NUCLEAR ROCKET OPERATIONS
SACRAMENTO PLANT

REPORT NO. RN-DR-0124

TO
AEC-NASA SPACE NUCLEAR PROPULSION OFFICE

PNEUMATIC INSTRUMENTATION LINES
AND THEIR USE IN MEASURING ROCKET
NOZZLE PRESSURE

MASTER

NERVA PROGRAM

CONTRACT SNP-1

JULY 1967

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PNEUMATIC INSTRUMENTATION LINES
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NOZZLE PRESSURE

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Program Manager
Nuclear Rocket Operations

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I. INTRODUCTION

Pressure measurements are required to determine nuclear rocket engine performance and to satisfy control system requirements. However, when the available pressure transducers are subject to radiation damage, it is necessary to place them behind radiation shields and couple them to the engine with pneumatic tubing. As a result, the frequency response of the measurement system is degraded. It is the intent of this report to describe the operational characteristics of pneumatic instrumentation lines and to discuss the infinite-line technique that was used to minimize the detrimental coupling effects of pneumatic tubing during the NERVA NRX/EST test series.

The overall objective of the NERVA (Nuclear Engine for Rocket Vehicle Applications) program is to develop a solid-core, nuclear-reactor powered engine for space flight vehicles. The NERVA NRX/EST engine was the first self-sustaining nuclear engine. The propellant feed system had a turbopump assembly that was driven with a portion of the engine exhaust drawn off at a hot-bleed port in the convergent nozzle section. That is, NERVA NRX/EST was a hot-bleed-cycle, nuclear rocket engine system. The overall objective of the NRX/EST test program was to demonstrate the practicability of a hot-bleed cycle nuclear engine system for both transient and steady-state operation.

The operation of the NERVA nuclear rocket engine is shown in Figure 1. The propellant (liquid hydrogen) is pumped to the reactor where it is heated and then exhausted through a convergent-divergent nozzle to produce propulsive thrust. The engine performance parameters used to evaluate nuclear rocket engine operation are thrust, specific impulse (thrust divided by propellant weight flow), and reactor power level (thermal energy generation rate).

Engine performance parameters can be calculated by inserting measured nozzle dimensions, thermodynamic gas parameters, propellant temperatures, and propellant pressures into thermodynamic equations. Since nozzle dimensions

ENGINE PERFORMANCE PARAMETERS	{	THRUST	$F = \dot{m} V_e + P_e A_e$
		SPECIFIC IMPULSE	$I_{SP} = F / g \dot{m}$
		POWER LEVEL	$P.L. = g \dot{m} c_p (T_c - T_i)$

WHERE: MASS PROPELLANT FLOW ($\dot{m}$) = $(P_c A_t) \left[\left(\frac{\gamma M}{R_c T_c} \right) \left(\frac{2}{\gamma + 1} \right) \right]^{1/2}$

A_e = NOZZLE EXIT AREA	P_e = NOZZLE EXIT PRESSURE
A_t = NOZZLE THROAT AREA	R_c = UNIVERSAL GAS CONSTANT
c_p = SPECIFIC HEAT, CONSTANT PRESSURE	T_c = NOZZLE CHAMBER TEMPERATURE
g = GRAVITATIONAL ACCELERATION	T_i = REACTOR INLET TEMPERATURE
M = MOLECULAR WEIGHT	V_e = NOZZLE EXIT VELOCITY
P_c = NOZZLE CHAMBER PRESSURE	γ = SPECIFIC HEAT RATIO

Figure 1 - Nuclear Rocket Engine Performance Parameters

and thermodynamic gas parameter values are established either by design or natural physical laws; thrust, specific impulse, and reactor power level can, to a large extent, be determined or controlled by measuring or controlling propellant gas temperatures and pressures in and around the reactor chamber and nozzle.

One of the most critical pressure environments is that existing within the convergent nozzle section, since it essentially determines or controls mass flow and thrust and can be a significant factor in determining or controlling reactor power level. In general, accurate measurement of pressure at the convergent nozzle station (Figure 1) is a prerequisite to any rocket test program where thrust and mass flow measurements are not made. However, the nuclear radiation environment aggravates the problem of obtaining accurate nozzle chamber pressure measurements.

Nuclear radiation can produce permanent or temporary changes in the physical and chemical properties of materials used in the fabrication of nozzle-mounted pressure transducers. In general, the changes (damage) produced during nuclear rocket testing result from neutron particle impingement and gamma (high-energy electromagnetic) irradiation.

Generally, both neutron and gamma radiation can damage organic compounds by disrupting chemical bonds and molecular structures. Inorganic crystalline materials, such as metals, are particularly susceptible to high-energy neutron particles that, upon entering the material, will collide with and displace atoms from their normal lattice positions. Such atomic displacements change the physical properties of components within the transducer and can result in transducer calibration variations. Neutron absorption by atomic nuclei can result in the transmutation of stable atoms into radioactive elements so that posttest handling of irradiated transducers becomes a problem.

TUBE CHARACTERISTICS:

LENGTH 325 FEET
 OUTER DIAMETER ... 0.187 INCHES
 INNER DIAMETER.... 0.131 INCHES
 WALL THICKNESS ... 0.028 INCHES

WALL MATERIALS:

FIRST 25 FEET, 304 STAINLESS STEEL
 DAMPING TUBE, 5052 ALUMINUM

Figure 2 - Infinite Line Configuration

Gamma irradiation of inorganic materials will produce ionization and electron excitation. When ceramics or metallic oxides are exposed to gamma radiation, their effectiveness as electrical insulators is reduced. When metals are similarly exposed, the rapidity of subsequent ion recombination and loss of electron excitation will release energy in the form of heat. Thus, one of the most marked effects of gamma radiation in metals is heat generation, which requires the provision of adequate structural cooling of the transducer.

Because of the difficulty in designing a flush-mounted pressure transducer of sufficient accuracy for nuclear rocket engine application, it was necessary to provide an interim technique for measuring nozzle chamber pressures to meet test schedules.

The solution was to place available pressure transducers in a less severe environment and couple them to the pressure source with pneumatic tubing. The infinite-line technique provided a method for improving the normally inadequate frequency response of transducer pneumatic-tube, pressure-measurement systems by reducing acoustic resonance effects.

The infinite-line configuration, shown in Figure 2, consisted of a long tube that was open at the pressure source (i.e., the nozzle chamber pressure environment) and closed at the other end. The tube was tapped to couple the pressure transducer to the nozzle pressure environment. Coupling length (25 feet) was established by the distance required to position the radiation-sensitive pressure transducer behind a radiation shield. Total length (325 feet) was sufficient to reduce line resonance to an acceptable level. In brief, additional tubing was used beyond the transducer tap station to partially absorb pressure waves traveling through the tube.

Experimental techniques were employed to confirm the theoretical model used to predict the transfer function of various infinite-line configurations. When the adequacy of the theoretical model was confirmed, two infinite-line

configurations were selected to support the NRX/EST test series. One had a single-tee tap, as shown in Figure 2, to accommodate a single pressure transducer; the other used a double-tee tap to accommodate two pressure transducers. The three pressure transducer outputs were averaged electronically to meet control system input requirements but were recorded individually to meet diagnostic data requirements.

II. SUMMARY

Engine control input requirements for the NERVA NRX/EST test series included a need for accurate nozzle chamber pressure information. However, the nuclear radiation environment at the proposed measurement station (convergent nozzle wall) exceeded the performance limits of available pressure transducers and the test schedule precluded the design and development of a new transducer capable of withstanding the radiation environment. Thus, it became necessary to develop a remote pressure-measurement system incorporating pneumatic tubing.

A pneumatic instrumentation line can be used to couple pressure transducers to a pressure source that originates within an environmental region that exceeds the transducer design tolerance. A pneumatic line also provides a means for obtaining pressure measurements when mechanical interface problems prevent direct coupling of the transducer to the pressure source. However, pneumatic lines have undesirable transfer function characteristics (attenuation and time delay of pressure information) that must be understood and accounted for. A survey of the literature describing the operating characteristics of pneumatic line systems indicated that most available literature can be grouped into two broad categories. The first category contains several excellent theoretical discussions whose actual or imagined complexity tends to discourage their use. The second category includes more acceptable approximation equations whose apparent simplicity encourages their adoption to describe systems having operational conditions different from those of the boundary conditions (often unstated) under which the approximations were derived. It has been the intent of this report to describe the meaning and applicability of formulas abstracted from two of the best currently available theoretical dissertations^{1,2} and to present approximations of these exact equations together with applicable

1. Iberall, Arthur S., Attenuation of Oscillatory Pressures in Instrument Lines, NBS Research Paper RP2115, Vol. 45, July 1950.
2. Bergh, H., and Tjrdeman, H., Theoretical and Experimental Results for the Dynamic Response of Pressure Measuring Systems, National Aero and Astronautical Research Institute, Technical Report NLR-TR F.238, Amsterdam, Jan. 1965.

boundary conditions. That is, this report is an attempt to explain the meaning of exact equations rather than to discuss their development.

To improve understanding of the transport process within pneumatic instrumentation lines, rather simple derivations were used to develop general expressions that describe gas pressure and velocity distributions in terms of a wave propagation coefficient and acoustic line impedances (Equations 12 and 13). These very general expressions were then equated to the rigorous Iberall equations¹. As a final step, the Bergh-Tijdeman segmented line equations were presented.

Comparison of theoretical Iberall and Bergh-Tijdeman formulas with experimental results has shown that the response characteristics of pressure-measuring systems can be predicted with an uncertainty of less than 5% through a spectral range that extends from 1 to 2000 Hz when:

- a. Circular instrumentation lines are used.
- b. The internal radius of the line is small when compared with its length.

$$L > \frac{80 R}{3\pi}$$

- c. The amplitude of pressure oscillations presented to the line entrance is small enough to insure laminar flow throughout the line length (Reynolds numbers less than 2000). By noting that the Reynolds number is $(DU \rho / \mu)$ and using Equation 75¹ and Equation 26 of this report, the limit imposed on the peak input oscillatory pressure amplitude is

$$P_o < j \frac{2000 \mu \omega}{yD} \left\{ \frac{[W J_o(W)] \cdot [\cosh(yL) + Q \sinh(yL)]}{[W J_o(W) - 2 J_1(W)] \cdot [\sinh y(L-x) + Q \cosh y(L-x)]} \right\}$$

when: $(0 < x < L)$.

1. Iberall, Arthur S., Attenuation of Oscillatory Pressures in Instrument Lines, NBS Research Paper RP2115, Vol. 45, July 1950.

d. An elastic gas is assumed where

$$1.0 > \mu/c\rho D$$

and the source frequency is such that

$$1.0 > \omega\mu/\rho c^2$$

By assuming the above boundary conditions, the following three important line configurations have been analytically described: the continuous line configuration, the volume terminated configuration, and the infinite-line configuration. Of the three configurations, the infinite-line is the most versatile, has the greatest potential for enabling accurate pressure information to be obtained with pneumatic tubing, and is the one adopted to meet NRX/EST pressure measurement requirements. However, regardless of their configuration, pneumatic lines should be eliminated whenever possible and the transducer should be coupled directly to the pressure source.

When pneumatic tubing must be used, both steady-state and transient pneumatic line response can be improved by using the largest practicable inner line radius, the smallest obtainable transducer cavity volume, and the shortest feasible line length between the transducer and the pressure source.

When working with volume-terminated lines where accurate transfer function predictions are not required, second-order approximations can be used to describe the transfer function up to the fundamental resonant pressure frequency value.

As a result of analytical pneumatic line response investigations, a mathematical pneumatic line model was obtained, a suitable line configuration was selected, transfer functions were predicted, and satisfactory pressure measurements were obtained during the NERVA NRX/EST test series.

III. CONCLUSIONS AND RECOMMENDATIONS

An analysis of recorded NRX/EST posttest data revealed good agreement between the three infinite-line data channel records. In addition, both infinite-line configurations performed as predicted and met all major test requirements.

As a result of analytical pneumatic line studies, several general statements can be made concerning the use of pneumatic tubing.

A. The infinite-line technique will improve the frequency response of pressure measurement systems utilizing pneumatic tubing by reducing acoustic resonance effects. This is accomplished by using additional tubing beyond the pressure transducer station to attenuate reflected sound waves.

B. Generally, pressure information obtained with a transducer utilizing the infinite-line technique is more satisfactory than that obtained from a transducer mounted on the end of a continuous length of tubing. However, a flush-mounted transducer provides the most satisfactory pressure information.

C. Pneumatic tubing should be eliminated whenever feasible, particularly when the acquired pressure information is to be used as input to a control system since the phase-lag portion of a pneumatic line transfer function tends to cause control instability.

D. If the pressure information is intended for diagnostic analysis and phase requirements are less demanding, data obtained from measurement systems that incorporate pneumatic lines may be completely acceptable.

E. System frequency response requirements are difficult to meet when pneumatic tube coupling is used to measure low-level-pressure environments.

F. Both steady-state and transient pneumatic line performance can be improved by using the largest practical inner line radius, the smallest obtainable transducer cavity volume, and the shortest feasible line length between the transducer and the pressure source.

G. It is essential that the transfer function of a pneumatic line be defined analytically if the intent is to: (1) correct pressure information sensed by the pressure transducer, (2) determine the overall operating characteristic of a measurement or control system utilizing pneumatic lines, (3) predict the transfer characteristics of various line configurations, or (4) determine the suitability of a given line configuration under varying temperature and pressure conditions.

H. The propagation of a unidirectional wave through circular pneumatic lines may be described by traveling wave equations. In particular, pressure perturbations produced by the passage of a unidirectional, sinusoidal, pressure wave at a distance (x) beyond the line entrance may be described by

$$P_{x,t} = (P_o e^{-yx}) \cos \omega t$$

while associated gas motions are described by

$$U_{x,t} = \left(\frac{P_o}{Z_o} e^{-yx} \right) \cos \omega t$$

and the wave propagation constants are

$$y = \frac{\omega}{c} \left\{ \frac{1 + \frac{2(\gamma - 1) J_1(E)}{E J_o(E)}}{\frac{2 J_1(W)}{W J_o(W)} - 1} \right\}^{1/2}$$

$$Z_o = \frac{-j \rho c}{\left\{ \left(\frac{2 J_1(W)}{W J_o(W)} - 1 \right) \cdot \left(1 + \frac{2 (\gamma - 1) J_1(E)}{E J_o(E)} \right) \right\}^{1/2}}$$

$$E = j^{3/2} R (\omega \rho c_p / k)^{1/2}$$

$$W = j^{3/2} R (\omega \rho / \mu)^{1/2}$$

I. The transfer-function for a pneumatic line whose open end is exposed to a sinusoidal pressure source and whose closed end is terminated with a line mounted pressure transducer is

$$\frac{P_{L,t}}{P_{o,t}} = \frac{1}{\cosh(yL) + Q \sinh(yL)}$$

where (Q) is defined by

$$Q = \frac{\omega \gamma V}{c m A} \left\{ \left(1 + \frac{2 (\gamma - 1) J_1(E)}{E J_o(E)} \right) \cdot \left(\frac{2 J_1(W)}{W J_o(W)} - 1 \right) \right\}^{-1/2}$$

J. Transfer function estimations for a line terminated with a pressure transducer can be obtained, at least for the first resonant mode, by employing the second-order approximation

$$\frac{P_{L,t}}{P_{o,t}} = \frac{1}{1 + j (2 \omega \delta / \omega_{pr}) + (\omega / \omega_{pr})^2}$$

where for short line lengths ($L < \rho \text{ cA}/52\mu$),

$$\omega_{pr} = \frac{c}{L \sqrt{\left(\frac{2}{\pi}\right)^2 + \frac{V}{AL} + \left(\frac{16 \gamma \mu L}{\pi \rho \text{ cA}}\right)^2}}$$

$$\delta = \frac{\frac{2(\gamma - 1 + \sigma)}{\sigma \pi} + \frac{2V}{LA}}{\sqrt{8} B \left[\left(\frac{2}{\pi}\right)^2 + \frac{V}{LA} \right]}$$

$$B = \left\{ \frac{c \rho R^2}{\mu L \sqrt{\left(\frac{2}{\pi}\right)^2 + \frac{V}{LA}}} \right\}^{1/2}$$

and for long line lengths ($L > \rho \text{ cA}/52\mu$),

$$\omega_{pr} = \frac{c}{L \sqrt{\left(\frac{2}{\pi}\right)^2 + \frac{V}{AL} + \left(\frac{16 \gamma \mu L}{\pi \rho \text{ cA}}\right)^2}}$$

$$\delta = \frac{\sqrt{8} \mu \gamma L}{c \rho R^2} \sqrt{\left(\frac{2}{\pi}\right)^2 + \frac{V}{AL}}$$

K. The transient response of a line terminated with a pressure transducer can be estimated by a Laplace transformation of a first-order transfer-function approximation as shown below

$$\left(\frac{P_{L,t}}{P_{o,t}} \right) (s) = \frac{e^{-st'}}{1 + \tau s}$$

where

$$t' = \left[\frac{2 \gamma L^2}{(1 + \gamma) c^2} \right]^{1/2}$$

$$\tau = \frac{8 \mu \gamma L^2}{\rho c^2 R^2} \left[\frac{1}{2} + \frac{V}{LA} \right]$$

L. The fundamental resonant frequency of a short line terminated with a pressure transducer can be approximated by

$$\omega_r = \frac{c}{L \sqrt{\left(\frac{2}{\pi}\right)^2 + \frac{V}{AL}}} ; \quad L < \frac{c \rho A}{1800 \mu}$$

M. When considering a line terminated with a pressure transducer, the frequency at which pressure fluctuations within the pressure transducer sensing port cavity phase lag those at the line entrance by ninety degrees may be approximated by

$$\omega_{pr} = \frac{c}{L \sqrt{\left(\frac{2}{\pi}\right)^2 + \frac{V}{AL} + \left(\frac{16 \gamma \mu L}{\pi \rho c A}\right)^2}}$$

N. The relative amount of viscous damping may be expressed as a function of line length as shown below

$$0 < L < \frac{\rho c A}{1800 \mu} ; \text{Essentially no viscous damping at resonance}$$

$$\frac{\rho c A}{1800 \mu} < L < \frac{\rho c A}{32 \mu} ; \text{Low viscous damping at resonance}$$

$$\frac{\rho}{32} \frac{cA}{\mu} < L < \frac{\rho}{52} \frac{cA}{\mu} ; \text{ Moderate viscous damping at resonance}$$

$$\frac{\rho}{52} \frac{cA}{\mu} < L < \frac{\rho}{8} \frac{cA}{\mu \gamma} ; \text{ High viscous damping at resonance}$$

$$\frac{\rho}{8} \frac{cA}{\mu \gamma} < L < \infty ; \text{ Excessively high viscous damping with complete suppression of acoustic line resonance.}$$

0. The operational characteristics of a inhomogeneous line may be described with homogeneous line equations. This is accomplished by dividing the inhomogeneous line into discrete, volume terminated, homogeneous line segments where the transfer function of each segment is described by

$$\frac{P_n}{P_{n-1}} = \left\{ \left[\cosh (yL) + Q \sinh (yL) \right]_n + \dots \right. \\ \left. \dots \cdot \frac{(yA)_{n+1}}{(yA)_n} \cdot \left[\frac{J_0(W)}{J_2(W)} \right]_n \cdot \left[\frac{J_2(W)}{J_0(W)} \right]_{n+1} \cdot \left[\frac{\sinh (yL)_n}{\sinh (yL)_{n+1}} \right] \cdot \left[\cosh (yL)_{n+1} - \frac{P_{(n+1)}}{P_n} \right] \right\}^{-1}$$

The pressure (P_n) is the sinusoidal pressure amplitude existing within the terminal volume of section (n), while (P_{n-1}) is the sinusoidal pressure amplitude within the cavity of the preceding section (n-1) that is presented to the entrance of section (n). By employing a general recursion equation, the pressures within any two terminal volumes may be related. As an example:

$$\frac{P_{(n+2)}}{P_{(n-1)}} = \frac{P_{(n)}}{P_{(n-1)}} \cdot \frac{P_{(n+1)}}{P_{(n)}} \cdot \frac{P_{(n+2)}}{P_{(n+1)}}$$

P. When a infinite-line configuration is composed of two homogeneous line sections, the infinite line transfer function may be described by

$$\frac{P_{d,t}}{P_{o,t}} = \left\{ \left[\cosh (y_1 d) + Q_1 \sinh (y_1 d) \right] + \dots \right. \\ \left. \dots \cdot \left[\frac{(yA)_2}{(yA)_1} \right] \cdot \left[\frac{J_o(W)}{J_2(W)} \right]_1 \cdot \left[\frac{J_2(W)}{J_o(W)} \right]_2 \cdot \left[\sinh (y_1 d) \cdot \tanh y_2 (L-d) \right] \right\}^{-1}$$

where subscript (1) refers to the line section extending from the line entrance to the transducer tap station, subscript (2) refers to the line section extending from the transducer tap station to the line termination, $P_{o,t}$ refers to pressure perturbations at the line entrance, and $P_{d,t}$ refers to pressure perturbations sensed a line mounted pressure transducer.

$$\text{PNEUMATIC LINE TRANSFER FUNCTION} = \frac{\text{PRESSURE PERTURBATION SENSED BY THE PRESSURE TRANSDUCER}}{\text{PRESSURE PERTURBATION AT THE LINE ENTRANCE}} = \frac{P_x \angle B}{P_o \angle A} = G \angle \theta$$

WHERE: $G = \text{AMPLITUDE RATIO: } (P_x/P_o)$

$\angle \theta = \text{PHASE LAG: } (\angle B - \angle A)$

Figure 3 - Pneumatic Line Transfer Function

IV. TECHNICAL DISCUSSION

The use of pneumatic tubing is generally indicated by a requirement to measure pressures that originate within environmental regions that exceed the design tolerances of available pressure transducers. Under these conditions, the pressure transducers are positioned where the environment is less severe and are coupled to the pressure source with a pneumatic line.

The signal flow diagram, Figure 3, shows that pressure information is affected by the instrument line and the pressure transducer before reaching the various system inputs. Therefore, it is essential that the energy-transfer characteristics of the line and transducer be defined to ensure the adequacy of pressure information delivered to data acquisition or control systems. The following discussions will be limited to defining the transfer characteristics of pneumatic instrumentation lines.

The parameter used to define signal alteration by any system element is the element transfer function. The pneumatic line transfer function relates the amplitude and phase of pressures existing at the line entrance to those sensed by the pressure transducer. Basically,

$$\text{Pneumatic Line Transfer Function} = \frac{\left(\begin{array}{c} \text{Pressure Perturbation Sensed} \\ \text{By the Pressure Transducer} \end{array} \right)}{\left(\begin{array}{c} \text{Pressure Perturbations} \\ \text{at the Line Entrance} \end{array} \right)} = \underline{G/\theta}$$

where G is the amplitude and θ is the phase difference between the actual and measured pressure environments. If the line transfer function is known, pressures sensed by the pressure transducer can be related to pressures existing at the line entrance by:

$$(\text{Source Pressure}) = \frac{1}{\underline{G/\theta}} (\text{Measured Pressure})$$

Thus, pressure information, as obtained from a line-mounted pressure transducer, can be modified to nullify pneumatic-tube-induced signal distortions and to reconstruct the original source pressure perturbations. In any event, it is essential that the line transfer function be defined analytically if the intent is to: (1) correct pressure information sensed by the pressure transducer; (2) determine the overall operating characteristic of a measurement or control system utilizing pneumatic lines; (3) predict the transfer characteristics of various line configurations; or (4) determine the suitability of a given line configuration when it is used under varying temperature and pressure conditions.

Pneumatic line transfer functions are complex; that is, pressure waves anywhere within the line differ in magnitude and phase from those entering the line. Analytical equations have been developed that describe the complex pneumatic line transfer function and have been programmed for computer solution^{1,2,3}.

The analytical expressions used in this report to describe oscillatory pressures in instrument lines are based on analytical investigations conducted by Iberall¹. Iberall developed a rigorous description of the input-output relation for a line terminated in a volume and subjected to sinusoidal pressure oscillations at the input. Bergh and Tijdeman² expanded on the work done by Iberall and derived a general recursion formula that describes the dynamic response of a series connection of N volume terminated line sections. Thus, the Bergh-Tijdeman equations can be used to describe pressure distributions

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1. Iberall, Arthus S., Attenuation of Oscillatory Pressures in Instrument Lines, NBS Research Paper RP 2115, Vol. 45, July 1950.
 2. Bergh, H., and Tijdeman, H., Theoretical and Experimental Results for the Dynamic Response of Pressure Measuring Systems, National Aero and Astronautical Research Institute, Technical Report NLR-TR F.238, Amsterdam, Jan. 1965.
 3. Watts, Geoffrey P., An Experimental Verification of a Computer Program for the Calculation of Oscillatory Pressure Attenuation in Pneumatic Transmission Lines, Los Alamos Scientific Laboratory Report LA-3199-MS, Feb. 1965.

within a pneumatic line system made up of N sections, each of which possesses unique dimensional and thermal properties.

Both the Iberall and Bergh-Tijdeman equations were developed on the premise that sinusoidal disturbances within the line are very small (laminar flow conditions prevail) and that the internal radius of the line is small compared to its length. In addition, the equations are limited to the motion of fluids in lines with circular cross-sections and to flow realms where fundamental flow equations can be applied; i.e., the equation of continuity, the equation of state, and an energy equation that gives the balance between thermal and kinetic energies, and the Navier-Stokes equations.

The derivation of the Iberall and Bergh-Tijdeman equations is of sufficient complexity that the overall objective may become obscured by the mathematical development of each step. To clarify the exact analytical solutions, the development of wave equations in this report will be based on simple, inexact, yet more descriptive expressions. The simplified solutions will then be related to the exact solutions, the significance of which will be discussed. Finally, approximation equations will be developed. This discussion emphasizes the meaning of the exact solutions rather than their development and is intended to provide a general understanding of pneumatic instrumentation line operation. However, it is recommended that the original documents^{1,2} be studied so that the inherent limitations of the exact solutions will be understood. That is, certain boundary conditions were imposed during the derivation of the exact solutions; and, if these boundary conditions are not satisfied by a system under consideration, the exact equations could yield invalid results.

Development of the simplified solutions will proceed in two steps. The first will discuss the phenomena of sound transmission within pneumatic

1. *ibid.*

2. *ibid.*

lines and the traveling wave concept. The second step will derive a simplified description of pressure distributions within pneumatic lines and then relate the simplified solutions to the more exact solutions. The significance of the exact solutions will then be discussed together with a description of the experimental work undertaken to confirm their validity.

Although there are mathematical techniques for describing the propagation of pressure perturbations through a fluid medium, the initial discussion will provide a qualitative description of the propagation mechanism. Further, this discussion will assume that the pressure environment presented to the line entrance consists of two pressure components, a static component (P_s) and a single oscillatory component ($P_o \cos \omega t$). Therefore, the source pressure may be described by

$$\text{Source Pressure} = P_s + P_o \cos (\omega t)$$

where (P_o) is the peak amplitude of the oscillatory component at the line entrance. The propagation and modification of this oscillatory component within the line is as follows.

A flat piston that is vibrating sinusoidally at the entrance of a gas-filled pneumatic line of infinite length (Figure 4) will compress, accelerate, and impart momentum to adjacent gas particles as it moves forward. As these particles move away from the piston they will collide with, compact, and tend to transfer their forward momentum to adjacent particles. These particles, in turn, will collide with other particles and so continue the process. In this manner, a compression wave will be propagated down the line through the gas medium away from the line entrance. When the piston retracts, a rarefaction occurs that causes particles to accelerate toward the piston, and the above process is repeated in the reverse direction, and so on, through successive cycles of the piston.

Figure 4 - Generating Traveling Waves with a Vibrating Piston

Thus, when pressure perturbations occur within a gas medium, the disturbance will not be contained. Pressure perturbations in one region will induce pressure variations in adjacent regions, which, in turn, will affect even more remote regions. Therefore, acoustic energy in the form of traveling pressure waves will propagate through a pneumatic line whenever an active source of acoustic energy is present within the line or at the line entrance.

The parameter used to describe the relationship between pressure and particle velocity is acoustic impedance (Z), which, for a given boundary, is defined as the complex ratio of effective acoustic pressure averaged over the boundary to the effective particle velocity through it. The boundary could be a hypothetical layer or plane in an acoustic medium or the moving surface of a mechanical structure. The units of acoustic impedance are dyne-sec/cm³.

The parameter used to describe the propagation velocity of a pressure perturbation through a gaseous medium is the speed of sound (c).

If it is assumed that an ideal gas is being used, the propagation velocity of sound through pneumatic instrumentation lines can be approximated by:

$$c_m = \sqrt{mP_s/\rho}$$

where (c_m) is the polytropic velocity of sound, (P_s) is the average absolute gas pressure, and (ρ) is the average gas density. The parameter (m) is the polytropic exponent and is defined by Equation 103 of Reference 1 as:

$$m = \frac{\gamma}{\left[1 + \frac{2(\gamma-1) J_1(E)}{E J_0(E)} \right]} \quad (1)$$

where:

$$E = \{-j R^2 \omega \rho / \mu\}^{1/2} \cdot \{c_p \mu / k\}^{1/2} = W \sigma$$

W = Shear Number

$$\sigma = (\text{Prandtl Number})^{1/2} = \{c_p \mu / k\}^{1/2}$$

j = $\sqrt{-1}$

R = Inner line radius, cm

ω = Oscillatory frequency, radians/sec

ρ = Mean gas density, gm/cm³

c_p = Mean gas specific heat at constant pressure, cal/gm - °K

γ = Specific heat ratio

k = Mean gas thermal conductivity, Cal/sec-cm-°K

The significance of W, the shear number, will be explained later. It will suffice here to note that (W) is proportional to the square root of the oscillatory frequency (ω).

The limiting oscillatory flow processes within any gas system are the reversible isothermal and adiabatic processes. A reversible isothermal process is one that occurs at a constant temperature where $PV = \text{constant}$. A reversible adiabatic process (isentropic process) is one that occurs without heat absorption or liberation where $PV^\gamma = \text{constant}$. More often the process is polytropic, in that it lies between the adiabatic and isothermal processes. A polytropic process is one in which heat enters and exits the system but not at a sufficient rate to maintain the system at a constant temperature. A polytropic process is one in which $PV^m = \text{constant}$, where (m) is the polytropic exponent, and ($1 < m < \gamma$).

Whenever a gas region is rapidly compressed, its temperature increases. Similarly, any gas region that undergoes a rapid pressure decline (rapid expansion) will be subject to a temperature drop. Therefore, gas temperatures at any point within an alternating sound field will rise and fall in relation to the mean ambient temperature. This variation occurs at the same frequency as the sound wave and is in phase with sound pressure.

In general, when a sound wave traverses a gaseous medium, the medium is alternately partitioned into warm zones of compression and cool zones of rarefaction. If there is sufficient time during each half cycle for a significant amount of heat to be transferred between adjacent warm and cool zones, thermal stabilization will be achieved, and the temperature of all gas zones will remain essentially at the ambient level. Therefore, the transmission of low frequency sound waves will occur isothermally. If there is insufficient time for a significant amount of heat to be exchanged between adjacent zones during each half cycle, the sound-wave-induced temperature gradients that exist between zones of compression and rarefaction will be unaltered. Therefore, high-frequency-sound oscillations will occur adiabatically.

The sound transport process depends on the oscillatory frequency. Since the shear number (W) is proportional to the square root of frequency, a small shear number ($W < 0.3$) will indicate the existence of an isothermal process and a large shear number ($W > 30$) will indicate the existence of an adiabatic process. For moderate shear numbers ($0.3 < W < 30$), the process is polytropic. The description of sound velocity within pneumatic instrumentation lines is summarized as follows:

Isothermal Propagation Velocity

When oscillatory frequencies are very small ($W < 0.3$), Equation 1 reduces to unity ($m = 1$); the alternating compression and rarefaction process occurs isothermally; and the propagation velocity of sound is approximated by:

$$c_n = \sqrt{\frac{P_s}{\rho}}$$

where c_n is commonly referred to as the Newtonian propagation velocity for elastic waves.

Polytropic Propagation Velocity

When oscillatory frequencies are such that ($0.3 < W < 30$), the alternating compression and rarefaction process occurs polytropically, and the propagation velocity of sound is approximated by:

$$c_m = \sqrt{\frac{mP_s}{\rho}}$$

where m is described by Equation 1.

Adiabatic Propagation Velocity

When oscillatory frequencies are very high ($W > 30$), Equation 1 reduces to the specific heat ratio ($m = \gamma$), the alternating compression and rarefaction process occurs adiabatically, and the propagation velocity of sound is approximated by:

$$c = \sqrt{\frac{\gamma P_s}{\rho}} = \sqrt{\gamma R_c T/M}$$

where: R_c = Universal gas constant = 8.3149×10^7 ergs/mole - °K
 M = Gas molecular weight, gm/mole
 T = Absolute mean gas temperature, °K

The most common transport process is adiabatic, and tabular listings for propagation values are nearly always for adiabatic sound transmission through unbounded gas media. Therefore, when considering sound transmission through instrumentation lines, an adiabatic transport process may be assumed and the propagation velocity designated by (c). When a polytropic flow process is encountered, some form of the ratio (mc^2/γ) will appear where (m) is described by Equation 1. For those limited cases where line equations are used

to describe isothermal transport processes ($W < 0.3$), the ratio c^2/γ , or its algebraic equivalent, will appear. Thus, subsequent equations will automatically account for a transition from adiabatic to isothermal flow and will allow the use of published adiabatic sonic velocity values for all flow realms. Equations 20, 23, and 29 are examples of expressions that make use of an algebraic form of the ratio c^2/γ to describe isothermal flow velocities.

Because of the finite propagation velocity of sound, pressure perturbations anywhere within the line will time-lag pressure perturbations at the line entrance by (x/c) seconds where (x) is distance from the entrance. If the source of acoustic energy were creating sinusoidal oscillations at the line entrance, pressure perturbations at any point within the pneumatic line would phase-lag the perturbations at the line entrance by $(\omega x/c)$ radians, where (ω) is the oscillatory frequency in radians per second. Therefore, if the time history of pressure fluctuations at the line entrance ($x = 0$) were described by $(\cos \omega t)$, then the time history of pressure alterations at a point x -centimeters from the line entrance would be $(\cos [\omega t - (\omega x/c)])$.

Another intrinsic property of traveling waves is their attenuation. As will be proven later, the pressure amplitude of a progressive sound wave tends to diminish exponentially with distance traveled. Thus, if the peak magnitude of a traveling pressure wave were (P_o) at the line entrance, then the peak magnitude of the traveling wave, after traveling a distance (x) beyond the line entrance, would be (P_x) where:

$$P_x = P_o e^{-ax}$$

The parameter (a) is defined as the attenuation constant. Sound attenuation results from (1) the energy expended in overcoming the viscous drag existing between neighboring gas particles; (2) the reduction of temperature-dependent pressure amplitudes by radiant and conductive heat flow between compression

and rarefaction zones; and (3) the molecular absorption and dispersion of energy in polyatomic gases as a result of transitional and vibrational energy exchanges between colliding molecules.⁴

By combining the effects of phase lag and amplitude attenuation, the character of a progressive pressure wave can be expressed as a function of time (t) and distance (x) beyond the line entrance.

$$P_{x,t} = (P_o e^{-ax}) \cos (\omega t - bx) \quad (2)$$

where:

- $P_{x,t}$ = pressure and amplitude of a traveling wave as a function of time and distance traveled beyond the line entrance, dynes/cm²
- P_o = peak pressure amplitude of a traveling wave at the line entrance (i.e., at $x = 0$), dynes/cm²
- a = attenuation constant, nepers/cm
- x = distance traveled beyond the line entrance, cm
- ω = oscillatory frequency, radians/sec
- t = time, sec
- b = (ω/c) = phase constant, radians/cm

This is a traveling wave equation with attenuations present. To demonstrate the validity of this statement, consider the motion of an observer who is able to maintain a position within a maximum compression zone. That is, the motion of the observer relative to the traveling wave is such that $\cos (\omega t - bx)$ always equals one. Thus,

$$\omega t - bx = 0; \quad x = \omega t / b .$$

4. L. L. Beranek, Acoustics, McGraw Hill Book Co., New York, 1954, p. 309.

The velocity of the observer's motion can be obtained by differentiating the above with respect to time

$$v = dx/dt = \omega/b = c.$$

Since the observer moves with the wave, it should be obvious that Equation 2 describes a pressure wave of diminishing amplitude traveling in the positive x-direction at the speed of sound (c).

It is often more convenient to express the traveling wave equation in the form of a rotating vector. This is accomplished by using the trigonometric identity:

$$\begin{aligned} e^{jx} &= \cos x + j \sin x; & j &= \sqrt{-1} \\ \text{or} \quad (\text{Re}) e^{jx} &= \cos x \end{aligned}$$

where Re indicates the real part of the complex number. Thus, the traveling wave equation can be expressed by:

$$P_{(x,t)} = \left(P_o e^{-ax} \right) \left\{ (\text{Re}) e^{j(\omega t - bx)} \right\}$$

In subsequent expressions the real part of e^{jx} is considered to be understood; that is, $(\text{Re})e^{jx} = e^{jx} = \cos x$. With this simplification, the traveling wave equation can be concisely written as:

$$P_{(x,t)} = \left(P_o e^{-yx} \right) e^{j\omega t} \quad (3)$$

where:

- y = a + jb = traveling wave propagation constant
- a = attenuation constant, nepers/cm
- b = phase constant, radians/cm

PRESSURE PERTURBATION AT A REFERENCE PLANE (THE LINE ENTRANCE):

$$P_{0,t} = P_0 \cos \omega t = P_0 e^{j\omega t}$$

PRESSURE PERTURBATION AT A DISTANCE (X) BEYOND THE REFERENCE PLANE:

$$P_{x,t} = P_0 e^{-ax} \cos (\omega t - bx) = (P_0 e^{-yx}) e^{j\omega t}$$

Figure 5 - Wave Propagation

The character of unidirectional wave propagation through a gas medium is summarized in Figure 5. In general, Equation 3 can be used to describe the propagation of a traveling wave through a homogeneous pneumatic line whose diameter (D) is both constant and small when compared to the wave length (λ) of pressure waves within the line ($\lambda > 6 D$). In particular, pressure perturbations at a distance (x) from the line entrance are related to those at the line entrance by Equations 2 and 3 when the propagation of a unidirectional wave down a cylindrical line of infinite length and constant cross section is considered.

Consideration may now be given to the reaction of a traveling wave as it encounters an acoustic discontinuity since infinitely long homogeneous lines are not physically possible. Acoustic discontinuities are produced by any abrupt variation in gas properties or line structure. A discontinuity in a pneumatic instrumentation line can result from an abrupt variation in the line temperature, a finite diameter variation, a hole tapped in the line wall to allow pressure transducer coupling, an interface between different gases, or the point of line termination.

Discontinuity effects can be described in terms of the kinetic and potential energy density components associated with a traveling wave. The kinetic energy component is initiated by the oscillatory motion of gas particles, and the potential energy component originates from the existence of compressive and rarefaction zones. Since traveling wave-induced gas motions and pressures are related by acoustic impedance, energy redistribution will occur whenever a traveling wave leaves one acoustic impedance region and enters another. That is, there must be a readjustment of energy relations as a wave impinges on a boundary plane separating different acoustic realms. Since energy cannot be added to the wave as it arrives at a realm boundary, the only way a new balance can be achieved is for some of the impinging energy to be expelled from the wave. This is what actually happens, and the expelled energy appears as a reflected wave. Therefore, when a wave traveling in one acoustic impedance region encounters a second acoustic impedance region, two

$$\left. \begin{aligned}
 P_{x,t} &= P_o e^{j\omega t} \left\{ \frac{(Z_o + Z_L) e^{y(L-X)} - (Z_o - Z_L) e^{-y(L-X)}}{(Z_o + Z_L) e^{yL} - (Z_o - Z_L) e^{-yL}} \right\} \\
 U_{x,t} &= \frac{P_o}{Z_o} e^{j\omega t} \left\{ \frac{(Z_o + Z_L) e^{y(L-X)} + (Z_o - Z_L) e^{-y(L-X)}}{(Z_o + Z_L) e^{yL} - (Z_o - Z_L) e^{-yL}} \right\}
 \end{aligned} \right\} 0 < X < L$$

WHERE: $P_{x,t}$ = LINE PRESSURE, dynes/cm²

$U_{x,t}$ = VELOCITY OF GAS MOTION, cm/sec

Figure 6 - Wave Reflection

new waves originate at the region interface boundary as shown in Figure 6. One wave continues into the second region while the other reflects back into the first region. The energy of the wave incident on the boundary is shared between the transmitted and reflected waves. The equations presented in Figure 6 will be derived in the following discussion.

Under certain conditions, the phase relationship between the incident and reflected waves is such that the reflected wave reinforces the incident wave. When reinforcement occurs, oscillatory pressure amplitudes within the line will exceed the peak amplitude of waves entering the line from the pressure source. Such a condition is referred to as acoustic line resonance and can seriously degrade the pressure data acquired from systems utilizing pneumatic instrumentation lines. The intensity of line resonance is governed by the magnitude of the reflected wave, which, in turn, is dictated by the character of the acoustic impedance mismatch boundary. To describe the resonant phenomena, it will be necessary to describe the process of wave reinforcement and the manner in which acoustic impedance influences wave motion within pneumatic lines through the development of analytical expressions. The development of analytical expressions can be simplified by limiting the discussion to the unidirectional wave motions normally encountered in pneumatic instrumentation lines.

The analytical development can be further simplified by assuming small perturbation amplitudes. Under this restriction, acceleration forces will dominate viscous-drag effects, and pressure gradients will be proportional to gas particle momentum variations. That is:⁵

$$-\frac{d(P_t)}{dx} = H_1 \frac{d(U_t)}{dt}$$

where: H_1 = constant of proportionality
 U_t = velocity of oscillatory gas motion = $U \cos \omega t = Ue^{j\omega t}$
 P_t = oscillatory gas pressure = $P \cos \omega t = Pe^{j\omega t}$

5. L. L. Beranek, Acoustics, McGraw-Hill Co., Inc., New York, 1954, p. 18, Eq. 2.4a.

By confining the discussion to sinusoidal variations, vector notation can be used and the imaginary quantity ($j\omega$) substituted for (d/dt) to yield:

$$\begin{aligned} - \frac{d(P_t)}{dx} &= j\omega H_1 U_t \\ - \frac{d(P_t)}{dx} &= Z U_t \end{aligned} \quad (4)$$

where (Z) is the acoustic impedance per unit line length.

If the peak magnitude of oscillatory flow velocity is observed at each end of an incremental line segment and gas motion is produced by an oscillatory pressure perturbation whose peak magnitude varies with time, the two observed peak velocity values will differ because of compressional effects. Stated analytically, the change in peak flow amplitude with distance ($-d(U_t)/dx$) is proportional to the rate of change in the peak oscillatory pressure amplitude.⁵ That is:

$$- \frac{d(U_t)}{dx} = H_2 \frac{d(P_t)}{dt}$$

By assuming sinusoidal variations and employing vector notation, the preceding expression becomes:

$$\begin{aligned} - \frac{d(U_t)}{dx} &= j\omega H_2 (P_t) \\ - \frac{d(U_t)}{dx} &= Y P_t \end{aligned} \quad (5)$$

where Y is defined as the acoustic admittance per unit line length.

6. L. L. Beranek, op. cit., p. 21, Eq. 2.14a.

Differentiating Equation 4 with respect to x yields:

$$-\frac{d^2 (P_t)}{dx^2} = Z \frac{d (U_t)}{dx}$$

However, the derivative $(d (U_t)/dx)$ is known from Equation 5 and substitution gives

$$\frac{d^2 (P_t)}{dx^2} - ZY (P_t) = 0$$

which is a second-order differential equation with constant coefficients whose general solution is

$$P_{x,t} = K_1 e^{-\sqrt{ZY} x} + K_2 e^{\sqrt{ZY} x} \quad (6)$$

where K_1 and K_2 are constants of integration. Continuing, Equations 4 and 5 can be transformed to yield:

$$U_{x,t} = N_1 e^{-\sqrt{ZY} x} + N_2 e^{\sqrt{ZY} x} \quad (7)$$

A necessary relation exists between pressure and flow velocity that is not included in Equations 6 and 7. This relation was implicit in differential Equations 4 and 5, but was lost in the simultaneous solution. It may be recovered by substituting Equations 6 and 7 into Equation 4 and equating like terms to obtain:

$$N_1 = \frac{K_1}{\sqrt{Z / Y}}$$

$$N_2 = -\frac{K_2}{\sqrt{Z / Y}}$$

The quantities $\sqrt{Z/Y}$ and $\sqrt{Z/Y}$ appear so often that they have been given special names and symbols.

$$\begin{aligned} \text{Characteristic line impedance} & \text{--- } Z_o = \sqrt{Z/Y} \\ \text{Propagation constant} & \text{----- } y = \sqrt{Z/Y} \end{aligned}$$

The reason for these designations will be clarified in the following discussion. Repeating Equations 6 and 7 with the Z and Y parameters eliminated yields

$$P_{x,t} = K_1 e^{-yx} + K_2 e^{yx} \quad (8)$$

$$U_{x,t} = (K_1 / Z_o) e^{-yx} - (K_2 / Z_o) e^{yx} \quad (9)$$

Integration constants K_1 and K_2 can be defined by imposing the general boundary conditions of Figure 6. Therefore:

at $x = 0$;

$$P_{o,t} = P_o e^{j\omega t} = K_1 + K_2 \quad (10)$$

at $x = L$;

$$P_{L,t} = K_1 e^{-yL} + K_2 e^{yL}$$

$$P_{L,t} = (Z_L) (U_{L,t})$$

$$U_{L,t} = (K_1 / Z_o) e^{-yL} - (K_2 / Z_o) e^{yL}$$

Eliminating $(U_{L,t})$ and $(P_{L,t})$ and collecting terms yields:

$$(Z_o - Z_L) K_1 e^{-yL} + (Z_o + Z_L) K_2 e^{yL} = 0 \quad (11)$$

Solving Equations 10 and 11 simultaneously to obtain expressions for K_1 and K_2 and then inserting these expressions into Equations 8 and 9 yields:

$$P_{x,t} = P_o e^{j\omega t} \left\{ \frac{(Z_o + Z_L) e^{y(L-x)} - (Z_o - Z_L) e^{-y(L-x)}}{(Z_o + Z_L) e^{yL} - (Z_o - Z_L) e^{-yL}} \right\} \quad (12)$$

$$U_{x,t} = \frac{P_o}{Z_o} e^{j\omega t} \left\{ \frac{(Z_o + Z_L) e^{y(L-x)} + (Z_o - Z_L) e^{-y(L-x)}}{(Z_o + Z_L) e^{yL} - (Z_o - Z_L) e^{-yL}} \right\} \quad (13)$$

Equations 12 and 13 describe pressure and flow velocities within a pneumatic line as the sum of two traveling waves expressed in exponential form. One represents an incident wave that becomes smaller as it travels away from the line entrance. The second represents a reflected wave that becomes smaller as it travels away from the line discontinuity where part or all of the incident wave was reflected as shown in Figure 6.

The equations define pressure and flow velocity distributions at any point between the line entrance and the first flow discontinuity. In addition, the equations describe the character of line resonance (when present) in terms of acoustic impedances.

Familiarity with the general operational characteristics of pneumatic instrumentation lines can be obtained by using Equations 12 and 13 to analyze two basic pneumatic line configurations: (1) a line terminated in its characteristic impedance, and (2) a line terminated with an arbitrary acoustic impedance (a finite volume terminated line).

A. A LINE TERMINATED IN ITS CHARACTERISTIC IMPEDANCE

The behavior of a line terminated in its characteristic impedance is found by referring to Equations 12 and 13 and replacing Z_L with Z_o . Performing this operation yields:

$$P_{x,t} = (P_o e^{-yx}) e^{j\omega t} \quad (14)$$

$$U_{x,t} = \left(\frac{P_o}{Z_o} e^{-yx} \right) e^{j\omega t} \quad (15)$$

Thus, flow perturbations within a line terminated in its characteristic impedance are characterized by unidirectional waves traveling from the line entrance to the line termination where complete wave absorption occurs. Since there is an absence of wave reflection, line resonance will never appear. It should also be noted that the wave equations for a line terminated in its characteristic impedance are identical to those for a line of infinite length. This becomes apparent by noting that Equations 12 and 13 are transformed into Equations 14 and 15 as the line length (L) approaches infinity. Therefore, a line terminated in its characteristic impedance simulates a line of infinite length. This is not surprising since the simplest method of obtaining a characteristic impedance termination is to attach a second line section that is identical in wall material and cross section configuration to the original line section. Since there are no reflective waves in a line terminated in its characteristic impedance, the second line section must be long enough to provide adequate incident wave absorption. However, it need not be infinitely long. In summary, Equations 14 and 15 may be used to describe the propagation of unidirectional sound waves through pneumatic instrumentation lines, the distribution of pressure and fluid motion within a line of infinite length, and the distribution of pressure and fluid motion within a line terminated in its characteristic impedance.

An analytical expression for the propagation constant (y) has been developed by Iberall (Equation 74 of Reference 1) that accounts for compressible flow, finite signal amplitudes, polytropic propagation, fluid inertia, fluid acceleration, and heat transfer within the line. The explicit restrictions under which the Iberall equation was developed are:

1. Laminar flow conditions prevail (i.e., Reynolds numbers less than 2000).
2. An elastic gas is assumed where

$$\frac{\mu}{cD\rho} < 1$$

and the oscillatory source frequency is such that

$$\frac{\omega\mu}{2c\rho} < 1$$

where (μ) is gas viscosity, (D) is the inside line diameter, (c) is the propagation velocity, (ρ) is gas density, and (ω) is oscillatory frequency.

3. The tube is long enough to render end-effect corrections insignificant.

Under the above restrictions, Iberall's theoretical analysis yields the following analytical expression for the propagation constant:

$$y = \frac{\omega}{c} \left\{ \frac{1 + \frac{2(\gamma - 1) J_1(E)}{E J_0(E)}}{\frac{2 J_1(W)}{W J_0(W)} - 1} \right\}^{1/2} \quad (16)$$

In the above expression, (J_0) and (J_1) are zero and first-order Bessel functions. The quantities (E) and (W) are arguments of the Bessel functions and are defined by:

$$W = j^{3/2} R (\omega \rho / \mu)^{1/2} = (1 - j) R (\omega \rho / 2\mu)^{1/2} \quad (17)$$

$$E = j^{3/2} R (\omega \rho c_p / k)^{1/2} = W\sigma \quad (18)$$

where $\sigma^2 = c_p \mu / k = \text{Prandtl Number}$
 $j = \sqrt{-1}$
 $R = \text{tube radii, cm}$
 $\omega = \text{oscillatory frequency, radians/sec}$
 $\rho = \text{gas density, gm/cm}^3$
 $\mu = \text{gas viscosity, dyne - sec/cm}^2$
 $c_p = \text{specific heat at constant pressure, cal/gm-}^\circ\text{K}$
 $k = \text{thermal conductivity, cal/sec - cm-}^\circ\text{K}$

As already explained, the parameter (W) is often referred to as the shear number and its magnitude indicates whether the transport process within the line is isothermal, polytropic, or adiabatic. However, its magnitude also describes the relative amount of viscous damping encountered by a sound wave traveling through a cylindrical line. To understand this, consider the forces opposing oscillatory flow within pneumatic lines. For steady-state and low frequency oscillatory flow, flow resistance is described by

$$- \frac{\Delta P / \Delta X}{U} = 8 \left(\frac{\mu}{R^2} \right)$$

where $\Delta P / \Delta X$ is the pressure gradient and U is flow velocity. At low frequencies, the preceding demonstrates that the forces producing motion are balanced by viscous resistance (fluid friction), which is proportional to μ / R^2 . When viscous resistance predominates (high viscous damping), the flow velocity is distributed parabolically across the line.

As the oscillatory frequency increases, the energy required to accelerate gas particles becomes significant and inertial forces must be added to the visous forces. The inertial impedance to sinusoidal flow oscillations can be approximated by

$$\frac{dP/dX}{U} = -j\omega\rho$$

Since inertial impedance is proportional to mass and frequency (accelerating forces), viscous retarding forces will control fluid motions near the line wall where oscillatory motions are small, while inertial forces will control fluid motions at the center of the line where oscillatory motions are large. When the inertial impedance is large, there is a flattening of the velocity profile at the center of the line. As the driving frequency increases, the center region of essentially constant velocity widens, and the velocity of oscillatory motion decreases more abruptly to zero at the line wall. For high frequencies, the velocity profile is very nearly constant across the line.

The type of velocity profile existing across the line depends on the ratio of inertial to viscous reactance as shown below:

$$W^2 = -j R^2 (\omega\rho/\mu) = \frac{-j\omega\rho}{\mu/R^2} \approx \frac{\text{Inertial Damping}}{\text{Viscous Damping}}$$

Therefore, the shear number (W) describes the character of flow within lines of any size. In addition, the transmission of oscillatory pressure waves through pneumatic lines depends on the relation of gas properties to line size and on the driving frequency. The significance of (W) is summarized as follows:

When (W) is small ($W < 0.5$), the line is acoustically narrow. Viscous frictional forces are much larger than inertial acceleration forces,

and the fluid velocity varies parabolically from zero at the line wall to a maximum at the center of the line.

When (W) is large ($W > 10$), the line is acoustically wide. Inertial acceleration forces are much larger than frictional viscous forces everywhere except within a thin boundary layer adjacent to the line wall. The flow velocity is essentially constant across the line, and the main body of gas behaves as a cylindrical core (piston) that oscillates as a unit along the central line axis.

It is the variation in flow profile with (W) that accounts for the mathematical complexity of Equation 16. Fortunately, the propagation constant equation can be simplified for small and large (W) values. For small values of any Bessel function argument (X), the Bessel functions may be approximated by:

$$J_0(X) = 1 - \frac{X^2}{4} + \frac{X^4}{64} \text{ ----}$$

$$J_1(X) = \frac{X}{2} - \frac{X^3}{16} + \frac{X^5}{384} \text{ ----}$$

When any Bessel function argument exceeds 10, the following identity may be assumed:

$$J_0(X) \approx j J_1(X).$$

With these approximations, the propagation constant for high and low viscous damping can be expressed in the following simplified forms:

For Very High Viscous Damping; $0 < W < 0.3$

$$y = \frac{\omega}{c} \left[\frac{8\gamma}{W^2} \right]^{1/2} = (1 + j) \frac{2}{cR} \left[\frac{\gamma\omega\mu}{\rho} \right]^{1/2} \quad (19)$$

For High Viscous Damping; $0.3 < W < 0.5$

$$y = \frac{\omega}{c} \left\{ j \frac{8\gamma\mu}{\omega\rho R^2} + \frac{1}{3} [3 (\sigma^2) (\gamma-1) - 4\gamma] \right\}^{1/2}$$

For Low Viscous Damping; $10 < W < 30$

$$y = \frac{\omega}{c} \left\{ -1 + j \frac{2}{W} \left(\frac{\gamma-1}{\sigma} + 1 \right) \right\}^{1/2}$$

For Very Low Viscous Damping; $30 < W < \infty$

$$y = \left(\frac{\gamma-1}{\sigma} + 1 \right) \left\{ \frac{\omega \mu}{2\rho c^2 R^2} \right\}^{1/2} + j \frac{\omega}{c} \approx j \frac{\omega}{c} \quad (20)$$

The expression describing the propagation constant for very high viscous damping (Equation 19) includes an algebraic form of the ratio (c^2/γ) so that the effective sonic velocity is that of an isothermal flow process. This is correct, since the propagation process is isothermal when very small shear numbers are encountered ($W < 0.3$). Therefore, the general propagation constant expression (Equation 16) describes a polytropic transport process that becomes isothermal or adiabatic as very small or very large shear numbers are encountered.

An expression will now be developed for the characteristic impedance Z_o . Since the specific acoustic impedance of any surface ($Z_{x,t}$) is defined as the complex ratio of effective acoustic pressure averaged over the surface to the effective particle velocity through it, the acoustic impedance anywhere within a line terminated in its characteristic impedance is found to be:

$$Z_{x,t} = (P_{x,t}/U_{x,t}) = Z_o.$$

That is, the "pressure-fluid velocity" relationship for a unidirectional wave traveling through a pneumatic line is governed by the line characteristic impedance. It was previously shown that:

$$\begin{array}{ll} \text{Characteristic line impedance} & - - - - Z_o = \sqrt{Z/Y} \\ \text{Propagation constant} & - - - - - y = \sqrt{ZY} \end{array}$$

where (Z) and (Y) are the acoustic impedance and admittance per unit line length, respectively. Consequently, Z_o may be defined by:

$$Z_o = \sqrt{Z/Y} = \frac{Z}{y} = Z/y$$

where (y) is defined by equation (16) and (Z) is defined by Equation 75 of Reference 1 as:

$$Z = \frac{j\omega\rho}{\left[1 - \frac{2J_1(W)}{WJ_o(W)}\right]} \quad (21)$$

Therefore,

$$Z_o = \frac{-j\rho c}{\left\{\left(\frac{2J_1(W)}{WJ_o(W)} - 1\right) \cdot \left(1 + \frac{2(\gamma-1)J_1(E)}{EJ_o(E)}\right)\right\}^{1/2}} \quad (22)$$

The expression for the characteristic impedance can be simplified for very small ($W < 0.3$) and very large ($W > 30$) wave shear numbers by employing the same techniques used to simplify the expression for the propagation constant.

For Very High Viscous Damping; $0 < W < 0.3$

$$Z_o = -j \frac{\rho c}{W} \left[\frac{8}{\gamma} \right]^{1/2} = (1 - j) \frac{2c}{R} \left[\frac{\rho \mu}{\gamma \omega} \right]^{1/2} \quad (23)$$

For High Viscous Damping; $0.3 < W < 0.5$

$$Z_o = -j \frac{\rho c}{3\gamma} \left\{ j \frac{72\gamma\mu}{\omega\rho R^2} - [12\gamma + 9\sigma^2(\gamma-1)] \right\}^{1/2}$$

For Low Viscous Damping; $10 < W < 30$

$$Z_o = \rho c \left\{ 1 + j \frac{2}{W} \left(\frac{\gamma-1}{\sigma} - 1 \right) \right\}^{1/2}$$

For Very Low Viscous Damping; $30 < W < \infty$

$$Z_o = \rho c + j \left(\frac{\gamma-1}{\sigma} - 1 \right) \sqrt{\frac{\mu\rho c^2}{2\omega R^2}} \approx \rho c \quad (24)$$

The characteristic line impedance (Z_o) for very low viscous damping approaches (ρc), which approximates the characteristic impedance of an unbounded gaseous medium. For very low viscous damping, pressure and particle velocity are in phase. That is, a region of maximum pressure is also a region of maximum particle velocity. Also, the direction of propagation for a wave traveling through a line is the direction of particle velocity in a compression zone when $W > 30$. Having analytically defined the propagation constant (γ) and characteristic line impedance (Z_o), Equations 14 and 15 may be used to describe the propagation of unidirectional waves through lines that are either infinitely long or terminated in their characteristic impedance as shown in Figure 7.

WAVE EQUATIONS

$$P_{x,t} = (P_0 e^{-\gamma x}) e^{j\omega t}$$

$$U_{x,t} = \left(\frac{P_0}{Z_0} e^{-\gamma x} \right) e^{j\omega t}$$

PROPAGATION CONSTANTS

$$\gamma = \left(\frac{-\omega}{c} \right) \left\{ \frac{1 + \frac{2(\gamma-1) J_1(E)}{E J_0(E)}}{\frac{2J_1(W)}{W J_0(W)} - 1} \right\}^{1/2}$$

$$W = (1 - j) R (\omega \rho / 2\mu)^{1/2}$$

$$E = W (Pr)^{1/2}$$

$$Z_0 = -j\rho c \left\{ \left(\frac{2J_1(W)}{W J_0(W)} - 1 \right) \cdot \left(1 + \frac{2(\gamma-1) J_1(E)}{E J_0(E)} \right) \right\}^{-1/2}$$

Figure 7 - Propagation of Acoustic Waves Through Pneumatic Instrumentation Lines

The parameter used to define signal alteration by any system element is the element transfer function. The transfer function for pneumatic instrumentation lines relates the amplitude and phase of pressures existing at the line entrance to those sensed by a line-mounted pressure transducer. If it were possible to attach a transducer to a line terminated with its characteristic impedance and to monitor internal line pressures without producing flow discontinuities, the transfer function (G/θ) of the resulting line-transducer system would be

$$\underline{G/\theta} = \frac{P_{d,t}}{P_{o,t}} = e^{-yd} \quad (25)$$

where (d) is the distance of the transducer tap station from the line entrance. Since (y) is equal to (a + jb), the above becomes

$$\underline{G/\theta} = e^{-ad} \underline{/ -bd}$$

In this manner, the magnitude of an incident wave diminishes as it propagates through a line. In addition, pressure perturbations everywhere within a line will phase-lag pressure perturbations at the line entrance. These characteristics are illustrated in Figure 8 where transducer-line, coupling-induced-flow discontinuities are assumed to be insignificant. If the amplitude ratio (G) and phase lag (θ) components of the transfer function were plotted for a line that is either infinitely long or terminated in its characteristic impedance, the plots would resemble those of Figure 9.

Although the curves of Figure 9 illustrate the dependency of the transfer function upon line diameter and source frequency variations, the transfer function also depends on temperature, pressure, line dimensions, and gas properties. The general nature of variations in the transfer function, relative to variations in parameter values, is presented in Table 1.

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ACOUSTIC
TRANSFER FUNCTION = $\frac{\text{PRESSURE PERTURBATION SENSED BY A LINE-MOUNTED PRESSURE TRANSDUCER}}{\text{PRESSURE PERTURBATION AT THE LINE ENTRANCE}}$

$$G \angle \theta = \frac{P_{d,t}}{P_{o,t}} = e^{-yd} = e^{-ad} \angle -bd$$

WHERE: a = ATTENUATION CONSTANT, nepers/cm
 b = PHASE CONSTANT, radians/cm

Figure 8 - Transfer Function of a Pneumatic Instrumentation Line Terminated in its Characteristic Impedance

DIAMETER DEPENDENT CONTINUOUS LINE TRANSFER FUNCTION

$$\text{TRANSFER FUNCTION} = \frac{P_{d,t}}{P_{o,t}} = G / \theta$$

$d = 25 \text{ FT.}$ $T = 500^\circ\text{F}$
 $D = \text{VARIABLE}$ $P = 560 \text{ PSIA}$

Figure 9 - Diameter Dependent Continuous Line Transfer Function

TABLE 1
DEPENDENCY OF WAVE PROPAGATION ON VARIOUS LINE PARAMETERS

Parameter Variation at Any Oscillatory Frequency	Transfer Function Variations	
	Amplitude Ratio Will Tend To	Phase Will Tend To be
An increase in absolute gas pressure (P)	Increase	Less lagging
An increase in absolute gas temperature (T)	Decrease	Less lagging
An increase in the inner line radius (R)	Increase	Less lagging
An increase in the "transducer-source" separation (x)	Decrease	More lagging

Both amplitude ratio and phase-lag characteristics are improved by increasing the inner line radius. Assuming the transducer is placed as close to the source as environmental conditions will permit, the line radius will be the only controllable variable, since all other parameters (pressure, temperature, gas properties, etc.) are usually fixed by the test environment.

Unlimited improvement of the line transfer function cannot be achieved by allowing the line radius to approach infinity. As the inner line diameter is increased and becomes very large compared with the wave length of wave motion within the line ($D \gg \lambda$), the propagation constant will asymptotically approach its unbounded value, which may be expressed analytically by:⁷

$$\text{Propagation constant for wave motion through an unbounded gas medium} = \frac{\omega^2}{2\rho c^3} \left[\frac{4\mu}{3} + \frac{(\gamma-1)k}{c_p} \right] + j \frac{\omega}{c} \approx j \frac{\omega}{c}$$

A limit exists beyond which the transfer function cannot be improved (reduced) by enlarging the inner line diameter. This limit is set by the propagation constant of an unbounded gas medium.

7. Herzfeld and Litovitz, Absorption and Dispersion of Ultrasonic Waves, Academic Press Inc., London, 1959, p. 44.

B. A LINE TERMINATED WITH A CAVITY

The cavity terminated line configuration (Figure 10) has special significance since it represents the pressure measurement system most commonly encountered when pneumatic instrumentation lines are used. The enclosed terminal volume represents the pressure port cavity found in most pressure transducers, and the line represents the pressure transmission tube used to couple the transducer to the pressure source.

There are three resonant frequencies associated with this measurement system: acoustic line resonance, acoustic transducer cavity resonance, and structural transducer resonance. Line resonance results from incident wave reflection at the line-cavity interface. Cavity resonance results from reflections within the cavity. Structural transducer resonance results from the excitation of mechanical resonances within the transducer, such as the vibrational resonance of a pressure-sensing diaphragm. Of the three possible resonant modes, line resonance is generally the lowest resonant frequency encountered. Consequently, line resonance usually limits the frequency response of measurements made with transducer-terminated pneumatic instrumentation lines. Therefore, this discussion will be limited to the description of line resonance. The discussion will be further simplified by eliminating all references to particle motion.

The distribution of pressure within pneumatic instrumentation lines terminated with a cavity of volume (V) is defined by Equation 12, which is repeated for convenience.

$$P_{x,t} = P_o e^{j\omega t} \left\{ \frac{(Z_o + Z_L) e^{y(L-x)} - (Z_o - Z_L) e^{-y(L-x)}}{(Z_o + Z_L) e^{yL} - (Z_o - Z_L) e^{-yL}} \right\} \quad (12)$$

$$P_{o,t} = P_o e^{j\omega t}$$

$$P_{x,t} = P_o e^{j\omega t} \left\{ \frac{\cosh y(L-x) + Q \sinh y(L-x)}{\cosh(yL) + Q \sinh(yL)} \right\}$$

$$P_{L,t} = P_o e^{j\omega t} \left\{ \frac{1}{\cosh(yL) + Q \sinh(yL)} \right\}$$

$$\text{WHERE: } \begin{cases} \text{PROPAGATION CONSTANT; } - - y = \frac{\omega}{c} \left\{ \frac{1 + \frac{2(\gamma-1)J_1(E)}{E J_0(E)}}{\frac{2J_1(W)}{W J_0(W)} - 1} \right\}^{1/2} \\ \text{REFLECTION COEFFICIENT; } - - Q = \frac{\omega \gamma}{\text{cm A}} \frac{V}{\left[\frac{1 + \frac{2(\gamma-1)J_1(E)}{E J_0(E)}}{\frac{2J_1(W)}{W J_0(W)} - 1} \right]} \end{cases}^{1/2}$$

$$\text{AND: } \begin{cases} W = R \sqrt{-j\omega\mu/\rho} \\ E = R \sqrt{-j\omega\mu c_p/k} \end{cases}$$

Figure 10 - Resonance Within Lines Terminated with a Finite Cavity Volume

In the above expression (Z_o) is the characteristic line impedance and (Z_L) is the acoustic impedance presented to the incident wave at the line-cavity interface. For convenience, hyperbolic functions are used to obtain a rearrangement of Equation 12 as shown below

$$P_{x,t} = P_o e^{j\omega t} \left[\frac{\cosh y(L-x) + Q \sinh y(L-x)}{\cosh (yL) + Q \sinh (yL)} \right] \quad (26)$$

where (y) is defined by Equation 16 and (Q) equals (Z_o/Z_L). The line transfer function relating pressures in the cavity to those at the line entrance is found from Equation 26 to be

$$\frac{P_{L,t}}{P_{o,t}} = \frac{1}{\cosh (yL) + Q \sinh (yL)} \quad (27)$$

An analytical expression for Q can be obtained by combining and manipulating Equations 74, 81, and 99 of Reference 1 to yield

$$Q = \frac{\omega \gamma V}{c m A} \left[\frac{1}{\left[1 + \frac{2(\gamma-1) J_1(E)}{E J_o(E)} \right] \cdot \left[\frac{2 J_1(W)}{W J_o(W)} - 1 \right]} \right]^{1/2} \quad (28)$$

where (V) is the cavity volume, (A) is the cross sectional line area, and (m) is the exponent of polytropic expansion in the terminal cavity ($1 < m < \gamma$).

As with any theoretical expression, Equations 26, 27, and 28 are applicable only under certain limiting conditions. Equations 26 and 27 are only valid when the excitation frequency is less than the fundamental resonant frequency of both the cavity and the transducer. In addition, it is assumed that line entrance effects were small enough to make the amplitude of a distortion-induced harmonic component negligible. Equation 28 was derived under the assumption that pressure-induced variations in the pressure transducer volume were very small and that equal mean pressures and temperatures

existed everywhere within the line-cavity system. The derivation of Equation 28 also assumed all of the boundary conditions imposed on the development of Equation 16, which defines the propagation constant (y).

Equation 28 accounts for the fact that the energy transport process within the cavity is polytropic by including the polytropic expansion exponent (m) for the cavity. The polytropic process within the line has been accounted for in Equations 16, 22, and 28.

Since the cavity encountered in most pressure transducers has the form of a short right cylinder, Equation 1 can often be used to estimate the value of the cavity polytropic coefficient (m) by letting R represent the inner cavity radius.

The parameter (Q) provides a measure of line resonance. When (Q) equals unity the line is terminated in its characteristic impedance, wave reflection at the line-cavity interface will not occur, and pressures within the line can be described by Equation 14 when the source frequency (ω) is less than the fundamental cavity resonant frequency. When the value of (Q) differs from unity, line resonance appears. In general, the value of (Q) can range from zero (zero cavity volume) to infinity (infinite cavity volume).

The expression for (Q) can be simplified to facilitate manual computations when the flow realm is characterized by very low or very high viscous damping. In simplifying (Q), the same approximations and assumptions are used as those employed to simplify the expression for (y).

1. For Very High Viscous Damping; $0 < W < 0.3$

$$Q = (1 + j) \frac{2V}{mcAR} \left[\frac{\gamma \omega \mu}{\rho} \right]^{1/2} = \frac{V}{mA} (y) \approx \frac{yV}{A} \quad (29)$$

where (y) is defined by Equation 19 and (m) is assumed to be very close to unity.

2. For High Viscous Damping; $0.3 < W < 0.5$

$$Q = \frac{\omega V}{3c\mu A} \left(j \frac{72\mu\gamma}{\omega\rho R^2} - [12\gamma + 9\sigma^2 (\gamma-1)] \right)^{1/2}$$

3. For Low Viscous Damping; $10 < W < 30$

$$Q = j \frac{\omega\gamma V}{c\mu A} \left[1 + j \frac{2}{W} \left(\frac{\gamma-1}{\sigma} - 1 \right) \right]^{1/2}$$

4. For Very Low Viscous Damping; $30 < W < \infty$

$$Q = j \frac{\omega\gamma V}{c\mu A} \left[1 + j \left(\frac{\gamma-1}{\sigma} - 1 \right) \sqrt{\frac{\mu}{2\omega\rho R^2}} \right] \approx j \frac{\omega\gamma V}{c\mu A} \approx j \frac{\omega V}{cA} \quad (30)$$

where (m) is assumed to be very close to (γ).

Although the equations of Figure 10 completely define the response of a pneumatic line terminated with a cavity, their complexity makes it difficult to visualize the dominate resonant phenomenon. Their complexity also discourages their use unless computers are available. It is possible, however, to simplify the line equations to obtain expressions that will yield valid results when applied under certain restricted conditions. It must be emphasized that simplified versions of Equations 16, 26, 27, and 28 should not be applied indiscriminately, and their use is discouraged by anyone not fully aware of their inherent limitations.

The first set of approximation equations will be for the limiting case of a line that is long enough to reduce the presence of the terminal volume to insignificance, yet short enough to insure the existence of a large shear number (the lowest resonant line frequency is high enough to insure that $W > 30$). If the presence of a terminal cavity is to have little or no effect

upon the line's response, a lower limit must be placed on the line length. In general, the presence of a terminal cavity may be ignored when

$$L > \frac{60V}{A} .$$

The requirement for large shear numbers places an upper limit on the line length which is:

$$L < \frac{c\rho A}{1800\mu}$$

Therefore, the approximation equations being developed for this unique case should be applied only when the line length satisfies the boundary conditions of Equation 31.

$$\frac{60V}{A} < L < \frac{c\rho A}{1800\mu} \quad (31)$$

Assuming that the cavity terminated pneumatic line system satisfies the boundary conditions of Equation 31, a zero terminal volume can be assumed and Equation 26 simplified to yield

$$P_{x,t} = \left[\frac{1}{\cosh(yL)} \right] \cdot \left[\cosh y(L-x) \right] \cdot \left[P_o e^{j\omega t} \right] \quad (32)$$

Equation 32 consists of three primary components. The first bracketed term is defined as the amplitude gain component (G). The second bracketed term describes pressure distributions along the line. The third term describes the amplitude and time history of pressure fluctuations presented to the line.

Since large (W) values are being assumed, Equation 20 can be used to describe the propagation constant. By taking the limiting case ($y = jb$), the amplitude gain component reduces to:

$$G = \frac{1}{\cos (bL)}$$

By inspection it is found that pressure oscillations everywhere within the line are largest (line resonance occurs) when

$$\cos (bL) = 0; \quad bL = \frac{\pi}{2} (2n - 1); \quad n = 1, 2, 3, - - - \infty$$

Similarly, resonant effects are minimized (a pseudo-antiresonant condition exists) when

$$\cos (bL) = 1; \quad bL = \pi n; \quad n = 1, 2, 3, - - - \infty$$

Since, ($b = \omega/c$), the frequencies at which line resonance is either accentuated or minimized are:

$$\left. \begin{array}{l} \text{Frequencies at which} \\ \text{line resonance occurs} \end{array} \right\} \omega_n = \frac{\pi c}{2L} (2n-1) \left. \begin{array}{l} \\ \\ \text{Frequencies at which} \\ \text{resonance effects} \\ \text{are minimized} \end{array} \right\} \omega_n = \frac{\pi c}{L} (n) \left. \begin{array}{l} \\ \\ \end{array} \right\} n = 1, 2, 3, - - - \infty \quad (33)$$

The expressions of Equation 33 are known as the organ-pipe equations and have been used extensively for estimating resonant frequencies within pneumatic instrumentation lines. However, they have often been improperly used to describe systems that do not satisfy the requirements of Equation 31.

The largest value assumed by the amplitude gain factor (G) is the resonant peak value (G_p). Although it is implied that (G_p) approaches infinity as ($\cos Lb$) approaches zero, the existence of some damping in any real system places a restriction on how large (G_p) may become. A good approximation of the magnitude assumed by (G_p) at resonance, at least for lines satisfying Equation 31, is

$$G = G_p = \frac{1}{\cosh(yL)} = \{j(-1)^{n-1} \sinh(aL)\}^{-1} \left. \vphantom{\frac{1}{\cosh(yL)}} \right\} n = 1, 2, 3, \dots \infty \quad (34)$$

when: $\omega_n = \frac{\pi c}{2L} (2n - 1)$

Similarly, the magnitude of the amplitude gain factor, when all resonant effects are suppressed (pseudo-antiresonant conditions exist), is

$$G = G_{\min} = \frac{1}{\cosh(yL)} = \{(-1)^n \cosh(aL)\}^{-1} \left. \vphantom{\frac{1}{\cosh(yL)}} \right\} n = 1, 2, 3, \dots \infty \quad (35)$$

when: $\omega_n = \frac{\pi c}{L} (n)$

For the special case being considered here ($W > 30$), the line attenuation constant (a) appearing in Equations 34 and 35 is defined by Equation 20 as

$$a = \left(\frac{\gamma - 1}{\sigma} + 1 \right) \left[\frac{\omega \mu}{2 \rho c^2 R^2} \right]^{1/2} \quad (36)$$

Continuing with the simplification of Equation 32, note that, at resonance, the pressure distribution component may be modified to yield:

$$\cosh y(L-x) = j (-1)^{n-1} \{ \sinh (aL - yx) \} ; \quad \omega_n = \frac{\pi c}{2L} (2n - 1)$$

By employing the preceding identities, Equation 32 can now be altered to yield a simple expression describing resonant pressure distributions within a line that is characterized by very low viscous damping and that is long enough to virtually eliminate the effects of a terminal cavity.

$$P_{x,t} = \{ |G_p| \sinh (aL - yx) \} P_o \cos (\omega_n t) \quad (37)$$

$$\text{where: } \frac{60V}{A} < L < \frac{cpA}{1800\mu}$$

$$\left. \begin{aligned} |G_p| &= \frac{1}{\sinh (aL)} = \frac{1}{(aL + \dots)} = \frac{\sigma c R}{(\gamma - 1 + \sigma)L} \left\{ \frac{2\rho}{\mu \omega_n} \right\}^{1/2} \\ W &> 30 \\ \omega_n &= \frac{\pi c}{2L} (2n - 1) \\ L &= \frac{\lambda}{4} (2n - 1) \end{aligned} \right\} \quad n = 1, 2, 3, \dots \infty$$

In addition to the above, an approximation for the standing wave envelope can be obtained by using Equation 37 and assuming zero line attenuation. Line pressure distributions, as predicted by Equation 37, are summarized in Figure 11. The restrictions imposed on the use of Equation 37 could be relaxed if the line were terminated with a pressure transducer possessing a zero cavity volume. If such a pressure transducer were used, the line length boundary restriction would become $(0 < L < cpA/1800\mu)$.

Although the measurement conditions under which the organ pipe approximation yields valid results are quite limited, the equations do provide an excellent overall description of the resonant phenomena. In particular, the standing wave pattern shown in Figure 11 is typical of that found in closed lines of any length.

STANDING-WAVE ENVELOPE APPROXIMATION:

$$|P_x| = |G_p| \sin \left[\frac{x \pi}{2L} (2n - 1) \right] ; \quad 0 < x < L$$

PRESSURE DISTRIBUTION APPROXIMATION:

$$P_{o,t} = P_o \cos (\omega_n t)$$

$$P_{x,t} = P_o \left\{ |G_p| \sinh (aL - yx) \right\} \cos (\omega_n t)$$

$$P_{L,t} = P_o |G_p| \cos \left[\omega_n t - \frac{\pi}{2} (2n - 1) \right]$$

WHERE:

$$\frac{60V}{A} < L < \frac{c\rho A}{1800\mu}$$

$$|G_p| = [\sinh (aL)]^{-1} = \frac{\sigma c R}{(\nu - 1 + \sigma) L} \left\{ \frac{2\rho}{\mu \omega_n} \right\}^{1/2}$$

$$\left. \begin{aligned} \omega_n &= \frac{\pi c}{2L} (2n - 1) \\ L &= \frac{\lambda}{4} (2n - 1) \end{aligned} \right\} n = 1, 2, 3, \dots \infty$$

Figure 11 - Pressure distributions Within Lines Satisfying the Boundary Condition $\left(\frac{60V}{A} < L < \frac{c\rho A}{1800\mu} \right)$

The resonant behavior illustrated in Figure 11 results from the reflection of traveling waves at the closed termination. The superposition of wave trains traveling in opposite directions produces stationary waves within the tube. Since gas motion is impeded at the closed end but not at the open end, a pressure antinode (a maximum pressure region) will always occur at the closed end, and a pressure node (a minimum pressure region) will always occur at the open end. The alternating distribution of nodes and antinodes is illustrated by the standing wave pattern in Figure 11.

Another resonant phenomenon described by the organ pipe equations is the harmonic relationship between the various resonant modes. In Equation 37 and Figure 11, the parameter (n) is the resonant mode number. When ($n = 1$), the fundamental mode is described and the corresponding frequency value (ω_1) is known as the fundamental resonant frequency (first harmonic); when ($n = 2$) the first overtone occurs (second harmonic); when ($n = 3$), the second overtone occurs (third harmonic), and so on. Line resonance can occur at any normal mode frequency (ω_n) when properly excited. Several modes can be excited simultaneously with the distribution of energy among the allowed modes being governed by line dimensions, gas properties, and the spectral content of the pressure source.

Figure 11 also indicates that resonant effects are most pronounced at the line termination where pressure transducers are normally attached. The end of the line is the worst possible place to mount a pressure transducer. The best measurement of any pressure environment is obtained at the line entrance where resonant effects are minimized. That is, the line should be eliminated when possible and the transducer coupled directly to the pressure environment (a flush-mounted pressure transducer configuration).

The primary effect produced by resonance within pneumatic instrumentation lines is the partitioning (quantitizing) of the spectral content of pressure information presented to the line entrance. That is, the transfer

function of a cavity terminated instrumentation line tends to exaggerate (amplify) or attenuate the relative amplitude of discrete frequency bands presented to the line entrance. As with other resonant phenomena, the quantizing effect is most simply described when the limitations of Equation 31 are satisfied. As a first step, the pressure perturbations at the transducer station ($P_{L,t}$) will be related to those at the line entrance ($P_{o,t}$). That is, the transfer function for the line-transducer system will be described. Assuming that the limitations of Equation 31 are satisfied, the general transfer function is approximated by letting the x-parameter of Equation 32 equal (L), which yields:

$$\left. \begin{aligned} \frac{P_{L,t}}{P_{o,t}} &= \frac{1}{\cosh (yL)} = G/\theta \\ \text{where: } G &= \{[\cosh (aL) \cos (bL)]^2 + [\sinh (aL) \sin (bL)]^2\}^{-1/2} \\ \theta &= -\tan^{-1} \{\tanh (aL) \tan (bL)\} \\ y = a + jb &= \frac{\omega}{c} \left\{ \frac{1 + \frac{2(\gamma - 1) J_1(E)}{E J_0(E)}}{\frac{2 J_1(W)}{W J_0(W)} - 1} \right\}^{1/2} \end{aligned} \right\} \quad (38)$$

For any resonant mode (n), the transfer function becomes:

$$\left(\frac{P_{L,t}}{P_{o,t}} \right)_{\max} = \{\sinh (aL)\}^{-1} / \underline{-\pi (2n-1)/2} ; \quad \omega_n = \frac{\pi c}{2L} (2n-1), \quad (39)$$

whereas, for every pseudo-antiresonant condition, the transfer function is:

$$\left(\frac{P_{L,t}}{P_{o,t}} \right)_{\min} = \{\cosh (aL)\}^{-1} / \underline{-\pi n} ; \quad \omega_n = \frac{\pi c}{L} (n) . \quad (40)$$

The attenuation constant (a) appearing in Equations 39 and 40 maybe described by Equation 36.

Equations 39 and 40 are the rearrangements of Equations 34 and 35, respectively. A Bode plot of the transfer function, as predicted by Equation 38, is presented in Figure 12.

The transfer function shown in Figure 12 does not represent any particular case but illustrates the general nature of resonance within lines satisfying the requirements of Equation 31. As a result, the amplitude-ratio values are arbitrary, and the roll-off of the upper and lower transfer-function loci was intentionally exaggerated for clarity.

From Figure 12, it is apparent that Equations 39 and 40 are loci for the maximum and minimum values assumed by the transfer function. Therefore, when dealing with a pneumatic instrumentation line that satisfies Equation 31, the transfer function can be simply plotted by the following five-step procedure.

1. Plot the upper and lower transfer-function loci with the aid of Equations 39 and 40.

2. Calculate the resonant mode frequencies from

$$\omega_n = \frac{\pi c}{2L} (n); n = 1, 2, 3, - - - \infty .$$

3. Plot the odd resonant mode frequency values ($n = 1, 3, 5, - - - \infty$) along the upper loci and the even resonant mode frequency values ($n = 2, 4, 6, - - - \infty$) along the lower loci.

4. Sketch in a sinusoidal curve whose peak values are coincident with the frequency points located on the upper and lower loci.

5. Plot the transfer function through the frequency range 0 to ω_1 with the aid of Equation 38.

TYPICAL TRANSFER FUNCTION
APPROXIMATION FOR A LINE
SATISFYING THE BOUNDARY
CONDITION: $\left(\frac{60V}{A} < L < \frac{\rho cA}{1800\mu} \right)$

$$\text{TRANSFER FUNCTION} = \frac{G/\theta}{\cosh(yL)} \approx \frac{1}{\cosh(yL)}$$

$$\text{WHEN } \left(\frac{\omega}{\omega_n} \right) \gg 1 \left\{ a \approx \left(\frac{\gamma - 1}{\sigma} + 1 \right) \left[\frac{\omega \mu}{2 \rho c^2 R^2} \right]^{1/2} \right.$$

Figure 12 - Typical Transfer Function Approximation for a Line
Satisfying the Boundary Condition $\left(\frac{60V}{A} < L < \frac{\rho cA}{1800\mu} \right)$

Although this procedure yields rather inaccurate amplitude ratios for all frequencies above ω_1 and not occurring at ω_n , the amplitude ratio is adequately described at frequencies below ω_1 . In general, accurate amplitude ratios above ω_1 are not required because the cutoff frequency of most systems is established by the point at which the transfer function exceeds some limiting value, such as the +3-db level. If accurate transfer-function information is required for all frequencies above ω_1 , Equation 38 should be used. If the best possible accuracy is desired, the presence of the terminal cavity must be accounted for and Equation 27 must be used to describe the transfer function.

The fundamental resonant frequency described by the organ-pipe equation is actually the undamped (natural) fundamental resonant frequency of a line satisfying the requirements of Equation 31. The subscript ($n = 1$) will be considered as the natural resonant mode number. Because the fundamental resonant mode is of primary interest, it will be assigned the special subscript (r). Thus, the fundamental resonant mode frequency for a line satisfying the conditions of Equation 31 is described by

$$\omega_r = \frac{\pi c}{2L} ; \quad \frac{60V}{A} < L < \frac{cpA}{1800\mu} \quad (41)$$

and will be defined as the organ-pipe, fundamental, undamped (natural), resonant frequency. It is also the frequency at which pressure fluctuations within the pressure-transducer cavity phase lag pressure fluctuations at the line entrance by -90° (phase lag = -90°).

Because there is always some damping in any real physical system, the fundamental resonant frequency predicted by the organ-pipe approximation will always exceed that actually encountered. However, the error will be small if the limitations of Equation 31 are satisfied. Approximations describing the natural fundamental resonant frequency for line systems not satisfying Equation 31 will now be developed.

When dealing with very short lines ($L < V/10A$), exceedingly low viscous damping will be encountered, and the limiting case for Equations 20 and 30 is applicable. That is:

$$y = j \frac{\omega}{c} ,$$

$$Q = j \frac{\omega V}{cA} .$$

Thus, the transfer function for a cavity-terminated line (Equation 27) becomes:

$$\frac{P_{L,t}}{P_{o,t}} = \frac{1}{\cos(\omega L/c) - (\omega V/cA) \sin(\omega L/c)} \cdot \begin{cases} L < V/10A \\ W > 30 \end{cases}$$

By inspection, $P_{L,t}$ will be largest (resonance will exist) when the denominator of the preceding expression is zero. In other words, resonance occurs when

$$\tan(\omega L/c) = \frac{cA}{\omega V} .$$

However, when dealing with a very short line, the approximation

$$\tan(x) = (x + \text{---})$$

can be used to obtain

$$\frac{\omega L}{c} = \frac{cA}{\omega V} ,$$

which, when rearranged, yields an expression that may be used to approximate the undamped (natural) fundamental resonant frequency of a short line terminated with a cavity

$$\omega_r = c \sqrt{\frac{A}{LV}} ; \quad L < \frac{V}{10A} . \quad (42)$$

Equation 42 is the well known Helmholtz resonator equation and was used by Helmholtz to detect the component frequencies present in complex musical tones.

A short line terminated with a cavity is often referred to as a Helmholtz resonator. The line-cavity system will resonate at several frequencies, of which the lowest is most pronounced and is described by Equation 42. The higher resonant modes result from the formation of standing-wave patterns within the cavity and are not harmonically related to the lowest resonant mode.

When comparing Equation 42 with experimental results, the predicted fundamental frequency will always exceed measured values. This discrepancy can be explained by recalling that, for very large shear numbers ($W > 30$), the flow velocity is essentially constant across the line, and the main body of gas behaves as a cylindrical core (piston) that oscillates as a unit along the central line axis. However, a sound wave exiting from the line entrance tends to expand spherically. Thus, there exists a transition region where waves leaving the line are converted from plane waves to spherical waves, with the spherical waves appearing to radiate from a point (ΔL) beyond the line entrance. In effect, the acoustic line length exceeds the physical line length by an amount (ΔL). When the physical length (L) in Equation 42 is replaced with the acoustic line length ($L + \Delta L$), the agreement between theoretical and experimental results is greatly improved. An expression for (ΔL) was derived by Rayleigh, who assumed that the line entrance was terminated with an infinite flange. Rayleigh's expression for (ΔL) is⁸:

$$\Delta L = \frac{8R}{3\pi} ,$$

8. Rayleigh, The Theory of Sound, Vol. II, Dover Publications, New York, 1945, pp 197-200.

HELMHOLTZ RESONATOR APPROXIMATION

$$\omega_r = c \sqrt{\frac{A}{LV}}$$

WHERE: $L < \frac{V}{10A}$

$$A = \pi R^2$$

HELMHOLTZ RESONATOR APPROXIMATION CORRECTED FOR END EFFECTS

$$\omega_r = c \sqrt{\frac{A}{V(L + \Delta L)}}$$

WHERE: $L < \frac{V}{10A}$

$$\Delta L = \frac{8R}{3\pi}$$

$$A = \pi R^2$$

Figure 13 - Helmholtz Resonator Approximation

so that the acoustic line length (L_a) becomes

$$L_a = L + \frac{8R}{3\pi} ,$$

where (R) is the inner line radius and (L) is the physical line length. The Helmholtz resonator equations are summarized in Figure 13.

Two expressions are now available, the organ-pipe approximation (Equation 41) and the Helmholtz resonator approximation (Equation 42), which can be used to describe the undamped fundamental resonant frequency for short and very short lines. However, the allowable line-length variation for both expressions is extremely limited. A simple approximation is needed to describe ω_r for intermediate line lengths.

When plotting Equations 41 and 42 on logarithmic graph paper, both appear as straight lines. Thus, an expression that would asymptotically approach both straight-line plots at $L = V/10A$ and $L = 60V/A$ should satisfy the need. The simplest method of obtaining such an approximation is to use an inverted root-sum-squared combination of inverted forms of Equations 41 and 42 to obtain:

$$\omega_r = \frac{1}{\sqrt{\left(\frac{2L}{\pi c}\right)^2 + \left(\frac{LV}{c^2 A}\right)}} ,$$

which, after some manipulation, gives:

$$\omega_r = \frac{c}{L \sqrt{\frac{4}{\pi^2} + \frac{V}{AL}}} ; \quad \frac{80R}{3\pi} < L < \frac{c\rho A}{1800\mu} . \quad (43)$$

The lower limit now being placed on the line length (L) is the point at which end effects become significant ($\Delta L = L/10$).

The lower limit can be removed by modifying Equation 43 to account for end effects as shown below

$$\omega_r = \frac{c}{\left[L + \frac{8R}{3\pi} \right] \sqrt{\frac{4}{\pi^2} + \frac{3\pi V}{A(3\pi L + 8R)}}} ; \quad 0 < r < \frac{cpA}{1800\mu} .$$

In most cases, instrumentation lines are long enough to render end effects insignificant. Therefore, Equation 43 will be used in future discussions to approximate the natural undamped frequency of a cavity-terminated pneumatic line system.

Equation 43 can be used in place of Equations 41 and 42. However, it must be recognized that Equation 43 describes the natural fundamental resonant frequency and, thus, describes the frequency at which pressure fluctuations within the pressure-transducer cavity phase lag those at the entrance by 90° ($\theta = -90^\circ$). The parameters required to describe the undamped fundamental resonant frequency are limited to the sonic velocity (speed of sound) and line dimensions. Equation 43 was developed for lines whose lengths were short enough to ensure the existence of large shear numbers ($W > 30$) and adiabatic sound propagation. However, for line lengths exceeding the upper limit imposed on Equations 41 and 43, the propagation process becomes polytropic (the speed of sound decreases). For very long lines, the process at the fundamental resonant frequency becomes isothermal.

For an isothermal process, the frequency at which pressure oscillations in the transducer cavity phase lag those at the line entrance by 90° is found by referring to the expression for the phase constant (b) in Equation 19. That is, for an isothermal process,

$$bL = \frac{2L}{cR} \left[\frac{\gamma\omega\mu}{\rho} \right]^{1/2} = \frac{\pi}{2}; \quad (W < 0.3),$$

which, when rearranged, yields

$$\omega_{pr} = \frac{\pi A \rho c^2}{16 \gamma \mu L^2}, \quad (44)$$

where the subscript pr indicates that the phase resonant frequency is being described.

Equation 44 describes phase resonance rather than natural resonance. The difference is that Equations 41 through 43 describe resonance within lines possessing virtually no viscous damping at the fundamental resonant frequency, whereas Equation 44 describes a resonant condition within lines characterized by very large viscous damping at the fundamental resonant frequency. Viewed another way, Equations 41 through 43 describe both the phase resonant frequency and the lowest frequency at which peak resonant pressure oscillations occur for acoustically short lines, whereas Equation 44 describes only the phase resonant frequency for a line with very large damping (long lines).

To obtain an expression that will approximate the phase resonant frequency for both short low-damped lines and long high-damped lines, Equations 41, 42, and 44 are combined by using inverted root-sum-squared summation techniques to obtain:

$$\omega_{pr} = \frac{c}{L \sqrt{\frac{4}{\pi^2} + \frac{V}{AL} + \left(\frac{16 \gamma \mu L}{\pi \rho c A} \right)^2}}. \quad (45)$$

LENGTH DEPENDENT TRANSFER FUNCTION
FOR A VOLUME TERMINATED, AIR FILLED LINE

$V = 0.8 \text{ CM}^3$
 $R = 0.14 \text{ CM}$
 $P = 14.7 \text{ PSIA}$
 $T = 80^\circ\text{F}$

CURVE IDENTIFICATION:

1. $L = 1000 \text{ CM}$
2. $L = 700 \text{ CM}$
3. $L = 300 \text{ CM}$
4. $L = 100 \text{ CM}$

Figure 14 - Length-Dependent Transfer Function for A
Volume-Terminated, Air-Filled Line

There are now two important resonant-frequency approximations for cavity-terminated pneumatic line systems. The first expression (Equation 43) approximates the natural undamped fundamental resonant frequency, and the second expression (Equation 45) approximates the phase resonant frequency. The expression for phase resonance is particularly useful for determining the upper frequency limitation imposed on control systems utilizing very long pneumatic instrumentation lines. That is, for very long lines, resonant induced pressure oscillations are very small and the control-system frequency response is often limited by the frequency at which the phase lag portion of the pneumatic-line transfer function approaches -90° .

It must be emphasized that Equations 43 and 45 are approximations derived by simplifying more exact expressions (Equations 26 and 28). If accurate predictions are required, Equations 26 and 28 must be used.

Thus far, approximations have been developed for the natural undamped fundamental resonant frequency (ω_r) and the phase resonant frequency (ω_{pr}). However, the frequency at which fundamental pressure oscillations are largest is, in general, of more interest. This frequency is known as the peak resonant pressure amplitude frequency (ω_p), and its value is always less than that of the phase resonance frequency (ω_{pr}). This discrepancy is illustrated in Figure 14, where the transfer function is plotted for several air-filled lines. As shown in Figure 14, the deviation between ω_{pr} and ω_p increases with increasing line lengths. To further illustrate this phenomenon, f_{pr} and f_p values are presented in Figure 15 as functions of line length where $f_{pr} = \omega_{pr}/2\pi$ and $f_p = \omega_p/2\pi$.

The reason for the large deviation of the resonant curves (curves 1 and 2 of Figure 15) from the curve representing the organ-pipe approximation (curve 4) becomes clear upon referring back to Equation 41 and noting that the lower limit placed on line length exceeds the upper line limit for this particular example. Thus, the organ-pipe approximation is invalid for all line lengths when dealing with the line-transducer system pictured in Figures 14 and 15. This example was purposely selected to illustrate the errors that

RESONANT FREQUENCIES FOR A AIR FILLED LINE

GAS: AIR
 INNER LINE RADIUS: . 0.14 CM
 CAVITY VOLUME: . . 0.80 CM³
 TEMPERATURE: . . . 80°F
 PRESSURE: 14.7 PSIA

CURVE 1. $f_p = \text{ACTUAL}$

CURVE 2. $f_{pr} = \text{ACTUAL}$

CURVE 3. $f_r = \frac{c}{2\pi} \sqrt{\frac{A}{VL}}$

CURVE 4. $f_r = \frac{c}{4L}$

CURVE 5. $f_r = \frac{A\rho c^2}{32\gamma\mu L^2}$

CURVE 6. $f_r = \frac{c}{2\pi L \sqrt{\frac{4}{\pi^2} + \frac{V}{AL}}}$

Figure 15 - Resonant Frequencies for an Air-Filled Line

can be incurred by the indiscriminate use of approximation equations and to emphasize the need for understanding the limited nature of their applicability.

Curves 1 and 2 of Figure 15 describe the frequency (f_p) at which resonant induced pressure oscillations are largest and the frequency (f_{pr}) at which pressure oscillations in the transducer cavity lag those at the line entrance by -90° . For very short lines, the peak resonant pressure frequency (f_p) and the phase resonant frequency (f_{pr}) are equal to each other and to the natural fundamental resonant frequency for the system (f_r). As the line length increases and the frequency of the lowest resonant mode decreases, a point is reached where viscous damping becomes noticeable and separation of the f_p and f_{pr} curves occurs. The point of separation occurs at approximately $W = 10$, whereas the separation becomes significant at $W = 4$. By assuming that $\omega = \pi c/2L$, the above transition points can be converted to the equivalent line-length approximations shown below:

- (1) Point at which damping effects appear at the lowest resonant mode:

$$W \approx 10; \quad L = \frac{\rho c A}{200\mu}.$$

- (2) Point at which a transition from low to moderate damping occurs at the lowest resonant mode:

$$W \approx 4; \quad L = \frac{\rho c A}{32\mu}.$$

These transition points are shown in Figure 15. The deviation between the peak resonant pressure frequency (f_p) and the phase resonant frequency (f_{pr}) becomes very large as the low-frequency flow enters a region of high viscous damping. Eventually, as the line length is increased, a point is reached where complete suppression of line resonance occurs. Thus, a unique line length exists, which, if exceeded, will completely suppress line resonance. The tendency for resonance still exists, but the amount of viscous damping in the line has reached a critical value that prevents further natural oscillations. However, the line will still transmit forced oscillations applied at the line entrance.

(a) TRANSFER-FUNCTION GAIN AMPLITUDE PLOTS

$$G = \frac{1}{\left[(1 - U^2)^2 + (2 \delta U)^2 \right]^{1/2}}$$

$$U = \omega / \omega_r$$

(b) TRANSFER-FUNCTION PHASE PLOTS

$$\theta = -\tan^{-1} \frac{2 \delta U}{(1 - U^2)}$$

$$U = \omega / \omega_r$$

Figure 16 - Second-Order-System Transfer Function Gain and Phase Plots

Peak resonant gain (amplitude ratio) values (G_p) correspond to peak resonant pressure frequency values (ω_p). The value of (G_p) was analytically defined by Equation 37 for systems satisfying the conditions of Equation 31. However, more general expressions are needed that will yield approximate ω_p and G_p values for both low- and high-damped line systems.

Considerable labor is required to obtain a point-by-point transfer-function plot for any system as a function of frequency. However, second-order systems are encountered with such regularity that normalized gain (amplitude ratio) and phase curves have been adopted (Figure 16). Therefore, the transfer function of a second-order system is simply obtained by calculating two parameters, δ and ω_r , and then referring to the appropriate gain and phase curves.

On inspection, the gain and phase curves describing the fundamental resonant mode for pneumatic lines are found to resemble those of a second-order system (compare the fundamental resonant curves of Figure 14 with the second-order system curves of Figure 16). Thus, if it were possible to obtain effective δ and ω_r parameter values for the higher-order line-transducer system, the second-order system curves could be adapted to obtain pneumatic-line transfer-function predictions. The two systems will now be discussed to see if their similarities are sufficient to allow use of effective δ and ω_r values.

The curves of Figure 16 describe the transfer function of a single-degree-of-freedom system with lumped parameters whose response is described by a linear second-order differential equation with constant coefficients. The requirement that the pneumatic line system be limited to a single degree of freedom is satisfied because the predominant gas motion at resonance is directed along the central line axis. Because the length of the pressure wave at the lowest resonant mode is four times the overall line length, the line can be treated as a lumped-parameter system with the gas

oscillating as a unit within the line. Finally, the requirement of a linear system with constant coefficients is satisfied if the discussion is restricted to small pressure disturbances so that fluid properties are nearly constant throughout the line. Having demonstrated the similarity between the two systems, the second-order system transfer function will be discussed in more detail.

A linear second-order system is described by a second-order differential equation with constant coefficients whose general form is:

$$\frac{d^2C}{dt^2} + 2\delta\omega_r \frac{dC}{dt} + \omega_r^2 C = \omega_r^2 F, \quad (46)$$

where:

- C = system output,
- F = system input (forcing function),
- δ = system damping ratio (damping coefficient),
- ω_r = undamped natural resonant frequency.

The response of a second-order system to a steady-state sinusoidal forcing function is described by the system transfer function:

$$G/\theta = \frac{1}{1 + j\omega(2\delta/\omega_r) - \omega^2(1/\omega_r^2)}, \quad (47)$$

where the amplitude-ratio (G) and phase-angle (θ) components are described by:

$$G = \frac{1}{\left\{ [1 - (\omega/\omega_r)^2]^2 + [2\delta\omega/\omega_r]^2 \right\}^{1/2}} ;$$

$$\theta = -\tan^{-1} \left[\frac{2\delta\omega/\omega_r}{1 - (\omega/\omega_r)^2} \right] .$$

It can be shown that the maximum (peak) resonant pressure gain amplitude (G_p) is described by:

$$G_p = \frac{1}{2\delta \sqrt{1 - \delta^2}} , \quad (48)$$

and that the frequency at which peak pressure resonance (ω_p) occurs is defined by:

$$\omega_p = \omega_r \sqrt{1 - 2\delta^2} . \quad (49)$$

Inspecting the curves of Figure 16 shows that the existence of viscous damping has no effect on the fundamental undamped resonant frequency (ω_r). However, the peak resonant pressure frequency (ω_p) is strongly affected by the size of δ .

The frequency range over which the second-order system transfer function (Equation 47) can be applied to approximate the response of pneumatic line systems is limited to frequencies slightly above ω_p , as the second-order system is characterized by one resonant mode whereas a pneumatic line system possesses an infinite number of resonant modes. The poor agreement between second-order and pneumatic-line transfer functions for frequencies exceeding ω_p becomes apparent when Figures 14 and 16 are compared. When adapting second-order system equations to obtain an expression that describes the response of a pneumatic line, the parameter δ becomes the viscous damping coefficient and is evaluated at ω_{pr} . The similarity between the transfer-function equation for a line-transducer system and a second-order system will now be demonstrated. Given

$$\frac{P_{L,t}}{P_{o,t}} = \frac{1}{\cosh(yL) + Q \sinh(yL)} \quad (27)$$

and an infinite series expansion for hyperbolic functions,

$$\text{Cosh}(x) = 1 + \frac{x^2}{2} + \frac{x^4}{24} + \dots,$$

$$\text{Sinh}(x) = x + \frac{x^3}{6} + \frac{x^5}{120} + \dots,$$

the transfer function for a pneumatic line can be expressed by

$$\frac{P_{L,t}}{P_{o,t}} = \frac{1}{(1 + 0.5(yL)^2 + \dots) + Q(yL + \dots)} \quad (50)$$

when the product (yL) is less than unity. When considering the case of very low viscous damping, the parameters y and Q can be approximated by

$$y = \left[\frac{\gamma-1}{\sigma} + 1 \right] \cdot \left[\frac{\omega \mu}{2 \rho c^2 R^2} \right]^{1/2} + j \frac{\omega}{c}, \quad (20)$$

$$Q = j \frac{\omega V}{cA} \left\{ 1 + j \left[\frac{\gamma-1}{\sigma} - 1 \right] \sqrt{\frac{\mu}{2 \omega \rho R^2}} \right\}. \quad (29)$$

Thus, the transfer function becomes

$$\frac{P_{L,t}}{P_{o,t}} = \frac{1}{1 + j \omega \frac{\omega L^2}{\sqrt{2} B c^2} \left[I + \frac{2V}{LA} \right] - \omega^2 \frac{L^2}{c^2} \left[\frac{1}{2} + \frac{V}{LA} \right]}, \quad (51)$$

where:

$$B = |W| = [R^2 \omega \rho / \mu]^{1/2},$$

$$I = \left[\frac{\gamma - 1 + \sigma}{\sigma} \right].$$

The agreement in form between Equations 47 and 51 is excellent and further supports the assertion that, for a limited frequency band, a line-transducer system can be described by second-order system equations and curves. By comparing Equations 47 and 51 and equating like terms, the following is obtained:

$$\omega_r = \frac{c}{L \sqrt{\frac{1}{2} + \frac{V}{LA}}},$$

$$\delta = \frac{\omega L \left(I + \frac{2V}{LA} \right)}{\sqrt{8} Bc \sqrt{\frac{1}{2} + \frac{V}{LA}}}.$$

Though already good, these expressions can be improved. An obvious improvement can be made in the expression for ω_r by replacing $1/2$ with $4/\pi^2$. It then becomes Equation 43, which was previously developed to approximate the natural resonant frequency for a pneumatic line system. Because $1/2$ was replaced by $4/\pi^2$ in the expression for ω_r , a similar substitution must be made in the expression for δ . The expression for δ can be further improved by multiplying the parameter I by $2/\pi$. The peak low-damped pressure amplitude for a line whose effective terminal volume is zero then becomes:

$$(G_p)_V = 0 = \frac{\sqrt{8} Bc}{2\omega L I} = \frac{\sigma c R}{(\gamma - 1 + \sigma)L} \left[\frac{2\rho}{\omega \mu} \right]^{1/2}, \quad \delta \ll 1;$$

which is identical to Equation 37. Finally, because the value of δ is being determined at ω_{pr} , which is identical to ω_r for lines with low viscous damping, the frequency parameter ω that appears in the expression for δ will be replaced with the expression for ω_r .

With these modifications, the equation describing δ for lines possessing very low viscous damping ($W > 30$) becomes

$$\frac{P_{(L,t)}}{P_{(0,t)}} = \frac{1}{1 + j\omega \left[\frac{2\delta}{\omega_{pr}} \right] - (\omega/\omega_{pr})^2} = G \angle \theta$$

$$G = \left\{ \left[1 - (\omega/\omega_{pr})^2 \right]^2 + \left[2\delta\omega/\omega_{pr} \right]^2 \right\}^{-1/2}$$

$$\theta = -\tan^{-1} \frac{2\delta\omega/\omega_{pr}}{1 - (\omega/\omega_{pr})^2}$$

$$G_p = \frac{1}{2\delta\sqrt{1-\delta^2}}$$

$$\omega_p = \omega_{pr} \sqrt{1-2\delta^2}$$

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SECOND-ORDER PARAMETER APPROXIMATIONS
FOR LOW VISCOUS DAMPING ($L < \rho cA/52\mu$)

$$\omega_{pr} = \frac{c}{L \sqrt{\frac{4}{\pi^2} + \frac{V}{LA} + \left[\frac{16\gamma\mu L}{\pi\rho cA} \right]^2}}$$

$$\delta = \frac{\frac{2(\gamma-1+\sigma)}{\sigma\pi} + \frac{2V}{LA}}{\sqrt{8} B \left[\frac{4}{\pi^2} + \frac{V}{LA} \right]}$$

WHERE: $B = \left\{ \frac{c\rho R^2}{\mu L \sqrt{\frac{4}{\pi^2} + \frac{V}{LA}}} \right\}^{1/2}$

SECOND-ORDER PARAMETER APPROXIMATIONS
FOR HIGH VISCOUS DAMPING ($L > \rho cA/52\mu$)

$$\omega_{pr} = \frac{c}{L \sqrt{\frac{4}{\pi^2} + \frac{V}{LA} + \left[\frac{16\gamma\mu L}{\pi\rho cA} \right]^2}}$$

$$\delta = \frac{\sqrt{8} L \mu \gamma}{c \rho R^2} \sqrt{\frac{4}{\pi^2} + \frac{V}{LA}}$$

Figure 17 - Second-Order Approximations for Pneumatic Lines

$$\delta = \frac{\frac{2(\gamma - 1 + \sigma)}{\sigma \pi} + \frac{2V}{LA}}{\sqrt{8} B \left[\frac{4}{\pi} + \frac{V}{LA} \right]}, \quad L < \rho c A / 52 \mu; \quad (52)$$

where

$$B = \left[\frac{c \rho R^2}{\mu L \sqrt{\frac{4}{\pi} + \frac{V}{LA}}} \right]^{1/2}.$$

Referring to Figure 16, it is readily seen that Equation 49 describes the deviation of ω_p from the frequency at which the phase angle is -90° . But, for pneumatic-line-transducer systems, the frequency at which a -90° phase lag occurs has been previously defined as the phase resonant frequency (ω_{pr}) and has been described by Equation 45. Therefore, the effective undamped resonant frequency (ω_r) will be described by Equation 45. A summation of the second-order approximation for lines possessing very low viscous damping values is presented in Figure 17.

A second-order approximation for lines with high viscous damping is found by referring to Equation 50 and assuming that

$$Q = yV/A, \quad (29)$$

$$y = \left[j \frac{8 \gamma \omega \mu}{\rho c \frac{2}{R^2}} \right]^{1/2}. \quad (19)$$

Thus, the transfer function for a pneumatic line-transfer system with high viscous damping becomes:

$$\frac{P_{L,t}}{P_{o,t}} = \frac{1}{1 + j\omega \left\{ \frac{8 \mu \gamma L}{\rho c \frac{2}{R^2}} \left[\frac{1}{2} + \frac{V}{LA} \right] \right\}}. \quad (53)$$

In comparing Equations 47 and 53, it becomes apparent that the component corresponding to ω^2/ω_r^2 is missing, thus implying that line resonance is completely suppressed when very low shear numbers are encountered ($W < 0.3$). However, the natural undamped resonant line frequency (ω_r) is an inherent line characteristic and, as such, always exists. Consequently, the undamped fundamental resonant frequency is still described by Equation 43.

The frequency term (ω) in Equation 53 was derived from an expression for y (Equation 19). By reviewing the development of Equation 44, it becomes apparent that if y were used to obtain the damped phase resonant frequency, ω_{pr} would be half the numerical value actually attained. Therefore, it would seem reasonable to assume that the resonant frequency at which δ of Equation 53 is being evaluated is half of what it should be. If the effective frequency is too low, the corresponding effective viscous damping parameter will be too large.

By reference to Equation 17, it is seen that the shear number (relative amount of viscous damping) is proportional to the square root of frequency. Then, if the effective frequency at which δ is to be evaluated is low by a factor of two, the effective viscous damping coefficient must be reduced (corrected) by a factor of $\sqrt{2}$. Therefore, by dividing the term for $2\delta/\omega_r$ by the square root of two, changing $1/2$ to $4/\pi^2$, and multiplying by ω_r , an expression for δ is obtained:

$$\delta = \frac{\sqrt{8} L \mu \gamma}{c \rho R^2} \sqrt{\frac{4}{\pi^2} + \frac{V}{AL}} \quad , \quad L > \rho c A / 52 \mu . \quad (54)$$

As before, the effective natural line resonance is described by Equation 45.

Values of G_p and ω_p , as calculated by the exact equations (Equations 16, 27, and 28), are compared to those obtained with second-order approximations in Table 2.

TABLE 2

COMPARING SECOND-ORDER APPROXIMATIONS TO EXACT THEORETICAL
PREDICTIONS FOR AN AIR-FILLED LINE

(Where Cavity Volume is 0.8 cu cm, Inner Line
Radius is 0.14 cm, Line Temperature is
80°F, and Line Pressure is 14.7 psia)

Viscous Damping at the Fundamental Resonant Mode Freq.	Line Length cm	Peak Resonant Freq (ω_p), radians/sec			Peak Amplitude Ratio (G_p)		
		Actual (Eq.27)	Low- Damp Approx'n	High- Damp Approx'n	Actual (Eq.27)	Low- Damp Approx'n	High- Damp Approx'n
Low Damping ($L < \frac{\rho c A}{52\mu}$)	3	5204.5	5315.8	5319.8	17.603	17.990	83.830
	10	2590.5	2652.0	2659.9	12.306	12.440	41.920
	30	1197.2	1259.6	1263.5	8.128	8.263	19.930
	100	415.1	468.5	471.4	4.550	4.882	7.497
High Damping ($L > \frac{\rho c A}{52\mu}$)	300	137.5	163.1	162.5	2.417	2.915	2.771
	700	48.8	61.6	53.7	1.322	1.962	1.321
	850	34.5	47.3	36.5	1.155	1.796	1.149
	1090	15.4	32.8	15.4	1.015	1.626	1.015

Table 2 demonstrates that the low-viscous-damping second-order approximations yield the best overall description of line resonance although the high-viscous-damping approximations are superior when describing resonance within long lines.

The dependency of G_p and ω_p on line length is illustrated in Figure 14. The dependency of the peak resonant pressure amplitude and resonant frequency on other line parameters is summarized in Table 3.

TABLE 3

DEPENDENCY OF RESONANT PRESSURE AND
FREQUENCY VALUES ON LINE PARAMETERS

Parameter Variation at Any Oscillatory Frequency	Resonant Variations	
	Resonant Pressures Will Tend to	Resonant Frequencies Will Tend to
An increase in absolute gas pressure (P)	Increase	Increase
An increase in absolute gas temperature (T)	Decrease	Increase
An increase in the inner line radius (R)	Increase	Increase
An increase in line length (L)	Decrease	Decrease
An increase in cavity volume (V)	Decrease	Decrease

The dependency of system response on effective second-order parameter values can be summarized as follows:

1. The amplitude ratio is unity, regardless of the amount of viscous damping present, when $(\omega/\omega_{pr}) = 0$.
2. If $0 < \delta < 0.707$, the amplitude ratio rises to a maximum value (G_p) as ω/ω_{pr} increases from zero with the peak value obtained before the ratio ω/ω_{pr} becomes unity.
3. If $\delta > 0.707$, the amplitude ratio decreases as ω/ω_{pr} increases from zero.
4. For all values of δ , the amplitude ratio approaches zero as ω/ω_{pr} approaches infinity.
5. The larger ω_{pr} is made, the faster the system time response becomes.
6. Both G_p and ω_p are reduced as δ is increased.

Having derived expressions for effective δ and ω_r parameters, it might appear reasonable to assume that these parameters could be inserted into Equation 46 and the transient response of a pneumatic line system to any

forcing function could be obtained. Such an assumption would be valid if the pneumatic line behaved as a linear system for all input functions; however, this is not the case. In almost every instance, the amplitude of transient pressure changes presented to the line entrance is sufficient to drive the line into a turbulent-flow realm where the line transfer function becomes nonlinear. Even if the transient-pressure input amplitude were small enough to ensure laminar flow, the question arises as to which viscous-damping expression should be used, because the spectral content of most input functions contains both low- and high-frequency components.

In general, most transient input fluctuations can be approximated by one of five functions; a nonrepetitive rectangular pulse, a nonrepetitive triangular pulse, a nonrepetitive half-sine pulse, a step function, or a ramp function. The amplitude of low-frequency spectral components exceeds the amplitude of high-frequency spectral components for all five functions. Thus, a high-viscous-damping approximation would seem appropriate as low-frequency oscillations enhance viscous effects. However, when the effective high-viscous-damping coefficient (Equation 54) is used to describe the transient response of short lines, the calculated amplitude of transient oscillations is much larger than that actually encountered. Thus, when using the expressions that have been presented in this report, the use of second-order equations for obtaining the transient response of pneumatic lines should be limited to those conditions where:

1. The line is acoustically long ($L > \rho c A / 52 \mu$);
2. The effective viscous damping coefficient (δ) is described by Equation 54;
3. The effective natural resonant frequency (ω_{pr}) is described by Equation 45;
4. The transient-pressure perturbation amplitude is not large enough to yield flow turbulence within the line.

Obviously, too many restrictions are placed on the use of second-order transient approximations. A mathematical model is needed that can be applied to both short and long lines and that will yield favorable results even when flow turbulence is encountered. Such a model would describe a low-frequency flow process but would not provide for the existence of a resonant condition. A first-order system satisfies this requirement.

The transfer function of a first-order system subjected to a steady-state, sinusoidal forcing function is:

$$G/\theta = \frac{1}{1 + j \omega \tau} \quad , \quad (55)$$

where τ is define as the first-order-system time constant. The form of Equation 55 is identical to that of Equation 53, which describes the transfer function of a pneumatic-line-transducer system with high viscous damping. By comparing Equations 53 and 55 and equating like terms, the effective time constant of a cavity-terminated line system is found to be

$$\tau = \frac{8\mu \gamma L^2}{\rho c^2 R^2} \left[\frac{1}{2} + \frac{V}{LA} \right]. \quad (56)$$

Thus, by using the general first-order-system differential equation (Equation 57) and the effective time constant (τ), an approximation of the pneumatic-line transient response can be obtained:

$$\tau \frac{dP_L}{dt} + P_L = P_O(t). \quad (57)$$

Equation 57 assumes that pressure variations presented to the line entrance will be instantaneously sensed by the line-mounted pressure transducer. However, instantaneous sensing is impossible because pressure disturbances are propagated with a finite velocity. In other words, the time (t')

required for the initial transient wave front to traverse the total line length must be accounted for. This time-delay factor is described by L/c , where L is line length and c is the propagation velocity of sound within the line. To determine whether the propagation process is adiabatic or isothermal, imagine that a step input is applied to the line entrance. The leading edge has the characteristics of a high-frequency input function, while the overall characteristic is that of a very-low-frequency input function. Thus, the effective propagation process is polytropic, and the speed of wave movement is described by the equation for polytropic propagation velocity. The value assigned to the polytropic coefficient (m) is an average value and is defined by

$$m = (1 + \gamma)/2,$$

so that the transport delay parameter becomes

$$t' = \left\{ \frac{2 \gamma L^2}{(1 + \gamma) c^2} \right\}^{1/2} \quad (58)$$

where c is now recognized as the adiabatic propagation velocity. Although Equation 58 was developed by considering a step input function, it is a general expression that can be applied to yield a time-delay approximation for any input function. To obtain a simple expression that will include a time-delay factor and provide a means for handling aperiodic input functions, it will be necessary to use a Laplace transformation of the first-order transfer-function equation to yield:

$$\frac{P_L(s)}{P_O(s)} = \frac{e^{-st'}}{1 + \tau s} \quad (59)$$

As an example of the kind of results that can be obtained by using Equation 59, consider the case where the line input pressure increases instantaneously from an initial value $(P_s)_o$ to a new value $(P_s)_f$ at time $t = 0$. Let the

$$\frac{P_L(S)}{P_o(S)} = \frac{e^{-st'}}{1 + \tau s}$$

$$t' = \left[\frac{2 \gamma L^2}{(1 + \gamma) c^2} \right]^{1/2}$$

$$\tau = \frac{8 \mu \gamma L^2}{\rho c^2 R^2} \left[\frac{1}{2} + \frac{V}{LA} \right]$$

WHEN $P_o(S)$ IS THE LAPLACE TRANSFORM OF A STEP INPUT FUNCTION, THE LINE RESPONSE IS APPROXIMATED BY:

$$\frac{(P_s)_L - (P_s)_o}{(P_s)_f - (P_s)_o} = (1 - e^{-(t - t')/\tau})$$

Figure 18 - First-Order Transient Response Approximations for Pneumatic Lines

input pressure remain constant for all other time values. Solving Equation 59 for this particular case yields

$$\frac{(P_s)_L - (P_s)_o}{(P_s)_f - (P_s)_o} = \left[1 - e^{-(t-t')/\tau} \right], \quad t \geq t';$$

where $(P_s)_L$ is the instantaneous pressure level within the cavity of a terminal-mounted transducer and t' and τ are described by Equations 58 and 56, respectively. By necessity, gas parameter values will be evaluated at the mean line temperature and pressure. The first-order step-input response approximation is illustrated in Figure 18.

To reiterate, a pneumatic instrumentation line can be used to couple a pressure transducer to a pressure source originating within an environmental region that exceeds the transducer design tolerance. A pneumatic line also provides a means for obtaining remote pressure measurements when mechanical interface problems prevent direct coupling of the transducer to the pressure source. When dealing with the cavity-terminated line configuration of Figure 10, certain general conclusions and recommendations can be made regarding the line-transducer system performance:

1. Both the steady-state and transient performance can be improved by using the largest practical inner line radius, the shortest permissible line length, and the smallest obtainable transducer cavity volume.
2. The amplitude ratio attained by the undamped response curve for a second-order system is approximately 1.04 when the system excitation frequency is two-tenths the peak resonant pressure frequency (i.e., $\omega = 0.2 \omega_p$). Since a pneumatic line resembles a second-order system, the line should be designed so the peak resonant pressure frequency (ω_p) is at least five times the required frequency response of the line-transfer system if amplitude linearity errors are not to exceed 5%.

$$\frac{P_n}{P_{n-1}} = \left\{ \left[\cosh(yL) + Q \sinh(yL) \right]_n + \left[\frac{(yA)_{n+1}}{(yA)_n} \right] \cdot \left[\frac{J_0(W)}{J_2(W)} \right]_n \cdot \left[\frac{J_2(W)}{J_0(W)} \right]_{n+1} \cdot \left[\frac{\sinh(yL)_n}{\sinh(yL)_{n+1}} \right] \cdot \left[\cosh(yL)_{n+1} - \frac{P_{n+1}}{P_n} \right] \right\}^{-1}$$

Figure 19 - Segmented Pneumatic Line Equation

C. SEGMENTED-LINE CONFIGURATIONS

All discussions so far have been restricted to the response of homogeneous lines (lines with uniform cross-sectional dimensions and gas properties). However, it is not difficult to find lines with temperature gradients and dimensional discontinuities. Most lines are characterized by some form of inhomogeneity.

It is easier to describe lines that are free of discontinuities than to describe lines that have discontinuities. Therefore, it is desirable to find some way of describing inhomogeneous lines with homogeneous line equations. One approach divides a line into many discrete, volume-terminated, homogeneous line sections as shown in Figure 19. When using the segmented-line configuration, the pressure in the terminal cavity of one section becomes the input pressure to the following section.

If all cavity volumes except the last one are set equal to zero and the length of each section approaches zero as the number of sections approaches infinity, the segmented line, in the limit, will resemble a line with continuously variable properties. However, the number of sections required to describe most pneumatic line systems is normally not large.

A general segmented-line equation has been developed by Bergh and Tijdeman (Equation 55 of Reference 2), which, after assuming that the cavity volumes are independent of pressure, can be expressed by:

$$\frac{P_n}{P_{n-1}} = \left\{ \left[\cosh(yL) + Q \sinh(yL) \right]_n + \dots \right. \quad (60)$$

$$\left. \dots \cdot \left[\frac{(yA)_{n+1}}{(yA)_n} \right] \cdot \left[\frac{J_0(W)}{J_2(W)} \right]_n \cdot \left[\frac{J_2(W)}{J_0(W)} \right]_{n+1} \cdot \left[\frac{\sinh(yL)_n}{\sinh(yL)_{n+1}} \right] \cdot \left[\cosh(yL)_{n+1} - \frac{P_{n+1}}{P_n} \right] \right\}^{-1},$$

where y is the propagation constant, Q is a reflection coefficient, A is the inner line cross-sectional area, L is the segment line length, W is the shear number, and n is the section number. Pressure (P_{n+1}) is the sinusoidal pressure amplitude existing within the terminal volume of section $n+1$, while P_n is the sinusoidal pressure amplitude within the cavity of the preceding section (n) that is presented to the entrance of section $n+1$. To summarize, Bergh and Tijdeman have developed a general recursion formula (Equation 60) for a series connection of n volume-terminated lines that relates the sinusoidal pressure disturbance in volume n to the pressure disturbances in the preceding volume ($n-1$) and the next volume ($n+1$).

When dealing with a single line section, the segmented line equation reduces to Equation 27. The parameters y , W , and Q in Equation 60 have the same meaning and are described by the same formula (i.e., Equations 16, 17, and 28, respectively) as those in Equation 27. This is as it should be, because the expressions for y , W , and Q were derived from the Iberall equations of Reference 1 and the Bergh-Tijdeman recursion equation is an extension of Iberall's original investigations.

The terminal pressure perturbations within any two line segment volumes may be related by using Equation 60 and a recursion formula. For example, if the pressure perturbation within the cavity of section $n-1$ is to be related to the pressure perturbation within the cavity of section $n+2$, a recursion formula is used to yield:

$$\frac{P_{(n+2)}}{P_{(n-1)}} = \frac{P_{(n)}}{P_{(n-1)}} \cdot \frac{P_{(n+1)}}{P_{(n)}} \cdot \frac{P_{(n+2)}}{P_{(n+1)}}.$$

The preceding recursion formula also relates pressure perturbations within the cavity of section $n+2$ to those at the entrance of section n .

As an example of the kind of systems that can be analyzed with the segmented-line and recursion formulas, consider the double-diameter systems of Figures 20 and 21. These line systems incorporate two homogeneous line segments where the terminal volume of the first segment is equal to zero. The segmented-line equation for section No. 2 (the section with the terminal volume) is

$$\frac{P_2}{P_1} = \left([\text{Cosh}(yL) + Q \text{ Sinh}(yL)]_2 \right)^{-1},$$

while the segmented-line equation for section No. 1 (the section incorporating the line entrance) is

$$\frac{P_1}{P_o} = \left\{ [\text{Cosh}(yL)]_1 + \dots \right. \\ \left. \dots \left[\frac{(yA)_2}{(yA)_1} \right] \cdot \left[\frac{J_o(W)}{J_2(W)} \right]_1 \cdot \left[\frac{J_2(W)}{J_o(W)} \right]_2 \cdot \left[\frac{\text{Sinh}(yL)_1}{\text{Sinh}(yL)_2} \right] \cdot \left[\text{Cosh}(yL)_2 - \frac{P_2}{P_1} \right] \right\}^{-1}.$$

By using the recursion formula,

$$\frac{P_2}{P_o} = \frac{P_1}{P_o} \cdot \frac{P_2}{P_1} = \frac{P_{L,t}}{P_{o,t}},$$

pressure perturbations within the transducer cavity ($P_{L,t}$) are related to pressure perturbations presented to the line entrance ($P_{o,t}$). That is, the recursion formula analytically defines the transfer function of the double-diameter line systems shown in Figures 20 and 21.

Figure 20 - Dual-Diameter Pneumatic Line Transfer Function,
Short Input Length

DUAL DIAMETER PNEUMATIC LINE
TRANSFER FUNCTION, LONG INPUT LENGTH

Figure 21 - Dual-Diameter Pneumatic Line Transfer Function,
Long Input Length

Figure 22 - Line Response Characteristics

These two cases represent a condition where the diameter of the line entrance section is fixed while the diameter of the line terminal section is variable. This condition would be encountered if the pressure environment to be monitored were surrounded by closely spaced structural components and a small line diameter was required to couple the environment to a pressure transducer located outside the test specimen envelope. Once the line had been routed to an accessible station, the test engineer would have the option of increasing the line diameter, decreasing the line diameter, or routing the line to the transducer station without changing the diameter. For those not intimately familiar with the operating characteristics of pneumatic lines, it might appear reasonable to assume that reducing the diameter of the accessible line section would always degrade the system frequency response, while increasing the diameter of the accessible line section might improve the system frequency response. Though these assumptions may appear reasonable, they are not necessarily correct. Figure 21 demonstrates that a small diameter reduction can, in some instances, improve the system frequency response (decrease low-frequency amplitude ratio and phase-lag values). Figures 20 and 21 demonstrate that an increase in the line terminal-section diameter degrades the system frequency response (increases low-frequency amplitude ratio and phase-lag values). These two special cases serve to illustrate the kind of erroneous conclusions that can be reached when working with pneumatic instrumentation lines. In general, diameter variations are not recommended unless the effects of such variations are fully appreciated.

D. INFINITE-LINE CONFIGURATION

Two primary line configurations have been discussed so far: a continuous line of infinite length and a closed line. The primary operating characteristics of these two configurations are shown in Figure 22.

If it were possible to attach a transducer to a continuous line at a distance (d) from the line entrance and monitor internal line pressures without producing a flow discontinuity, the resulting line-transducer

transfer-function amplitude ratio would be as illustrated in Figure 22A. A continuous line configuration behaves as though the line were terminated with its characteristic impedance at the transducer tap station. Since a continuous line has no discontinuities, wave reflections will not occur, and the amplitude ratio will decrease continuously and smoothly from unity to zero as the source frequency increases from zero to infinity. The amplitude-dependent frequency bandwidth of a continuous-line system will extend from zero to the cutoff frequency (ω_{co}) where the amplitude-ratio curve intersects the -3-db level (has dropped to 70.7% of its zero-frequency value). That is, the system frequency response is often defined as that frequency at which the transfer-function amplitude ratio exceeds the ± 3 -db band portrayed in Figure 22.

When a closed line configuration is used, pressure waves traveling through the line from the line entrance will encounter a very large flow discontinuity at the line termination. The discontinuity is so large that total wave reflection almost occurs. The discontinuity results from the large acoustic impedance produced by the termination wall. Assuming that the termination wall represents a pressure transducer diaphragm, the curves of Figure 22B will represent the amplitude ratio of a terminal-mounted zero-volume transducer line system. The peaks and valleys that appear in the transfer-function plot for the closed line configuration represent a resonant condition that arises from the superposition of incident and reflected pressure waves. Very often, the amplitude of resonant induced pressure oscillations will exceed the upper 3-db band limit as shown in Figure 22B. The point at which the amplitude-ratio curve exceeds the 3-db level is normally defined as the amplitude-dependent cutoff frequency (ω_{co}) and represents the upper frequency-response limit for a closed line configuration.

The cutoff frequency of a continuous line configuration normally exceeds the cutoff frequency of a closed line configuration. If the peak amplitude of resonant pressure oscillations in the closed line were reduced to a value below the upper 3-db limit, the frequency response of the closed

line configuration would extend to a point where the -3-db limit was exceeded. In general, resonant pressure amplitude reductions can be achieved by reducing the effective size of the acoustic-impedance mismatch that exists at the measurement station (at $x = d$).

It can be shown that the effective acoustic line impedance presented to incident pressure waves at a distance d from the line entrance will decrease from infinity ($Z_L = \infty$) and approach the characteristic line impedance value ($Z_L = Z_0$) as the closed termination is moved from the measurement station ($x = d$) to infinity ($x = \infty$). However, perfect impedance matching (complete resonance suppression) is not required. It is only necessary to reduce the amplitude of resonant pressure oscillations at the measurement station (at $x = d$) to a value that is below the upper 3-db limit. That is, it is only necessary to reduce the size of the effective acoustic-impedance mismatch encountered by an incident wave at $x = d$. This is accomplished by extending the overall line length and attaching a line-mounted pressure transducer to measure the internal line pressure at the measurement station (at $x = d$), as shown in Figure 22C.

Although the concept of impedance matching was used to arrive at the infinite-line configuration illustrated in Figure 22C, the same result could have been achieved by considering the resonant standing-wave pattern within a closed line configuration. As shown in Figure 11, the amplitude of fundamental resonant pressure oscillations decreases as distance from the line termination increases. In other words, the amplitude of fundamental resonance pressure oscillations decrease as the measurement station is moved away from the fundamental resonant pressure antinode. Therefore, when the measurement station is to be located at a predetermined distance (d) from the line entrance and a closed line configuration is to be used, resonant pressure levels at the measurement station can be reduced, and the line-transducer frequency response can be improved by removing the transducer from the end of the line, capping the end of the line, and increasing the overall line length (L) until acceptable line resonant pressures exist at the measurement station. A pressure

A. SEGMENTED LINE EQUATION FOR A TWO SECTION LINE WITH \$V_2 = 0\$

$$N = 1; \frac{P_{d,t}}{P_{o,t}} = \left\{ \left[\text{Cosh } (yL) + Q \text{ Sinh } (yL) \right]_1 + \left[\frac{(yA)_2}{(yA)_1} \right] \cdot \left[\frac{J_0(W)}{J_2(W)} \right]_1 \cdot \left[\frac{J_2(W)}{J_0(W)} \right]_2 \cdot \left[\frac{\text{Sinh } (yL)_1}{\text{Sinh } (yL)_2} \right] \cdot \left[\text{Cosh } (yL)_2 - \frac{P_{L,t}}{P_{d,t}} \right] \right\}^{-1}$$

$$N = 2; \frac{P_{L,t}}{P_{d,t}} = \frac{1}{\text{Cosh } (yL)_2}$$

B. INFINITE - LINE EQUATION

$$\frac{P_{d,t}}{P_{o,t}} = \left\{ \left[\text{Cosh } (y d) + Q \text{ Sinh } (y d) \right]_1 + \left[\frac{(yA)_2}{(yA)_1} \right] \cdot \left[\frac{J_0(W)}{J_2(W)} \right]_1 \cdot \left[\frac{J_2(W)}{J_0(W)} \right]_2 \cdot \left[\text{Sinh } (y_1 d) \cdot \text{Tanh } y_2 (L - d) \right] \right\}^{-1}$$

C. INFINITE - LINE EQUATION FOR A HOMOGENEOUS LINE

$$\frac{P_{d,t}}{P_{o,t}} = \left\{ \left[\text{Cosh } (y d) + Q \text{ Sinh } (y d) \right] + \left[\text{Sinh } (y d) \cdot \text{Tanh } y (L - d) \right] \right\}^{-1}$$

Figure 23 - Infinite-Line Configuration and Equation

transducer is then attached to the line to monitor internal line pressures at the measurement station (at $x = d$). Thus, the infinite-line configuration is once more achieved.

When a homogeneous infinite-line configuration is used and the transducer-line coupling-induced discontinuities are small, the system transfer function may be approximated by manipulating Equation 26 to obtain:

$$\frac{P(d,t)}{P(o,t)} = \frac{\cosh y (L - d)}{\cosh (yL)} . \quad (61)$$

However, transducer-line coupling effects can be large and should be accounted for by using the segmented line formula (Equation 60) to yield an infinite-line formula.

The development of the infinite-line formula assumes that two homogeneous line sections are jointed to form a two-section segmented-line configuration as shown in Figure 23. The segmented-line equation for section one is

$$\begin{aligned} \frac{P_{d,t}}{P_{o,t}} = & \left\{ \left[\cosh(yL) + Q \sinh(yL) \right]_1 + \dots \right. \\ & \left. \dots \left[\frac{(yA)_2}{(yA)_1} \right] \cdot \left[\frac{J_o(W)}{J_2(W)} \right]_1 \cdot \left[\frac{J_2(W)}{J_o(W)} \right]_2 \cdot \left[\frac{\sinh(yL)_1}{\sinh(yL)_2} \right] \cdot \left[\cosh(yL)_2 - \frac{P_{L,t}}{P_{d,t}} \right] \right\}^{-1} \end{aligned} \quad (62)$$

while the segmented-line equation for section two is

$$\frac{P_{L,t}}{P_{d,t}} = \left\{ \cosh(yL)_2 \right\}^{-1} . \quad (63)$$

Figure 24 - Transfer Function of an Infinite-Line Possessing a Temperature Gradient

By replacing the last term of Equation 62 with Equation 63 and noting that $L_1 = d$ and $L_2 = L - d$, an infinite-line equation is obtained.

$$\frac{P_{d,t}}{P_{o,t}} = \left\{ \cosh(y_1 d) + Q_1 \sinh(y_1 d) + \dots \right. \quad (64)$$

$$\left. \dots \left[\frac{(yA)_2}{(yA)_1} \right] \cdot \left[\frac{J_o(W)}{J_2(W)} \right]_1 \cdot \left[\frac{J_2(W)}{J_o(W)} \right]_2 \cdot \left[\sinh(y_1 d) \cdot \tanh y_2 (L-d) \right] \right\}^{-1}.$$

If the line is homogeneous everywhere except at the transducer tap station, the infinite-line formula (Equation 64) can be simplified to yield:

$$\frac{P_{d,t}}{P_{o,t}} = \left\{ \cosh(yd) + Q \sinh(yd) + \left[\sinh(yd) \cdot \tanh y(L-d) \right] \right\}^{-1}. \quad (65)$$

If the transducer-line coupling-induced discontinuity is known to be small ($Q \approx 0$), the infinite-line formula can be simplified even further to yield Equation 61. A summary of the infinite-line technique is presented in Figure 23.

A comparison of Figures 19 and 23 shows that the segmented-line configuration assumes a series-connected volume, whereas the infinite-line configuration assumes a tapped volume. In general, the effect of a tapped volume will be the same as for a series volume if the tube used to couple the cavity to the line is very short.

To further demonstrate the versatility of the segmented-line equation in describing the response of an infinite-line configuration, consider the line described in Figure 24. Although it may not be obvious, there are three dominant resonant mechanisms contributing to the transfer-function plot for this system: terminal-induced resonance, tap-induced resonance, and

Figure 25 - Individual Transfer Function Resonant Modes for the Line Configuration of Figure 24

temperature-gradient-induced resonance. These resonant modes have been mathematically isolated in Figure 25. When the curves of Figures 24 and 25 are compared, it is seen that the various resonant modes tend to be additive. The temperature gradient was accounted for by dividing the line section possessing the temperature gradient into ten equal-length isothermal sections.

The dependency of the infinite-line transfer function on line dimensions and gas properties is illustrated in Figures 26 through 31.

By referring to Figures 8 and 23, it seems reasonable to assume that if the reflected waves were completely absorbed before reaching the transducer tap station, line resonance effects would no longer appear and the line would, in effect, behave as a continuous line. This assumption is confirmed by Figure 26, which demonstrates that, for high frequencies where a condition of complete resonance suppression is approached, the infinite-line transfer function tends to assume the characteristics of the continuous-line transfer function illustrated in Figure 9.

The standing-wave envelopes shown in Figure 11 suggest that the amplitude of the fundamental resonant mode will decrease as the attachment station of a line-mounted pressure transducer is moved from the line termination toward the line entrance. This observation is confirmed by Figure 30, which demonstrates that, for a fixed overall line length, the amplitude of the fundamental resonant mode decreases as the transducer tap station is moved toward the line entrance. Figure 28 demonstrates another resonant phenomenon unique to the infinite-line configuration. That is, the fundamental resonant mode appears smaller than the amplitude of higher harmonics when long lines are used. In particular, for the infinite-line configuration described in Figure 28, the peak-to-peak amplitude-ratio variation produced by the third resonant mode exceeds that produced by all other modes. The relative amplitude of resonant modes at the measurement station depends on line dimensions and gas properties and can, in part, be implied by the standing-wave patterns of Figure 11. When the measurement station is close to the line

DIAMETER-DEPENDENT-VARIATIONS IN
THE INFINITE-LINE TRANSFER FUNCTION

$D = \text{VARIABLE}$ $V = 0.4 \text{ CU. CM.}$ $P = 560 \text{ PSIA}$
 $L = 325 \text{ FT.}$ $\text{TAP} = 25 \text{ FT.}$ $T = 500^\circ\text{F}$

Figure 26 - Diameter-Dependent Variations in the Infinite-Line Transfer Function

Figure 27 - Line-Length-Dependent Variations in the Infinite-Line Transfer Function

TEMPERATURE-DEPENDENT-VARIATIONS IN
THE INFINITE-LINE TRANSFER FUNCTION

D = 0.131 IN. V = 0 CU. CM. P = 560 PSIA
L = 325 FT. TAP = 25 FT. T = VARIABLE

Figure 28 - Temperature-Dependent Variations in the Infinite-Line Transfer Function

PRESSURE-DEPENDENT VARIATIONS IN THE
INFINITE-LINE TRANSFER FUNCTION

$D = 0.131$ IN. $V = 0.4$ CU. CM.
 $L = 325$ FT. $TAP = 25$ FT.

$P = \text{VARIABLE}$
 $T = 500^{\circ}\text{F}$

Figure 29 - Pressure-Dependent Variations in the Infinite-Line Transfer Function

TAP-DISTANCE-DEPENDENT VARIATIONS IN THE
INFINITE-LINE TRANSFER FUNCTION

$D = 0.131$ IN. $V = 0.4$ CU. CM. $P = 560$ PSIA
 $L = 325$ FT. TAP = VARIABLE $T = 500^\circ\text{F}$

Figure 30 - Tap-Distance-Dependent Variations in the Infinite-Line
Transfer Function

Figure 31 - Cavity-Volume-Dependent Variations in the Infinite-Line Transfer Function

entrance, it will be located near the antinode regions produced by harmonics of the fundamental resonant mode. Thus, the line-mounted pressure transducer will tend to be more sensitive to the harmonic resonant modes when the distance between the transducer and the line entrance is a small fraction of the overall line length.

Figure 30 demonstrates that pressure measurements are improved as the distance separating the pressure source and the transducer is decreased. The best pressure measurement is obtained at the line entrance. That is, eliminate the line whenever possible and couple the transducer directly to the pressure source.

Figure 31 shows the dependency of the infinite-line transfer function on the transducer cavity volume. Note that amplitude-ratio rolloff increases as the cavity size increases. Therefore, it is important to minimize the cavity size when the best possible frequency response is desired.

When designing an infinite-line pressure-measurement system, the inner line radius, the overall line length, and the size of the transducer cavity are normally the only controllable variables, because all other parameters (line pressure, line temperature, gas properties, and the minimum source-transducer coupling distance) are usually fixed by the test environment. Generally, the infinite line is homogeneous, and its design may adhere to the following three-step procedure:

1. Determine the minimum allowable separation distance between the transducer and the pressure source. Determine the effective volume coupled to the line at the transducer tap station.
2. Determine the lowest probable line pressure and the highest possible line temperature that could be encountered during the test and assume that these values coexist throughout the test period. Assume an infinite line length beyond the transducer tap station and then increase the inner line radius from zero until analytical calculations (Equation 66) indicate that the frequency-response requirements are satisfied under the assumed line pressure and temperature conditions.

$$\frac{P_{d,t}}{P_{o,t}} = \left\{ \cosh(yd) + Q \sinh(yd) + \sinh(yd) \right\}^{-1} \quad (66)$$

3. Determine the highest probable line pressure and the lowest possible line temperature and assume that these values coexist throughout the test period. Decrease the total line length (L) from infinity until analytical calculations (Equation 65) indicate that a further reduction in line length would yield a system incapable of satisfying the frequency-response requirements under the assumed temperature and pressure conditions.

If the line is not homogeneous, the preceding three-step design procedure can still be applied except that either Equation 60 or 64 would be required for steps 2 and 3. If temperature-gradient-induced inhomogeneities are encountered, the highest possible temperature for each line segment will be assumed in step 2 and the lowest possible values will be assumed for step 3.

The inner line radius should be no larger than is absolutely necessary to satisfy overall system frequency-response requirements. A large line radius will increase the handling problems because the line will be more rigid and difficult to shape and route. When nuclear radiation is part of the test environment, the greater mass associated with larger line diameters will tend to increase the magnitude of nuclear-radiation-induced line heating.

The overall line length should be no longer than absolutely necessary because the transient response of a pneumatic line system is degraded as the overall line length is increased. There is also the problem of handling and storing long line lengths.

With the use of the infinite-line technique, the desirable characteristics of both long lines (small oscillation amplitudes) and short lines (small reductions in transmitted signal strength) can be combined to

extend the system frequency response. Be reviewing Figure 22, it is seen that the cutoff frequency (ω_{co}) for the infinite-line configuration is almost equal to that of a continuous line and may be thought of as a wave-propagation-limited frequency value. That is, the frequency response of an infinite line may be thought of as the frequency at which the amplitude of a pressure wave traveling through the line is reduced to 70.7% of its original value by the time it reaches the measurement station. From Figures 9 and 26, it is apparent that decreasing the inner diameter of the line will result in smaller oscillation amplitudes and a lower propagation-limited frequency (a smaller ω_{co} value). Decreasing the overall line length (with inner diameter held constant) would produce larger oscillation excursions but would not produce any change in the propagation-limited-frequency value. In other words, the frequency response of a properly designed infinite-line configuration is essentially independent of total line length but is strongly influenced by the inner line diameter, coupling length, and gas properties.

E. EXPERIMENTAL CONFIRMATION

Most of the equations appearing in this report can be traced to the expressions derived by Iberall¹. Expressions for the polytropic exponent, m (Equation 1); propagation constant, γ (Equation 16); shear number, W (Equation 17); characteristic line impedance, Z_0 (Equation 22); and the reflection coefficient, Q , for a volume-terminated line (Equation 28) were all abstracted or derived from the expressions contained in Iberall's paper. In addition, the segmented-line equation developed by Bergh and Tijdeman could be considered as an extension of Iberall's investigations.

Of all the expressions that have been presented, the Bergh-Tijdeman segmented-line formula (Equation 60) is the most valuable. Simplifications of the segmented-line formula will yield the infinite-line formula (Equation 64) and the transfer function of a volume-terminated line (Equation 27, which was initially derived by Iberall).

Extensive experimental verification of Iberall's volume-terminated-line transfer-function formula (Equation 27) and the Bergh-Tijdeman segmented-line formula (Equation 60) has been performed and reported by G. P. Watts⁹ and H. Bergh and H. Tijdeman². In general, the experimental verification of theoretical predictions has been excellent. The experimental investigations conducted by Watts⁹ indicated that, when end effects are insignificant, Iberall's equations agree to within 2% of actual test data through a 10- to 2000-Hz frequency range. Experimental investigations conducted by Bergh and

1. Iberall, Arthur S., Attenuation of Oscillatory Pressures in Instrument Lines, NBS Research Paper RP2115, Vol. 45, July 1950.
9. Watts, G. P., The Response of Pressure Transmission Lines, 20th Annual Instrumentation Society of America Conference and Exhibit, October 1965, Los Angeles.
2. Bergh, H., and Tijdeman, H., Theoretical and Experimental Results for the Dynamic Response of Pressure Measuring Systems, National Aero and Astronautical Research Institute, Technical Report NLR-TR F.238, Amsterdam, January 1965.

Figure 32 - CFDTs Infinite-Line Configuration

Tijdeman showed that the segmented-line equation agreed to within 2 to 5% of experimental data when single- and double-section lines were used through a frequency range of 10 to 200 Hz.

Although excellent experimental confirmation of theoretical predictions was obtained at frequencies above 10 Hz, the control system for the NRX/EST nuclear rocket test series required an accurate definition of the infinite-line transfer function below 10 Hz.

The first opportunity for experimental confirmation was in a cold-flow nozzle test conducted at the Aerojet-Sacramento test facility. This test, designated CFDTs, simulated engine startup and was to confirm control-system performance prior to NRX/EST testing. Because the purpose of the CFDTs setup was to establish the adequacy of infinite-line measurement techniques and to confirm the theoretical prediction model, the same infinite-line configuration was used as for NRX/EST.

Because of flow asymmetry within the nozzle during startup, a second transducer was mounted on the infinite-line and positioned as close to the nozzle as possible. The final CFDTs infinite-line configuration is shown in Figure 32. The output of the close-coupled transducer was assumed to represent the nozzle chamber pressure. It was further assumed that use of the close-coupled tap would not alter the infinite-line transfer function within the control-system operational frequency band.

A review of the recorded data revealed a transient-pressure perturbation at 191.7 sec during CFDTs No. 1.2-02-NNS-010 that would lend itself to spectral analysis. A tracing of the recorded data is included in Figure 32. Because available plant facilities could not analyze pulse data of this magnitude and duration, Fourier analysis by numerical integration was performed.

Figure 33 - CFDTs 1.2-02-NNS-010 Test Results

The Fourier analysis is based on the fact that any repetitive wave, regardless of form, can be synthesized by summing an infinite number of sinusoidal waves of different frequencies and magnitudes. The technique used to yield a sinusoidal summation, which would synthesize the two pressure pulses, is described in¹⁰. The pulse duration (0.33 sec) was assumed to represent the half-wave interval (180 degrees) of a repetitive wave possessing half-wave symmetry about the line bias-pressure level. The half-wave pulses were subdivided into 64 equal parts, and numerical integration techniques were used to yield a Fourier series for both pulses. Knowing the Fourier series and the pulse transport time delay (0.064 sec), the amplitude ratio and phase angle were calculated and compared with theoretical prediction curves as shown in Figure 33. The theoretical prediction curves were calculated to correspond to the test conditions that existed during the pulse period.

As shown in Figure 33, the agreement between the experimental points and the theoretical curves at the fundamental pulse frequency ($f_1 = 1.515$ Hz) was extremely good in view of the approximate nature of numerical integration techniques and the amount of noise present. The only discrepancy that could be considered excessive was the amplitude-ratio deviation occurring at the third harmonic ($f_3 = 4.415$ Hz). As yet, the existence of the large third-harmonic error has not been satisfactorily explained. The extent of agreement between experimental and theoretical values was encouraging, because the pulse was a transient phenomenon whereas the theoretical model assumed that steady-state conditions prevailed. The good agreement implied that transient effects produced by a pressure pulse of the type encountered during CFDTS 1.2-02-NNS-010 startup would be suppressed by the infinite line.

The second test, which confirmed the correctness of the theoretical model, employed the sinusoidal pressure generator shown in Figure 34. The generator consisted of a hydraulically driven piston whose motion within a

10. Skilling, H. H., Electrical Engineering Circuits, John Wiley and Sons, Inc., pp. 439-444.

Figure 34 - Infinite-Line Low Frequency Test

cylinder generated sinusoidal pressure oscillations. The generator frequency was variable from 0.1 to 40.0 Hz. By using control feedback circuitry, the piston was forced to follow the sinusoidal signal emanating from an audio oscillator with a position linearity uncertainty that did not exceed 0.2% of the full-scale piston stroke.

Line bias pressure was controlled with a helium reservoir while line temperature was maintained by placing the sinusoidal generator, helium reservoir, and infinite line within an environmental test cell whose temperature was held at 72°F. The pressure transducers used for this test had a frequency response that was essentially flat to 300 Hz and had an accuracy of $\pm 0.5\%$ of full scale. In an attempt to minimize measurement uncertainty, zero suppression techniques were used to electrically nullify line bias-pressure signals, and shunt calibration was used to calibrate each transducer-amplifier-recorder data channel.

The test results are presented in Figure 35, where the points represent experimental data and the curves represent theoretical predictions. Since the theoretical predictions assume laminar flow within the line, oscillatory amplitudes were generally held to less than 1% of the line bias pressure to prevent line turbulence. In general, line turbulence was easily discerned by the appearance of a distorted sine wave on the oscilloscope monitor. As based on the equipment used and estimated oscillographic reading errors, the estimated uncertainty of experimentally derived data was $\pm 8.00\%$ of the data point. However, the average deviation between measured and predicted values was $\pm 4.77\%$ of the data point, which was well within the expected experimental error.

Comparison of the theoretical Iberall and Bergh-Tijdeman formulas and experimental results has shown that the response characteristics of pneumatic-line pressure-measuring systems can be predicted with an uncertainty of less than 5% from 1 to 2000 Hz. Because excellent confirmation of the exact

Figure 35 - Infinite-Line Low Frequency Test Results

equations has been achieved, simplifications of the exact equations should also yield valid results when properly applied. That is, because the approximation equations developed in this report agree well with the exact equations from which they were derived, the approximation equations can be used to good advantage under the stated boundary limitations. Confidence in the approximation equations has been increased by several of the equations being independently derived and experimentally confirmed by other investigators.

The organ-pipe (Equation 33) and Helmholtz (Equation 42) resonant equations have been known for many years and need no further verification. The expression defining the attenuation constant for small viscous damping (Equation 20) has been derived (Equation 77 of Reference 11) and experimentally confirmed by Nichols.

The upper and lower transfer-function loci (Equations 39 and 40) have been derived and experimentally confirmed by Karam and Franke¹². When working with gases where $\gamma \approx 1.4 \approx \sqrt{2}$, the effective second-order damping coefficient for high viscous damping (Equation 54) can be simplified to yield

$$\delta = \frac{4\mu L}{c\rho R^2} \sqrt{\frac{4}{\pi^2} + \frac{V}{AL}},$$

which, if $4/\pi^2$ were replaced with $1/2$, is identical to an expression (Equation 2 of Reference 13) derived and experimentally confirmed by Hougen, Martin, and Walsh. The effective first-order approximation formula (Equation 53), from which the effective system time constant was derived (Equation 56), is identical to Equations 50 and 52 of Reference 1 when very long and very short line lengths are assumed, respectively.

11. Nichols, N. B., The Linear Properties of Pneumatic Transmission Lines, ISA Transactions: 1 (5-14), 1962.
12. Karam, J. T., and Franke, M. E., The Frequency Response of Pneumatic Lines, 7th Joint Automatic Control Conference, Seattle, August 1966.
13. Hougen, J. O., Martin, O. R., and Walsh, R. A., Dynamics of Pneumatic Transmission Lines, Control Engineering, September 1963, pp 114-117.

Although other similarities to the approximation equations developed here may be available, the sample provided should add some measure of confidence to approximate solutions. However, it must be emphasized that the approximate solutions are simplifications of more exact expressions and must be used with discretion. The exact equations should be used whenever possible. The approximation equations should never be used unless their inherent limitations are fully understood.

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VI. SYMBOL LIST

a	=	Attenuation constant, nepers/cm
A	=	πR^2 = Inner cross-sectional line area, cm^2
b	=	Phase constant, radians/cm
B	=	$ W $ = Absolute shear-number value
c	=	Adiabatic velocity of sound, cm/sec
c_n	=	Newtonian velocity of sound, cm/sec
c_m	=	Polytropic velocity of sound, cm/sec
c_p	=	Mean gas specific heat at constant pressure, cal/gm-°K
C	=	Second-order system response function
d	=	Separation distance between a line mounted transducer and the line entrance, cm
D	=	Inner line diameter, cm
E	=	$W\sigma$
F	=	Second-order system forcing function
G	=	Amplitude ratio (amplitude gain), db
G_p	=	Amplitude-ratio (amplitude gain) value at resonance, db
$G_{\min}$	=	Amplitude-ratio (amplitude gain) value at pseudo-antiresonant frequencies, db
$\left. \begin{matrix} H_1 \\ H_2 \end{matrix} \right\}$	=	Proportionality constants
I	=	Mathematical constant
j	=	$\sqrt{-1}$
J_0	=	Bessel function of order zero
J_1	=	Bessel function of order one
k	=	Mean gas thermal conductivity, cal/sec-cm-°K
$\left. \begin{matrix} K_1 \\ K_2 \end{matrix} \right\}$	=	Integration constants
L	=	Physical line length, cm
L_a	=	Acoustic line length, cm
m	=	Polytropic exponent

M	= Gas molecular weight, gm/mole
$\left. \begin{matrix} N_1 \\ N_2 \end{matrix} \right\}$	= Integration Constants
P	= Pressure, dynes/cm ²
P _s	= Static mean pressure, dynes/cm ²
(P _s) _o	= Initial static mean pressure, dynes/cm ²
(P _s) _f	= Final static mean pressure, dynes/cm ²
(P _s) _L	= Instantaneous static mean pressure at the line termination, dynes/cm ²
P _o	= Peak amplitude of the oscillatory pressure component at the line entrance, dynes/cm ²
P _x	= Peak amplitude of an oscillatory pressure component at a distance (x) beyond the line entrance, dynes/cm ²
P _t	= Time-dependent pressure variations, dynes/cm ²
P _{o,t}	= Time-dependent pressure variation at the line entrance, dynes/cm ²
P _{x,t}	= Time-dependent pressure variation at any distance (x) beyond the line entrance, dynes/cm ²
P _{d,t}	= Time-dependent pressure variation at the point where inner line pressures are to be measured, dynes/cm ²
P _{L,t}	= Time-dependent pressure variations at the line termination, dynes/cm ²
Q	= Reflection coefficient for a volume-terminated line (defined by Equation 28)
R	= Inner line radius, cm
s	= Laplace-transform operator
t	= Time, sec
t'	= Transport delay time, sec
T	= Absolute mean temperature, °K
U	= Peak velocity amplitude of oscillatory gas motion, cm/sec
U _t	= Time-dependent velocity of oscillatory gas motion, cm/sec
U _{o,t}	= Time-dependent velocity of gas motion at the line entrance, cm/sec
U _{x,t}	= Time-dependent velocity of gas motion at a distance (x) from the line entrance, cm/sec
v	= Velocity, cm/sec
V	= Cavity volume size, cm ³

W	=	Shear number (defined by Equation 17)
x	=	Arbitrary distance from line entrance, cm
y	=	Propagation constant (defined by Equation 16)
Y	=	Acoustic admittance per unit line length, $\text{cm}^3/\text{sec-dynes}$
Z	=	Acoustic impedance per unit line length, dynes-sec/cm^3
Z_o	=	Characteristic line impedance (defined by Equation 22), dyne-sec/cm^3
Z_L	=	Line impedance at a line discontinuity boundary, dyne-sec/cm^3

Greek Characters

γ	=	Ratio of specific heat
δ	=	Viscous-damping coefficient for second-order systems
θ	=	Phase lag angle, degrees
λ	=	Wave length, cm
μ	=	Mean gas viscosity, dyne-sec/cm^2
π	=	3.14159
ρ	=	Mean gas density, gm/cm^3
σ	=	$\sqrt{\text{Prandtl Number}} = \sqrt{c\rho\mu/k}$
τ	=	Effective first-order system response time, sec^{-1}
ω	=	Oscillatory frequency, radians/sec
ω_n	=	Resonant frequency, radians/sec
ω_r	=	Natural undamped fundamental resonant frequency, radians/sec
ω_p	=	Peak resonant pressure amplitude frequency, radians/sec
ω_{pr}	=	Phase resonant frequency, radians/sec
ω_{co}	=	System cutoff frequency, radians/sec