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New York University
Courant Institute of Mathematical Sciences
Magneto-Fluid Dynamics Division

MF-80   COO-3077-61

STABILITY CRITERIA AND THE MAXIMUM GROWTH RATE

IN MAGNETOHYDRODYNAMICS

G.O. Spies

July 1974

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ABSTRACT

Variational methods, originally developed for deriving
stability criteria from energy principles, are well suited
for obtaining bounds for the maximum growth rate in arbi-
trary equilibria. An upper bound is derived which corre-
sponds to a sufficient stability criterion, and a lower
bound is derived which corresponds to a necessary criterion.
Each of these bounds equals the actual maximum growth rate
in special cases.

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I. INTRODUCTION

There are two traditional approaches to the problem of plasma stability: the energy principle, and the normal mode analysis. The advantage of the energy principle is its flexibility which allows using direct variational methods to derive stability criteria with a wide range of applicability. On the other hand, this flexibility is traded against information: the energy principle tells us whether an equilibrium is exponentially stable or not, without giving any precise information about the nature of instabilities. From the point of view of controlled fusion, this is a serious drawback because exponential stability (which is difficult to achieve in practice) is an unnecessarily restrictive requirement. For instance, unstable modes with small growth rates may be ignored for practical purposes.

A normal mode analysis yields more information than the energy principle, but is so involved that it has been carried out only for special equilibria. As a consequence, no results seem available which are general enough to yield an intuitive picture of the relation between the geometry of the equilibrium and the nature of instabilities.

In the present paper an approach is used which is intermediate between the energy principle and the normal mode analysis regarding both complexity and obtainable information. This approach is based on a modified energy principle.[1] Using techniques known from numerous energy prin-

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ciple analyses, it yields bounds for the maximum growth rate. More precisely, any process leading to a sufficient stability criterion according to the energy principle yields an upper bound, and any process leading to a necessary criterion yields a lower bound. The huge amount of work already invested in evaluating the ordinary energy principle thus gains new practical significance.

Even though this method applies to any systems governed by a self-adjoint set of equations (e.g., the guiding center plasma), we restrict ourselves to ideal magnetohydrodynamics. We neither try to give an account of what one can achieve with this method (which would involve reviewing the existing energy principle calculations), nor do we try to give answers to practical questions arising from present fusion experiments (which would involve restricting consideration to certain modes which have been observed), but rather give two particularly simple (though somewhat academic) illustrations, which, however, include arbitrary geometries.

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II. THE MODIFIED ENERGY PRINCIPLE

The equations of ideal magnetohydrodynamics, when linearized about some static equilibrium, are

$$\rho_o \frac{\partial \chi_1}{\partial t} = -\nabla p_1 + J_1 \times B_o + J_o \times B_1 \quad , \tag{1}$$

$$\frac{\partial B_1}{\partial t} = \text{curl} \, (\chi_1 \times B_o) \quad , \tag{2}$$

$$\frac{\partial p_1}{\partial t} = -\chi_1 \cdot \nabla p_o - \gamma p_o \, \text{div} \, \chi_1 \quad , \tag{3}$$

$$\text{div} \, B_1 = 0 \, , \qquad J_1 = \text{curl} \, B_1 \quad , $$

where the quantities with a subscript one (perturbing quantities) are viewed as unknowns, and those with a subscript zero are known functions in space, subject to the equilibrium equations

$$J_o \times B_o = \nabla p_o \, , \qquad J_o = \text{curl} \, B_o \, , \qquad \text{div} \, B_o = 0 \, . \tag{4}$$

For a normal mode analysis, one Fourier-decomposes the perturbing quantities in time, thus replacing $\partial/\partial t$ by $i\omega$. Eliminating B_1, J_1, and p_1, one then obtains

$$\omega^2 \rho u = F u \quad , \tag{5}$$

$$F u = B \times \text{curl curl} \, (u \times B) - J \times \text{curl} \, (u \times B)$$
$$- \nabla (u \cdot \nabla p + \gamma p \, \text{div} \, u) \, , \tag{6}$$

where u is the Fourier transform of χ_1, and subscripts have been omitted. Since F is a symmetric (i.e., formally self-adjoint) operator[2,3] in the Hilbert space of square integrable vector functions u satisfying appropriate boundary conditions (e.g., $u_n = 0$ at a toroidal boundary), the spectrum of Eq. (5) is real, and its point eigenvalues ω^2 are obtain-

able from the variational principle

$$\omega^2 = W/N , \qquad \delta(W/N) = 0 , \tag{7}$$

where

$$W[\chi] = \frac{1}{2}(\chi, F\chi) , \qquad N[\chi] = \frac{1}{2}(\chi, \rho\chi) .$$

An equilibrium is called "exponentially unstable" if the equations of motion (1)-(3) have a solution whose kinetic energy, defined by

$$K(t) = N[v_1(\chi, t)]$$

satisfies the inequality

$$K(t) \geq K_o \exp(2\sigma t) \tag{8}$$

for some positive numbers σ and K_o and for large enough t. This is the case if, and only if, the spectrum of Eq. (5) extends below the origin. For instance, if $\omega^2 = -\sigma^2 < 0$ is a point eigenvalue, equality holds in (8), and σ is the growth rate of an unstable eigenmode. The "maximum growth rate" is defined as the supremum of all numbers σ for which an inequality of the form (8) holds, or equivalently, as minus the smallest number ω^2 belonging to the spectrum. Since the spectrum always includes the origin, the maximum growth rate is at least zero. In an exponentially stable equilibrium it is zero. It may also be infinite. However, this cannot happen unless the mass density is zero somewhere in the plasma domain, or the pressure gradient is unbounded.[2] It should be pointed out that the maximum growth rate need not correspond to an eigenvalue. In axially symmetric equilibria with $J \cdot B = 0$, for instance, it corresponds to an accumulation point of eigenvalues which is not an eigenvalue itself, thus belonging to the continuous spectrum.[4]

The ordinary energy principle asserts that positivity of the quadratic functional W is necessary and sufficient for exponential stability. Correspondingly, the modified energy principle asserts that positivity of the functional

$$W_\sigma = W + \sigma^2 N \tag{9}$$

is necessary and sufficient for $\sigma \geq \sigma_{max}$. Since this is equivalent to

$$-\sigma_{max}^2 = \inf (W/N) , \tag{10}$$

general theorems (viz., the fact that the operator ξ has a self-adjoint extension whenever this infimum exists,[5] and the spectral theorem for self-adjoint operators) can be used to prove it. However, a more elementary demonstration is more to the point in the present context.

Necessity of the modified energy principle follows from a demonstration of necessity of the ordinary energy principle[6]: One introduces a displacement vector ξ by

$$\frac{\partial}{\partial t} \xi(\chi,t) = \chi_1(\chi,t) ,$$

so that Eqs. (2) and (3) can be integrated to yield

$$B_1 = \text{curl} (\xi \times B_o) + \hat{B}$$

$$p_1 = -\xi \cdot \nabla p_o - \gamma p_o \text{ div } \xi + \hat{p} ,$$

where $\hat{B}(\chi)$ and $\hat{p}(\chi)$ are constants of integration. Restricting consideration to perturbations which are accessible from the equilibrium through an actual motion, one then puts $\hat{B} = 0$ and $\hat{p} = 0$ to obtain

$$\rho \partial^2 \xi / \partial t^2 + F\xi = 0 . \tag{11}$$

From this equation, the energy conservation law

$$N[\partial \xi / \partial t] + W[\xi] = E = \text{const} \tag{12}$$

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and the virial theorem

$$\frac{d^2}{dt^2} N[\xi] - 2N[\partial\xi/\partial t] + 2W[\xi] = 0$$

are derived by taking its scalar product with $\partial\xi/\partial t$ and with ξ, respectively. It follows from these two relations that a solution of Eq. (11) exists whose kinetic energy $K(t) = N[\partial\xi/\partial t]$ satisfies the inequality (8) if a trial vector function η can be found such that $W_\sigma[\eta] < 0$.

Sufficiency of the ordinary energy principle is widely believed to follow from the energy conservation (12) because $W \geq 0$ implies $N[v_1] \leq E$, and hence boundedness of $K(t)$. However, this is incomplete because Eq. (11), and hence also the energy conservation law (12), hold only for the subclass of accessible perturbations. Indeed, it has been shown that the kinetic energy (of inaccessible perturbations) can grow quadratically in time even if $W \geq 0$.[7] Sufficiency of the ordinary energy principle for <u>exponential</u> stability to arbitrary perturbations follows, in the special case $\sigma = 0$, from the following demonstration of sufficiency of the modified energy principle. Differentiating Eqs. (2) and (3) with respect to time, and substituting the resulting expressions into Eq. (1) one obtains

$$\rho\partial^2\chi/\partial t^2 + F\chi = 0 \quad . \tag{13}$$

The velocity vector of arbitrary perturbations thus satisfies the same equation as the displacement vector of accessible perturbations. However, not every solution of Eq. (13) corresponds to a solution of the original system (1)-(3) because of the differentiation. Equation (13) thus could not

be used for proving necessity (which involves exhibiting one solution of the original system with certain properties); but it is suitable for proving sufficiency (which involves showing that all solutions have a certain property). Equation (13) implies the conservation law

$$N[\frac{\partial \chi}{\partial t}] + W[\underset{\sim}{v}] = C = const \quad , \tag{14}$$

which is analogous to energy conservation, but which cannot be identified with it (for inaccessible perturbations, there is no energy conservation law involving only first order quantities). Assuming now that $W_{\bar{\sigma}} \geq 0$ for some $\bar{\sigma} > 0$, we have

$$N[\partial \chi / \partial t] \leq C + \bar{\sigma}^2 N[v] \quad . \tag{15}$$

Making use of the relation between χ and $\underset{\sim}{\xi}$ we have

$$\frac{dK}{dt} = (\chi, \rho \partial \chi / \partial t) \quad , \tag{16}$$

and making use of Schwartz' inequality we have

$$(\chi, \rho \partial \chi / \partial t)^2 \leq 4N[\chi]N[\partial \chi / \partial t] \quad . \tag{17}$$

Combining the three relations we conclude that the kinetic energy satisfies the differential inequality

$$\frac{dK}{dt} \leq 2(\bar{\sigma}^2 K^2 + CK)^{1/2} \quad . \tag{18}$$

Since $\bar{\sigma}^2 K^2 + CK < \sigma^2 K^2$ for large enough K whenever $\sigma > \bar{\sigma}$, an inequality of the form (8) cannot hold if $\sigma > \bar{\sigma}$. Hence $\sigma_{max} \leq \bar{\sigma}$. For $\bar{\sigma} = 0$ the inequality (18) implies that K is bounded by a quadratic function of time, which because of the above mentioned possibility of a quadratic growth of K, shows that the present estimate is optimal.

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Determining σ_{max} essentially amounts to a normal mode
analysis. The difficulties of such an analysis are avoided
if one merely wants to calculate bounds for σ_{max}. A lower
bound can be obtained from a necessary criterion for $W_\sigma \geq 0$
by determining the smallest number σ for which the criterion
is satisfied. Correspondingly, an upper bound can be ob-
tained from a sufficient criterion. Criteria for $W_\sigma \geq 0$ are
obtained in the same manner as those for $W \geq 0$: Narrowing
the admissibility class of displacements ξ or dropping neg-
ative terms in W_σ yields a necessary criterion, while widen-
ing the class or dropping positive terms yields a sufficient
criterion. In particular, one can use various procedures
already existing in the literature, thus obtaining bounds
for σ_{max} which correspond to well known stability criteria.

III. LOWER BOUND FOR THE MAXIMUM GROWTH RATE

We consider arbitrary toroidal equilibria [Eqs. (4)], in which the plasma is either in contact with a perfectly conducting rigid well, or surrounded by a nonconducting vacuum which terminates at a conducting well. The pressure surfaces form sets of nested toroids. Since our analysis will turn out to be local at magnetic field lines, we may ignore the complications arising from disconnected pressure surfaces, thus restricting attention to just one set of nested toroidal surfaces.

These surfaces can be labelled by their volume v. The fluxes ψ and χ the long and the short way of the magnetic field, as well as the fluxes ϕ and η of the current density can then be introduced as functions of v. These fluxes are related to the pressure through

$$\dot{p} = \dot{\eta}\psi - \dot{\phi}\chi \quad , \tag{19}$$

where dots denote derivatives with respect to v. Two angle-like variables θ and ζ increasing by unity once the short and the long way around, respectively, can be constructed[8-10] such that (v,θ,ζ) is a coordinate system with Jacobian unity, $\nabla v \cdot (\nabla\theta \times \nabla\zeta) \equiv 1$, and such that the fields $\underline{J}$ and $\underline{B}$ have the simple representations

$$\underline{J} = \dot{\eta}\underline{e}_\theta + \dot{\phi}\underline{e}_\zeta \; , \qquad \underline{B} = \dot{\chi}\underline{e}_\theta + \dot{\psi}\underline{e}_\zeta \quad . \tag{20}$$

Here $\underline{e}_i = \partial\underline{\chi}/\partial i$ ($i = v,\theta,\zeta$) are the covariant basis vectors. These satisfy the relations

$$\underline{e}_v = \nabla\theta \times \nabla\zeta \; , \qquad \underline{e}_\theta = \nabla\zeta \times \nabla v \; , \qquad \underline{e}_\zeta = \nabla v \times \nabla\theta \; ,$$

$$\nabla v = \underline{e}_\theta \times \underline{e}_\zeta \; , \qquad \nabla\theta = \underline{e}_\zeta \times \underline{e}_v \; , \qquad \nabla\zeta = \underline{e}_v \times \underline{e}_\theta \; .$$

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The metric tensor $g_{ik} = \mathbf{e}_i \cdot \mathbf{e}_k$ is, apart from the fluxes, the
only equilibrium quantity which will enter our results. It
can be determined[10] from the fields $\mathbf{J}$ and $\mathbf{B}$ (and their
fluxes) through the relations (20), and through

$$e_v = \nabla v / |\nabla v|^2 - k_\zeta \mathbf{e}_\theta + k_\theta e_{\sim \zeta} \ ,$$

where the coefficients k_i are determined by integrating the
relations

$$\nabla v \times \nabla k_i = \mathrm{curl} \ (\mathbf{e}_i \times \nabla v / |\nabla v|^2) \ \ .$$

If all magnetic field lines are closed, the rotation
number $\mu = \dot{\chi}/\dot{\psi}$ is a rational constant. Interchanges, that
is displacement vectors which leave the magnetic field
unperturbed,

$$Q = \mathrm{curl} \ (\zeta \times B) = 0 \ \ ,$$

may then be unstable, and the corresponding necessary sta-
bility criterion is $I \geq 0$, where

$$I = - \Omega + \frac{\gamma p}{\beta^2} \Omega^2 \ , \qquad \Omega = - p \dot{q}/q \ , \qquad q = \oint d\ell/B \ . \qquad (21)$$

Considering only interchanges in W_σ, we now derive a lower
bound for σ_{max} (denoted by σ_I) which corresponds to the sta-
bility criterion $I \geq 0$.

Since derivatives along field lines will play an impor-
tant role, it is convenient to replace the coordinate θ by a
linear combination of θ and ζ which is constant on a field
line. Such a combination is

$$w = \theta - \mu\zeta \ \ .$$

In the coordinate system (v,w,ζ), $B = \dot{\psi}\mathbf{e}_\zeta$, and $\mathbf{B}\cdot\nabla = \dot{\psi}\partial/\partial\zeta$.
(Note that in the new system $\mathbf{e}_\zeta$ is not the same vector as in
the old one, whereas $\mathbf{e}_v$ remains unchanged, and $\mathbf{e}_w = \mathbf{e}_\theta$.)

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Introducing contravariant components of ξ by writing

$$\xi = X\xi_v + Y\xi_w + Z\xi_\zeta \quad,$$

one finds that $Q = 0$ is equivalent to

$$\frac{\partial X}{\partial \zeta} = 0 \quad, \qquad \frac{\partial Y}{\partial \zeta} = 0 \quad, \qquad \frac{\partial}{\partial v}(\dot\psi X) + \frac{\partial}{\partial w}(\dot\psi Y) = 0 \quad.$$

For interchanges, the first two components of ξ are thus constrained by

$$X = \frac{1}{\dot\psi}\frac{\partial \Phi}{\partial w} \quad, \qquad Y = -\frac{1}{\dot\psi}\frac{\partial \Phi}{\partial v} \tag{22}$$

where Φ is an arbitrary function of v and w, periodic in w.

Our variational functional can be written as[3]

$$W_\sigma = W_F + W_S + W_V + \sigma^2 N \quad, \tag{23}$$

where

$$W_F = \frac{1}{2}\int d^3\tau [\,|Q|^2 + (J\times\xi)\cdot Q + (\xi\cdot\nabla p)\,\text{div }\xi \\ + \gamma p(\text{div }\xi)^2\,] \quad. \tag{24}$$

The surface term W_S and the vacuum term W_V (which are present only if a vacuum surrounds the plasma) are not needed because they are zero if one considers, as we shall do, displacements which are nonzero only within the plasma. For interchanges, Eq. (23) then becomes

$$W_\sigma = \frac{1}{2}\int dv\,dw\,d\zeta[\,IX^2 + \gamma p\Big(\frac{\partial Z}{\partial \zeta}\Big)^2 + \sigma^2\rho|\xi|^2\,] \quad. \tag{25}$$

Minimizing this with respect to Z leads to the Euler equation

$$\gamma p\,\partial^2 Z/\partial \zeta^2 = \sigma^2\rho(g_{v\zeta}X + g_{w\zeta}Y + g_{\zeta\zeta}Z) \quad. \tag{26}$$

This is an ordinary differential equation for Z with v and w as parameters, subject to the periodicity condition

$$Z(\zeta+n) = Z(\zeta)$$

(singlevaluedness of Z). Here n is the number of circuits the long way of a (closed) field line ($\mu = m/n$, m and n are

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relative primes). Since the homogeneous equation (obtained by putting X = Y = 0) has no nontrivial periodic solution if $\sigma^2 > 0$, the full equation has exactly one such solution for every given X and Y. Since we are not able to write down this solution explicitly, we have to take recourse to esti- mates and limiting cases. However, it is convenient to dis- cuss the procedure leading to the criterion for $W_\sigma \geq 0$ in general terms: Since X and Y are independent of ζ, the so- lution of Eq. (26) is a linear combination of X and Y with coefficients depending only on equilibrium quantities (and σ). Inserting this solution into Eq. (25), and performing the ζ-integration, one obtains an expression of the form

$$W_\sigma = \frac{1}{2} \int dvdw (W_{xx} X^2 + 2W_{xy} XY + W_{yy} Y^2) , \qquad (27)$$

where the components of the matrix $W_{\alpha\beta}$ are functions of v, w, and σ.

The positivity of the matrix $W_{\alpha\beta}$, i.e.

$$W_{xx} \geq 0 , \qquad W_{yy} \geq 0 , \qquad W_{xx} W_{yy} - W_{xy}^2 \geq 0$$

is necessary and sufficient for $W_\sigma \geq 0$. Because of the con- straint (22), only sufficiency is obvious. To demonstrate necessity, we observe the constraint by writing

$$W_\sigma = \frac{1}{2} \int dvdw \frac{1}{\psi^2} [W_{xx} (\frac{\partial\Phi}{\partial w})^2 - 2W_{xy} \frac{\partial\Phi}{\partial w} \frac{\partial\Phi}{\partial v} + W_{yy} (\frac{\partial\Phi}{\partial v})^2] ,$$

and assume that the matrix $W_{\alpha\beta}$ is nonpositive in some sub- region of the (v,w) plane. After performing a change of variables such that W_{xy} is zero at some point (v_0, w_0) of this sub-region, we can conclude that one of the diagonal ele- ments, say W_{yy}, is negative in a neighborhood of this point. A function Φ which is nonzero only in an ϵ-neighborhood of

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this point, and whose v-derivative is large compared to its w-derivative (e.g., which is a function of the single variable $(v-v_o)^2 + \varepsilon^2(w-w_o)^2$), will then render W_σ negative for sufficiently small ε. For the following estimates we thus note that the constraint (22) may be ignored whenever this is convenient.

We now derive both a lower bound and an upper bound for σ_I, and then show that each of these bounds is approached in limiting cases. For the lower bound, we further restrict the class of displacements by $\partial Z/\partial\zeta = 0$. Equation (25) then can be written as

$$W_\sigma = \frac{1}{2} \int dvdw \ W_{ik} X^i X^k$$

where $X^1 = X$, $X^2 = Y$, $X^3 = Z$,

$$W_{ik} = \begin{pmatrix} I & o & o \\ o & o & o \\ o & o & o \end{pmatrix} + \sigma^2 \langle \rho g_{ik} \rangle \quad ,$$

and

$$\langle ... \rangle = \frac{1}{n} \int_o^n d\zeta \ ... = \frac{1}{q} \oint \frac{d\ell}{B} \ ...$$

is the usual field line average. The positivity of the 3×3 matrix W_{ik} is easily seen to be equivalent to the positivity of its determinant, and hence to

$$I \ \det_{w\zeta} \langle \rho g_{ik} \rangle + \sigma^2 \det \langle \rho g_{ik} \rangle \geq 0 \ ,$$

where $\det_{w\zeta}...$ is the indicated 2×2 sub-determinant. The smallest value of σ for which this holds (denoted by $\underline{\sigma}_I$) is a lower bound for σ_I and is given by

$$\underline{\sigma}_I^2 = \max \frac{-I \ \det_{w\zeta} \langle \rho g_{ik} \rangle}{\det \ \langle \rho g_{ik} \rangle} \quad . \tag{28}$$

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For the upper bound (denoted by $\bar{\sigma}_I$) we drop the posi-
tive second term in the integrand of Eq. (25), thus dealing
with the same expression for W_σ as in the derivation of $\underline{\sigma}_I$.
But since there now is no constraint on Z, the condition for
W_σ is less restrictive, and leads to

$$\bar{\sigma}_I^2 = \max \frac{-I \langle \rho g^{vv}/g_{\zeta\zeta} \rangle}{\det_{vw} \langle \rho g^{ik}/g_{\zeta\zeta} \rangle} \tag{29}$$

where $g^{ik} = \nabla i \cdot \nabla k$ is the contravariant metric tensor.

To show that the bounds $\underline{\sigma}_I$ and $\bar{\sigma}_I$ are actually ap-
proached by σ_I, we solve the Euler equation (26) in the two
limiting cases $c^2 \ll 1$ and $c^2 \gg 1$, where c^2 is a character-
istic value of the quantity $n^2 \sigma^2 \rho g_{\zeta\zeta}/\gamma p$. In both cases the
solution can be expanded in powers of the respective small
parameter. Substituting its lowest order into Eq. (25), and
calculating σ_I in the same manner as before, we find $\sigma_I \approx \underline{\sigma}_I$
if $c^2 \ll 1$, and $\sigma_I \approx \bar{\sigma}_I$ if $c^2 \gg 1$.

In terms of the original coordinates (v, θ, ζ), Eqs. (28)
and (29) are

$$\underline{\sigma}_I^2 = \max \frac{-I \det_{\theta\zeta} \langle \rho g_{ik} \rangle}{\det \langle \rho g_{ik} \rangle} \tag{30}$$

$$\bar{\sigma}_I^2 = \max \frac{-I \langle \rho g^{vv}/B^2 \rangle}{D} \quad, \tag{31}$$

where

$$\begin{aligned}
D = \dot{\psi}^2 & \left(\langle \rho g^{vv}/B^2 \rangle \langle \rho g^{\theta\theta}/B^2 \rangle - \langle \rho g^{v\theta}/B^2 \rangle^2 \right) \\
-2\dot{\psi}\dot{\chi} & \left(\langle \rho g^{vv}/B^2 \rangle \langle \rho g^{\theta\zeta}/B^2 \rangle - \langle \rho g^{v\theta}/B^2 \rangle \langle \rho g^{v\zeta}/B^2 \rangle \right) \\
+\dot{\chi}^2 & \left(\langle \rho g^{vv}/B^2 \rangle \langle \rho g^{\zeta\zeta}/B^2 \rangle - \langle \rho g^{v\zeta}/B^2 \rangle^2 \right) \quad .
\end{aligned}$$

Since $\underline{\sigma}_I \leq \sigma_I \leq \sigma_{max}$, $\underline{\sigma}_I$ is a useful estimate for σ_{max} whenever the criterion $I \geq 0$ is violated. On the other hand, $\bar{\sigma}_I$ has no practical significance in general because it can be either smaller or larger than σ_{max}. However, $\sigma_I \gtrsim \sigma_{max}$, and hence $\sigma_{max} \lesssim \bar{\sigma}_I$ if the magnetic field is sufficiently close to the vacuum field of external currents (low β). For if one expands the modified energy principle in powers of β (similar to an expansion of the ordinary energy principle[11]), one finds, in the zeroth order, $W_\sigma^{(o)} \geq 0$, and $W_\sigma^{(o)} = 0$ if, and only if, $\sigma_{max}^{(o)} = 0$ and $\underset{\sim}{Q}^{(o)} = 0$. Using this in the first order one exactly repeats the above calculation, thus obtaining

$$\sigma_{max}^2 = \sigma_I^2 + 0(\beta^2) , \qquad \sigma_I^2 = 0(\beta) .$$

The bound σ_I is still valid if the magnetic field has small shear. For if one puts $\mu = \mu_o + \varepsilon\mu_1(v)$, where μ_o is a rational constant, and then considers displacements with $\underset{\sim}{Q} = 0(\varepsilon)$, one finds $\sigma_I + 0(\varepsilon) \leq \sigma_{max}$, where σ_I is computed for the unperturbed closed line equilibrium ($\varepsilon = 0$). This naive conclusion should not be invalidated by the fact that the low shear limit is singular because results obtained by assuming closed field lines seem to be always more optimistic than the low shear limit of results obtained for sheared equilibria, with regard to both the stability threshold[12,13] and the maximum growth rate.[14]

In a straight circular cylinder, the coordinates (v,θ,ζ) are essentially the familiar cylindrical coordinates, that is $v \sim r^2$, $\theta \sim \phi$, $\zeta \sim z$. In particular, they are or-

thogonal, so that the off-diagonal elements of the metric
tensor are zero. Since equilibrium quantities are constant
on field lines in this case, the averages may be omitted in
Eqs. (30) and (31). Using det g_{ik} = det g^{ik} = 1, we then
find that $\underline{\sigma}_I = \bar{\sigma}_I = \sigma_I$, and

$$\sigma_I \simeq \max \frac{-I|\nabla v|^2}{\rho} . \tag{32}$$

This expression agrees with the maximum growth rate of res-
onant modes[14] [i.e., for modes with k+mµ = 0, where
$\xi \sim \exp i(m\theta + k\zeta)$] in the limit k → ∞, m → ∞, if the latter
is evaluated for p << B^2.

In general, $\underline{\sigma}_I$ and $\bar{\sigma}_I$ do not coincide, but are not
drastically different. In particular, they both have the
property that the expression to be maximized vanishes at the
magnetic axis (where v = 0) because $e_v \sim 1/v$, $e_\theta \sim v$, $e_\zeta \sim 1$.
From this we conclude that violation of the interchange sta-
bility criterion just near the axis does not cause any seri-
ous growth. Since the mass density tends to be small near
the edge of the plasma, violation of the criterion near
the edge tends to cause the largest growth.
These statements will be confirmed by the upper bound for
σ_{max} to be derived in the next Section.

Let us finally discuss the question under what circum-
stances σ_I is close to $\underline{\sigma}_I$ or to $\bar{\sigma}_I$. A rough estimate shows
that

$$c^2 \sim \sigma^2 L^2/v_s^2 = \sigma^2 L^2/\beta v_A^2$$

where L is the length of a field line, v_s is the sound speed,
and v_A is the Alfvén speed. Hence, c^2 is small (and $\sigma_I \gtrsim \underline{\sigma}_I$)

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if the interchange criterion is only slightly violated or if the Alfvén speed is large, and c^2 is large (and $\sigma_I \approx \bar{\sigma}_I$) if β is small or if the field lines are very long (e.g., close upon themselves after very many circuits around the torus). For a family of slightly unstable equilibria with different constant rotation numbers μ, but otherwise similar parameters, σ_I varies between $\underline{\sigma}_I$ and $\bar{\sigma}_I$ as μ is varied, depending on whether μ is a low order rational or not. This is a rather subtle geometrical effect which does not occur in cylindrical symmetry. Whether this effect has to do with the highly complex, and as yet unexplained, behavior of the WIIa stellarator (which is a low β, low shear device in which the rotation number can be varied) remains to be seen.

IV. UPPER BOUND FOR THE MAXIMUM GROWTH RATE

We now consider more general equilibria in that we allow for varying rotation numbers. If a vacuum surrounds the plasma we assume, however, that the plasma current density drops to zero towards the plasma vacuum interface, so that the surface term in Eq. (23) is identically zero. Since the vacuum term is always positive,[3] we then have

$$W_\sigma \geq \frac{1}{2} \int d^3\tau [|\underset{\sim}{Q}|^2 + (\underset{\sim}{J} \times \underset{\sim}{\xi}) \cdot \underset{\sim}{Q} + (\underset{\sim}{\xi} \cdot \nabla p) \, \text{div} \, \underset{\sim}{\xi}$$
$$+ \gamma p (\text{div} \, \underset{\sim}{\xi})^2 + \sigma^2 \rho |\underset{\sim}{\xi}|^2] \tag{33}$$

in all cases.

We introduce, similar to those in Section III, components of $\underset{\sim}{\xi}$ according to

$$\underset{\sim}{\xi} = x \underset{\sim}{e}_v + y \underset{\sim}{J} + z \underset{\sim}{B} \quad .$$

The identity [15]

$$(\underset{\sim}{J} \times \underset{\sim}{\xi}) \cdot \underset{\sim}{Q} + (\underset{\sim}{\xi} \cdot \nabla p) \, \text{div} \, \underset{\sim}{\xi} = -\Omega x^2 + 2(\underset{\sim}{J} \times \underset{\sim}{e}_v) \cdot \underset{\sim}{Q} x + \text{div} (xz p \underset{\sim}{B})$$

is then used to rewrite Eq. (33) as

$$W_\sigma \geq \frac{1}{2} \int d^3\tau [|\underset{\sim}{Q} + (\underset{\sim}{J} \times \underset{\sim}{e}_v) x|^2 + Kx^2$$
$$+ \gamma p (\text{div} \, \underset{\sim}{\xi})^2 + \sigma^2 \rho |\underset{\sim}{\xi}|^2] \quad . \tag{34}$$

Here

$$K = -\Omega - |\underset{\sim}{J} \times \underset{\sim}{e}_v|^2 \, , \qquad \Omega = \dot{n}\ddot{\psi} - \dot{\phi}\ddot{\chi} \quad .$$

The quantity Ω reduces to that in Eq. (21) if all field lines are closed. Dropping the first and the third term in Eq. (34), and using Schwartz' inequality

$$|\underset{\sim}{\xi}|^2 |\nabla v|^2 \geq (\underset{\sim}{\xi} \cdot \nabla v)^2$$

for the last term, we see that

$$K + \sigma^2 \rho / |\nabla v|^2 \geq 0$$

is sufficient for $W_\sigma \geq 0$. The expression

$$\sigma_K^2 = \max \frac{-K|\nabla v|^2}{\rho} \qquad (35)$$

thus is an upper bound for σ_{max}^2 which corresponds to the
sufficient stability condition[16] $K \geq 0$.

The stability condition $K \geq 0$ is violated near a mag-
netic axis ($\nabla v = 0$) whenever $\underset{\sim}{J} \neq 0$ there, because then
$|\underset{\sim}{J} \times \underset{\sim}{e}_v|^2 = 0(1/v)$. The bound (35) shows that the growth rate
still remains finite in this case because the term $|\nabla v|^2$
annihilates the singularity in the stability criterion.
Moreover, the expression to be maximized in Eq. (35) van-
ishes at the magnetic axis whenever $\underset{\sim}{J} = 0$ there, so that vi-
olation of the criterion $K \geq 0$ just near the axis leads only
to small growth rates. This is, for instance, the case in
stellarator equilibria (which are usually defined as to have
no toroidal net current, that is, by $\phi \equiv 0$). This indicates
that estimates of a critical β in stellarators which were
obtained by an evaluation of the criterion $K \geq 0$ at the
magnetic axis, can be significantly improved if one allows
for instabilities with a small growth rate, that is, if one
replaces this criterion by $\sigma_K \leq \sigma_0$, where σ_0 is some given
number corresponding to practical requirements. More gener-
ally, results arising from evaluating any stability criteri-
on at the magnetic axis are subject to some doubt as to
their relevance because one has to consider the whole plasma
domain in order to obtain an upper bound for σ_{max}.

To give a specific (though trivial) example, we consider the straight Z-pinch. In cylindrical coordinates (r, ϕ, z), the magnetic field has only a ϕ component $B(r)$, and the equilibrium equation is

$$\frac{d}{dr} \frac{1}{2} r^2 B^2 + r^2 \frac{dp}{dr} = 0 \quad .$$

Our bounds for σ_{max} turn out to be given by

$$\sigma_I^2 = \max \frac{-B^2(f+2)(f+2g)}{\rho r^2(1-g)} \tag{36}$$

$$\sigma_K^2 = \max \frac{-2B^2 f}{\rho r^2} \quad , \tag{37}$$

where

$$f = \frac{1}{B^2} r \frac{dp}{dr} \quad , \qquad g = \frac{\gamma p}{B^2 + \gamma p} \quad .$$

It has been shown[4] that maximum growth appears in the limit $k \to \infty$, where $\xi \sim \exp i(m\phi + kz)$, and that σ_{max} can be computed from

$$(m^2 + \lambda^2 + 2f)(m^2 g + \lambda^2) + 4g\lambda^2 = 0 \quad , \qquad \lambda^2 = \sigma^2 \rho r^2 / B^2$$

by solving for σ^2, and then maximizing with respect to both m^2 and r. Maximum growth can be either resonant ($m=0$) or off-resonant ($m \neq 0$), depending (in an extremely complex manner) on the functions $f(r)$ and $g(r)$. A sufficient condition for resonant maximum growth is[14]

$$f + 2g(1+g) \leq 0 \quad , \tag{38}$$

and σ_{max} is then given by

$$\sigma_{max}^2 = \max \frac{-2B^2(f+2g)}{\rho r^2} \quad .$$

Obviously, $\sigma_{max} \approx \sigma_I$ if $|f| \ll 1$ and $g \ll 1$, while $\sigma_{max} \approx \sigma_K$ if $g \ll |f|$ (at the maximizing value of r). The former is the case at low β (as we have already seen in more general-

ity), and the latter is the case if the pressure gradient is large at the plasma edge.

For instance, the pressure profile

$$p = p_o\left(1 - \frac{r^n}{a^n}\right) , \qquad 0 \leq r \leq a$$

leads to

$$B^2 = \frac{2np_o r^n}{(n+2)a^n} ,$$

and $f = -(n+2)/2 = $ const, whereas $g(r)$ decreases from $g = 1$ at the axis ($r=0$) to $g = 0$ at the wall ($r=a$). For $n \geq 6$ the inequality (38) holds so that σ_{max} is given by Eq. (39). Since B^2/r^2 is increasing, while f is a negative constant and g is decreasing, maximum growth appears at the wall (at least as long as ρ is not increasing), and $\sigma_{max} = \sigma_K$ in this case.

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