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**STATIC STRESS ANALYSIS OF AXISYMMETRIC SOLIDS WITH
MATERIAL AND GEOMETRIC NONLINEARITIES BY THE
FINITE ELEMENT METHOD**

M. Hartzman

January 19, 1973

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STATIC STRESS ANALYSIS OF AXISYMMETRIC SOLIDS WITH MATERIAL AND GEOMETRIC NONLINEARITIES BY THE FINITE ELEMENT METHOD

Abstract

A technique for analyzing axisymmetric solids subjected to axisymmetric static loads is described. The material properties are described by elastic-plastic incremental stress-strain relations of the Prandtl-Reuss type. The complete formulation includes geometric effects due to assumed large displacements and rotations. The technique therefore incorporates both material and geometric nonlinearities. A solution is obtained by applying the finite element method and using an incremental-iterative numerical algorithm. The technique was applied to the problem of an internally pressurized thick cylinder where considerable deformation occurred. Comparison with an alternate numerical solution indicates reasonable agreement.

Introduction

The finite element method has in recent years been applied successfully to the solution of problems involving elastic-plastic material behavior and large displacements which give rise to geometric nonlinear effects in the relevant field equations. A comprehensive review of the literature prior to 1969 is given in the reports by Yaghmai¹ and Oden.² Additional papers have since appeared,³⁻⁷ with emphasis on particular problems. Considerable research in this field is presently being conducted at various universities and laboratories.

This report describes the development of a computer program for the analysis of axisymmetric solids subjected to axisymmetric loads where material and geometric nonlinearities exhibit significant effects. The theory on which this program is based parallels and extends that developed in the comprehensive work by Yaghmai.¹

Theoretical Analysis

Problems involving geometric and material nonlinearities are incremental in nature; their solution requires an incremental approach.

In considering the description of nonlinear deformation, it is important to define clearly the various geometric quantities entering into the analysis. In applying the

finite element method the principle of virtual work is the basic tool by which we obtain the necessary algebraic equations for solution.

GEOMETRY

We consider two configurations of a deformable body in the process of being deformed from an undeformed to a final state. The configuration at the beginning of a load increment will be termed the initial configuration. The configuration at the end of the load increment will be termed the current configuration. The configuration at time = 0 is termed the undeformed configuration.

The volume, boundary surface and the coordinates of the material points of the body in that order in the various configurations are denoted as follows (see Fig. 1 for plane representation):

undeformed configuration:	$\bar{v}, \bar{a}, \bar{z}_i$	
initial configuration:	v, a, z_i	$(i = 1, 3)$
current configuration:	V, A, Z_i

Fig. 1. Various stages in the deformation process.

PRINCIPLE OF VIRTUAL WORK

A generic point $\bar{p}$ in the undeformed body will occupy positions p and P in the initial and current configurations respectively. The displacement vectors between these positions are shown in Fig. 1.

Since the geometry is known in the initial configuration, it will be considered as the reference configuration. The principle of virtual work in the current configuration referred to the initial configuration is given by the following expression:^{*}

$$\int_a 2t^i \delta u_i da + \int_v 1\rho 2f^i \delta u_i dv = \int_v (1\tau^{ij} + s^{ij}) \delta \epsilon_{ij} dv. \quad (1)$$

The same expression for virtual work in the initial configuration is given as

$$\int_a 1t^i \delta u_i da + \int_v 1\rho 1f^i \delta u_i dv = \int_v 1\tau^{ij} \delta e_{ij} dv, \quad (2)$$

where

$$\delta \epsilon_{ij} = \delta e_{ij} + \delta \eta_{ij}. \quad (3)$$

Subtracting Eq. (2) from Eq. (1) and introducing Eq. (3) we get the incremental expression of the principle of virtual work in the initial configuration:

$$\int_a t^i \delta u_i da + \int_v 1\rho f^i \delta u_i dv = \int_v (1\tau^{ij} \delta \eta_{ij} + s^{ij} \delta \epsilon_{ij}) dv, \quad (4)$$

where[†]

$$\epsilon_{ij} = \frac{1}{2} (u_{i|j} + u_{j|i} + u^k|_i u_{k|j}), \quad (5)$$

$$\eta_{ij} = \frac{1}{2} (u^k|_i u_{k|j}) \quad (6)$$

$$t^i = 2t^i - 1t^i \quad (7)$$

$$f^i = 2f^i - 1f^i \quad (8)$$

^{*} Indicinal notation is used. A repeated diagonal superscript-subscript combination indicates summation.

[†] A vertical slash indicates covariant differentiation.

and*

- t_i^1 = incremental traction acting/unit area
- f_i^1 = incremental body force/unit mass
- $1\tau_{ij}$ = Cauchy stress
- s_{ij}^1 = P-K (Piola-Kirchhoff) incremental stress
- ϵ_{ij}^1 = incremental Lagrangian strain from the initial to the current configurations.
- η_{ij} = nonlinear component of ϵ_{ij}^1
- u_i^1 = incremental displacements from the initial to the current configuration
- 1ρ = material density.

The P-K stress in the current configuration is given as:

$$2s_{ij}^2 = 1\tau_{ij}^1 + s_{ij}^1. \quad (9)$$

These stresses act in the current configuration but are measured/unit area in the initial configuration. To obtain the Cauchy stresses in the current configuration we use the transformation

$$2\tau_{mn}^2 = \frac{2\rho}{1\rho} \frac{\partial Z^m}{\partial z^1} \frac{\partial Z^n}{\partial z^1} 2s_{ij}^1. \quad (10)$$

This transformation is necessary since the current configuration becomes the initial configuration for the next load increment. Likewise the Lagrangian strains in the current configuration may be obtained from

$$2\epsilon_{ij}^2 = 1\epsilon_{ij}^1 + \xi_{ij} \quad (11)$$

where

$$\xi_{ij} = \frac{\partial z^k}{\partial z^1} \frac{\partial z^\ell}{\partial z^1} \epsilon_{k\ell}^1 \quad (12)$$

ξ_{ij} = Lagrangian strain difference

$2\epsilon_{ij}^2$ = Lagrangian strains from the undeformed to the current configurations.

In some applications it is convenient to transform the current Lagrangian strains into current Eulerian strains. This is accomplished through the transformation

$$2e_{ij}^2 = \frac{\partial z^k}{\partial Z^1} \frac{\partial z^\ell}{\partial Z^1} 2\epsilon_{k\ell}^2.$$

* Incremental quantities and quantities with left handed superscript 1 exist in the current configuration but are referred to the initial configuration. Quantities with left handed superscript 2 exist in and are referred to the current configuration.

An incremental version can also be calculated as

$${}^2e_{ij} = \frac{\partial z^r}{\partial Z^i} \frac{\partial z^s}{\partial Z^j} [{}^1e_{rs} + \epsilon_{rs}]$$

where

${}^2e_{ij}$ = Eulerian strains in the current configuration

${}^1e_{ij}$ = Eulerian strains in the initial configuration.

EQUILIBRIUM CHECK

Once the incremental calculations are performed, we can check for equilibrium in the deformed geometry by applying the principle of virtual work in that configuration, i.e.

$$\delta W = \int_A {}^2t_i^i \delta u_i dA + \int_V {}^2\rho f_i^i \delta u_i dV - \int_V {}^2\tau^{ij} \delta e_{ij} dV, \quad (13)$$

where now

${}^2\tau^{ij}$ = Cauchy stresses in the current configuration

e_{ij} = Linear part of the Eulerian strain

and all integration is performed over the current geometry (which is assumed known from the solution of the equilibrium equations). If the body is exactly in equilibrium, then $\delta W = 0$. However, due to the numerical approximations involved, $\delta W \neq 0$ and an unbalance will exist. We can define pseudotractions required for balance as

$$\delta W = \int_A r^i \delta u_i dA.$$

These tractions can be made to vanish by iterating over a load increment, or can be carried as additional loads in the succeeding calculations. Their effect should be small if this equilibrium check is performed at every cycle.

From the derivation of incremental principle of virtual work we see that Eq. (13) represents equilibrium at the beginning of the next load increment. Thus, we can introduce this equilibrium check directly into the calculations by considering the equilibrium of a generic current configuration, given by Eq. (1). This equation can be expressed in the following two forms:

$$\begin{aligned} \int_A {}^1t_i^i \delta u_i da + \int_V {}^1\rho f_i^i \delta u_i dv + \left[\int_A {}^1t_i^i \delta u_i da + \int_V {}^1\rho f_i^i \delta u_i dv - \int_V {}^1\tau^{ij} \delta e_{ij} dv \right] \\ = \int_V ({}^1\tau^{ij} \delta \eta_{ij} + s^{ij} \delta \epsilon_{ij}) dv \quad (14) \end{aligned}$$

where the expression in the brackets is Eq. (2). This is the same expression as shown by Washizu⁸ and used by Hofmeister.⁹ Alternatively,

$$\int_A 2t^i \delta u_i da + \int_V {}^1\rho 2f^i \delta u_i dv - \int_V {}^1\tau^{ij} \delta e_{ij} dv = \int_V ({}^1\tau^{ij} \delta \eta_{ij} + s^{ij} \delta \epsilon_{ij}) dv.$$

This expression was used by Farhoomand.¹⁰ It is obtained from the first expression by using the definitions shown by Eqs. (7) and (8).

INCREMENTAL STRESS-STRAIN RELATIONS

We assume that the incremental stress-strain relations may be written in the form

$$\bar{s}^{ij} = \bar{C}^{ijk\ell} \xi_{k\ell} + \bar{B}^{ij} \phi, \quad (15)$$

where

$\bar{s}^{ij}$ = incremental stress in the current configuration referred to the undeformed configuration,

ϕ = temperature increment,

$\bar{C}^{ijk\ell}$ = material stiffness tensor } { referred to the undeformed
 $\bar{B}^{ij}$ = thermal stiffness tensor } configuration.

For combination with the virtual work expression, Eq. (15) must first be transformed to quantities referred to the initial configuration. Thus we can write

$$\bar{s}^{ij} = \frac{\bar{\rho}}{\rho} \frac{\partial \bar{z}^i}{\partial z^k} \frac{\partial \bar{z}^j}{\partial z^\ell} s^{k\ell}. \quad (16)$$

Substituting Eqs. (12) and (16) into (15) we get

$$s^{ij} = \frac{1}{\rho} \left(\frac{\partial z^i}{\partial \bar{z}^k} \frac{\partial z^j}{\partial \bar{z}^\ell} \bar{C}^{k\ell mn} \frac{\partial z^r}{\partial \bar{z}^m} \frac{\partial z^s}{\partial \bar{z}^n} \right) \epsilon_{rs} + \frac{1}{\rho} \frac{\partial z^i}{\partial \bar{z}^k} \frac{\partial z^j}{\partial \bar{z}^\ell} \bar{B}^{k\ell} \phi$$

or

$$s^{ij} = C^{ijrs} \epsilon_{rs} + B^{ij} \phi, \quad (17)$$

where

$$C^{ijrs} = \frac{1}{\rho} \frac{\partial z^i}{\partial \bar{z}^k} \frac{\partial z^j}{\partial \bar{z}^\ell} \bar{C}^{k\ell mn} \frac{\partial z^r}{\partial \bar{z}^m} \frac{\partial z^s}{\partial \bar{z}^n}, \quad B^{ij} = \frac{1}{\rho} \frac{\partial z^i}{\partial \bar{z}^k} \frac{\partial z^j}{\partial \bar{z}^\ell} \bar{B}^{k\ell}. \quad (18)$$

For an elastic-plastic material $\bar{C}^{ijkl}$ will in general depend on the stresses and material properties. A practical form of this tensor is derived in Appendix A and is given in curvilinear components as

$$\bar{C}^{ijkl} = \mu(\bar{g}^{ik} \bar{g}^{jl} + \bar{g}^{jk} \bar{g}^{il}) + \lambda(\bar{g}^{ij} \bar{g}^{kl}) - \beta \frac{2\mu}{3} \frac{1_{s,ij} 1_{s,kl}}{s^2 \{1 + H'/3\mu\}}, \quad (19)$$

where

$\bar{g}^{ij}$ = contravariant components of the metric tensor in the undeformed configuration

λ, μ = material properties

H' = slope of uniaxial-stress—plastic-strain curve

s = current yield stress

$\beta = \begin{cases} 1 & \text{if loading} \\ 0 & \text{if neutral loading or unloading} \end{cases}$

$1_{s,kl}$ = deviatoric stress tensor.

The quantity s represents the current yield stress and can be obtained if a s - ϵ^P curve is available. At yielding, s is calculated from

$$s = \frac{3}{2} \sqrt{1_{s,ij} 1_{s,ij}}$$

and the equivalent plastic strain ϵ^P is defined as

$$\epsilon^P = \sum \Delta \epsilon^P$$

where it can be shown (see Appendix A) that

$$\Delta \epsilon^P = \frac{1_{s,ij} \xi_{ij}}{s[1 + H'/3\mu]}.$$

The equations listed above are given in tensor components. For practical calculations they must be expressed in physical components.

REDUCTION TO AXISYMMETRIC SOLIDS SUBJECTED TO AXISYMMETRIC LOADS

This section presents a summary of the pertinent equations of previous sections in cylindrical coordinates. For this purpose the curvilinear coordinates are defined, using Cartesian notation, as

$$\begin{array}{lll} \bar{z}_1 = \bar{r} & z_1 = r & Z_1 = R \\ \bar{z}_2 = \bar{z} & z_2 = z & Z_2 = Z \\ \bar{z}_3 = \bar{\theta} & z_3 = \theta & Z_3 = \Theta \end{array}$$

In the following expressions, the components of stress, strain and displacement tensors are physical components. Furthermore, matrix notation will be used where convenient.

The incremental principle of virtual work is expressed as

$$\int_a \{\delta u\}^T \{^2t\} da + \int_v {}^1\rho \{\delta u\}^T \{^2f\} dv - \int_v \{\delta e\}^T \{^1\tau\} dv = \int_v \left[\{\delta \eta\}^T \{^1\tau\} + \{\delta \epsilon\}^T \{s\} \right] dv \quad (20)$$

where

$$\begin{aligned} \{u\} &= \{u_r \ u_z\} \\ \{^2t\} &= \{^2t_r \ ^2t_z\} \\ \{^2f\} &= \{^2f_r \ ^2f_z\} \\ \{\eta\} &= \{\eta_{rr} \ \eta_{zz} \ \eta_{\theta\theta} \ \eta_{rz}\} \\ \{\tau\} &= \{\tau_{rr} \ \tau_{zz} \ \tau_{\theta\theta} \ \tau_{rz}\} \\ \{s\} &= \{s_{rr} \ s_{zz} \ s_{\theta\theta} \ s_{rz}\} \\ \{\epsilon\} &= \{\epsilon_{rr} \ \epsilon_{zz} \ \epsilon_{\theta\theta} \ \epsilon_{rz}\} \end{aligned}$$

The strain components are defined as follows:

$$\{\epsilon\} = \{e\} + \{\eta\}$$

where*

$$\left. \begin{aligned} e_{rr} &= u_{r,r} \\ e_{zz} &= u_{z,z} \\ e_{\theta\theta} &= u_r/r \\ e_{rz} &= (u_{r,z} + u_{z,r}) \end{aligned} \right\} \quad (21)$$

$$\left. \begin{aligned} \eta_{rr} &= \frac{1}{2} [(u_{r,r})^2 + (u_{z,r})^2] \\ \eta_{zz} &= \frac{1}{2} [(u_{r,z})^2 + (u_{z,z})^2] \\ \eta_{\theta\theta} &= \frac{1}{2} (u_r/r)^2 \\ \eta_{rz} &= [u_{r,r} u_{r,z} + u_{z,r} u_{z,z}] \end{aligned} \right\} \quad (22)$$

* A comma indicates partial differentiation, i.e. $u_{r,r} \equiv [\partial(u_r)/\partial r]$.

The stress-strain relations, Eq. (17), may also be expressed in matrix form:

$$\{s\} = [C] \{\epsilon\} + \{B\} \phi \quad (23)$$

$$= \frac{1}{\rho} [T] [\bar{C}] [T]^T \{\epsilon\} + \frac{1}{\rho} [T] \{\bar{B}\} \phi$$

where the matrices $[\bar{C}]$ and $\{\bar{B}\}$ take the following forms:

$$[\bar{C}] = 2\mu \begin{bmatrix} A - ({}^1s'_{rr})^2/D & & & \text{Symmetrical} \\ B - {}^1s'_{rr} {}^1s'_{zz}/D & A - ({}^1s'_{zz})^2/D & & \\ B - {}^1s'_{rr} {}^1s'_{\theta\theta}/D & B - {}^1s'_{\theta\theta} {}^1s'_{zz}/D & A - ({}^1s'_{\theta\theta})^2/D & \\ - {}^1s'_{rr} {}^1s'_{rz}/D & - {}^1s'_{zz} {}^1s'_{rz}/D & {}^1s'_{\theta\theta} {}^1s'_{rz}/D & \frac{1}{2} - ({}^1s'_{rz})^2/D \end{bmatrix}$$

where

$$A = \frac{1 - \nu}{1 - 2\nu}, \quad B = \frac{\nu}{1 - 2\nu}, \quad \mu = \frac{E}{2(1 + \nu)}$$

ν = Poisson's ratio

E = Elastic modulus

$$D = \frac{2}{3} (s)^2 \{1 + H'/3\mu\}$$

and

$$\{\bar{B}\} = \begin{Bmatrix} C - \frac{2}{3} s F' {}^1s'_{rr}/D \\ C - \frac{2}{3} s F' {}^1s'_{zz}/D \\ C - \frac{2}{3} s F' {}^1s'_{\theta\theta}/D \\ - \frac{2}{3} s F' {}^1s'_{rz}/D \end{Bmatrix}$$

where

$$C = \frac{E\alpha}{1 - 2\nu}, \quad F' = \frac{\partial s}{\partial T}$$

α = coefficient of thermal expansion.

In terms of physical components the matrix $[T]$ has the form^{*}

$$[T] = \begin{bmatrix} (1 + u_{r,r})^2 & (u_{r,z})^2 & 0 & 2(1 + u_{r,r})u_{r,z} \\ (u_{z,r})^2 & (1 + u_{z,z})^2 & 0 & 2u_{z,r}(1 + u_{z,z}) \\ 0 & 0 & \left(1 + \frac{u_r}{r}\right)^2 & 0 \\ (1 + u_{r,r})u_{z,r} & (u_{r,z})(1 + u_{z,z}) & 0 & u_{r,z}u_{z,r} + (1 + u_{r,r})(1 + u_{z,z}) \end{bmatrix}.$$

Substituting Eqs. (23) into (20) we get

$$\begin{aligned} \int_a \{\delta u\}^T \{^2 t\} da + \int_v {}^1 \rho \{\delta u\}^T \{^2 f\} dv - \int_v \{\delta e\}^T \{^1 \tau\} dv + \int_v \{\delta \epsilon\}^T \{B\} \phi dv \\ = \int_v \left[\{\delta \eta\}^T \{^1 \tau\} + \{\delta \epsilon\}^T [C] \{\epsilon\} \right] dv. \quad (24) \end{aligned}$$

The second term on the right-hand side of this expression contains nonlinear terms which give rise to higher order matrix products, involving products of (unknown) displacements. To avoid iterations necessary for calculating these displacements, the expression is linearized by disregarding the nonlinear terms. This is valid as long as the incremental strains are small. Thus, we can write

$$\int_v \{\delta \epsilon\}^T [C] \{\epsilon\} dv \approx \int_v \{\delta e\}^T [C] \{e\} dv.$$

The first term on the right can also be written in expanded form as

$$\int_v \{\delta \eta\}^T \{^1 \tau\} dv \approx \int_v \{\delta u^i\}^T [\bar{\tau}] \{u^i\} dv,$$

where

$$\{u^i\} \equiv \{u_{r,r} \ u_{r,z} \ u_{r/r} \ u_{z,r} \ u_{z,z}\} \quad (25)$$

^{*}The T matrix is based on the undeformed geometry.

$$[\bar{\tau}] = \begin{bmatrix} 1\tau_{rr} & 1\tau_{rz} & & & \\ 1\tau_{rz} & 1\tau_{zz} & & & \\ & & 1\tau_{\theta\theta} & & \\ & & & 1\tau_{rr} & 1\tau_{rz} \\ & & & 1\tau_{rz} & 1\tau_{zz} \end{bmatrix}$$

Thus Eqs. (24) become on substitution

$$\begin{aligned} \int_a \{\delta u\}^T \{^2t\} da + \int_v 1\rho\{\delta u\}^T \{^2f\} dv - \int_v \{\delta e\}^T [\{^1\tau\} - \{B\}\epsilon] dv \\ = \int_v \{\delta u\}^T [\bar{\tau}]\{u\} dv + \int_v \{\delta e\}^T [C]\{e\} dv. \end{aligned} \quad (26)$$

The Cauchy stresses are obtained from the stress transformation Eqs. (10). For small incremental strains these relations can be linearized as shown in Appendix B. Thus in matrix notation Eqs. (10) can be expressed as

$$\begin{Bmatrix} \tau_{rr} \\ \tau_{zz} \\ \tau_{\theta\theta} \\ \tau_{rz} \end{Bmatrix} = \begin{Bmatrix} s_{rr} \\ s_{zz} \\ s_{\theta\theta} \\ s_{rz} \end{Bmatrix} + \begin{bmatrix} s_{rr} & 2s_{rz} & -s_{rr} & 0 & -s_{rr} \\ -s_{zz} & 0 & -s_{zz} & 2s_{rz} & s_{zz} \\ -s_{\theta\theta} & 0 & s_{\theta\theta} & 0 & -s_{\theta\theta} \\ 0 & s_{zz} & -s_{rz} & s_{rr} & 0 \end{bmatrix} \begin{Bmatrix} u_{r,r} \\ u_{r,z} \\ u_{r/r} \\ u_{z,r} \\ u_{z,z} \end{Bmatrix}$$

or

$$\{^2\tau\} = \{^2s\} + [\bar{s}]\{u\}. \quad (27)$$

Likewise, Eqs. (12) can also be linearized and expressed in matrix form as

$$\begin{Bmatrix} \xi_{rr} \\ \xi_{zz} \\ \xi_{\theta\theta} \\ \xi_{rz} \end{Bmatrix} = \begin{Bmatrix} \epsilon_{rr} \\ \epsilon_{zz} \\ \epsilon_{\theta\theta} \\ \epsilon_{rz} \end{Bmatrix} + \begin{bmatrix} 2\epsilon_{rr} & 0 & 0 & 2\epsilon_{rz} & 0 \\ 0 & 2\epsilon_{rz} & 0 & 0 & 2\epsilon_{zz} \\ 0 & 0 & 2\epsilon_{\theta\theta} & 0 & 0 \\ \epsilon_{rz} & \epsilon_{rr} & 0 & \epsilon_{zz} & \epsilon_{rz} \end{bmatrix} \begin{Bmatrix} u_{r,\bar{r}} \\ u_{r,\bar{z}} \\ u_{r/\bar{r}} \\ u_{z,\bar{r}} \\ u_{z,\bar{z}} \end{Bmatrix}$$

or

$$\{\xi\} = \{\epsilon\} + [\bar{\epsilon}]\{u\}. \quad (28)$$

Expressions (27) and (28) each contain a displacement gradient vector. The two vectors are calculated on different geometries, however.

APPLICATION OF THE (DISPLACEMENT) FINITE ELEMENT METHOD

The finite element method is adequately described in Refs. 11 and 12. Thus Eq. (22), which is written for the entire deformable body, is also valid for an assemblage of regions or elements into which the body might be divided. In this report we will use isoparametric quadrilateral elements, as described in Refs. 11 and 13. The displacement field within each element can be expressed in terms of generalized displacements by interpolation functions, such that

$$\{u\} = [h]\{u_c\}, \quad (29)$$

where

$[h]$ = matrix whose coefficients depend on the element geometry
 $\{u_c\}$ = incremental corner displacements.

The (r, z) coordinates of each material point within the element can also be expressed in terms of the corner coordinates by the same interpolating functions as shown by Eqs. (29), i.e.

$$\{r\} = [h]\{r_c\}, \quad (30)$$

where

$\{r_c\}$ = corner coordinates
 $\{r\}$ = $\{r \ z\}$.

With these definitions we can express the linear strains as

$$\{e\} = [H]\{u_c\}. \quad (31)$$

Likewise, we can also express the derivatives of the displacements in terms of the corner displacements as

$$\{u'\} = [G]\{u_c\}. \quad (32)$$

The forms of these matrices are derived in Appendix C. Substituting Eqs. (29), (31), and (32) into (26) we get:

$$\begin{aligned} \int_a [h]^T \{2t\} da + \int_v 1 \rho [h]^T \{2f\} dv - \int_v [H]^T \{1 \tau\} - \{B\} \phi dv \\ = \left[\int_v [G]^T \{1 \tau\} [G] dv + \int_v [H]^T [C] [H] dv \right] \{u_c\}. \end{aligned} \quad (33)$$

Equations (33) are thus expressed for each element as

$$\{\tilde{2}_T\} + \{\tilde{2}_F\} - \{\tilde{G}\} = \left[[k]_G + [k]_E \right] \{\tilde{U}\} . \quad (34)$$

By considering interelement compatibility and the boundary conditions, Eqs. (34) can be assembled by standard methods into a system of global equilibrium equations for the entire structure. Thus we get

$$\{\tilde{2}_T\} + \{\tilde{2}_F\} - \{\tilde{G}\} = \left[[K]_G + [K]_E \right] \{\tilde{U}\} . \quad (35)$$

On solving this system of equations, the Cauchy stresses and the strains at the end of the load increment can be calculated. The geometry is updated from the relations

$$\{\Sigma \tilde{U}\}^{n+1} = \{\Sigma \tilde{U}\}^n + \{\tilde{U}\} .$$

In Eqs. (35) the external nodal forces are related to the surface tractions which are assumed to act in the current configuration but are referred to the initial configuration. These tractions usually depend on the applied pressure, which is referred to the current configuration, however. A transformation or load correction must therefore be performed when calculating the nodal forces due to these tractions, as pointed out by Oden.¹⁴ For formulations where the geometry is periodically updated this correction is unnecessary as long as incremental geometry changes are small.

Numerical Results

BRIEF DESCRIPTION OF THE COMPUTER PROGRAM

The theory presented above is implemented in practical application by the computer program EPSOLA. This program uses an incremental-iterative scheme for generating solutions by the following procedure:

- a. The entire load is applied and the elastic solution is calculated.
- b. The load is reduced so that yielding occurs in the highest stressed element.
- c. Loads are then applied incrementally until the maximum external load is reached.
- d. For each increment of load, an incremental elastic solution is first obtained. This indicates which elements might be experiencing either initial or additional yielding during the current load increment, or internal unloading. If yielding is indicated the material stiffness matrix is modified to include plastic effects. The equivalent stress in the elements is then reduced to correspond to that on the equivalent-stress—equivalent-plastic-strain curve for the various materials. Additional iterations are performed, during which the geometry is

updated, and the calculated equivalent stress compared with the corresponding equivalent stress from the stress-strain curve. Equilibrium is checked at each iteration; convergence is achieved when a norm ratio of displacements becomes less than a prescribed minimum.

Although it is not strictly necessary, a load correction provision is also included after the first iteration to allow for the possibility of large incremental changes in geometry.

- e. The stresses, strains and displacements are thus obtained for the entire load, taking into account the boundary conditions. Residual stresses can also be obtained by unloading in a number of increments.

A provision of EPSOLA also exists for analyzing problems subjected to simultaneous gravitational loads and pressure. The elastic stresses due to gravity are calculated initially. The pressure required to just cause yielding in the highest stressed element can be obtained if a reduction factor r is first calculated:

$$r = \frac{[4(\bar{\sigma}_p)^2 (\sigma_y^2 - \bar{\sigma}_G^2) + 9(\sigma'_{ij} \sigma'_{ijG})]^{1/2} - 3\sigma'_{ijG}}{2(\bar{\sigma}_p)^2},$$

where the subscripts p and G indicate pressure and gravity respectively, and

$$\sigma \equiv \left(\frac{3}{2} \sigma'_{ij} \sigma'_{ij} \right)^{1/2}$$

$$\sigma'_{ij} \equiv \sigma_{ij} - \frac{1}{3} \sigma_{kk} \delta_{ij}$$

$$\sigma_y = \text{initial yield stress.}$$

This formula is obtained by solving for r from

$$\sigma_y^2 = \frac{3}{2} (\sigma'_{ijG} + r\sigma'_{ijp}) (\sigma'_{ijG} + r\sigma'_{ijp}).$$

The reduction factor lowers the total applied pressure to the point when yielding will just occur under the combination of the reduced stresses and the stresses due to gravity alone.

The code EPSOLA uses the QM4 isoparametric element (see Appendix C) as described by Dougherty,¹³ with provision for inclined boundaries. Stresses and strains are calculated at the centers of the elements.

NUMERICAL EXAMPLE

The literature contains few examples of complete solutions of elastic-plastic axisymmetric deformation including large strains. One such example, described in Ref. 15, treats the deformation of thick-walled tubes under internal pressure.

Fig. 2. Finite element model of a thick-walled aluminum cylinder. Nodes are restrained against horizontal movement. (Not to scale)

Figure 2 shows the finite element model for the specific example given in Ref. 15. It consists of a 17SO aluminum tube with wall ratio (o.d./i.d. ratio) = 2.0. The tube is modeled by 20 elements. The internal pressure at which yielding first occurred was calculated at 4.995 ksi (actual 4.77 ksi) as described above. Additional load was then applied in increments of 400 psi until the curve leveled out (around 18.6 ksi).

For each load increment, a number of interactions were performed until the radial stress at the inner element converged sufficiently as shown in Fig. 3. Beyond 18.6 ksi the calculated behavior became erratic, indicating ill-conditioning of the equations which implied physical instability of the cylinder. Also, as this critical load was approached, the number of iterations/cycle increased progressively, making the process very time consuming.

Figure 4 compares calculated and experimental results from Ref. 15 with the

Fig. 3. Engineering stress-strain curve for 17SO aluminum (based on true-stress - true-strain data from Ref. 15).

Fig. 4. Comparison of calculated results with theoretical and experimental results from Ref. 15.

result computed by the present technique, plotted as internal pressure vs external tangential strain. The calculated and computed results are seen to agree reasonably well, but disagree with the experimental result. As pointed out in Ref. 15, however, this disagreement appears to be due to the anisotropy of the material rather than to any significant discrepancy.

Conclusion

We have presented a method for static stress analysis of axisymmetric elastic-plastic solids with large deformation. We have also described a computer program based on this theory and presented a numerical result. This result compares favorably with available analytical and experimental results. It indicates, however, that additional effort is required in devising a more efficient numerical method of solving the resulting systems of nonlinear algebraic equations.

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Appendix A

Derivation of Incremental Elastic-Plastic Stress-Strain Relations

Small Deformation Formulation*

The basic assumptions on which the present derivation is based are:

- a. The material is initially isotropic.
- b. A yield function f exists.
- c. The total strain increments are composed of elastic, plastic and thermal components.
- d. The plastic strain increments are proportional to the components of the gradient of the yield function (normality rule).
- e. The material is plastically incompressible.

Assumptions (a) and (b) imply that the yield condition is a function of the invariants of the stress tensor. In general, the yield function can be expressed as

$$f = f(\sigma_{ij}, \epsilon_{ij}^P, T),$$

where

$$\begin{aligned} \epsilon_{ij}^P &= \text{plastic strain components} \\ T &= \text{temperature,} \end{aligned}$$

or as

$$f = f(\sigma_{ij}, s),$$

where

$$\begin{aligned} s &= \text{yield strength in tension} \\ &= s(\epsilon_{ij}^P, T). \end{aligned}$$

Yielding is specified by the conditions

$$\begin{aligned} f &= 0 \\ \dot{f} &= \frac{\partial f}{\partial \sigma_{ij}} \dot{\sigma}_{ij} + \frac{\partial f}{\partial s} \dot{s} = 0 \end{aligned} \tag{A1}$$

where

$$\dot{s} = \frac{\partial s}{\partial \epsilon_{ij}^P} \dot{\epsilon}_{ij}^P + \frac{\partial s}{\partial T} \dot{T}.$$

* An alternate derivation is given in Y. Yamada, N. Yoshimura, and T. Sakurai, Int. J. Mech. Sci. **10**, 343 (1968).

From assumption (c) we can express the total strain increments as

$$\dot{\epsilon}_{ij} = \dot{\epsilon}_{ij}^E + \dot{\epsilon}_{ij}^P + \dot{\epsilon}_{ij}^T, \quad (A2)$$

where^{*}

$$\begin{aligned} \dot{\epsilon}_{ij}^E &= \text{elastic strain increment} \\ \dot{\epsilon}_{ij}^P &= \text{plastic strain increment} \\ \dot{\epsilon}_{ij}^T &= \text{thermal strain increment.} \end{aligned}$$

These components can be further defined as follows:

The elastic strain increments are related to the stress increments by Hookes Law:

$$\dot{\epsilon}_{ij}^E = \frac{1}{2G} \dot{\sigma}_{ij} - \frac{\nu}{E} \dot{\sigma}_{kk} \delta_{ij}. \quad (A3)$$

The thermal strain increments are given as

$$\dot{\epsilon}_{ij}^T = \alpha \dot{T} \delta_{ij}. \quad (A4)$$

According to assumption (d) the plastic strain increments are taken as

$$\dot{\epsilon}_{ij}^P = \dot{\Lambda} \frac{\partial f}{\partial \sigma_{ij}}. \quad (A5)$$

Here

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Λ = (unknown) proportionality coefficient

$\dot{\sigma}_{ij}$ = incremental stress tensor

σ_{ij} = total stress tensor

α = coefficient of thermal expansion

$\dot{T}$ = temperature rate

ν = Poisson's ratio } {assumed independent
 E = Young's modulus } {of temperature

$G = E/(2(1 + \nu))$

Substituting Eqs. (A3-A5) into (A2) we get

$$\dot{\epsilon}_{ij} = \frac{1}{2G} \dot{\sigma}_{ij} - \frac{\nu}{E} \dot{\sigma}_{kk} \delta_{ij} + \alpha \dot{T} \delta_{ij} + \dot{\Lambda} \frac{\partial f}{\partial \sigma_{ij}}. \quad (A6)$$

* A dot above a symbol indicates an increment.

For practical application it is desirable to express the stress increments in terms of the total strain increments, since these are the only ones which can be expressed in terms of displacement increments. To invert Eq. (A6) we first rewrite it as

$$\dot{\sigma}_{ij} - \frac{\nu}{1+\nu} \dot{\sigma}_{kk} \delta_{ij} = 2G \dot{\epsilon}_{ij} - 2G \alpha \dot{T} \delta_{ij} - 2G \dot{\lambda} \frac{\partial f}{\partial \sigma_{ij}}. \quad (A7)$$

We solve for $\dot{\sigma}_{kk}$ by letting $j = i$ and summing. Thus

$$\dot{\sigma}_{kk} = \left(\frac{E}{1-2\nu} \right) \left[\dot{\epsilon}_{kk} - 3\alpha \dot{T} - \dot{\lambda} \frac{\partial f}{\partial \sigma_{kk}} \right]. \quad (A8)$$

Substituting back into (A7) we get:

$$\dot{\sigma}_{ij} = 2G \dot{\epsilon}_{ij} + \frac{2G\nu}{(1-2\nu)} \dot{\epsilon}_{kk} \delta_{ij} = \frac{E\alpha \dot{T}}{1-2\nu} \delta_{ij} - 2G \dot{\lambda} \left[\frac{\nu}{1-2\nu} \frac{\partial f}{\partial \sigma_{kk}} \delta_{ij} + \frac{\partial f}{\partial \sigma_{ij}} \right]. \quad (A9)$$

The solution for $\dot{\lambda}$ is obtained as follows:

Multiply (A6) by $\partial f / \partial \sigma_{ij}$ and add:

$$\frac{\partial f}{\partial \sigma_{ij}} \dot{\epsilon}_{ij} = \frac{1}{2G} \frac{\partial f}{\partial \sigma_{ij}} \dot{\sigma}_{ij} - \frac{\nu}{E} \dot{\sigma}_{kk} \frac{\partial f}{\partial \sigma_{mm}} + \alpha \dot{T} \frac{\partial f}{\partial \sigma_{mm}} + \dot{\lambda} \frac{\partial f}{\partial \sigma_{ij}} \frac{\partial f}{\partial \sigma_{ij}}. \quad (A10)$$

From (A1) we get, at yielding,

$$\frac{\partial f}{\partial \sigma_{ij}} = - \frac{\partial f}{\partial s} \left(\frac{\partial s}{\partial \epsilon_{ij}^P} \dot{\epsilon}_{ij}^P + \frac{\partial s}{\partial T} \dot{T} \right) = - \frac{\partial f}{\partial s} \left(\frac{\partial s}{\partial \epsilon_{ij}^P} \dot{\lambda} \frac{\partial f}{\partial \sigma_{ij}} + \frac{\partial s}{\partial T} \dot{T} \right). \quad (A11)$$

Substituting (A8 and A11) into Eqs. (A10) we get

$$\begin{aligned} \frac{\partial f}{\partial \sigma_{ij}} \dot{\epsilon}_{ij} = & - \frac{1}{2G} \frac{\partial f}{\partial s} \left(\frac{\partial s}{\partial \epsilon_{ij}^P} \dot{\lambda} \frac{\partial f}{\partial \sigma_{ij}} + \frac{\partial s}{\partial T} \dot{T} \right) - \frac{\nu}{1-2\nu} \left[\dot{\epsilon}_{kk} - 3\alpha \dot{T} - \dot{\lambda} \frac{\partial f}{\partial \sigma_{kk}} \right] \\ & + \alpha \dot{T} \frac{\partial f}{\partial \sigma_{mm}} + \dot{\lambda} \frac{\partial f}{\partial \sigma_{ij}} \frac{\partial f}{\partial \sigma_{ij}}. \end{aligned}$$

Collecting terms and solving for $\dot{\lambda}$ we get:

$$\dot{\lambda} = \frac{\frac{\partial f}{\partial \sigma_{ij}} \dot{\epsilon}_{ij} + \frac{\partial f}{\partial \sigma_{mm}} \left[\left(\frac{\nu}{1-2\nu} \right) \dot{\epsilon}_{kk} - \frac{1+\nu}{1-2\nu} \alpha \dot{T} \right] + \frac{1}{2G} \frac{\partial f}{\partial s} \frac{\partial s}{\partial T} \dot{T}}{\frac{\partial f}{\partial \sigma_{ij}} \frac{\partial f}{\partial \sigma_{ij}} + \frac{\nu}{1-2\nu} \frac{\partial f}{\partial \sigma_{kk}} \frac{\partial f}{\partial \sigma_{mm}} - \frac{1}{2G} \frac{\partial f}{\partial s} \frac{\partial s}{\partial \epsilon_{ij}^P} \frac{\partial f}{\partial \sigma_{ij}}}. \quad (A12)$$

The quantity $\frac{\partial s}{\partial \epsilon_{ij}^P}$ is obtained from Eq. (A2):

$$\begin{aligned}\dot{s} &= \frac{\partial s}{\partial \epsilon_{ij}^P} \dot{\epsilon}_{ij}^P + \frac{\partial s}{\partial T} \dot{T} \\ &= \frac{\partial s}{\partial \epsilon^P} \frac{\partial \epsilon^P}{\partial \epsilon_{ij}^P} \dot{\epsilon}_{ij}^P + \frac{\partial s}{\partial T} \dot{T}\end{aligned}$$

where ϵ^P = equivalent plastic strain.

The quantity $\partial s / \partial \epsilon^P$ is therefore the slope H' of the $s - \epsilon^P$ curve which can be obtained from a uniaxial stress-strain curve at constant temperature. Therefore

$$\dot{s} = H' \frac{\partial \epsilon^P}{\partial \epsilon_{ij}^P} \dot{\epsilon}_{ij}^P + \frac{\partial s}{\partial T} \dot{T} \quad (A13)$$

However, we can also write $\dot{s}$ as

$$\begin{aligned}\dot{s} &= \frac{\partial s}{\partial w^P} \dot{w}^P + \frac{\partial s}{\partial T} \dot{T} \\ &= \frac{\partial s}{\partial \epsilon^P} \frac{\partial \epsilon^P}{\partial w^P} \dot{w}^P + \frac{\partial s}{\partial T} \dot{T}\end{aligned}$$

where $\dot{w}^P$ = plastic work increment

$$\begin{aligned}&\approx \sigma'_{ij} \dot{\epsilon}_{ij}^P \\ &= s'_{\epsilon} \dot{\epsilon}^P\end{aligned}$$

where

$$\sigma'_{ij} = \sigma_{ij} - \frac{1}{3} \sigma_{kk} \delta_{ij}$$

and

$$\dot{\epsilon}_{kk}^P = 0.$$

Thus

$$\dot{s} = \frac{H'}{s} \sigma'_{ij} \dot{\epsilon}_{ij}^P + \frac{\partial s}{\partial T} \dot{T} \quad (A14)$$

Comparing (A13) and (A14) we get

$$\frac{\partial \epsilon^P}{\partial \epsilon_{ij}^P} = \frac{\sigma'_{ij}}{s} \quad (A15)$$

and

$$\frac{\partial s}{\partial \epsilon_{ij}} = \frac{H'}{s} \sigma'_{ij}.$$

Equations (A9), (A12), and (A15) define the desired stress-strain relations. For the von Mises yield condition we take

$$f = J_2 - \frac{(s)^2}{3}, \quad (A16)$$

where

$$J_2 = \frac{1}{2} \sigma'_{ij} \sigma'_{ij}.$$

At yielding

$$J_2 = \frac{(s)^2}{3}.$$

From (A16) we get

$$\frac{\partial f}{\partial \sigma'_{ij}} = \sigma'_{ij}$$

$$\frac{\partial f}{\partial s} = -\frac{2}{3} s$$

$$\frac{\partial f}{\partial \sigma_{kk}} = 0.$$

Therefore the expression for $\dot{\lambda}$ becomes

$$\dot{\lambda} = \frac{\sigma'_{ij} \dot{\epsilon}_{ij} - \frac{1}{3G} s \frac{\partial s}{\partial T} \dot{T}}{\sigma'_{ij} \sigma'_{ij} + \frac{s}{3G} \frac{H'}{s} \sigma'_{ij} \sigma'_{ij}} = \frac{\sigma'_{ij} \dot{\epsilon}_{ij} - \frac{1}{3G} s \frac{\partial s}{\partial T} \dot{T}}{\frac{2}{3} (s)^2 [1 + H'/3G]}.$$

Equations (A9) thus become

$$\dot{\sigma}_{ij} = 2G \dot{\epsilon}_{ij} + \left(\frac{2G\nu}{1-2\nu} \dot{\epsilon}_{kk} - \frac{E\alpha \dot{T}}{1-2\nu} \right) \delta_{ij} - \beta 2G \sigma'_{ij} \left[\frac{\sigma'_{mn} \dot{\epsilon}_{mn} - \frac{1}{3G} s \frac{\partial s}{\partial T} \dot{T}}{\frac{2}{3} (s)^2 [1 + H'/3G]} \right]$$

or

$$\dot{\sigma}_{ij} = 2G \left[\dot{\epsilon}_{ij} + \frac{\nu \delta_{ij}}{1-2\nu} \dot{\epsilon}_{kk} - \beta \frac{\sigma'_{ij} \sigma'_{mn} \dot{\epsilon}_{mn}}{\frac{2}{3} (s)^2 [1 + H'/3G]} \right] - \left[\frac{E\alpha \delta_{ij}}{1-2\nu} - \frac{\beta}{s[1 + H'/3G]} \sigma'_{ij} \right] \dot{T}. \quad (A17)$$

These are the same relations as developed by Yamada, except for the inclusion of the thermal terms.

In Eq. (A17) the factor β , introduced to indicate loading or unloading, is specified as follows:

$$\begin{aligned}\beta &= 1 \text{ if } f = 0 \text{ and } df > 0 \\ \beta &= 0 \text{ if } f = 0 \text{ and } df = 0 \\ &\text{or } f < 0.\end{aligned}$$

Hence the plastic terms are included only upon loading, and unloading occurs elastically.

The equivalent plastic strain increment is obtained as follows:

$$\begin{aligned}\dot{\epsilon}^P &= \frac{\sigma'_{ij} \dot{\epsilon}_{ij}^P}{s} = \frac{\sigma'_{ij}}{s} \lambda \frac{\partial f}{\partial \sigma_{ij}} \\ &= \frac{\sigma'_{ij} \sigma'_{ij}}{s} \lambda = \frac{2}{3} s \lambda \\ &= \frac{1}{s} \frac{\left(\sigma'_{ij} \dot{\epsilon}_{ij} - \frac{s}{3G} \frac{\partial s}{\partial T} \dot{T} \right)}{[1 + H'/3G]},\end{aligned}$$

also obtained by Yamada except for the thermal term.

The term H' can be shown to be related to the uniaxial stress-strain curve by the relation

$$H' = \frac{EE^*}{E - E^*}$$

where

E^* = slope of the stress-strain curve at the current value of s .

Equations (A17) can also be expressed as

$$\dot{\sigma}_{ij} = C_{ijk\ell} \dot{\epsilon}_{k\ell} - B_{ij} \dot{T} \quad (A18)$$

where

$$C_{ijk\ell} = G(\delta_{ik} \delta_{j\ell} + \delta_{jk} \delta_{i\ell}) + \lambda(\delta_{ij} \delta_{k\ell}) = 3G\beta \frac{\sigma'_{ij} \sigma'_{k\ell}}{(s)^2 [1 + H'/3G]} \quad (A19)$$

$$B_{ij} = \frac{E\alpha}{1 - 2\nu} \delta_{ij} - \beta \frac{\partial s}{\partial T} \frac{\sigma'_{ij}}{s [1 + H'/3G]} \quad (A20)$$

$$\lambda = \frac{2G\nu}{1 - 2\nu}.$$

Equations (A18) are valid in Cartesian coordinates only.

Large Deformation Formulation

Equations (A17) or (A18) can be applied to problems where large displacements occur, provided they are correctly interpreted. For large displacements, the geometry with respect to which the stresses and strains in the deformed body are specified must be indicated.

Following Yaghmai* we base the equations on the undeformed geometry, with suitable transformations to the initial geometry as needed.

In terms of quantities referred to the undeformed geometry, we can express Eqs. (A18) as

$$\begin{aligned}\bar{s}_{ij} &= \tau_{ijk\ell} \xi_{k\ell} + \bar{B}_{ij} \phi \\ \tau_{ijk\ell} &= G (\delta_{ik} \delta_{jl} + \delta_{jk} \delta_{il}) + \lambda (\delta_{ij} \delta_{kl}) - 2G\beta \frac{{}^1s'_{ij} {}^1s'_{kl}}{\frac{2}{3}(s)^2[1 + H'/3G]} \\ \bar{B}_{ij} &= \frac{E\lambda}{1-2\nu} \delta_{ij} - \beta \frac{\partial s / \partial T} {s[1 + H'/3G]} {}^1s'_{ij}\end{aligned}\tag{A21}$$

These equations are based on

$$\begin{aligned}r &= \frac{1}{2} \left({}^1s'_{ij} {}^1s'_{ij} \right) - \frac{(s)^2}{3} \\ H' &= \partial s / \partial {}^1\epsilon_{ij}^P \\ {}^1\epsilon^P &= \frac{{}^1s'_{ij} \xi_{ij}}{s[1 + H'/3G]}\end{aligned}\tag{A22}$$

and

$${}^1s'_{ij} = {}^1s_{ij} - \frac{1}{3} \delta_{ij} s_{kk}$$

where

${}^1s_{ij}$ = stresses in the initial configuration referred to the undeformed configuration.

ξ_{ij} = Lagrangian strain difference referred to the undeformed configuration.

$\bar{s}_{ij}$ = incremental stresses in the current configuration referred to the undeformed configuration.

ϕ = temperature difference.

*S. Yaghmai, Incremental Analysis of Large Deformations in Mechanics of Solids with Applications to Axisymmetric Shells of Revolution, Structural Engineering Laboratory, University of California, Berkeley, Rept. SESM 68-17 (1968).

These equations are still valid only in Cartesian coordinates. In curvilinear coordinates the corresponding equations are given by Eqs. (19) in the report.

The corresponding uniaxial stress-strain equation to be used in Eqs. (A21) and (A22) are based on the initial area. It is, however, convenient to approximate Eqs. (A21) and (A22) by using the available Cauchy stresses referred to the initial configuration. In this case the corresponding uniaxial stress-strain curve is then based on the current area, i.e., "true stress."

Appendix B

Calculations of Approximate Transformation for Stresses and Strains in Cylindrical Coordinates

Cauchy-Piola Stress Transformation

As shown by Fung,^{*} the relation between the Cauchy and Piola stress terms at a point is given as

$$2_{\tau mn} = \frac{2}{I\rho} \frac{\partial Z^m}{\partial z^k} \frac{\partial Z^n}{\partial z^l} 2_{s kl} . \quad (B1)$$

In general curvilinear coordinates the deformation gradients may be expressed in terms of the tensor displacement components. Let

$$Z^l = z^l + u^l .$$

Then

$$\frac{\partial Z^l}{\partial z^j} = \delta_j^l + u^l_{|j} . \quad (B2)$$

Substituting into Eqs. (B1) we get

$$2_{\tau mn} = \frac{2}{I\rho} \left[2_{s mn} + 2_{s ml} u^n_{|l} + 2_{s kn} u^m_{|k} + u^m_{|k} u^n_{|l} 2_{s kl} \right] . \quad (B3)$$

Now, the following relation can be shown to be true[†]

$$\frac{2}{I\rho} = (1 + 2I + 4II + 8III)^{-\frac{1}{2}} \quad (B4)$$

where I, II, and III are the invariants of the Lagrangian strain tensor from the initial to the current configuration. For small incremental strains, Eq. (B4) can be approximated by

$$\frac{2}{I\rho} = (1 - I_e) \approx 1 - u^k_{|k} .$$

^{*}Y.C. Fung, Foundations of Solid Mechanics (Prentice Hall, Inc., New Jersey, 1965).

[†]A. Eringen, Nonlinear Theory of Continuous Media (McGraw Hill, New York, 1962).

Thus, if higher order terms are neglected, Eqs. (B3) reduce to

$$2_{\tau mn} = \left(1 - u^k|_k\right) 2_{s mn} + 2_{s m \ell} u^{\ell}|_n + 2_{s kn} u^m|_k .$$

The components in this expression are tensor components. The equivalent expression in physical components is:

$$2_{\tau(mn)} = \left(1 - u^{(k)}|_{(k)}\right) 2_{s(mn)} + 2_{s(m \ell)} u^{(n)}|_{(\ell)} + 2_{s(kn)} u^{(m)}|_{(k)} . \quad (B5)$$

It can be shown that in cylindrical coordinates:

$$u^{(1)}|_{(1)} = u_{1,1}, \quad u^{(2)}|_{(1)} = u_{2,1}, \quad u^{(3)}|_{(3)} = u_1/z_1, \quad (B6)$$

$$u^{(1)}|_{(2)} = u_{1,2}, \quad u^{(2)}|_{(2)} = u_{2,2}, \quad \text{all other } u^{(i)}|_{(j)} = 0.$$

In standard notation:

$$z_1 = r, \quad z_2 = z, \quad z_3 = \theta$$

$$u_1 = u, \quad u_2 = v, \quad u_3 = w$$

where u_i now indicates physical components. Equations (B5) may now be expressed in the more convenient matrix form as

$$\begin{Bmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{33} \\ \tau_{12} \end{Bmatrix} = \begin{Bmatrix} s_{11} \\ s_{22} \\ s_{33} \\ s_{12} \end{Bmatrix} + \begin{bmatrix} s_{11} & 2s_{12} & -s_{11} & 0 & -s_{11} \\ -s_{22} & 0 & -s_{22} & 2s_{12} & s_{22} \\ -s_{33} & 0 & s_{33} & 0 & -s_{33} \\ 0 & s_{22} & -s_{12} & s_{11} & 0 \end{bmatrix} \begin{Bmatrix} u_{1,1} \\ u_{1,2} \\ u_1/z_1 \\ u_{2,1} \\ u_{2,2} \end{Bmatrix} \quad (B7)$$

or

$$\{\tau\} = \{s\} + [\bar{S}] \{u'\}$$

where symmetry of the stress tensor was taken into account. From Appendix C we get the relations

$$\{u'\} = [G] \{u\}, \quad (B8)$$

Thus, Eqs. (B7) become

$$\{\tau\} = \{s\} + [\bar{S}] [G] \{u\} . \quad (B9)$$

Strain Transformation

An approximate expression may also be obtained for the transformation

$$\xi_{mn} = \frac{\partial z^k}{\partial \bar{z}^m} \frac{\partial z^l}{\partial \bar{z}^n} \epsilon_{kl} \quad (B10)$$

by using Eqs. (B2). Thus, by neglecting higher order terms we get similar expressions to Eqs. (B5) in terms of physical components:

$$\xi_{(mn)} = \epsilon_{(mn)} + \epsilon_{(mn)} u^{(l)} \Big|_{(n)} + \epsilon_{(kn)} u^{(k)} \Big|_{(m)} .$$

These equations can be expressed in matrix form, taking into account the symmetry of the strain tensor:

$$\begin{Bmatrix} \xi_{11} \\ \xi_{22} \\ \xi_{33} \\ \xi_{12} \end{Bmatrix} = \begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \epsilon_{12} \end{Bmatrix} + \begin{bmatrix} 2\epsilon_{11} & 0 & 0 & 2\epsilon_{12} & 0 \\ 0 & 2\epsilon_{12} & 0 & 0 & 2\epsilon_{22} \\ 0 & 0 & 2\epsilon_{33} & 0 & 0 \\ \epsilon_{12} & \epsilon_{11} & 0 & \epsilon_{22} & \epsilon_{12} \end{bmatrix} \begin{Bmatrix} u_{1,1} \\ u_{1,2} \\ u_{1,\bar{z}_1} \\ u_{2,1} \\ u_{2,2} \end{Bmatrix} \quad (B11)$$

or

$$\{\xi\} = \{\epsilon\} + [\bar{\epsilon}] \{u'\} .$$

By using Eqs. (B8) we get the desired approximation in terms of the physical components of the displacements:

$$\{\xi\} = \{\epsilon\} + [\bar{\epsilon}] [G] \{u\} . \quad (B12)$$

[G] here is based on the undeformed geometry. Equations (B11) are valid as long as the strains remain relatively small. If not, the higher order terms may have to be calculated. This can always be done (at the expense of computation time) since the displacement field is completely known at the beginning of a load increment.

Appendix C

Development of the Isoparametric Quadrilateral Element

The development of the isoparametric quadrilateral element follows that by Dougherty.*

A generic quadrilateral can be defined by two coordinate systems as depicted in Fig. C1.

Fig. C1. Coordinate systems defining a generic quadrilateral.

The (r, z) coordinates form an orthogonal coordinate system while the (s, t) coordinates (also called natural coordinates) may be skewed and are defined such that s and t vary from -1 to 1 .

The (r, z) coordinates of any point within the element can be expressed in terms of the (s, t) coordinates by interpolating functions which depend on (s, t) :

$$r(s, t) = h_i(s, t)r_i \quad z(s, t) = h_i(s, t)z_i \quad i = 1, 4 \quad (C1)$$

where

$$\begin{aligned} h_1 &= (1 - s)(1 - t)/4 & h_3 &= (1 + s)(1 + t)/4 \\ h_2 &= (1 + s)(1 - t)/4 & h_4 &= (1 - s)(1 + t)/4 \end{aligned}$$

The displacement fields within the element can also be expressed by the same interpolating functions in terms of the nodal displacements as

$$u_r(s, t) = h_i u_{ri} \quad u_z(s, t) = h_i u_{zi} \quad i = 1, 4 \quad (C2)$$

* V. P. Dougherty, E. L. Wilson, and R. L. Taylor, Stress Analysis of Axisymmetric Solids Utilizing Higher Order Quadrilateral Finite Elements, Structural Engineering Laboratory, University of California, Berkeley, California (1969).

or

$$\{u\} = [h] \{u_c\}$$

where

$$[h] = \begin{bmatrix} h_1 & 0 & h_2 & 0 & h_3 & 0 & h_4 & 0 \\ 0 & h_1 & 0 & h_2 & 0 & h_3 & 0 & h_4 \end{bmatrix} \quad (C3)$$

and

$$\{u_c\} = \{u_{r1} \ u_{z1} \ u_{r2} \ u_{z2} \ u_{r3} \ u_{z3} \ u_{r4} \ u_{z4}\}.$$

The partial derivatives of u_r and u_z are calculated by applying the chain rule.

Thus

$$\begin{Bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial z} \end{Bmatrix} = \frac{1}{J} \begin{bmatrix} \frac{\partial z}{\partial t} & -\frac{\partial z}{\partial s} \\ -\frac{\partial r}{\partial t} & \frac{\partial r}{\partial s} \end{bmatrix} \begin{Bmatrix} \frac{\partial}{\partial s} \\ \frac{\partial}{\partial t} \end{Bmatrix} \quad (C4)$$

where

$$J = J(s, t) = \frac{\partial r}{\partial s} \frac{\partial z}{\partial t} - \frac{\partial r}{\partial t} \frac{\partial z}{\partial s}.$$

From Eq. (C1) we obtain

$$\begin{aligned} J &= (h_{i,s} h_{j,t} - h_{i,t} h_{j,s}) r_i z_j \\ &= p_{ij} r_i z_j. \end{aligned} \quad (C5)$$

In terms of the orthogonal coordinates of the element corners J takes the form:

$$J = [(r_{13} z_{24} - r_{24} z_{13}) + s(r_{34} z_{12} - r_{12} z_{34}) + t(r_{23} z_{14} - r_{14} z_{23})]/8$$

where

$$\begin{aligned} r_{ij} &\equiv r_i - r_j \\ z_{ij} &\equiv z_i - z_j. \end{aligned} \quad (C6)$$

The derivatives of u_r and u_z are now obtained:

$$\begin{aligned} u_{r,r} &= u_{r,s} s_{,r} + u_{r,t} t_{,r} = p_{ij} z_j u_{ri} \\ u_{r,z} &= u_{r,s} s_{,z} + u_{r,t} t_{,z} = p_{ij} r_j u_{ri} \\ u_{z,r} &= u_{z,s} s_{,r} + u_{z,t} t_{,r} = p_{ij} z_j u_{zi} \\ u_{z,z} &= u_{z,s} s_{,z} + u_{z,t} t_{,z} = p_{ij} r_j u_{zi}. \end{aligned} \quad (C7)$$

Likewise, the quotient u_r/r can also be expressed as

$$u_r/r = h_i u_{ri}/r. \quad (C8)$$

Equations (C7) and (C8) may be expressed in the matrix form

$$\{u'\} = [G] \{u_c\}$$

where

$$\{u'\} = \{u_{r,r} \quad u_{r,z} \quad u_r/r \quad u_{z,r} \quad r_{z,z}\}$$

$$[G] = \begin{bmatrix} Y_1 & 0 & Y_2 & 0 & Y_3 & 0 & Y_4 & 0 \\ X_1 & 0 & X_2 & 0 & X_3 & 0 & X_4 & 0 \\ G_1 & 0 & G_2 & 0 & G_3 & 0 & G_4 & 0 \\ 0 & Y_1 & 0 & Y_2 & 0 & Y_3 & 0 & Y_4 \\ 0 & X_1 & 0 & X_2 & 0 & X_3 & 0 & X_4 \end{bmatrix} \quad (C9)$$

$$Y_i = p_{ij} z_j$$

$$X_i = p_{ij} r_j$$

$$G_i = h_i/r.$$

Likewise, the linear components of the strain tensor may also be expressed in matrix form as follows:

$$\{e\} = [H] \{u_c\}$$

where

$$\{e\} = \{e_{rr} \quad e_{zz} \quad e_{\theta\theta} \quad e_{rz}\}$$

$$[H] = \begin{bmatrix} Y_1 & 0 & Y_2 & 0 & Y_3 & 0 & Y_4 & 0 \\ 0 & X_1 & 0 & X_2 & 0 & X_3 & 0 & X_4 \\ G_1 & 0 & G_2 & 0 & G_3 & 0 & G_4 & 0 \\ X_1 & Y_1 & X_2 & Y_2 & X_3 & Y_3 & X_4 & Y_4 \end{bmatrix} \quad (C10)$$

The matrices (C9) and (C10) correspond to the matrices in Eqs. (31) and (32) in the main text.

The development to this point describes the basic quadrilateral called Q4. Other elements (such as the Q8) are obtained by simply extending the summation range.

The QM4 element is obtained from Q4 by assuming that the shear strain is constant over the element and evaluated at its center. Thus $[H]$ becomes:

$$[H] = \begin{bmatrix} Y_1 & 0 & Y_2 & 0 & Y_3 & 0 & Y_4 & 0 \\ 0 & X_1 & 0 & X_2 & 0 & X_3 & 0 & X_4 \\ G_1 & 0 & G_2 & 0 & G_3 & 0 & G_4 & 0 \\ X_5 & Y_5 & X_6 & Y_6 & X_7 & Y_7 & X_8 & Y_8 \end{bmatrix}$$

where

$$\begin{aligned} X_{i+4} &= X_i(r, s) \\ Y_{i+4} &= Y_i(r, s) \end{aligned} \bigg|_{r, s = 0} \quad i = 1, 4$$