Appendix references to specific equations in Kourganoff's Basic Methods in Transfer Problems will be denoted by K followed by the number of the equation as found in this text.

the integral over y is just the Laplace transform of λy which is

$$-\frac{\gamma}{\gamma + \lambda n \tau}.$$

Further, for $\alpha > 1$, Eq. (C10) becomes

$$-\gamma \int_0^\infty h^{\alpha-1} e^{-\frac{\gamma}{\lambda n \tau} h} d h = \frac{\gamma}{\lambda n \tau} \int_0^\infty h^{\alpha-1} e^{-\frac{\gamma}{\lambda n \tau} h} d h = -\gamma \int_0^\infty h^{\alpha-1} e^{-\frac{\gamma}{\lambda n \tau} h} d h$$

$$-\int_0^\infty h^{\alpha-1} e^{-\frac{\gamma}{\lambda n \tau} h} d h + \lambda n \tau \int_0^\infty h^{\alpha-1} e^{-\frac{\gamma}{\lambda n \tau} h} d h = \int_0^\infty h^{\alpha-1} e^{-\frac{\gamma}{\lambda n \tau} h} d h \quad (C11)$$

These integrals are directly related to the $E_1(x)$ and $E_2(x)$ integrals for negative arguments. Integrating by parts, it can be shown that

$$\int_a^\infty h^{\alpha-1} e^{-\frac{\gamma}{\lambda n \tau} h} d h = -E_1(-a) \lambda n \tau + E_1(-\delta) \lambda n \delta + E_2(-\delta) - E_2(-a).$$

(C12)

Combining these results and taking the limit as $\epsilon \rightarrow 0$, we obtain for $\alpha > 1$

$$I_{1,1}^{(1)}(h, \alpha) = -\frac{\alpha}{2} N_1(h) + \frac{\alpha}{2} E_2^{(1)}(h) + \frac{\alpha}{2} (\gamma + \lambda n \tau) E_1(h)$$

$$+ \frac{\alpha}{2} e^{-\frac{\gamma}{\lambda n \tau} h} \left\{ [-\gamma - \lambda n \tau] E_1(-h) (\alpha - 1) - E_2^{(1)}(-h) (\alpha - 1) \right\}$$

(C13)

APPENDIX D

The Integral $I_{(z)}^{(j+1)}(h, \alpha)$

Again with a change in the meaning of the indices, we rewrite Eq. (73) in the form

$$I_{(z)}^{(j)}(h, \alpha) = \int_h^\infty d\tau \int_0^1 \frac{\gamma \ell}{\lambda} e^{(\gamma - \alpha)\tau} e^{-\gamma h} F_m(h - \tau, -\alpha)$$

$$= e^{-\alpha h} \int_h^\infty d\gamma \int_0^1 \frac{\gamma \ell}{\lambda} e^{(\alpha - \gamma)\tau} F_m(\tau, -\alpha) d\tau \quad (D1)$$

$$= e^{-\alpha h} \int_h^\infty d\tau e^{\alpha \tau} E \ell(\tau) F_m(\tau, -\alpha).$$

As in Appendix C recursion formulae can be obtained for this integral by partial integration:

$$I_{(z)}^{(j)}(h, \alpha) = e^{-\alpha h} \int_h^\infty d\tau F_m(\tau, -\alpha) \frac{d}{d\tau} F \ell(\tau, \alpha)$$

$$= e^{-\alpha h} F_m(h, -\alpha) F \ell(h, \alpha) - I_{(z)}^{(j-1)}(h, -\alpha), \quad (D2)$$

and

$$I_{(z)}^{(j)}(h, \alpha) = e^{-\alpha h} \int_h^\infty d\tau E \ell(\tau) \frac{d}{d\tau} e^{\alpha \tau} F_m(\tau, -\alpha) \left\{ \frac{\alpha}{e^{\alpha \tau} F_m(\tau, -\alpha)} \right\} \\ = \frac{E \ell(h)}{F_m(h)} \left\{ F_m(h, -\alpha) - e^{-\alpha h} F_m(h, 0) \right\} + \frac{\alpha}{I_{(z)}^{(j-1)}(h, 0)} \left\{ I_{(z)}^{(j)}(h, \alpha) - e^{-\alpha h} I_{(z)}^{(j-1)}(h, 0) \right\}$$

$$(D3) \quad (j > 1),$$

where

$$I_{(z)}^{(j)}(0) = F_m + I(0) \left\{ E \ell + I(0) - F \ell + I(h) \right\} - G_{\ell, m+1}(h). \quad (D4)$$

The integral

$$G_{\ell, m+1}(h) = \int_h^\infty d\tau E \ell(\tau) F_m + I(\tau) = \frac{h E \ell(h) F_m + I(h) + F \ell(h, -1) + F_m + I(h, -1)}{\ell + m} \quad (D5)$$

is discussed by Chandrasekhar.*

It is necessary therefore to investigate only $I_{(2)}^{1,m}(h, \alpha)$, where

$$I_{(2)}^{1,m}(h, \alpha) = e^{-\alpha h} \int_h^{\infty} dt e^{\alpha t} E_1(t) F_m(t, -\alpha) \quad (D6)$$

Making use of the recursion relation (B2), this can be rewritten

$$I_{(2)}^{1,m}(h, \alpha) = e^{-\alpha h} \left\{ \frac{\alpha}{1} F_1(h, \alpha) - G_{m,1}(h) - \frac{\alpha}{1} I_{(2)}^{1,m-1}(h, \alpha), (m > 1). \right\} \quad (D7)$$

The recursion relations (D3) and (D7) permit the evaluation of any integral, $I_{(2)}^{1,m}(h, \alpha)$, in terms of

$$I_{(2)}^{1,1}(h, \alpha) = e^{-\alpha h} \int_h^{\infty} dt e^{\alpha t} E_1(t) F_1(t, -\alpha).$$

Making use of the definition of F_1 , this becomes

$$I_{(2)}^{1,1}(h, \alpha) = e^{-\alpha h} \left\{ \frac{\alpha}{1} \int_h^{\infty} dt [E_1(t)]^2 + \frac{\alpha}{1} \int_h^{\infty} dt E_1(t) \int_h^{\infty} dt e^{\alpha t} E_1(t) \right\}$$

where**

$$\int_h^{\infty} dt [E_1(t)]^2 = 2 \int_h^{\infty} dt E_1(t) \int_h^{\infty} dt E_1(2t).$$

Next consider

$$\int_h^{\infty} dt E_1(t) \int_{-\infty}^{\infty} d\beta \frac{e^{-\beta t}}{\beta + \alpha} = -E_1(h) \frac{E_1(h) - e^{-\alpha h} E_1[(1 + \alpha)h]}{\alpha}$$

$$+ \int_{-\infty}^{\infty} d\beta \frac{\beta(\beta + \alpha)}{\beta(1 + \beta)} \ln(1 + \beta) + \int_{-\infty}^{\infty} d\beta \frac{\beta(\beta + \alpha)}{\beta(1 + \beta)} E_1[h(1 + \beta)].$$

*Chandrasekhar, Radiative Transfer, Oxford (1950), p. 376.

**J. Le Caine, A Table of Integrals Involving the Functions $E_n(x)$, N.R.C. No. 1553; Eqs. (3.2.1) and (3.3.1).

where

$$\int_{-\infty}^{\infty} \frac{d\beta}{\beta} \ln(1+\beta) = \left\{ \frac{1}{\pi^2} \left[\frac{\alpha}{1-\alpha} + \sum_{n=1}^{\infty} \left(\frac{n}{1-\alpha} \right)^2 + \frac{1}{\pi^2} \left[\frac{\alpha}{1-\alpha} + \sum_{n=1}^{\infty} \left(\frac{n}{1-\alpha} \right)^2 \right] \right] \right\}, \quad (-1 < \alpha \leq 0)$$

(Le Caine, Eqs. (B.1.2) and (B.3.1).) In view of this limitation on negative values of α , it may be necessary to evaluate the set of integrals, $I_{m,h}^{(2)}(\alpha)$, for positive α and use Eq. (C2) to obtain the corresponding set for negative values of α . Finally

$$\int_{-\infty}^{\infty} \frac{d\beta}{\beta} E_1[h(1+\beta)] = \frac{2\alpha}{1} [E_1(h)]^2 - \frac{\alpha}{1} [hE_1(2h)] \ln \frac{|\alpha+1|}{2} + h M_0 \left(2h, \frac{\alpha-1}{2} \right),$$

where we have used Eq. (B17).

Combining these results, we have for $\alpha \gg 0$

$$(D8) \quad \left\{ \begin{aligned} & \frac{1}{\alpha} \ln |\alpha+1| E_1(h, \alpha) - \frac{\alpha}{1} [2 \ln 2 - E_2(h) E_1(h) - e^{-h} E_1(h) + 2 E_1(2h)] \\ & - \frac{1}{\alpha} [E_1(h)]^2 - e^{-\alpha h} E_1(h) E_1[h(1+\alpha)] \left[\frac{1}{\pi^2} + \frac{\alpha}{2} \ln \frac{2}{1+\alpha} + \sum_{n=1}^{\infty} \left(\frac{n}{1-\alpha} \right)^2 \right] \\ & + \frac{2\alpha}{1} E_1(h) \ln \frac{(\alpha+1)}{2} + h M_0 \left(2h, \frac{\alpha-1}{2} \right) \end{aligned} \right\}$$