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SECOND ORDER EFFECTS AS RELATED TO
CRITICAL COOLANT FLOW VELOCITIES AND
REACTOR PARALLEL PLATE FUEL ASSEMBLIES

by

M. W. Wambsganss, Jr.

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Reactor Engineering Division

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NOMENCLATURE

<u>Symbol</u>	<u>Description</u>	<u>Symbol</u>
a	Width (normal to flow) of plate	N
b	Length of plate (parallel to flow) between supports	T
c	Plate thickness	U
h	Initial channel depth (undeflected plate position)	V_{CR}
k	Design constant, see eq. (43)	α
ℓ	Dummy variable (in Case I equal to b; in Case II equal to a)	β
p	Instantaneous channel pressure	γ
r	Dummy variable	δ
t	Time	δ_{CR}
v_{CR}	(Miller's) critical flow velocity (first-order considerations), see eq. (23)	λ
w	Plate deflection	μ
x	Transverse (normal to flow) displacement coordinate	ν
y	Longitudinal (parallel to flow) displacement coordinate	ξ
z	Dummy variable (in Case I equal to w; in Case II equal to $\frac{1}{a} \int_0^a w dx$), see eq. (9)	ρ
A	Instantaneous channel cross-sectional area	ρ_f
D	Flexural rigidity of the plate, see eq. (1)	ψ
E	Young's modulus for plate material	<u>Subscripts</u>
F	Equivalent spring force, see eq. (18)	0
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SECOND-ORDER EFFECTS AS RELATED TO CRITICAL COOLANT-FLOW VELOCITIES AND REACTOR PARALLEL-PLATE FUEL ASSEMBLIES

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M. W. Wambsganss, Jr.

ABSTRACT

Second-order effects, removed by the linearization process in the analyses of Miller,¹ Rosenberg and Youngdahl,³ and others, are retained in a stability analysis of the equation of motion of a flat fuel plate in a reactor parallel-plate fuel assembly. The second-order terms generate an additional stability criterion in the form of an upper bound (critical deflection) on the amplitude of quasistatic deflections for stable oscillations, or neutral stability, about the undeflected fuel plate position. The critical flow velocities derived in the earlier studies,^{1,3} and again as a consequence of this analysis, are upper bounds to insure local stability about the undeflected plate position. A constant of proportionality, relating the critical deflection to the initial channel depth, is designated as a design constant. When assigned a value, the design constant defines new critical coolant-flow velocities which are lower than those first arrived at by Miller.

I. INTRODUCTION

The collapse of reactor fuel plates, subjected to water flow, was reportedly¹ observed as far back as 1948² and has been encountered on numerous occasions since then. Today's higher-power reactors are even more susceptible to this type of failure since design specifications often call for thinner fuel plates and higher coolant-flow velocities--both destabilizing effects. This requires that the designer of a reactor core be all the more aware of the type of fuel-plate instability which can lead to plate divergence, and the more serious consequences of flow blockage and resulting fuel melting.

Miller¹ was the first to present a theoretical analysis predicting the flow velocity at which collapse occurs. His analysis, applying a method of "neutral equilibrium," whereby pressure forces and plate-restoring forces are balanced, led to a formulation of critical coolant-flow velocities for various types of support. Rosenberg and Youngdahl³ later formulated a

dynamic model describing the motion of a fuel plate in a parallel-plate assembly. They then solved the resulting eigenvalue problem and obtained the critical coolant-flow velocities of Miller. It should be noted that Rosenberg and Youngdahl also showed that consideration of fluid inertia, in the case of periodically supported plates, leads to a different interpretation of "critical" velocity.

Miller, and Rosenberg and Youngdahl, employ the same basic assumptions in their analyses. Both linearize the pressure-drop expression, using a first-order approximation in which higher-order terms are neglected. In this paper, the second-order terms are retained in an attempt to assess their influence on conclusions regarding stability and the definition of the critical coolant velocity.

The approach taken involves an analysis of the stability of solutions to the describing differential equation of motion. Two methods of stability analysis, both leading to the same results, are employed. The one, heuristic in nature, utilizes inherent properties associated with the potential energy of conservative systems. The second, more formal, method is based on an analysis of the singular points of the describing equations.

The stability analysis, including second-order terms in the series expansion representing the pressure drop, gives the same critical velocities arrived at by the earlier investigators. These represent upper bounds to insure local stability about the undeflected plate position. However, in the extreme environment of a nuclear reactor core, a fuel plate cannot be assumed to remain undeflected, and the question arises as to what maximum steady-state or quasistatic deflections can be experienced without the subsequent occurrence of plate divergence. A first-order approximation will not give this information. However, the inclusion in this study of second-order effects provides the answer in the form of an upper bound on the amplitude of deflection for stable oscillations (neutral stability) about the undeflected plate position.

Selecting a value for a design constant, k , which represents the maximum anticipated fuel-plate deflection as a fraction of initial channel depth, new critical coolant-flow velocities are defined. For example, for $k = 0.5$, the new critical velocities vary (for different types of supports) from 0.633 to 0.852 times the original (Miller's) critical velocities. From theoretical considerations then, the new critical velocities (defined for $k = 0.5$) insure neutral stability about the undeflected plate position, for all steady-state and quasistatic displacements up to one-half the initial channel depth.

It should be noted that other studies of both theoretical and experimental nature, in addition to those already cited, have been performed. These studies and their relation to this work are discussed later.

II. MATHEMATICAL MODEL

A. Modeling Assumptions

The modeling assumptions used in the derivation of equations are essentially the same as those of the earlier investigators.^{1,3} They will be repeated here only to make the paper autonomous:

1. The plates are homogenous, isotropic, elastic, initially flat, and uniformly spaced.
2. The coolant is inviscid and incompressible; one-dimensional flow is assumed.
3. The plates can be deflected locally without significant redistribution of flow among the coolant channels; the mass flow in all channels is the same and constant; no cross flow exists between channels.
4. Linearized bending theory applies; the deflection takes on a preferred direction such that the curvature in the perpendicular direction is small.
5. The deflection pattern is symmetric, such that the deflections of adjacent plates are equal in magnitude but opposite in direction.

Fig. 1. Schematic of Reactor Parallel-plate Fuel Assembly

where

$$D \equiv \text{flexural rigidity} = Ec^3/12(1-\nu^2);$$

$$\nabla^2 \nabla^2 = \left(\frac{\partial^4}{\partial x^4} + 2 \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4} \right).$$

B. Equation of Motion

The schematic, used in the derivation of equations, and illustrating the direction of the coordinate system and defining some of the parameters, is given in Fig. 1. Under the given modeling assumptions, the following two equations form the mathematical model describing the motion of a fuel plate in the parallel-plate assembly:

1. Fuel-plate motion:

$$D \nabla^2 \nabla^2 w + \rho c \frac{\partial^2 w}{\partial t^2} = \Delta p(t), \quad (1)$$

2. Pressure differential:

$$\Delta p(t) = p_2 - p_1 = \frac{1}{2} \rho_f A_0^2 v_0^2 \left(\frac{1}{A_1^2} - \frac{1}{A_2^2} \right). \quad (2)$$

The pressure differential expression is obtained from an application of Bernoulli's Theorem and the continuity equation, both referenced to section 0-0 in Fig. 1.

Two different methods of fabrication are considered for the parallel-plate fuel assemblies:

Case I: Periodic supporting at locations along the length of the plate (in the direction of flow);

Case II: Continuous supporting along the longitudinal edges (in the direction of flow).

In each case, the type of edge fixity is further broken down to a consideration of the commonly assumed fixed and pinned edges, and for Case I also a cantilever support.

Case I: Periodic Support along the Length. In this case the y-direction is assumed to be the preferred direction for deflection. Accordingly, the curvature in the transverse or x-direction is assumed to be small. Therefore, the equation of motion, (1), can be written as

$$D \frac{\partial^4 w}{\partial y^4} + \rho c \frac{\partial^2 w}{\partial t^2} = \Delta p(t). \quad (3)$$

The instantaneous cross-sectional areas of the deflected channels are given by

$$\left. \begin{aligned} A_1(w,t) &= a(h - 2w); \\ A_2(w,t) &= a(h + 2w), \end{aligned} \right\} \quad (4)$$

which, upon substitution into (2), allow writing for the pressure differential,

$$\Delta p(t) = \frac{1}{2} \rho_f v_0^2 \left[\left(1 - \frac{2w}{h} \right)^{-2} - \left(1 + \frac{2w}{h} \right)^{-2} \right]. \quad (5)$$

Case II: Continuous Support along Longitudinal Edges. For this case the x-direction is assumed the preferred direction for deflection, and the equation of motion, (1), becomes

$$D \frac{\partial^4 w}{\partial x^4} + \rho c \frac{\partial^2 w}{\partial t^2} = \Delta p(t). \quad (6)$$

The instantaneous cross-sectional areas are represented by the integral expressions

$$\left. \begin{aligned} A_1(w, t) &= \int_0^a (h - 2w) dx; \\ A_2(w, t) &= \int_0^a (h + 2w) dx. \end{aligned} \right\} \quad (7)$$

After substituting (7) into (2), the pressure differential can be written as

$$\Delta p(t) = \frac{1}{2} \rho_f v_0^2 \left[\left(1 - \frac{2}{ah} \int_0^a w dx \right)^{-2} - \left(1 + \frac{2}{ah} \int_0^a w dx \right)^{-2} \right]. \quad (8)$$

Note that the governing equations are of the same form for both cases, that is,

$$\left. \begin{aligned} D \frac{\partial^4 w}{\partial \xi^4} + \rho c \frac{\partial^2 w}{\partial t^2} &= \Delta p(t); \\ \Delta p(t) &= \frac{1}{2} \rho_f v_0^2 \left[\left(1 - \frac{2z}{h} \right)^{-2} - \left(1 + \frac{2z}{h} \right)^{-2} \right], \end{aligned} \right\} \quad (9)$$

where for Case I: ($\xi = y, z = w$); Case II: ($\xi = x, z = \frac{1}{a} \int_0^a w dx$).

Expanding the right-hand side of the pressure-drop relationship in series form gives

$$\Delta p(t) = \frac{4 \rho_f v_0^2 z}{h} \left(1 + \frac{8}{h^2} z^2 + \frac{48}{h^4} z^4 + \dots \right) \quad (10)$$

for $z/h < 1$. The first two approximations to (10) can be written as

$$\Delta p(t) \approx \begin{cases} \frac{4 \rho_f v_0^2}{h} z, & \text{first-order approximation,} \\ \frac{4 \rho_f v_0^2}{h} z \left(1 + \frac{8}{h^2} z^2 \right), & \text{second-order approximation.} \end{cases} \quad (11)$$

From (11), to a first-order approximation,

$$\Delta p(t) \approx \begin{cases} \frac{4\rho_f v_0^2}{h} w, & \text{for Case I,} \\ \frac{4\rho_f v_0^2}{h} \frac{1}{a} \int_0^a w dx, & \text{for Case II.} \end{cases} \quad (12)$$

These are, in effect, the approximations used in the earlier studies.^{1,3}

Including second-order terms (the approximation to be used in this analysis) the following expressions result:

$$\Delta p(t) \approx \begin{cases} \frac{4\rho_f v_0^2}{h} w \left(1 + \frac{8}{h^2} w^2 \right), & \text{for Case I,} \\ \frac{4\rho_f v_0^2}{h} \left[1 + \frac{1}{a^2} \left(\int_0^a w dx \right)^2 \right] \left(\frac{1}{a} \int_0^a w dx \right), & \text{for Case II.} \end{cases} \quad (13)$$

Reverting to general terms, the differential equation of fuel-plate motion, including second-order effects, can be written, by substituting (11) into (9), as

$$\frac{\partial^2 w}{\partial \xi^4} + \mu^2 \frac{\partial^2 w}{\partial t^2} = Kz \left(1 + \frac{8}{h^2} z^2 \right), \quad (14)$$

where

$$\mu^2 = \rho c / D;$$

$$K = 4\rho_f v_0^2 / Dh.$$

Assuming that the fuel plate responds in its fundamental normal mode of free vibration, a solution to (14) can be represented as

$$w(\xi, t) = \psi(\xi)T(t), \quad (15)$$

where $T(t)$ is the generalized coordinate; $\psi(\xi)$ is the fundamental normal mode, satisfying the relationship

$$\frac{d^4 \psi}{d\xi^4} = \lambda^4 \psi; \quad (16)$$

and λ in (16) is the fundamental eigenvalue of the corresponding free-vibration problem. Application of Galerkin's method by substituting (15) into (14), multiplying through by $\psi(\xi)$, and integrating from 0 to ℓ , gives, using relationships given by (13) and (16),

$$\ddot{T} + M^2 T - N^2 T^3 = 0, \quad (17)$$

where,

$$M^2 = \frac{1}{\mu} \left\{ \lambda^4 - K \frac{\int_0^1 \psi(\bar{\xi}) \left\{ \int_0^1 \psi(\bar{\xi}) d\bar{x} \right\} d\bar{\xi}}{\int_0^1 \psi^2(\bar{\xi}) d\bar{\xi}} \right\}^{1/2};$$

$$N^2 = \frac{1}{\mu h} \left\{ 8K \frac{\int_0^1 \psi(\bar{\xi}) \left\{ \int_0^1 \psi(\bar{\xi}) d\bar{x} \right\}^3 d\bar{\xi}}{\int_0^1 \psi^2(\bar{\xi}) d\bar{\xi}} \right\}^{1/2};$$

$$\bar{\xi} = \xi/\ell; \quad \bar{x} = x/\ell.$$

For Case I: ($\xi = y, \ell = b$); for Case II: ($\xi = x, \ell = a$).^{*} Equation (17) is the equation describing the fuel-plate motion. It is valid for both Cases I and II and general edge fixities, or boundary conditions, as represented by the corresponding free-vibration mode. The stability of fuel-plate motion can, therefore, be studied by investigating the stability of solutions to eq. (17).

^{*}According to the notation employed, viz.,

$$\xi = \begin{cases} y, & \text{Case I} \\ x, & \text{Case II} \end{cases},$$

the integrals, of the type used in eq. (17) are evaluated as

$$\int_0^1 \psi(\bar{\xi}) \left(\int_0^1 \psi(\bar{\xi}) d\bar{x} \right)^k d\bar{\xi} = \begin{cases} \int_0^1 \psi^k(\bar{y}) d\bar{y}, & \text{Case I} \\ \left[\int_0^1 \psi(\bar{x}) d\bar{x} \right]^k, & \text{Case II} \end{cases}$$

III. STABILITY ANALYSIS

A. Considerations of Potential Energy

Observe that the governing differential equation of motion, eq. (17), is identical to the equation of motion of a unit mass on a soft spring, the soft spring being represented by the relationship

$$F = M^2 T - N^2 T^3, \quad (18)$$

where M^2 and N^2 are, for the moment, assumed positive constants. The potential energy stored in the deflected "spring" is

$$U = \int_0^T F dT = \frac{1}{2} M^2 T^2 - \frac{1}{4} N^2 T^4. \quad (19)$$

The force and potential energy as functions of displacement are sketched, from (18) and (19), in Figs. 2a and 2b.

Fig. 2. Force, Potential Energy, and Phase-plane Representation of Fuel-plate Motion

For a conservative system with a single degree of freedom, for which potential energy varies only with displacement and kinetic energy only with velocity, the following holds true:⁴

1. Equilibrium or singular points are located where (dU/dT) vanishes.
2. Solution curves have zero slope at values of T corresponding to singular points.
3. If the point is a local minimum, the singularity is a vortex (neutrally stable); if a local maximum, the singularity is a saddle (unstable).

With these properties, the solution curves given in Fig. 2c were sketched.

Note that the origin is locally stable if $M^2 > 0$, which, from (17), requires that

$$\lambda^4 > K \left[\frac{\int_0^1 \psi(\bar{\xi}) \left\{ \int_0^1 \psi(\bar{\xi}) d\bar{x} \right\} d\bar{\xi}}{\int_0^1 \psi^2(\bar{\xi}) d\bar{\xi}} \right]. \quad (20)$$

In terms of coolant flow velocity, the criterion for a stable vortex at $T = 0$ can be expressed as

$$v_0 < \left[\alpha \frac{Ec^3 h}{(1 - \nu^2) \rho_f l^4} \right]^{1/2}, \quad (21)$$

where

$$\alpha = \left\{ \frac{(\lambda l)^4 \int_0^1 \psi^2(\bar{\xi}) d\bar{\xi}}{48 \int_0^1 \psi(\bar{\xi}) \left[\int_0^1 \psi(\bar{\xi}) d\bar{x} \right] d\bar{\xi}} \right\}; \quad (22)$$

for Case I, $l = b$; for Case II, $l = a$. Therefore, a "critical coolant velocity" above which plate divergence takes place and below which local stability at the undeflected plate position exists, can be defined as

$$v_{CR} = \left[\alpha \frac{Ec^3 h}{(1 - \nu^2) \rho_f l^4} \right]^{1/2}. \quad (23)$$

As will be demonstrated later, when α is evaluated for specific edge fixities, the critical velocities represented by (23) are identical with those arrived at by Miller and by Rosenberg and Youngdahl. The stability criterion is then

$$v_0 < v_{CR}. \quad (24)$$

The second-order effects give rise to a second stability criterion, this one regarding the amplitude of displacement. That is, despite the fact that (24) is satisfied, instability in the form of divergence can still occur if the amplitude, $|T|$, is greater than $|M/N|$; or if the displacement, T , and velocity, $\dot{T}$, are such that at some instant of time the point $(T, \dot{T})$ falls outside the shaded stability region shown in Fig. 2c. The stability criterion for quasistatic deflections can be written as

$$|T| < \left| \frac{M}{N} \right| \quad (25)$$

To express (25) in terms of the actual maximum displacement, define the new generalized coordinate

$$\delta(t) = \psi_{\max} T(t). \quad (26)$$

Equation (26) leads to

$$w(\xi, t) = \frac{\psi(\xi)}{\psi_{\max}} \delta(t). \quad (27)$$

Therefore,

$$w_{\max} = \delta_{\max}. \quad (28)$$

Using (26) to rewrite (25), one obtains

$$|\delta| < \psi_{\max} \left| \frac{M}{N} \right|, \quad (29)$$

or in terms of coolant velocity, the stability criterion

$$\delta < \beta \left[\left(\frac{v_{CR}}{v_0} \right)^2 - 1 \right]^{1/2} h, \quad (30)$$

where

$$\beta = \left\{ \frac{\psi_{\max}^2 \int_0^1 \psi^2(\bar{\xi}) d\bar{\xi}}{8 \int_0^1 \psi(\bar{\xi}) \left[\int_0^1 \psi(\bar{\xi}) dx \right]^3 d\bar{\xi}} \right\}^{1/2} \quad (31)$$

Hence, a "critical deflection" can be defined from (30) to be

$$\delta_{CR} = \beta \left[\left(\frac{v_{CR}}{v_0} \right)^2 - 1 \right]^{1/2} h. \quad (32)$$

Note that β , as with α in (22), is a function of the fundamental-mode shape and, therefore, of the boundary conditions or edge fixities. The second stability criterion can be written as

$$\delta < \delta_{CR}. \quad (33)$$

Having selected a coolant flow velocity satisfying (24), eq. (32) can be used to calculate the maximum allowable quasistatic deflection. Alternatively, the core designer may anticipate a maximum deflection which could occur during reactor operation, and use it to define a new critical coolant-flow velocity insuring neutral stability for all foreseen deflections. These applications, particularly the alternative use of (32) to define new critical velocities, will be discussed later.

B. Analysis of Singular Points

Before proceeding to an evaluation of critical velocities and deflections for specific edge fixities, consider the more formal approach to a stability analysis, that being an analysis of the singular points of the governing equation of motion. Defining $S = \dot{T}$, eq. (17) can be written as the two first-order equations

$$\left. \begin{aligned} \dot{S} &= -M^2T + N^2T^3 = P(T,S); \\ \dot{T} &= S = Q(T,S), \end{aligned} \right\} \quad (34)$$

from which

$$\frac{dS}{dT} = \frac{P(T,S)}{Q(T,S)} \quad (35)$$

follows. Singularities are those values of T and S for which both P and Q simultaneously vanish. A singularity is always a point of equilibrium, which may be stable or unstable. Consideration of (34) reveals three singularities, all at $S = 0$:

$$\left. \begin{aligned} (T_{S1}, S_{S1}) &= [0, 0]; \\ (T_{S2}, S_{S2}) &= \left[\left(\frac{M^2}{N^2} \right)^{1/2}, 0 \right]; \\ (T_{S3}, S_{S3}) &= \left[-\left(\frac{M^2}{N^2} \right)^{1/2}, 0 \right]. \end{aligned} \right\} \quad (36)$$

The nature of solutions near a singularity may be explored by substituting into (35),

$$T = T_S + u; \quad S = S_S + v, \quad (37)$$

where u and v are small variations. Making the substitution and linearizing, one obtains

$$\frac{dv}{du} \approx \frac{(-M^2 + 3N^2 T_S^2) u}{v} \quad (38)$$

Equation (38) is of the form

$$\frac{dv}{du} = \frac{cu + dv}{au + bv}, \quad (39)$$

from which the roots of the characteristic equation, or eigenvalues, at the singular points can be found⁴ from

$$(\xi_1, \xi_2) = \frac{1}{2} \{ (a+d) \pm [(a+d)^2 + 4(bc - ad)]^{1/2} \}. \quad (40)$$

The type of singularity is then determined from the nature of the characteristic roots.

From (38), (39), and (40), the characteristic roots are found to be

$$(\xi_1, \xi_2) = \pm [-M^2 + 3N^2 T_S^2]^{1/2}. \quad (41)$$

Substituting, in turn, the singularities given by (36), gives

$$\left. \begin{aligned} (\xi_1^1, \xi_2^1) &= \pm (-M^2)^{1/2}; \\ (\xi_1^2, \xi_2^2) &= \pm (2M^2)^{1/2}; \\ (\xi_1^3, \xi_2^3) &= \pm (2M^2)^{1/2}. \end{aligned} \right\} \quad (42)$$

In analyzing the singular points, there are two cases to consider:

I. $M^2 > 0$

$$\left. \begin{aligned} (\xi_1^1, \xi_2^1): & \text{ both pure imaginary} \Rightarrow (T_{S1}, S_{S1}): \text{ vortex (neutrally stable);} \\ (\xi_1^2, \xi_2^2): & \text{ both real, opposite sign} \Rightarrow (T_{S2}, S_{S2}): \text{ saddle (unstable);} \\ (\xi_1^3, \xi_2^3): & \text{ both real, opposite sign} \Rightarrow (T_{S3}, S_{S3}): \text{ saddle (unstable).} \end{aligned} \right\}$$

These results agree with those obtained from a consideration of potential energy ($M^2 > 0$). Refer to the phase-plane diagram of Fig. 2c for a sketch of the solution curves.

II. $M^2 < 0$. For this case the singularity at the origin becomes an unstable saddle since the characteristic roots from (42) are both real and of opposite sign. The other two singular points are physically unrealizable since they are imaginary (assuming, as is the case, $N^2 > 0$).

Therefore, as to be expected, the formal analysis of singular points leads to the same results as the heuristic approach considering potential energy.

IV. CRITICAL VELOCITIES AND DEFLECTIONS

From the preceding stability analyses, a critical velocity has been defined as an upper bound for local stability about the undeflected position of the fuel plate. The inclusion of second-order effects gave rise to a second stability criterion in the form of an upper bound on quasistatic deflections. Defined as a critical deflection, it is a function of the ratio of coolant flow velocity to the critical velocity. The equations defining the critical velocities and deflections, eq. (23), (22), (32), and (31), are reproduced below:

$$v_{CR} = \left[\alpha \frac{Ec^3 h}{(1 - \nu^2) \rho_f l^4} \right]^{1/2}; \quad (23)$$

$$\delta_{CR} = \beta \left[\left(\frac{v_{CR}}{v_0} \right)^2 - 1 \right]^{1/2} h, \quad (32)$$

where

$$\alpha = \left[\frac{(\lambda l)^4 \int_0^1 \psi^2(\bar{\xi}) d\bar{\xi}}{48 \int_0^1 \psi(\bar{\xi}) \left\{ \int_0^1 \psi(\bar{\xi}) d\bar{x} \right\} d\bar{\xi}} \right]; \quad (22)$$

$$\beta = \left\{ \frac{\psi_{\max}^2 \int_0^1 \psi^2(\bar{\xi}) d\bar{\xi}}{8 \int_0^1 \psi(\bar{\xi}) \left[\int_0^1 \psi(\bar{\xi}) d\bar{x} \right]^3 d\bar{\xi}} \right\}^{1/2} \quad (21)$$

and for Case I: ($l=b$, $\xi=y$); Case II: ($l=a$, $\xi=x$).

Table I has been constructed⁵ as an aid in evaluating the mode-dependent constants, α and β , for commonly assumed edge fixities. The

integral $\int_0^1 \psi^4(r) dr$ was evaluated using numerical integration techniques..

By use of the information in Table I, the mode-dependent constants α and β have been evaluated and are tabulated in Table II.

TABLE I. Modal Properties (Fundamental)

Edge Fixities	$(\lambda \ell)$	$(\lambda \ell)^2$	$(\lambda \ell)^4$	$\int_0^1 \psi(r) dr$	$\int_0^1 \psi^2(r) dr$	$\int_0^1 \psi^4(r) dr$	$\psi_{\max}$
Pinned	π	9.870	97.42	$2/\pi = 0.637$	0.5	0.375	1.0
Fixed	4.73	22.37	500.6	0.831	1.0	1.85	1.588
Cantilever	1.875	3.516	12.36	0.783	1.0	2.47	2.0

TABLE II. Mode-dependent Constants

Deflection $\rightarrow$	Case I (Longitudinal)			Case II (Transverse)		
Edge Fixities	α	β	$\gamma(k=0.5)$	α	β	$\gamma(k=0.5)$
Pinned	$\pi^4/48 = 2.03$	0.409	0.633	2.5	0.616	0.777
Fixed	10.42	0.414	0.638	15.1	0.814	0.852
Cantilever	0.258	0.450	0.670	-	-	-

Fig. 3. Critical Deflection as a Function of Coolant to Critical Velocity Ratio

The critical velocities obtained by substituting the values of α from Table II into eq. (23) can be compared and seen to be identical with the corresponding critical velocities derived by Miller,¹ and by Rosenberg and Youngdahl.³

To assess the sensitivity of the critical deflection to the ratio of coolant flow velocity to critical velocity, the curve of Fig. 3 is presented as plotted from eq. (32). An illustration of a use of the curve of Fig. 3 follows. Having selected a coolant flow velocity, say, based on heat transfer requirements, we compute the critical velocity, corresponding to the type of fuel-plate edge fixities, from eq. (23) using Table II. If the selected flow velocity is less than the computed critical velocity, local stability about the unde-

flected plate position is guaranteed. Forming the ratio of coolant to critical velocity, one can then enter the curve of Fig. 3 and, using β from Table II, determine as a percentage of initial channel depth, the maximum allowable quasistatic displacement below which stable oscillations are sustained and above which plate divergence occurs.

If on some rational basis a fraction of initial channel depth is selected as the maximum anticipated deflection of a fuel plate, the stability criteria given by (24) and (33) can be combined to define new critical coolant-flow velocities. The maximum anticipated deflection corresponds to the critical deflection and, hence, the design constant

$$k = \delta_{CR}/h \quad (43)$$

evolves. The new critical velocities can be derived in terms of the original (Miller's) critical velocities by substituting (43) into (32) and solving for the coolant flow velocity:

$$V_{CR} = \gamma v_{CR} \quad (44)$$

where

$$\gamma = \left[1 + (k/\beta)^2 \right]^{-1/2} \quad (45)$$

In general, the new critical coolant-flow velocities can be expressed as

$$V_{CR} = \left[\frac{\alpha}{1 + (k/\beta)^2} \frac{Ec^3h}{(1 - \nu^2) \rho_f \ell^4} \right]^{1/2} \quad (46)$$

where α and β depend on the edge fixities of the plate; and k is a design constant. Note that $V_{CR} = v_{CR}$ if $k = 0$.

The significance of the design constant k is illustrated in Fig. 4, where solution curves for possible fuel-plate motions are sketched on the

Fig. 4. Solution Curves and Their Relationship to the Design Constant k .

phase plane. For a selected value of k , eq. (46) defines critical coolant-flow velocities which represent upper bounds for neutral stability about the undeflected plate position for all quasistatic displacements up to $w = kh$. It can be seen that the solutions are divergent for $w > kh$.

On the basis of assumption (5), that is, that the deflection pattern is a symmetric one in which the deflection of adjacent plates are equal in magnitude but opposite in direction, the maximum allowable deflection is one-half the initial channel depth. Taking this to be the critical deflection gives, according to (43), $k = 0.5$. The mode-dependent constant γ has been computed for $k = 0.5$ and the previously considered edge fixities, and is tabulated in Table II. From the values given in Table II, it can be seen that, for $k = 0.5$, the new critical flow velocities vary from 0.633 to 0.852 times those first arrived at by Miller.

V. DISCUSSION

The theoretical analyses of Miller and of Rosenberg and Youngdahl have previously been discussed. Additional, more detailed, theoretical studies have been performed by Johannson,⁶ Friedrich of Bettis Atomic Power Laboratory, and Scavuzzo.⁷ Miller's analysis, carried out under the most restrictive assumptions, is the simplest and most easily interpreted and applied. Johannson expanded on Miller's approach by including the effects of fluid friction and flow redistribution. However, the pressure-drop term was again linearized. Scavuzzo reports that in an unpublished document, C. M. Friedrich considered large deflections using the nonlinear hydraulic loading term. Scavuzzo in his doctoral dissertation⁸ extended Friedrich's work by including the effects of fluid friction and flow redistribution.

From the above abbreviated literature review, it appears that this author's study to include second-order effects has already been performed and also surpassed with the previous considerations of fluid friction and flow redistribution. Certainly in Scavuzzo's analysis, the second-order terms considered here are carried implicitly in his nonlinear pressure-drop relationship. However, the results of these more detailed studies inherently require presentation in a form sufficiently complicated to preclude their ready use and application by core designers seeking a good estimate and understanding of the phenomenon of a critical coolant-flow velocity. For example, inclusion of fluid friction effects requires a priori knowledge of entrance and exit losses as well as the friction factor, quantities which are generally obtained empirically. As a further example, Scavuzzo's end result is an integral equation, the solution to which is obtained by an iterative procedure on the digital computer. It is because of these complexities and the fact that experimentation and analysis have

shown, as Scavuzzo⁷ points out, that "Even though some of the assumptions made by Miller are significant, equations presented in his paper give a good indication of the coolant velocities at which plate motion can be expected," that Miller's results are still frequently applied and referenced.

This is often the case in design, where a simple approximate analysis is more frequently used and referenced than a more accurate and complicated analysis. Hence, this paper was prepared to give expressions more inclusive than those of Miller's, without compromising the simplicity afforded by his relationships. Also, from a consideration of the solution curves sketched on the phase-plane diagrams, one obtains a better understanding and feel for the critical coolant-flow velocities as related to parallel-plate fuel assemblies and governed by the assumptions of the analysis.

BIBLIOGRAPHY

1. D. R. Miller, "Critical Flow Velocities for Collapse of Reactor Parallel-plate Fuel Assemblies," *ASME Journal of Engineering for Power*, 82, 83-95 (1960).
2. W. K. Stromguist and O. Sisman, *High Flux Reactor Fuel Assemblies--Vibration and Water Flow*, ORNL-50 (June 15, 1948).
3. G. S. Rosenberg and C. K. Youngdahl, "A Simplified Dynamic Model for the Vibration Frequencies and Critical Coolant-flow Velocities for Reactor Parallel-plate Fuel Assemblies," *Nucl. Sci. Eng.* 13(2), 91-102 (1962).
4. W. J. Cunningham, *Introduction to Nonlinear Analysis*, McGraw-Hill Book Co., Inc., New York (1958).
5. R. E. D. Bishop and D. C. Johnson, *Vibration Analysis Tables*, Cambridge University Press, Cambridge (1956).
6. E. B. Johansson, *Hydraulic Instability of Reactor Parallel-fuel-plate Assemblies*, KAPL-M-EJ-9 (July 13, 1959).
7. R. J. Scavuzzo, *Hydraulic Instability of Flat Parallel-plate Assemblies*, WAPD-T-1679 (Feb 1964).
8. R. J. Scavuzzo, *Hydraulic Instability of Flat Parallel-plate Assemblies*, Ph.D. Dissertation, University of Pittsburgh (1962).

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