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ON OPERATOR SOLUTIONS
OF
BOUNDARY - VALUE PROBLEMS

by

Oswald Wyler

DECEMBER 1959

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ON OPERATOR SOLUTIONS
OF
BOUNDARY-VALUE PROBLEMS

by

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December 1959

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TABLE OF CONTENTS

	Page
INTRODUCTION	5
I. ALGEBRAIC THEORY	9
1. Inner Product Spaces	9
2. Operators on E	10
3. Statement of the Problem	11
4. General Theorems on Operator Solutions	13
5. A Class of Operator Solutions	14
6. Another Class of Operator Solutions	15
7. Green Operators	16
8. Proper Green Operators	17
9. Properties of Proper Green Operators	19
10. Equivalent Operator Solutions	20
11. Null Spaces of Green Operators	21
12. Special Cases	23
II. APPLICATIONS	25
13. Notations	25
14. Associated Boundary-Value Problems	26
15. Preliminary Considerations	27
16. Associated Operator Solutions	28
17. Relations Between Associated Operator Solutions	30
18. Characterizations of Associated Operator Solutions	31
REFERENCES	33

ON OPERATOR SOLUTIONS OF BOUNDARY-VALUE PROBLEMS

INTRODUCTION

The simplest way of stating an inhomogeneous boundary-value problem for differential equations is perhaps the following:

(I) Given u in E , find x in F such that $Lx = u$.

Here E is a certain functional space, L is a differential operator, with its domain and range contained in E , and F is the set of all elements of the domain of L which satisfy the given boundary conditions. If the boundary conditions are linear and homogeneous, then F is a subspace of E . This is the only case discussed in this report.

In many cases, E is an inner product space, and it is possible to define an adjoint operator L^* of L , and an adjoint problem:

(I*) Given v in E , find y in F^* such that $L^*y = v$.

Typically, we have $(y, Lx) = (L^*y, x)$ for any x in F and any y in F^* , and (I) can be solved, for u in E , if and only if u is orthogonal to the solution space of the homogeneous adjoint problem:

(H*) Find y in F^* such that $L^*y = 0$.

Then the first stage of the theory of (I) consists of proving this statement, and of finding a solution of (I) by an operator on E or, as we shall say, an operator solution of (I). By this we mean an operator G on E such that Gv is a solution of (I) whenever v is orthogonal to the solution space of (H*).

In most cases, it is found that an operator G on E exists which has not only this property, but also the additional property that G^*v is a solution of (I*) whenever v is orthogonal to the solution space of the problem:

(H) Find x in F such that $Lx = 0$.

In other words, an operator solution G on E exists such that the adjoint operator G^* of G is an operator solution of (I^*) . If this is the case, then the further theory is completely symmetrical with respect to the adjoint problems (I) and (I^*) .

The next stage of the theory of (I) and (I^*) is a discussion of properties of operator solutions, of relations between operator solutions, and if possible, a usable description of the class of all operator solutions. If (H) and (H^*) have only the trivial zero solution, then this part of the theory is particularly simple: there cannot be more than one operator solution of (I) or of (I^*) . If E is a functional space, this operator usually is an integral operator, defined by a kernel function or matrix. The kernel of this operator then is called Green's function or matrix for (I) .

If the solution spaces of (H) and of (H^*) are nontrivial, then several operator solutions of (I) may (and do) exist. Thus the second stage of the theory of (I) , as outlined in the preceding paragraph, must be studied more closely. We do this in Chapter I of this report, under the assumptions outlined above, and the additional assumption that the solution spaces of (H) and of (H^*) are finite-dimensional. The theory of operator solutions of (I) then turns out to be purely algebraic.

If (H) or (H^*) has nonzero solutions, then one can use, instead of Green's function or matrix, a class of kernels of operator solutions of (I) with similar properties. These are called generalized Green's functions or matrices. This class of kernels is defined in the literature in two essentially different ways. In our general theory, the integral operators with these kernels become what we call Green operators, and proper Green operators, corresponding to the two ways of defining generalized Green's functions. We give special attention to the properties of these operators.

There are two classes of boundary-value problems of type (I) for ordinary differential equations. In the first class, L is a scalar differential operator of order n , acting on real- or complex-valued functions of class C_n . In the second class, L is of the form $Lx = Px + Q\dot{x}$, where P and Q are square matrices of order n , and L acts on differentiable functional columns of length n . We shall call these classes the class of scalar problems and the class of vector problems, respectively.

It was shown in an earlier report by the same author² that all assumptions of Chapter I of this report are satisfied by both classes of problems mentioned in the preceding paragraph. It was also shown that every pair (Is) , (Is^*) of adjoint scalar

problems is connected with a pair (Iv), (Iv*) of adjoint vector problems in such a way that the solutions of (Is) and (Is*) are obtained from special solutions of (Iv) and (Iv*). We have recapitulated this connection in Section 14 of this report, and the connection between solutions of (Is) and of (Iv) is restated as Theorem 11. In Chapter II of this report, we study the connections between operator solutions of scalar problems and vector problems which are associated in this way.

Operator solutions for boundary-problems for operators of the form $Lx = Px + Qx$ have been studied by W. T. Reid in Reference 1. Reid considered only problems satisfying the following two conditions: $Q = I_n$, and the number of independent boundary conditions for (I) (which determine F) is equal to the order n of the operator L . All results of Reference 1 remain valid in the general theory considered in Chapter I of this report.

CHAPTER I

ALGEBRAIC THEORY

1. Inner Product Spaces

We denote by K a field which is either the field of real numbers or the field of complex numbers. For c in K , we denote by $\bar{c}$ the conjugate complex of c , as usual ($\bar{c} = c$ if c is real).

A vector space E over K is called an inner product space if an inner or scalar product, denoted by (u, v) , is defined for any two vectors u and v in E , with the following properties.

- (a) For u and v in E , (u, v) is defined and in K .
- (b) We have $(u, c_1 v_1 + c_2 v_2) = c_1 (u, v_1) + c_2 (u, v_2)$ for any vectors u, v_1 and v_2 in E , and any scalars c_1 and c_2 in K .
- (c) $(v, u) = \overline{(u, v)}$ for any two vectors u and v in E .
- (d) $(u, u) > 0$ for any vector $u \neq 0$ in E .

In defining inner product spaces, it is customary to require that (u, v) is linear in the first argument. We have changed this here because of the applications of our theory to spaces of functional columns, where the natural definition of an inner product is $(u, v) = \int_a^b u^* v$. This is linear in the second argument.

If (u, v) is linear in one argument, then it follows from (c) that (u, v) is conjugate linear in the other argument. Thus it follows from (b) and (c) that $(c_1 u_1 + c_2 u_2, v) = \bar{c}_1 (u_1, v) + \bar{c}_2 (u_2, v)$ for any vectors u_1, u_2 , and v in E and any scalars c_1 and c_2 in K .

The following "shorthand" notation will be helpful. If U is a row of vectors in E , of length m , and A an (m, n) -matrix with entries in K , then UA is defined as a matrix product in the usual way. UA is a row of vectors in E , of length n .

If $U = \|\|u_i\|\|$ and $v = \|\|v_j\|\|$ are rows of vectors in E , of lengths m and n , we put $(U, V) = \|\|(u_i, v_j)\|\|$. This is an (m, n) -matrix with entries in K .

The algebraic laws for this shorthand notation are easily established; we note only a few of them. If $A = \|\|a_{ij}\|\|$ is a matrix with entries in K , we denote by $A^* = \|\|\bar{a}_{ji}\|\|$ the conjugate transpose matrix of A . Then we have $(V, U) = (U, V)^*$ for any two rows of vectors in E , and $(UA, V) = A^*(U, V)$ and $(V, UA) = (V, U)A$ if UA is defined.

We say that two vectors u and v in E , or two rows U and V of vectors in E , are orthogonal if $(u, v) = 0$ or $(U, V) = 0$, respectively.

We have discussed matrices with zero rows or columns in Reference 2, Section 9. We shall use such matrices, and rows of vectors in E of length zero, in the same way as other matrices and rows of vectors. The algebraic laws for the algebra of matrices and rows of vectors in E remain the same if we do this, and we avoid lengthy discussions of special cases.

2. Operators on E

Let E be an inner product space, over the field K . Two linear transformations G, H of E into itself are called adjoint if $(v, Gu) = (Hv, u)$ for any two vectors u, v in E . It is easily seen that a linear transformation of E into itself cannot have more than one adjoint transformation.

For the purposes of this paper, we define an operator on E as a linear transformation of E into itself which has an adjoint linear transformation. We denote the adjoint transformation of an operator G by G^* . It is easily seen that then G^* is an operator, with $(G^*)^* = G$.

Operators on E form a ring of transformations; i.e., the sum $G + H$ and the iteration product HG ("first G , then H ") of two operators G, H are operators. Adjunction defines an involutive antiautomorphism of this ring; i.e., $(G + H)^* = G^* + H^*$, $(HG)^* = G^*H^*$, and $(G^*)^* = G$ for any two operators G and H on E .

If $U = \|\|u_i\|\|$ is a row of vectors in E , and G is an operator on E , we put $GU = \|\|Gu_i\|\|$. Then $(U, GV) = (G^*U, V)$ for any two rows U and V of vectors in E , and any operator G on E .

If U and V are two rows of vectors in E , of the same length, we define an operator $[U, V]$ on E by putting

$$(2.1) \quad [U, V]u = U(V, u)$$

for any vector u in E . Then $[U, V]$ obviously is a linear transformation of E into itself. To show that $[U, V]$ is an operator, we prove:

$$(2.2) \quad [U, V]^* = [V, U].$$

We have indeed

$$\begin{aligned} (v, [U, V]u) &= (v, U(V, u)), \\ ([V, U]v, u) &= (V(U, v), u) = (U, v)^*(V, u) = (v, U)(V, u), \end{aligned}$$

for any two vectors u and v in E .

The "operator product" $[U, V]$ is distributive on both sides, and satisfies an "associative" law: If UA is defined and a row of the same length as V , then $[UA, V] = [U, VA^*]$.

If U and V are rows of vectors in E , of the same length, then $G[U, V] = [GU, V]$ and $[U, V]G = [U, G^*V]$ for any operator G on E . It can be shown that an operator G on E is of the form $[U, V]$ if and only if the range of G is finite-dimensional. Thus these operators form an ideal of the ring of operators on E .

3. Statement of the Problem

Algebraically, an inhomogeneous boundary-value problem with homogeneous boundary conditions is of the following type. A linear transformation L is given which maps a subspace F of a vector space E into E , and we want to find the range $L(F)$ of L , and the solutions of the equation $Lx = u$, with x in E , for a given vector u in E . We formulate this problem briefly as follows.

(I) Given u in E , find x in F such that $Lx = u$.

The null space of L is defined as the space of all vectors x in F such that $Lx = 0$. We denote it by $N(L)$. If $Lx_0 = u$, x_0 in F , then $Lx_1 = u$ for a vector x_1 in F if and only if $x_1 - x_0$ is in $N(L)$.

Assume that E is an inner product space, and that two linear transformations L and L^* of subspaces F and F^* of E into E are given. We call L and L^* adjoint operators if they satisfy the following condition.

$$(1) \quad (y, Lx) = (L^*y, x) \text{ for any vectors } x \text{ in } F \text{ and } y \text{ in } F^*.$$

If this condition is satisfied, and if $u = Lx$, x in F , then $(y, u) = (L^*y, x) = 0$ for any vector y in $N(L^*)$. Thus (I) can only be solved if u is orthogonal to $N(L^*)$ (i.e., to every vector in $N(L^*)$).

If L and L^* are adjoint operators, and if the problems

(I) Given u in E , find x in F such that $Lx = u$;

(I*) Given v in E , find y in F^* such that $L^*y = v$;

satisfy the following conditions:

(2) (I) has a solution whenever u is orthogonal to $N(L^*)$;

(2*) (I*) has a solution whenever v is orthogonal to $N(L)$;

then we call (I) and (I*) adjoint problems.

We shall assume from now on that two adjoint problems (I) and (I*) in an inner product space E are given. We define an operator solution of (I) as an operator G on E such that Gu is in F , and $LGu = u$, whenever u in E is orthogonal to $N(L^*)$. In other words, we require that Gu is a solution of (I) for all vectors u in E which are orthogonal to $N(L^*)$. Operator solutions of (I*) are defined in the same way.

We shall discuss operator solutions of (I) and of (I*) under the following additional assumptions.

(3) $N(L)$ and $N(L^*)$ are finite-dimensional vector spaces.

(4) (I) has an operator solution.[†]

Under these assumptions, the theory of operator solutions of (I) and of (I*) turns out to be purely algebraic. We note that, because of (3), we can reformulate (2) and (2*) as follows: The ranges of L and of L^* are the orthogonal complements in E of $N(L^*)$ and of $N(L)$, respectively.

[†]The author found, after completion of the present report, that (4) follows from (1), (1*), (2), (2*), and (3). This result is a consequence of a more general theory to be discussed in a later report.

4. General Theorems on Operator Solutions

The following notations shall be used throughout this chapter. We denote by m and n the dimensions of $N(L)$ and of $N(L^*)$. We let X and Y be rows of vectors in E such that the entries of X form a basis of $N(L)$, and the entries of Y form a basis of $N(L^*)$. We denote by X_0 and Y_0 two fixed rows of vectors in E such that $(X, X_0) = I_m$ and $(Y, Y_0) = I_n$. Such rows exist since (X, X) and (Y, Y) are nonsingular matrices. A vector u of E is orthogonal to $N(L^*)$ if and only if $(Y, u) = 0$.

Let now G and G_1 be operator solutions of (I). If we put

$$H = G - G_1 - [Z, Y], \text{ with } Z = GY_0 - G_1Y_0,$$

then H is an operator on E , and $HY_0 = 0$. If u in E is orthogonal to $N(L^*)$, then $LG_u = LG_1u = u$, Gu and G_1u are in F , and $[Z, Y]u = 0$. Thus Hu is in $N(L)$ and hence of the form $X\alpha$ for a column α with entries in K . But then $\alpha = (X_0, Hu) = (H^*X_0, u)$.

If $u_0 = u - Y_0(Y, u)$, then $(Y, u_0) = 0$, and $Hu_0 = Hu$, for any vector u in E . Thus $Hu = X(X_0, Hu) = X(H^*X_0, u)$ for any vector u in E , and $H = [X, W]$ with $W = H^*X_0$.

Conversely, let G_1 be an operator solution of (I), and let

$$(4.1) \quad G = G_1 + [Z, Y] + [X, W],$$

where Z and W are rows of vectors in E . If u in E is orthogonal to Y , then G_1u is a solution of (I), and $Gu - G_1u = X(W, u)$ is in $N(L)$. Thus Gu is a solution of (I), and G is an operator solution of (I). This completes the proof of the following theorem.

THEOREM 1. -- Let G_1 be an operator solution of (I). Then any operator solution G of (I) is of the form (4.1), where Z and W are rows of n and m vectors in E , respectively. Conversely, any operator G of this form is an operator solution of (I).

Let now G be an operator solution of (I), and let u and v in E be vectors such that $(Y, u) = 0$ and $(X, v) = 0$. Then $v = L^*y$ for some y in F^* , $LG_u = u$, and we have

$$(u, G^*v) = (Gu, v) = (Gu, L^*y) = (LG_u, y) = (u, y).$$

Thus $y - G^*v$ is in the orthogonal complement $N(L^*)$ of the range of L , and hence G^*v is in F^* , and $L^*G^*v = L^*y = v$. This shows that G^* is an operator solution of (I^*) . This, and an analogous converse argument, proves the following theorem.

THEOREM 2. -- An operator G on E is an operator solution of (I) if and only if the adjoint operator G^* of G is an operator solution of (I^*) .

With this theorem, the assumptions of Section 3 become completely symmetric with respect to (I) and (I^*) , and thus every theorem or formula for (I) has an obvious counterpart for (I^*) , and vice versa. We shall use the same number for the two theorems or formulas, with an asterisk after this number if the theorem or formula for (I^*) is used.

5. A Class of Operator Solutions

In this section we study operator solutions of (I) which map E into F .

LEMMA 1. -- Let Y_1 be a row of n vectors in E such that $(Y, Y_1) = I_n$. Then an operator solution G of (I) maps E into F if and only if the entries of GY_1 are in F .

PROOF. -- For u in E , let $u_1 = u - Y_1(Y, u)$. Then $(Y, u_1) = 0$, and thus Gu_1 is in F . If the entries of GY_1 are in F , it follows that Gu is in F . The converse is obvious.

Let now G be an operator solution of (I) which maps E into F . Then LG is defined. For u in E , let $u_0 = u - Y_0(Y, u)$. Then $(Y, u_0) = 0$, and hence $LG u_0 = u_0$. Thus we have

$$(5.1) \quad LGu = u - Y_1(Y, u),$$

for any u in E , and $Y_1 = Y_0 - LGY_0$.

Conversely, let G be an operator on E mapping E into F , and assume that G satisfies an equation of this type. Then G obviously is an operator solution of (I) , and $Y_1 = Y_0 - LGY_0$. Since $(Y, LGY_0) = (L^*Y, GY_0) = 0$, we have $(Y, Y_1) = I_n$. This completes the proof of the following theorem.

THEOREM 3. -- If G is an operator solution of (I) which maps E into F , then G satisfies an equation of the form (5.1), where Y_1 is a row of n vectors in E such that $(Y, Y_1) = I_n$. Conversely, an operator G on E which maps E into F and satisfies an equation of the form (5.1) always is an operator solution of (I).

How much do these properties characterize G ? The answer is given by

THEOREM 4. -- Let Y_1 be a row of n vectors in E such that $(Y, Y_1) = I_n$. Then there are operators on E which map E into F and satisfy (5.1) for the given row Y_1 . If G_1 is such an operator, then an operator G on E satisfies these conditions if and only if there is a row W of m vectors in E such that $G = G_1 + [X, W]$.

PROOF. -- Let G be an operator solution of E , and let $G_1 = G + [Z, Y]$ with $Z = -GY_1$. Then $G_1 Y_1 = 0$, and thus G_1 is an operator solution of (I) which maps E into F , by Theorem 1 and Lemma 1. For u in E , let $u_1 = u - Y_1(Y, u)$. Then $(Y, u_1) = 0$, and thus $LG u_1 = u_1$. Since $G_1 Y_1 = 0$, we have $G_1 u = G_1 u_1$, which shows that G_1 satisfies (5.1), for the given row Y_1 .

Let now G and G_1 be operators on E which satisfy the given conditions. Then $Cu - G_1 u$ is in $N(L)$ for any u in E and hence of the form $X\alpha$, where α is a constant column of length m . Then

$$\alpha = (X_0, Cu - G_1 u) = (G^* X_0 - G_1^* X_0, u),$$

and thus $G - G_1 = [X, W]$ with $W = G^* X_0 - G_1^* X_0$.

Finally, $[X, W]$ maps E into F for any row W of m vectors in E , and $L[X, W] = 0$. Thus G maps E into F , and $LG = LG_1$, for $G = G_1 + [X, W]$ if G_1 maps E into F . This completes the proof.

6. Another Class of Operator Solutions

In this section we study operator solutions G of (I) such that G^* maps E into F^* .

If G is such an operator solution, then

$$(6.1) \quad L^* G^* v = v - X_1(X, v)$$

for v in E , with $X_1 = X_0 - L^*G^*X_0$, by Theorem 3* and its proof. For x in F , GLx is in F , and $LGLx = Lx$, since $(Y, Lx) = (L^*Y, x) = 0$. Thus $GLx = x - X\alpha$ for some column α of length m with entries in the field K , and then

$$\alpha = (X_0, x - GLx) = (X_0 - L^*G^*X_0, x).$$

Thus we have

$$(6.2) \quad GLx = x - X(X_1, x)$$

for any x in F with the same row X_1 of vectors in E .

Conversely, let G be an operator on E which satisfies (6.2). If u in E satisfies $(Y, u) = 0$, then $u = Lx$ for some x in F , and it follows from (6.1) that Gu is in F and $LGu = Lx = u$. Thus G is an operator solution of (I). Using (6.1) with x replaced by X , we see that $(X_0, GLX) = I_m - (X_1, X)$. Since $LX = 0$, we conclude that $(X_1, X) = (X, X_1) = I_m$.

Now let G_1 be an operator on E such that G_1^* maps E into F^* and satisfies (6.1). Such an operator exists by Theorem 4*, and G_1 is an operator solution of (I) by Theorems 3* and 2. It follows that G_1 satisfies (6.2). If G is any operator satisfying (6.2), then $GLx = G_1Lx$ for any x in F . For u in E , let $u_0 = u - Y_0(Y, u)$. Then $(Y, u_0) = 0$, $u_0 = Lx$ for some x in F , and thus $Gu_0 = G_1u_0$. It follows that $Gu = G_1u + Z(Y, u)$, and hence $G = G_1 + [Z, Y]$, for $Z = GY_0 - G_1Y_0$. By Theorem 4*, G^* maps E into F^* , and G satisfies (6.1). This completes the proof of the following theorem.

THEOREM 5. -- An operator G on E satisfies (6.2) for some row X_1 of m vectors in E , if and only if G^* maps E into F^* , and G satisfies (6.1) for the same row X_1 of vectors in E .

We note again that an operator G satisfying (6.2) exists only if $(X, X_1) = I_m$, that any such operator is an operator solution of (I), and that the difference of two such operators is described by Theorem 4*.

7. Green Operators

An operator solution G of (I) is called a Green operator for the problem (I) if G maps E into F and G^* maps E into F^* .

It follows immediately from Theorem 2 that an operator G on E is a Green operator for (I) if and only if G^* is a Green operator for (I*). The theory of Sections 5 and 6 applies in particular to Green operators.

The existence of Green operators will be established in the next section. To characterize the difference of two Green operators, we prove first the following lemma.

LEMMA 2. -- Let G_1 be an operator solution of (I) which maps E into F , and let G be an operator of the form (4.1). Then G maps E into F if and only if the entries of Z are in F .

PROOF. -- We have $GY_0 = G_1Y_0 + Z + X(W, Y_0)$, and thus the entries of GY_0 are in F if and only if the entries of Z are. Lemma 2 now follows immediately from Theorem 1 and Lemma 1.

Using Theorem 1 and Lemmas 2 and 2*, we immediately obtain the following theorem.

THEOREM 6. -- Let G_1 be a Green operator for (I). Then any Green operator G for (I) is of the form (4.1), where Z is a row of n vectors in F , and W is a row of m vectors in F^* . Conversely, any operator G of this form is a Green operator for (I).

8. Proper Green Operators

THEOREM 7. -- Let X_1 and Y_1 be rows of m and n vectors in E , respectively, such that $(X, X_1) = I_m$ and $(Y, Y_1) = I_n$. There is an operator G on E which satisfies the following three groups of properties and is completely characterized by any one of these three groups of properties.

- (i) G maps E into F , G satisfies (5.1) for the given row Y_1 of vectors in E , and $G^*X_1 = 0$.
- (ii) G satisfies (6.2) for the given row X_1 of vectors in E , and $GY_1 = 0$.
- (iii) G is an operator solution of (I), $GY_1 = 0$, and $G^*X_1 = 0$.

PROOF. -- Let G_1 be an operator which maps E into F and satisfies (5.1). By Theorem 4, any operator G with these properties is of the form $G = G_1 + [X, W]$, and then $G^*X_1 = G_1^*X_1 + W$. This shows that there is exactly one operator G which satisfies group (i) of the requirements.

The theorem will be proved if we show that the three groups of requirements for G are logically equivalent. To do this, we prove the following lemma. The form of the hypothesis of this lemma is dictated by its use in Chapter II.

LEMMA 3. -- Let Y_1 be a row of n vectors in E such that $(Y, Y_1) = I_n$, and let G be an operator solution of (I) for which $LG Y_1$ is defined. Then $LG u$ is defined for all u in E , and G satisfies (5.1) for the given row Y_1 of vectors in E if and only if $LG Y_1 = 0$.

PROOF. -- For u in E , let $u_1 = u - Y_1(Y, u)$. Then $(Y, u_1) = 0$, and thus $LG u_1 = u_1$, and $LG u$ is defined if $LG Y_1$ is. If $LG Y_1 = 0$, then $LG u = u_1$ for any u in E ; i.e., (5.1) is satisfied. Conversely, $LG Y_1 = 0$ follows immediately from (5.1). This proves the lemma.

We continue with the proof of the theorem. If G satisfies (iii), then G is a Green operator by Lemmas 1 and 1*, and it follows from Lemmas 3 and 3* and Theorem 5 that G satisfies (i) and (ii).

If G satisfies (i), then $LG Y_1 = 0$. Thus the entries of GY_1 are in $N(L)$, and there is an (m, n) -matrix T with entries in the field K such that $GY_1 = XT$. Then $T = (X_1, GY_1) = (C^*X_1, Y_1) = 0$, and hence $GY_1 = 0$. G^* maps E into F^* by Lemma 1. Thus G satisfies (6.1) by Lemma 3*, and hence (6.2) is satisfied by Theorem 5. It follows that G satisfies (ii) and (iii).

If G satisfies (ii), then G^* satisfies (i*) by Theorem 5 and hence (iii*). Thus G satisfies (iii), which completes the proof of the theorem.

Using (iii) and Lemmas 1 and 1*, we see that the operator G characterized by the properties of Theorem 7 is a Green operator. We call G the proper Green operator for (I) defined by X_1 and Y_1 (in this order).

If X_1 and Y_1 satisfy the hypothesis of Theorem 7, then G is the proper Green operator for (I) defined by X_1 and Y_1 if and only if G^* is the proper Green operator for (I*) defined by Y_1 and X_1 . This is an immediate consequence of Theorem 7 (iii).

We discuss the role of the auxiliary matrices X_1 and Y_1 in the characterization of a proper Green operator more fully in Section 11.

9. Properties of Proper Green Operators

Let G be a Green operator, and let G_0 be the proper Green operator defined by X_0 and Y_0 . By Theorem 6, there is a row Z of n vectors in F , and a row W of m vectors in F^* , such that

$$(9.1) \quad G = G_0 + [Z, Y] + [X, W].$$

We shall establish conditions for G to be a proper Green operator.

It follows from (9.1) that $LG_u = LG_0 U + (LZ)(Y, u)$ for u in E , and $L^*G^*v = L^*G_0^*v + (L^*W)(X, v)$ for v in E . Thus G satisfies (5.1) and (6.1) with

$$(9.2) \quad X_1 = X_0 - L^*W, \quad Y_1 = Y_0 - LZ.$$

Let G_1 be the proper Green operator defined by X_1 and Y_1 . By Theorems 4 and 4* there are rows Z_1 and W_1 of vectors in E such that

$$G = G_1 + [Z_1, Y] = G_1 + [X, W_1].$$

Computing GY_1 and G^*X_1 in both ways, we obtain $Z_1 = X(W_1, Y_1)$ and $W_1 = Y(Z_1, X_1)$. Thus there is an (m, n) -matrix T with entries in the field K such that

$$(9.3) \quad G = G_1 + [XT, Y] = G_1 + [X, YT^*].$$

Then $GY_1 = XT$ and $G^*X_1 = YT^*$.

Using (6.2) for G_0 , and $G_0 Y_0 = 0$, we have

$$G_0 Y_1 = -G_0 LZ = X(X_0, Z) - Z,$$

and hence, using (9.1),

$$GY_1 = G_0 Y_1 + Z + X(W, Y_1) = X[(W, Y_1) + (X_0, Z)].$$

Thus $GY_1 = 0$ if and only if $(X_1, GY_1) = (W, Y_1) + (X_0, Z) = 0$. Using (9.2), we obtain

$$(X_1, GY_1) = (W, Y_0) + (X_0, Z) - (W, LZ).$$

We recall in this connection that $(W, LZ) = (L^*W, Z)$.

Using $GY_1 = XT$, which follows from (9.3), and applying G to both sides of (5.1), we obtain

$$(9.4) \quad GLG = G - [XT, Y] = G - [X, YT^*] = G_1.$$

This completes the proof of the following theorem.

THEOREM 8. -- Let G be a Green operator, defined by (9.1). Then the following five statements are logically equivalent.

- (i) $G(Y_0 - LZ) = 0$.
- (ii) $G*(X_0 - L*W) = 0$.
- (iii) $(W, \bar{Y}_0) + (X_0, Z) = (W, LZ) = (L*W, Z)$.
- (iv) $GLG = G$.
- (v) G is a proper Green operator.

We note in connection with (iv) that $LGL = L$ for any Green operator G . This is an immediate consequence of (6.2).

10. Equivalent Operator Solutions

In this section we discuss the relation between two operator solutions G and G_1 which provide the same solutions of (I) and of (I*). This means that $Gu = G_1u$ for all u in E such that $(Y, u) = 0$, and that $G*v = G_1*v$ for all v in E such that $(X, v) = 0$. If two operators G and G_1 on E satisfy these two requirements, then we shall call them equivalent with respect to the problem (I).

We do not restrict this concept to operator solutions of (I), but it is obvious that an operator equivalent with respect to (I) to an operator solution of (I) is itself an operator solution of (I). It is equally obvious that equivalence of operators with respect to (I), as defined above, is indeed an equivalence relation, and that two operators G and G_1 on E are equivalent with respect to (I) if and only if G^* and G_1^* are equivalent with respect to (I*).

To obtain conditions for the equivalence of two operators with respect to (I), we first prove the following lemma.

LEMMA 4. -- Let G and G_1 be two operators on E . Then $Gu = G_1u$ for all u in E such that $(Y, u) = 0$ if and only if there is a row Z of vectors in E , of length n , such that $G = G_1 + [Z, Y]$.

PROOF. -- If $G = G_1 + [Z, Y]$, then obviously $Gu = G_1u$ for all u in E such that $(Y, u) = 0$. Assume conversely that G and G_1 satisfy this relation, and let $u_0 = u - Y_0(Y, u)$ for u in E . Then $(Y, u_0) = 0$, and thus $Gu_0 = G_1u_0$. It follows immediately from this that $G = G_1 + [Z, Y]$, with $Z = GY_0 - G_1Y_0$. This proves the lemma.

If G is equivalent to G_1 with respect to (I), it follows from Lemmas 4 and 4* that there are rows Z and W with entries in E such that

$$G = G_1 + [Z, Y] = G_1 + [X, W].$$

Computing GY_0 in both ways, we obtain $Z = X(W, Y_0)$. Thus there is a matrix T with entries in the field K such that

$$(10.1) \quad G = G_1 + [XT, Y] = G_1 + [X, YT^*].$$

Conversely, G is equivalent to G_1 with respect to (I) by Lemmas 4 and 4* if (10.1) is satisfied. Thus we have the following theorem.

THEOREM 9. -- Two operators G and G_1 on E are equivalent if and only if they satisfy a relation of the form (10.1), where T is an (m, n) -matrix with entries in the field K .

Comparing (10.1) with (9.3) and its derivation, we immediately have the following theorem.

THEOREM 10. -- An operator G on E is a Green operator for the problem (I) if and only if G is equivalent with respect to (I) to a proper Green operator for (I).

11. Null Spaces of Green Operators

The role of the rows X , Y , X_1 , and Y_1 of vectors in E in the results of Sections 5, 6, and 8 needs some further discussion. It will be sufficient to discuss Y and Y_1 .

If an operator G on E maps E into F and satisfies (5.1), then it follows immediately from (5.1) and Lemma 3 that the null space of the linear transformation LG is the subspace of E generated by the entries of Y_1 . We shall denote this subspace by S .

It follows from $(Y, Y_1) = I_n$ that S is n -dimensional, and that no nonzero vector of S is orthogonal to the space $N(L^*)$ generated by the entries of Y . In other words, the sum of S and of the orthogonal complement of $N(L^*)$ is direct. Now this orthogonal complement is the range $L(F)$ of L , and S and $N(L^*)$ have the same finite dimension. It follows that E is the direct sum of S and $L(F)$.

Conversely, assume that S is a subspace of E such that E is the direct sum of S and of the orthogonal complement $L(F)$ of $N(L^*)$, and let U be a row of vectors in S which form a basis of S . Then the length of U is the common dimension n of S and $N(L^*)$. If $(Y, U) = T$, then T is nonsingular, for otherwise S would contain nonzero vectors orthogonal to $N(L^*)$, contrary to our assumption. If we put $Y_1 = UT^{-1}$, then the entries of Y_1 form a basis of S , and $(Y, Y_1) = I_n$.

Changing Y means to replace Y with a row YT^* such that the entries of YT^* form a basis of $N(L^*)$. Then T is a nonsingular matrix with entries in K . If we replace Y with YT^* and Y_1 with $Y_1 T^{-1}$, then the relation $(Y, Y_1) = I_n$ and the term $Y_1(Y, u)$ of (5.1) remain unchanged, and the entries of $Y_1 T^{-1}$ form a basis of S .

We state the result of this discussion as a scholium.

SCHOLIUM. -- The class of operators G which map E into F and satisfy (5.1) for a given row Y_1 of vectors in E such that $(Y, Y_1) = I_n$, depends only on the subspace S of E which is generated by the entries of Y_1 , and not otherwise on Y or Y_1 . S may be any subspace of E such that E is the direct sum of S and of the range $L(F)$ of L .

Let now G be a Green operator of the form (9.3). Then G satisfies (5.1), and S is the null space of LG . Thus the null space of G is contained in S . Now $GY_1 = XT$, and since the entries of X are linearly independent, it follows that the null space of G consists of all vectors $Y_1 \alpha$ of E , where α is a column with n entries in the field K such that $T\alpha = 0$. The dimension of this null space thus is $n - r$, where r is the rank of T . It follows in the same way that $m - r$ is the dimension of the null space of G^* .

It follows from these considerations that each of the following two statements is logically equivalent to the other and to the five statements of Theorem 8.

(vi) The null space of G is n -dimensional.

(vii) The null space of G^* is m -dimensional.

12. Special Cases

Special cases occur if $m = 0$ or $n = 0$, or $m = n$. A brief discussion of these cases will be sufficient.

a. $m = 0$. This means that L is one-to-one, and that (I) has at most one solution for u in E . All terms involving X or X_1 become zero and can be omitted from the formulas of this chapter. G^* maps E into F^* for any operator solution G of (I), and (I*) can be solved for any vector v in E . An operator solution G of (I) is a Green operator if and only if G maps E into F , and any Green operator of (I) is a proper Green operator.

b. $n = 0$. This case is exactly analogous to the preceding one, with the roles of (I) and of (I*) interchanged. In fact, it is Case 1*.

c. $m = n$. This is the most common case in applications. In this case, we have the so-called Alternative: Either (I) has exactly one solution for any vector u in E , or (I) does not always have a solution, and never a unique solution.

d. $m = n = 0$. This is the first case of the Alternative. In this case, (I) has exactly one operator solution, and this operator solution is a proper Green operator.

A special case of a different kind arises if the problem (I) is self-adjoint, i.e., if $F^* = F$ and $L^* = L$. This implies $m = n$. If we put $X = Y$ and $X_1 = Y_1$, then the proper Green operator G defined by X_1 and Y_1 becomes self-adjoint; i.e., $G^* = G$. Thus any self-adjoint problem has self-adjoint operator solutions. A problem may, however, have self-adjoint operator solutions, and even self-adjoint proper Green operators, without being self-adjoint, if m and n are both positive.

CHAPTER II

APPLICATIONS

13. Notations

In applying the theory of Chapter I to the problems discussed in Reference 2, we shall use as much as possible the notations of Reference 2, except that we shall use asterisks throughout to denote adjointness.

Let K again be the field of real numbers or the field of complex numbers. The continuous functions of one real variable with domain $[a, b]$ and with values in K form a vector space over K . We shall denote this space by $\mathcal{C}_0$. If we put $(u, v) = \int_a^b \bar{u}v$ for u and v in $\mathcal{C}_0$, then $\mathcal{C}_0$ becomes an inner product space. It is easily seen that then $(\phi, \psi) = \int_a^b \phi^* \psi$ for any two rows ϕ, ψ with entries in $\mathcal{C}_0$.

If $g(t, s)$ is an integrable function of two real variables such that the equations

$$(Gu)(t) = \int_a^b g(t, s)u(s)ds, \quad \text{and}$$

$$(G^*v)(t) = \int_a^b \overline{g(s, t)}v(s)ds,$$

define mappings G and G^* of $\mathcal{C}_0$ into itself, then G and G^* are adjoint operators on $\mathcal{C}_0$. If this is the case, we call G the integral operator on $\mathcal{C}_0$ with kernel function $g(t, s)$.

As in Reference 2, we denote by $\mathcal{C}$ the space of all columns of a given length n with entries in $\mathcal{C}_0$. If we put $(u, v) = \int_a^b u^*v$ for u and v in $\mathcal{C}$, then $\mathcal{C}$ becomes an inner product space. Rows with entries in $\mathcal{C}$ are matrices with columns in $\mathcal{C}$, and $(U, V) = \int_a^b U^*V$ for any matrices U and V with columns in $\mathcal{C}$.

If $G(t, s)$ is an integrable matrix function of two real variables such that the equations:

$$(\mathcal{G}u)(t) = \int_a^b G(t, s)u(s)ds, \quad \text{and}$$

$$(\mathcal{G}^*v)(t) = \int_a^b G^*(s, t)v(s)ds,$$

define mappings $\mathcal{G}$ and $\mathcal{G}^*$ of $\mathcal{C}$ into itself, then $\mathcal{G}$ and $\mathcal{G}^*$ are adjoint operators on $\mathcal{C}$. If this is the case, we call $\mathcal{G}$ the integral operator on $\mathcal{C}$ with kernel matrix $G(t, s)$.

If ϕ and ψ are rows of the same length with entries in $\mathcal{C}_0$, then $[\phi, \psi]$ is the "degenerate" integral operator with kernel function $\phi(t)\psi^*(s)$. If U and V are matrices with the same number of columns in $\mathcal{C}$, then $[U, V]$ is the "degenerate" integral operator with kernel matrix $U(t)V^*(s)$.

We denote by $\mathcal{D}$ (as in Reference 2) the subspace of $\mathcal{C}$ consisting of all differentiable columns in $\mathcal{C}$, and by $\mathcal{D}_0$ the subspace of $\mathcal{C}_0$ consisting of all functions of class C_n in $\mathcal{C}_0$. Using the notations of Reference 2, Section 5, we have $\hat{\xi}^* \tilde{\phi} = \xi^* \phi$, and hence

$$(13.1) \quad (\hat{\xi}, \tilde{\phi}) = (\xi, \phi)$$

for any row ξ with entries in $\mathcal{D}_0$ and any row ϕ with entries in $\mathcal{C}_0$.

14. Associated Boundary-Value Problems

If we consider a pair of adjoint boundary-value problems:

(Iv) Given u in $\mathcal{C}$, find x in $\mathcal{B}$ such that $\mathcal{L}x = u$;

(Iv*) Given v in $\mathcal{C}$, find y in $\mathcal{B}^*$ such that $\mathcal{L}^*y = v$;

defined as in Reference 2, Section 3, then the discussion of Reference 2, Sections 7 and 8, shows that these problems satisfy the conditions of Section 3 of this report, and that one operator solution of (Iv) is an integral operator. Any operator on $\mathcal{C}$ which is of the form $[U, V]$ has a continuous kernel matrix. It follows that all operator solutions of (Iv) are integral operators, and that all kernel matrices of these operators have the same discontinuities. These discontinuities are described in Reference 2, Theorem 5 (2). Comparing the conditions of Theorem 7(i) with those of Reference 2, Theorem 5, we see that the generalized functions of Green, as defined in Reference 2, are the kernel matrices of proper Green operators.

Now let L be a scalar differential operator, defined as in Reference 2, Section 5, and $\mathcal{L}$ the canonically associated vector operator (Ref. 2, Sect. 6). Because of the similarity of Reference 2, (2.2), and Reference 2, (6.7), the discussion of adjoint boundary conditions is essentially the same for the operators L and $\mathcal{L}$. We shall assume from now on that (Iv) and (Iv*) are adjoint "vector" boundary-value problems for the operator $\mathcal{L}$ which is canonically associated with the scalar

differential operator L . We denote by $\mathcal{B}_0$ the space of all functions x in $\mathcal{D}_0$ such that $\hat{x}$ is in $\mathcal{B}$, and by $\mathcal{B}_0^*$ the space of all functions y in $\mathcal{D}_0$ such that $\hat{y}$ is in $\mathcal{B}^*$. Then the scalar boundary-value problems

(Is) Given u in $\mathcal{C}_0$, find x in $\mathcal{B}_0$ such that $Lx = u$;

(Is*) Given v in $\mathcal{C}_0$, find y in $\mathcal{B}_0^*$ such that $L^*y = v$;

are adjoint.

If two pairs of adjoint boundary-value problems are given in this way, we call (Iv) and (Iv*) the "vector" boundary-value problems associated with the scalar boundary-value problems (Is) and (Is*).

The following theorem is an immediate consequence of Reference 2, Theorem 2, Corollary.

THEOREM 11. -- Let u_1 be a function in $\mathcal{C}_0$. Then all solutions of (Iv) for $\tilde{u}_1$ are of the form $\hat{x}_1$, where x_1 is a solution of (Is) for u_1 , and a function x_1 in $\mathcal{B}_0$ is a solution of (Is) for u_1 if and only if $\hat{x}_1$ is a solution of (Iv) for $\tilde{u}_1$.

This theorem sums up the basic connection between solutions of (Is) and solutions of (Iv). Our aim in this chapter is to establish and discuss similar relations between operator solutions of (Is) and operator solutions of (Iv).

15. Preliminary Considerations

We shall denote by ξ and η rows of functions in $\mathcal{D}_0$ such that the entries of ξ and η form bases of the solution spaces of the homogeneous problems $Lx = 0$, x in $\mathcal{B}_0$, and $L^*y = 0$, y in $\mathcal{B}_0^*$, respectively, and by ξ_0 and η_0 rows of functions in $\mathcal{C}_0$ such that $(\xi, \xi_0) = I_\mu$ and $(\eta, \eta_0) = I_\nu$. These rows thus replace the rows X , Y , X_0 , and Y_0 of Chapter I for the problems (Is) and (Is*). Then the columns of the matrices $\hat{\xi}$ and $\hat{\eta}$ form bases of the solution spaces of the homogeneous problems $\mathcal{L}x = 0$, x in $\mathcal{B}$, and $\mathcal{L}^*y = 0$, y in $\mathcal{B}^*$, by Theorems 11 and 11*, and $(\hat{\xi}, \tilde{\xi}_0) = I_\mu$ and $(\hat{\eta}, \tilde{\eta}_0) = I_\nu$ by (13.1). Thus the matrices $\hat{\xi}$, $\hat{\eta}$, $\tilde{\xi}_0$, and $\tilde{\eta}_0$ replace the rows X , Y , X_0 , and Y_0 of Chapter I for the problems (Iv) and (Iv*).

Let $\mathcal{G}_0$ be the integral operator on $\mathcal{C}$ with the kernel matrix $\Gamma(t, s)$ of Reference 2, Section 8, let $\Gamma(t, s) = \|\gamma_{ik}(t, s)\|$ and $\gamma(t, s) = \gamma_{11}(t, s)$, and let G_0 be the integral operator on $\mathcal{C}_0$ with the kernel function $\gamma(t, s)$. Then $\mathcal{L}\mathcal{G}_0\tilde{u} = \tilde{u}$ for any u in $\mathcal{C}_0$, and thus $\mathcal{G}_0\tilde{u} = \hat{x}$, and $u = Lx$, for some x in $\mathcal{D}_0$, by Theorem 11.

It is readily seen that then $x = G_0 u$. If $(\eta, u) = 0$, then $(\hat{\eta}, \tilde{u}) = 0$ by (13.1), and $\mathcal{G}_0 \tilde{u}$ is in $\mathcal{B}$ by Reference 2, Section 8. As $\mathcal{G}_0 \tilde{u} = \hat{x}$ for $x = G_0 u$, it follows that $G_0 u$ is in $\mathcal{B}_0$. Thus G_0 is an operator solution of (Is), and we have the following lemma.

LEMMA 5. -- The operators $\mathcal{G}_0$ and G_0 are operator solutions of the problems (Iv) and (Is), respectively. $\mathcal{G}_0$ maps $\mathcal{C}$ into $\mathcal{D}$, and G_0 maps $\mathcal{C}_0$ into $\mathcal{D}_0$. For u in $\mathcal{C}_0$ and $x = G_0 u$, we have $Lx = u$ and $\mathcal{G}_0 \tilde{u} = \hat{x}$.

Lemma 5* is proved in the same way as Lemma 5. It follows from Lemmas 5 and 5* that the problems (Is) and (Is*) satisfy the conditions of Section 3.

If $\mathcal{L}$ is canonically associated with L , as we have assumed, then the matrices U and V of Reference 2, Section 2 are of the form $U = \hat{\phi}$, $V = \hat{\psi}$, by Theorem 11, where the entries of ϕ and of ψ form bases of the solution spaces of the equations $Lx = 0$ and $L^*y = 0$, respectively. Using this in Reference 2, (8.1), we see readily that $\gamma(t, s)$ is given by the same formula, with U and V replaced by ϕ and ψ , and that we have

$$(15.1) \quad \gamma_{ik}(t, s) = \frac{\partial^{i-1}}{\partial t^{i-1}} \frac{\partial^{k-1}}{\partial s^{k-1}} \gamma(t, s),$$

for t, s in $[a, b]$ and $t \neq s$, and for $1 \leq i \leq n$ and $1 \leq k \leq n$.

16. Associated Operator Solutions

We shall say that an operator $\mathcal{G}$ on $\mathcal{C}$ is of class D if both $\mathcal{G}$ and $\mathcal{G}^*$ map $\mathcal{C}$ into $\mathcal{D}$, and that an operator G on $\mathcal{C}_0$ is of class D if both G and G^* map $\mathcal{C}_0$ into $\mathcal{D}_0$.

It follows immediately from this definition that $\mathcal{G}_0$ and G_0 are of class D, and that all Green operators for (Iv) and for (Is) are of class D.

We shall use $D\zeta$ as an alternate notation for $\hat{\zeta}$ if ζ is a row with entries in $\mathcal{D}_0$. Then the last part of Lemma 5 can be stated as follows: For u in $\mathcal{C}_0$, we have $LG_0 u = u$, and $\mathcal{G}_0 \tilde{u} = DG_0 u$.

If G is an operator solution of class D for (Is), and $\mathcal{G}$ an operator solution of (Iv) such that $\mathcal{G}\tilde{u} = DGu$ and $\mathcal{G}^*\tilde{v} = DG^*v$ for any functions u and v in $\mathcal{C}_0$, then we shall say that $\mathcal{G}$ is associated with G .

By Lemmas 5 and 5*, $\mathcal{G}_0$ is associated with G_0 in this sense.

Any operator solution of (Is) is of the form

$$(16.1) \quad G = G_0 + [\zeta, \eta] + [\xi, \theta],$$

where ζ and θ are rows with entries in $\mathcal{C}_0$, by Theorem 1 and Lemma 5. G obviously is of class D if the entries of ζ and of θ are in $\mathcal{D}_0$. Conversely, $G\eta_0 = G_0\eta_0 + \zeta + \xi(\theta, \eta_0)$, and the entries of this are in $\mathcal{D}_0$ only if the entries of ζ are. A similar argument for G^* shows that the entries of θ are in $\mathcal{D}_0$ too if G is of class D.

If the entries of ζ and θ in (16.1) are in $\mathcal{D}_0$, then

$$(16.2) \quad \mathcal{G} = \mathcal{G}_0 + [\hat{\zeta}, \hat{\eta}] + [\hat{\xi}, \hat{\theta}]$$

defines an operator solution of (Iv), by Theorem 1. For u in $\mathcal{C}_0$, we have $[\zeta, \eta]u = \zeta(\eta, u)$, and

$$[\hat{\zeta}, \hat{\eta}]\tilde{u} = \hat{\zeta}(\hat{\eta}, \tilde{u}) = \hat{\zeta}(\eta, u) = D([\zeta, \eta]u),$$

using (13.1). Similarly, $[\hat{\xi}, \hat{\theta}]\tilde{u} = D([\xi, \theta]u)$, and thus we have $\mathcal{G}\tilde{u} = DGu$.

$\mathcal{G}^*\tilde{v} = DG^*v$, for v in $\mathcal{C}_0$, follows in the same way. Thus $\mathcal{G}$ is associated with G .

The following lemma shows that $\mathcal{G}$ is the only operator solution of (Iv) which is associated with G .

LEMMA 6. -- If $\mathcal{G}$ and $\mathcal{G}_1$ are two operator solutions of (Iv) such that $\mathcal{G}\tilde{u} = \mathcal{G}_1\tilde{u}$ and $\mathcal{G}^*\tilde{v} = \mathcal{G}_1^*\tilde{v}$ for any functions u and v in $\mathcal{C}_0$, then $\mathcal{G} = \mathcal{G}_1$.

PROOF. -- By Theorem 1, there are matrices Z and W with columns in $\mathcal{C}$ such that $\mathcal{G} = \mathcal{G}_1 + [Z, \hat{\eta}] + [\hat{\xi}, W]$. Then $\mathcal{G}\tilde{\eta}_0 = \mathcal{G}_1\tilde{\eta}_0 = \mathcal{G}_1\tilde{\eta}_0 + Z + \hat{\xi}(W, \tilde{\eta}_0)$, and hence $Z = -\hat{\xi}(W, \tilde{\eta}_0)$. It follows that $\mathcal{G} = \mathcal{G}_1 + [\hat{\xi}, W_1]$, with $W_1 = W - \hat{\eta}(W, \tilde{\eta}_0)^*$. But then $\mathcal{G}^*\tilde{\xi}_0 = \mathcal{G}_1^*\tilde{\xi}_0 = \mathcal{G}_1^*\tilde{\xi}_0 + W_1$, and $W_1 = 0$. Thus $\mathcal{G} = \mathcal{G}_1$.

We state our results in the following theorem.

THEOREM 12. -- Let G be an operator solution of class D for (Is). There is exactly one operator solution $\mathcal{G}$ of (Iv) which is associated with G . G is of the form (16.1), where ζ and θ are rows with entries in $\mathcal{D}_0$, and then $\mathcal{G}$ is given by (16.2).

17. Relations Between Associated Operator Solutions

THEOREM 13. -- Let G be an operator solution of class D for (Is) , let $\mathcal{G}$ be the operator solution of (Iv) associated with G , and let ξ_1 and η_1 be rows with entries in $\mathcal{C}_0$ such that $(\xi, \xi_1) = I_\mu$ and $(\eta, \eta_1) = I_\nu$. Then we have the following relations between G and $\mathcal{G}$:

(a) G^* is an operator solution of class D for (Is^*) , and $\mathcal{G}^*$ is the operator solution of (Iv^*) associated with G^* .

(b) G maps $\mathcal{C}_0$ into $\mathcal{B}_0$ if and only if $\mathcal{G}$ maps $\mathcal{C}$ into $\mathcal{B}$.

(c) $LGU = u - \eta_1(\eta, u)$ for all u in $\mathcal{C}_0$ if and only if $\mathcal{L}\mathcal{G}u = u - \hat{\eta}_1(\eta, u)$ for all u in $\mathcal{C}$.

(d) $GLx = x - \xi(\xi_1, x)$ for all x in $\mathcal{B}_0$ if and only if $\mathcal{G}\mathcal{L}x = x - \hat{\xi}(\tilde{\xi}_1, x)$ for all x in $\mathcal{B}$.

(e) G is a Green operator for (Is) if and only if $\mathcal{G}$ is a Green operator for (Iv) .

(f) G is the proper Green operator for (Is) defined by ξ_1 and η_1 if and only if $\mathcal{G}$ is the proper Green operator for (Iv) defined by $\tilde{\xi}_1$ and $\tilde{\eta}_1$.

PROOF. -- Part (a) follows immediately from the definitions and Theorem 2.

Since $\mathcal{G}\tilde{\eta}_1 = DG\eta_1$, $G\eta_1$ is in $\mathcal{B}_0$ if and only if $\mathcal{G}\tilde{\eta}_1$ is in $\mathcal{B}$. We obtain (b) from this by using Lemma 1 for G and for $\mathcal{G}$.

By Lemma 3, G satisfies the first equation of (c) if and only if $LG\eta_1 = 0$, and $\mathcal{G}$ satisfies the second equation of (c) if and only if $\mathcal{L}\mathcal{G}\tilde{\eta}_1 = 0$. Now $\mathcal{G}\tilde{\eta}_1 = DG\eta_1$, and $\mathcal{L}Dx = \tilde{u}$ if $Lx = u$, for any x in $\mathcal{D}_0$, by Theorem 11. Thus $LG\eta_1 = 0$ if and only if $\mathcal{L}\mathcal{G}\tilde{\eta}_1 = 0$. This proves (c).

Part (d) follows immediately from (b*) and (c*), with Theorem 5.

Part (e) follows immediately from (b) and (b*).

Since $\mathcal{G}^*\tilde{\xi}_1 = DG^*\xi_1$, we have $G^*\xi_1 = 0$ if and only if $\mathcal{G}^*\tilde{\xi}_1 = 0$. Using this together with (b) and (c), and applying Theorem 7, (i), to G and to $\mathcal{G}$, we obtain (f).

18. Characterizations of Associated Operator Solutions

THEOREM 14. -- If $\mathcal{G}$ is an operator solution of (Iv), then the following three statements are logically equivalent:

(i) $\mathcal{G}\tilde{u}$ and $\mathcal{G}^*\tilde{v}$ are of the form $\hat{z}$, with z in $\mathcal{D}_0$, for any functions u and v in $\mathcal{C}_0$.

(ii) $\mathcal{G}$ is of class D, and if $G(t, s) = \|g_{ik}(t, s)\|$ is the kernel matrix of $\mathcal{G}$, then

$$(18.1) \quad g_{ik}(t, s) = \frac{\partial^{i-1}}{\partial t^{i-1}} \frac{\partial^{k-1}}{\partial s^{k-1}} g_{11}(t, s)$$

for t, s in $[a, b]$ and $t \neq s$, and for $1 \leq i \leq n$ and $1 \leq k \leq n$.

(iii) $\mathcal{G}$ is associated with an operator solution of class D for (Is).

PROOF. -- We show that (i) implies (iii), that (iii) implies (ii), and finally that (ii) implies (i).

Assume first that $\mathcal{G}$ satisfies (i). There are matrices Z and W with columns in $\mathcal{C}$ such that $\mathcal{G} = \mathcal{G}_0 + [Z, \hat{\eta}] + [\hat{\xi}, W]$ by Theorem 1. Then $\mathcal{G}\tilde{\eta}_0 = D\mathcal{G}_0\eta_0 + Z + \hat{\xi}(W, \tilde{\eta}_0)$, using Lemma 5, and the columns of this are of the form $\hat{z}$, with z in $\mathcal{D}_0$, only if the columns of Z are of this form. Thus we can put $Z = \hat{\zeta}$, where ζ is a row with entries in $\mathcal{D}_0$. We see in the same way, computing $\mathcal{G}^*\tilde{\xi}_0$, that $W = \hat{\theta}$ for a row θ with entries in $\mathcal{D}_0$. But then $\mathcal{G}$ is of the form (16.2) and thus associated with an operator G of the form (16.1). This shows that (i) implies (iii).

If $\mathcal{G}$ satisfies (iii), then $\mathcal{G}$ is of the form (16.2), with the entries of ζ and of θ in $\mathcal{D}_0$. Thus the columns of $\hat{\zeta}$ and of $\hat{\theta}$ are in $\mathcal{D}$, and it follows that $\mathcal{G}$ is of class D. If we replace $G(t, s)$ in (18.1) by the kernel matrix $\hat{\zeta}(t)\hat{\eta}^*(s)$ of $[\hat{\zeta}, \hat{\eta}]$, then (18.1) obviously is satisfied. The same is true for $[\hat{\xi}, \hat{\theta}]$. Using these two results together with (15.1), we see that $G(t, s)$ satisfies (18.1). This shows that (iii) implies (ii).

Finally, assume that $\mathcal{G}$ satisfies (ii). Let $\mathcal{K} = \mathcal{G} - \mathcal{G}_0$, and let $K(t, s) = G(t, s) - \Gamma(t, s)$ be the kernel matrix of $\mathcal{K}$. Using (15.1), we see that

(18.1) is true with $G(t, s)$ replaced by $K(t, s)$. Let u be in $\mathcal{C}_0$, and let $\tilde{\mathcal{K}}u = \|x_i\|$. $K(t, s)$ is continuous, and thus we have

$$\begin{aligned} x_i(t) &= \int_a^b k_{i1}(t, s)u(s)ds = \int_a^b \frac{\partial^{i-1}}{\partial t^{i-1}} k_{11}(t, s)u(s)ds \\ &= \frac{d^{i-1}}{dt^{i-1}} \int_a^b k_{11}(t, s) = x_1^{(i-1)}(t) \end{aligned}$$

for t in $[a, b]$ and $1 \leq i \leq n$. Thus $\tilde{\mathcal{K}}u = \hat{x}_1$. Since $\mathcal{K}$ is of class D, x_1 must be in $\mathcal{D}_0$. Using Lemma 5, we see that $\tilde{\mathcal{G}}u$ is of the form $\hat{z}$ with z in $\mathcal{D}_0$. Using the same argument for $\mathcal{G}^*$, we see that $\mathcal{G}$ satisfies (i). Thus (iii) implies (i).

Theorem 14 does not only characterize the operator solutions of (Iv) which are associated with operator solutions of class D for (Is), but also gives the connection between kernel matrices and functions of associated operator solutions. We state this as a separate theorem.

THEOREM 15. -- Let $g(t, s)$ be the kernel function of an operator solution G of class D for (Is), and let $G(t, s) = \|g_{ik}(t, s)\|$ be the kernel matrix of the operator solution $\mathcal{G}$ of (Iv) which is associated with G . Then $g(t, s) = g_{11}(t, s)$, and $G(t, s)$ satisfies (18.1).

PROOF. -- The only new statement of the theorem, that $g(t, s) = g_{11}(t, s)$, is obvious. The rest of the theorem follows from Theorem 14.

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