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Functionals Additive on a Symmetric Set

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FUNCTIONALS ADDITIVE ON A SYMMETRIC SET

by

E. D. Cashwell and C. J. Everett

ABSTRACT

A Galois correspondence $S \rightarrow TS$, $T \rightarrow ST$ is set up between a lattice S of symmetric sets S of pairs (X, Y) of vectors of Hilbert space H , and a lattice T of subspaces T of continuous functionals t on H , TS being the space of functionals additive on S , and ST the additivity domain of the space T . This viewpoint suggests many deep problems, some of which are discussed. Perhaps the most interesting is the characterization of those (momentum) probability densities consistent with the relativistic elastic collision relation. Some significant positive results are (1) the only pairs on which all functionals "through the origin" are additive satisfy $|X| |Y| = 0$; (2) the only pairs on which all homogeneous functionals are additive are collinear; (3) if t is a functional additive on all orthogonal pairs, and on all unidirectional pairs, or on all oppositely directed pairs, then t is linear.

I. THE LATTICE S OF SYMMETRIC SETS

Let H be a real Hilbert space, with vectors 0 , X , Y , ..., scalars $0, 1, a, b, \dots$, inner product XY , and norm $|X| = \sqrt{XX} = \sqrt{X^2}$. By a symmetric set is meant a set of pairs (X, Y) of (H, H) such that $(X, Y) \in S$ implies $(Y, X) \in S$.

A whole family of symmetric sets may be defined by binary symmetric relations involving the inner product, such as $XY = k$, or $X^2 + XY + Y^2 = 0$. Thus the relations $X^2 + Y^2 > 0$, $|X| |Y| = 0$, and $X^2 + Y^2 = 0$ respectively define the sets

$$S^P = (H, H) \supset S^A = (H, 0) \cup (0, H) \supset S^Z = \{(0, 0)\}.$$

Similarly $S^C = \{(X, Y); XY = c |X| |Y|\}$, c real, denotes the set consisting of S^A together with all "equi-angular" pairs of non-zero vectors, for which $XY/|X| |Y| = c = \cos \theta$. In particular, $S^0 = \{(X, Y); XY = 0\}$ is the set of all orthogonal pairs; $S^1 = \{(X, Y); XY = |X| |Y|\} = S^A \cup \{(X, aX); a > 0, X \neq 0\}$ the set of "unidirectional" pairs; $S^{-1} = \{(X, Y); XY = -|X| |Y|\} = S^A \cup \{(X, aX); a < 0, X \neq 0\}$ that of

all "oppositely directed" pairs; and finally, $S^L = \{(X, Y); |XY| = |X| |Y|\} = S^1 \cup S^{-1}$ will denote the set of all "collinear" pairs.

We observe that the class S of all symmetric sets S such that $S^P \supset S \supset S^A$ forms a lattice under set inclusion $\supset$, set intersection $\cap$, and set union $\cup$.

II. THE LATTICE T OF FUNCTION SPACES

The linear space F , of all real valued continuous functionals t on H , contains the subspace $T^0 = \{t; t(0) = 0\}$ of all such functionals "through the origin" and the subspace $T^L \subset T^0$ of all linear such functionals, which, according to the representation theorem of F. Riesz,¹ has the form $T^L = \{t; t(X) \equiv \sum x_i B_i\}$.

As a companion to the lattice S , we introduce the class T of all linear subspaces T of F such that $T^L \subset T \subset T^0$, which forms a lattice under set inclusion and intersection, and linear closure $(T_1 \vee T_2)$ of set union.

For example, T contains all spaces of type $T^{(m)}$
 $= \{t; t(X) = \sum_{i=1}^m a_i t_i(X) + BX; a_i \text{ real}, B \in H\}$, where
 $t_1, \dots, t_m \in T^0$ form a basis for the linear quotient
space $T^{(m)}/T^\ell$. Of this kind (with $m = 1$) are the
spaces $T^{X^2} = \{t; t(X) = aX^2 + BX, a \text{ real}, B \in H\}$, and
 $T^{|X|} = \{t; t(X) = a|X| + BX, a \text{ real}, B \in H\}$.

We also note the presence in T of the space $T^\pm$
 $= \{t; t(aX) \equiv at(X), a \text{ real}\} \subset T^\pm = \{t; t(aX) \equiv at$
 $(X), a > 0\}$. The space $T^\pm$ of "homogeneous" func-
tionals contains such non-linear ones as $t(X) \equiv$
 $(BX) |CX|/|X|$, $B, C \in H$, $t(0) \equiv 0$, while T^+ contains
in addition, for example, all norm functionals $n(X)$.

III. THE CORRESPONDENCE $S \rightarrow TS$

We next define a mapping $S \rightarrow TS$ on S into T ,
where $T(S) = \{t; t(X+Y) = t(X) + t(Y), (X, Y) \in S\}$
is the space of all t additive on S . Obviously we
have the property (T) $S_1 \supset S_2$ implies $TS_1 \subset TS_2$,
and from $S^P \supset S \supset S^A$ follows $T^\ell = TS^P \subset TS \subset TS^A =$
 T^0 . Moreover, $T(S_1 \cup S_2) = T(S_1) \cap T(S_2)$, and
 $T(S_1 \cap S_2) \supset T(S_1) \vee T(S_2)$, although equality may
fail in the latter.

It is known² that $TS^0 = T^{X^2}$, and it is easy to
show, by a standard argument,³ that $TS^1 = T^+ \supset$
 $T^\pm = TS^{-1}$.

We note in passing the interesting consequence
that every continuous functional which is additive
on orthogonal pairs, and also additive on either
unidirectional pairs, or on oppositely directed
pairs, is necessarily linear. For we have
 $T(S^0 \cup S^1) = T(S^0) \cap T(S^1) = T^{X^2} \cap T^+ = T^\ell$, and
 $T(S^0 \cup S^{-1}) = T(S^0) \cap T(S^{-1}) = T^{X^2} \cap T^\pm = T^\ell$. Sim-
ilarly, one may show that $TS^L = T^+$.

On the other hand, we observe that $T^0 = TS^A =$
 $T(S^1 \cap S^{-1}) \not\subset T(S^1) \vee T(S^{-1}) = T^+ \vee T^\pm = T^+$.

Figure 1 indicates these results and others to come.

IV. THE CORRESPONDENCE $T \rightarrow ST$

Having considered mapping $S \rightarrow TS$, with property
(T), it is natural to study the related correspond-
ence $T \rightarrow ST$ of T into S , where $ST = \{(X, Y);$
 $t(X + Y) = t(X) + t(Y), t \in T\}$, i.e., the common
domain of additivity of the functionals t of T . For
this we have the property

$$(S) \quad T_1 \subset T_2 \text{ implies } ST_1 \supset ST_2,$$

Fig. 1. Galois correspondence for the fundamental sets.

and from $T^\ell \subset T \subset T^0$ follows $S^P = ST^\ell \supset ST \supset ST^0 = S^A$.

We include a simple proof that $ST^0 \subset S^A$, sig-
nifying that the only pairs (X, Y) , on which all
functionals through the origin are additive, are
those for which $|X| |Y| = 0$. For, if

$$(X_1, Y_1) \in S(T^0),$$

then the functional $t(X) = |X| |X_1 + Y_1 - X| \in T^0$,
and hence

$$0 = t(X_1 + Y_1) = t(X_1) + t(Y_1) = 2|X_1| |Y_1|.$$

It is clear that $S(T_1 \vee T_2) = S(T_1) \cap S(T_2)$,
and $S(T_1 \cap T_2) \supset S(T_1) \cup S(T_2)$, where equality may
again fail.

It is readily verified that $ST^{X^2} = S^0$ (since
 $(X + Y)^2 = X^2 + Y^2$ iff $XY = 0$), and that $ST^{|X|} = S^1$
(since $|X + Y| = |X| + |Y|$ iff $XY = |X| |Y|$).

Although it will appear obvious later (since
 $S^1 = ST^{|X|}$ is "closed"), we show directly here that

$ST^+ = S^1$. For, from $T^{|X|} \subset T^+$ follows $S^1 = ST^{|X|} \supset ST^+$. But also, $S^1 \subset ST^+$, and hence $S^1 = ST^+$.

We next prove that $ST^{\pm} = S^L$, that is to say, the only pairs (X_1, Y_1) , on which all homogeneous functionals are additive, are collinear. We have to show that $(X_1, Y_1) \in S(T^{\pm})$ implies $(X_1, Y_1) \in S^L$. If $X_1 = 0$, or $Y_1 = 0$, or $X_1 + Y_1 = 0$, this is obvious, and we assume the contrary. Define $Y_1 = aX_1$, where $a = (X_1 Y_1) / |X_1|^2$, and $Y_1 = Y_1 - Y_1$, with $X_1 Y_1 = 0$. If $Y_1 = 0$, we again have $(X_1, Y_1) \in S^L$. If we suppose $Y_1 \neq 0$, we are led to a contradiction. For, as noted in §2, $T^{\pm}$ contains the functional $t(X) = (X_1 X) |Y_1 X| / |X|$, and its additivity on (X_1, Y_1) implies the equation

$$(1+a)|aX_1 + Y_1| = a|(1+a)X_1 + Y_1|,$$

where both norms are positive. Squaring shows that $a = -\frac{1}{2}$, which is impossible in the preceding equation.

Finally, we note that

$$S^P = ST^L = S(T^{X^2} \cap T^+) \not\subset S(T^{X^2}) \cup S(T^+) = S^0 \cup S^1.$$

V. THE GALOIS CONNECTION $S \sim T$

Since the two mappings $S \rightarrow TS$ and $T \rightarrow ST$ obviously also satisfy the condition

$$(L) \quad S \subset STS \text{ and } T \subset TST,$$

they have the formal properties (T, S, L) of a Galois correspondence⁴ between the lattices S and T .

It follows that $S^* \equiv STS$ and $T^* \equiv TST$ have the nature of closure operators ($A \subset A^*$, $A^{**} = A^*$, $A \subset B$ implies $A^* \subset B^*$). A set S of S is therefore said to be closed in case $S^* = S$, and it is trivial that S is closed iff $S = ST$ for some T ; similarly for a set T of T .

These concepts help to clarify the relations between the sets and spaces we have considered. A glance at Fig. 1, which indicates the relations established, shows that the sets S^P , S^L , S^1 , S^0 , and S^A are closed, while S^{-1} , $S^0 \cup S^1$, and $S^0 \cup S^{-1}$ are not. Similarly, the spaces T^L , $T^{\pm}$, T^+ , T^{X^2} , and T^0 are closed, but $T^{|X|}$ is not.

VI. PROBLEMS AND COMMENTS

The spaces $T^{(m)}$ of §2 which are closed, i.e., with $TST^{(m)} = T^{(m)}$, are of particular interest

algebraically, since every functional t which is additive on the common domain of additivity of $t_1, \dots, t_m$ is then (modulo T^L) a linear combination of the t_i . It is natural therefore to propose

Problem 1. Does there exist, for every integer m , a closed space $T^{(m)}$? We have no example for any $m > 1$. Certainly for $t_1(X) = X^2$, $t_2(X) = |X|$, the corresponding $T^{(2)}$ is not closed, since $TST^{(2)} = T^0$.

Even for $m = 1$, we know only of the isolated case $T^{X^2} = TST^{X^2}$, signifying that every functional additive when X^2 is additive must be of form $aX^2 + BX$. Certainly $T^{|X|}$ is not closed, and we ask in

Problem 2. For what functionals $t_1(X)$ is $T^{(1)}$ closed?

Suggested by the known special case ($Z_0 = 0$), we may propose

Problem 3. If $t_1(X) = (X - Z_0)^2$, is the corresponding space $T^{(1)}$ closed? It is apparent here that $S(T) = S_k \equiv \{(X, Y); XY = k^2\}$ where $k^2 = \frac{1}{2}Z_0^2$. Note that every set S_k is closed. (In the present instance we must, of course, abandon the restriction to functions through the origin.)

One may consider a host of other specific functionals, e.g., $t_1(X) = (Z_0 X)^2$, with $S(T^{(1)}) = \{(X, Y); (XZ_0)(YZ_0) = 0\}$ having an obvious geometric meaning.

The closed property of T^{X^2} has an interesting physical consequence (Ref. 2), the relativistic version of which we formulate in another specific

Problem 4. If $k(X) = (X^2 + 1)^{1/2} - 1$, is the corresponding $T^{(1)}$ space closed?

If true, then every (momentum-vector probability-density) functional $q(X)$, $q(0) > 0$, such that $q(X)q(Y) = q(Z)q(W)$ for all X, Y, Z, W such that $X + Y = Z + W$ and $k(X) + k(Y) = k(Z) + k(W)$ (conservation of momentum and kinetic energy) is of the form $q(X) = q(0)\exp\{ak(X) + BX\}$ (Maxwell-Jüttner density). The relation $k(X + Y) = k(X) + k(Y)$ is an analogue of the relation $(X + Y)^2 = X^2 + Y^2$, and as such, appears as the relativistic version of orthogonality. In H_3 , it has a simple geometric interpretation, involving the momentum ellipsoid, in place of the Pythagorean theorem in the non-relativistic sphere.⁵ Specifically, such a pair X, Y are determined by opposite points of an ellipse with one vertex at the origin, and having semi-axes

in the relation $a^2 = b^2 + b^4$. (A derivation of the Maxwell-Jüttner density may be found in Ref. 6.)

The general position of the sets S^c in our scheme of things is not clear. Thus we do not know the nature of TS^c for any $c \neq 0, \pm 1$, nor whether S^c is closed for any $c \neq 0, 1$. Perhaps more interesting is our final

Problem 5. If $TS = T^{(1)}$ for some $T^{(1)}$ space, is S necessarily closed?

Again we have only the example $TS^0 = T^X$. The converse is false, since S^1 is closed, but $TS^1 = T^+$ is not a $T^{(1)}$ (not all norms are of form $a|X|+bX$).

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