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SOLIDS BY THE FINITE ELEMENT METHOD

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NONLINEAR DYNAMIC ANALYSIS OF AXISYMMETRIC SOLIDS BY THE FINITE ELEMENT METHOD^{*}

Mark Hartzman[†]

ABSTRACT

A method for calculating the dynamic response of deformable axisymmetric solids, subjected to time-dependent axisymmetric loads is described. The nonlinearities considered in this analysis include material nonlinearity (elastic-plastic behavior) and geometric nonlinearity, which includes finite deformation. The finite-element method is applied to approximate the continuum by a lumped-mass system connected by axisymmetric elements. The equations of motion are solved by applying a step-by-step numerical technique. The analysis is illustrated by application to the collapse of a built-in spherical dome with varying thickness and to the impact of a cylinder against a rigid wall. Close agreement is obtained between the results from the present technique and results obtained from the literature.

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INTRODUCTION

In this investigation, the finite-element technique was applied to obtain the approximate nonlinear dynamic response of axisymmetric solid bodies subjected to time-dependent thermal and mechanical axisymmetric fields. The type of nonlinearities examined include both material nonlinearity, i.e., elasto-plastic response, and geometric nonlinearity, which includes finite strains and the effect of the finite rigid-body rotation on the stress and strain states. The objective was to develop a numerical procedure for obtaining these solutions by using existing descriptions of the material properties.

The application of finite-element methods to nonlinear dynamic problems of solids is very recent. In 1968, Constantino [1]¹ examined wave propagation in axisymmetric and plane solids where the nonlinear material behavior (elastic-plastic response) was taken into account. More recently, Oden [2] presented a general theory of finite elements in which he outlined procedures for solutions of problems of nonlinear solid and fluid mechanics. Hartzman and Hutchinson [3] applied the method to the solution of plain strain problems involving both geometric and material nonlinearities (elastic-plastic constitutive relations). Farnoomand [4] also applied this method to axisymmetric granular solids (soils) where geometric nonlinearity was also taken into account.

The method described here can be classified as Eulerian-Lagrangian. It is incremental and is based on the application of the principle of virtual

¹Numbers in brackets indicate References listed at the end of the report.

work, in conjunction with certain assumptions concerning 1) the displacement distribution within the body and 2) the replacement of the mass of the body by a system of lumped masses.

The continuum is thus approximated by a lumped-mass system, where the masses are interconnected by massless elements. The partial differential equations describing the motion of the body are replaced by a set of ordinary differential equations in time. Solutions of the equations are then obtained by applying a numerical algorithm.

NOMENCLATURE

- σ_{ij} = components of the Cauchy stress tensor.
- ϵ_{ij} = components of the Almansi strain tensor.
- n_i = components of the displacement vector.
- a^i = components of the acceleration vector.
- T^i = components of the surface traction vector.
- F^i = components of the body force vector.
- ρ = instantaneous material density.
- V = instantaneous volume.
- S = instantaneous surface.
- U_{in} = components of the n th nodal displacement vector.
- $\phi_n(X)$ = n th space-dependent interpolation function.
- X^i = instantaneous (Eulerian) coordinate system.
- v^i = components of the velocity vector.
- $(R^f)_n^m$ = defined by Eq. (5).
- $(p^f)_n^m$ = defined by Eq. (6).

M_n = lumped mass at point n .

$\ddot{U}_n^t$ = acceleration components of point n .

c_{ij} = linear components of the Almansi strain tensor.

λ, G = Lamé constants.

f = yield function.

S_{ij} = deviatoric components of the stress tensor.

E = modulus of elasticity.

E^a = tangential modulus.

ΔT = temperature increment.

ω = natural frequency.

g_{ij} = components of the metric tensor.

Other symbols are defined where they first appear.

THEORETICAL ANALYSIS

Introduction

The motion of a solid body that experiences large linear and angular displacements can be specified at a given instant in terms of the configuration of time $t = 0$ or at the present time. Figure 1 shows a deformable body in the process of deformation and motion from the undeformed state at time $t = 0$ to the instantaneous deformed state at time t^a .² In this paper, the deformation and motion will be referred to the current configuration, in what is commonly referred to as an Eulerian formulation [5]. The approach in describing problems involving geometric and material nonlinearity is

²Superscripts a refer to particular instants in time.

incremental; thus, two other configurations are also shown in Fig. 1: the configuration at $t = t^{s-1}$ prior to the instantaneous configuration, and the next future configuration at $t = t^{s+1}$ after the instantaneous configuration. The objective of the analysis is to predict the motion of the body from t^s to t^{s+1} , based on knowledge of the state of the body at t^s and t^{s-1} .

Formulation by the Principle of Virtual Work

The volume, the boundary surface, and the coordinates of the material points of the body are described at every instant by coincident cylindrical coordinate systems: $X(X^1, X^2, X^3)$, where $X^1 = R$, $X^2 = Z$, $X^3 = \theta$ (see Fig. 1 for the plane representation). These quantities are denoted as follows:

undeformed configuration	$(t = 0): \bar{v}, \bar{a}, \bar{x}^i$
previous configuration	$(t = t^{s-1}): v, a, x^i \quad (i = 1, 3)$
instantaneous configuration	$(t = t^s): V, A, X^i$
future configuration	$(t = t^{s+1}): \bar{v}, \bar{a}, \bar{x}^i$

According to an Eulerian formulation, the geometry in the instantaneous configuration is assumed known and therefore becomes the reference configuration. For axisymmetric deformation, the instantaneous position of the particles in the body is specified by a displacement field $u^i(x, t)$, $(i = 1, 2)$, which describes the displacement experienced by a particle located at (X^1, X^2) from its position at $(\bar{X}^1, \bar{X}^2)$. Figure 1 also shows the total and incremental displacements experienced by a generic point p during the deformation process.

The principle of virtual work referred to the instantaneous configuration is given by the following expression³:

$$\int_V \rho a^i \delta u_i dv = \int_{S_A} T^i \delta u_i dS + \int_V F^i \delta u_i dv - \int_V \sigma^{ij} \delta e_{ij} dv, \quad (1)$$

where:

u_i = components of the displacement vector

a^i = components of the acceleration vector

T^i = components of the surface traction vector

F^i = components of the body force vector

σ^{ij} = components of the Cauchy stress tensor

e_{ij} = linear components of the Almansi strain tensor ϵ_{ij}

ρ = instantaneous material density.

The tensor ϵ_{ij} is defined as (see Ref. 5)⁴

$$\epsilon_{ij} = \frac{1}{2} (u_i{}^j + u_j{}^i - u^k{}_i u_k{}^j),$$

whence:

$$e_{ij} = \frac{1}{2} (u_i{}^j + u_j{}^i).$$

³Indicial notation is used. A repeated diagonal superscript-subscript combination indicates summation. A superscript indicates a contravariant component; covariant components are indicated by subscripts.

⁴A vertical slash indicates covariant differentiation with respect to the instantaneous geometry.

For application of the displacement finite element method, the body is divided by a mesh into M regions or elements. The intersections of the mesh lines are called nodes. The geometry of each region can be defined by suitable interpolation formulas in terms of the nodal coordinates.

Assuming that the body has been so divided, Eq. (1) can then be rewritten in the following form:

$$\sum_V^M \rho a^i \delta u_i dV = \sum_{S_a}^M T^i \delta u_i dS + \sum_V^M F^i \delta u_i dV - \sum_V^M \sigma^{ij} \delta \epsilon_{ij} dV, \quad (2)$$

where M = number of regions. The surface integrals are evaluated only for those regions containing loaded external surfaces.

Let the displacement field within the $\underline{n}$ th region be expressed in terms of the nodal displacements as follows:

$$u_l(X, t) = \sum_n^N \phi_n(X) U_{ln}(t), \quad (3)$$

where:

U_{ln} = components of the $\underline{n}$ th nodal displacement vector

$l = 1, 2$

n = number of nodes

$\phi_n(X)$ = $\underline{n}$ th space-dependent interpolation function.

Taking the variation of Eq. (3), substituting into Eq. (2), and considering that the varied nodal displacements δU_{ln} are independent, we get the following set of equations of motion:

$$\sum_m \int_V \rho a^i \frac{\partial u_i}{\partial U_{ln}} dV = \sum_m \int_{S_a} T^i \frac{\partial u_i}{\partial U_{ln}} dS + \sum_m \left(\int_V F^i \frac{\partial u_i}{\partial U_{ln}} dV - \int_V \sigma^{ij} \frac{\partial e_{ij}}{\partial U_{ln}} dV \right). \quad (4)$$

For axisymmetric motion there are $2 \times N$ equations. The acceleration in the instantaneous configuration is given as (see Ref. 6)

$$a^i(X, t) = \frac{\partial v^i}{\partial t} + v^i |_k v^k,$$

where v^i = components of the velocity vector.

Also, define quantities $(R^\ell)_n^m$ and $(P^\ell)_n^m$ as follows:

$$(R^\ell)_n^m = \left(\int_V F^i \frac{\partial u_i}{\partial U_{ln}} dV - \int_V \sigma^{ij} \frac{\partial e_{ij}}{\partial U_{ln}} dV \right)^m \quad (5)$$

$$(P^\ell)_n^m = \left[\int_{S_a} T^i \frac{\partial u_i}{\partial U_{ln}} dS \right]^m. \quad (6)$$

Substituting into Eq. (4), we get

$$\sum_m \int_V \rho \left[\frac{\partial v^i}{\partial t} + v^i |_k v^k \right] \frac{\partial u_i}{\partial U_{ln}} dV = \sum_m (P_n^\ell + R_n^\ell)^m, \quad \begin{matrix} \ell = 1, 2 \\ n = 1, N \end{matrix} \quad (7)$$

The evaluation of the integrals on the left side based on Eq. (3) generates some very complicated, highly nonlinear coupled equations involving the displacements, velocities, and acceleration. The solution of such equations becomes highly uneconomical for practical purposes. A considerable reduction in computation time and effort is therefore achieved when the left side of Eq. (7) is replaced by the change in momentum associated with a mass which is concentrated at the point n .⁵ Thus, Eq. (7) is rewritten as

$$M_n \ddot{U}_n^\ell = \sum_m (p_n^\ell + R_n^\ell)^m, \quad \begin{matrix} \ell = 1, 2 \\ n = 1, N \text{ (no sum on } n), \end{matrix} \quad (8a)$$

where

M_n = mass at point n

$\ddot{U}_n^\ell$ = acceleration components of point n .

(The summation is now performed only over the elements adjacent to the n th mass.) The lumped mass is defined (for four-node quadrilaterals; see Fig. 2) as follows:

$$M_n = \frac{1}{4} \sum_{M^i} \int \rho dV,$$

where M^i refers to the particular elements associated with the node n , i.e., we sum one quarter of each of the masses of the elements surrounding

⁵As an intermediate step, approximate consistent mass expressions can also be obtained, from which the lumped-mass approach would follow.

a given node.⁶ The total mass of the body is conserved, since, for each region m , the mass is conserved:

$$\int_V \rho dV = \int_{\bar{V}} \bar{\rho} d\bar{V}.$$

Hence, each mass M_n is evaluated once and for all at the beginning of a computation.

Physically, Eq. (8a) may be interpreted as describing the approximation of the continuous body by a lumped-mass system. The masses are connected by three-dimensional massless elements, with the internal and external distributed force system being replaced by equivalent forces concentrated at the corners of the elements.

If an explicit numerical solution scheme for solving the equations of motion is adopted, Eq. (8a) forms the basis of a direct method for obtaining the dynamic response of a body. Thus, if the instantaneous right-hand side of Eq. (8a) is known, i.e., the equivalent external and internal forces have been calculated, the accelerations and the incremental displacements can also be calculated. The internal forces depend on the incremental displacements through the stress and strain states within the elements. Therefore, if suitable stress-strain relations are adopted, the internal forces for the next instant are also completely determined and available for calculating the next acceleration. These forces can be calculated element-by-element, contrary to the direct stiffness-method procedures, where stiffness and mass matrices are first assembled and the incremental

⁶If a quadrilateral reduces to a triangle, a different procedure is used.

displacements obtained by solution of the resulting set of simultaneous equations. Thus, considerations of bandwidth, etc., need not be taken into account, a convenience from a practical point of view.

Numerical Integration of the Equation of Motion

The set of equations in Eq. (8a) represents a set of second-order ordinary nonlinear differential equation in which the mass matrix is diagonal. An explicit algorithm for the numerical integration of such equations is described in Ref. 3.

We assume that the acceleration components $\ddot{U}_n^1, \ddot{U}_n^2, \dots, \ddot{U}_n^{s-1}$ have already been determined. If the external and internal forces are also assumed now, we can calculate the acceleration $\ddot{U}_n^s$ from Eq. (8a). The velocity and incremental displacement that this mass experiences are obtained from the following relations:

$$\dot{U}_n^s = \dot{U}_n^{s-1} + \frac{\Delta t}{12} [\ddot{U}_n^s + 8\ddot{U}_n^{s-1} - \ddot{U}_n^{s-2}] \quad (8b)$$

$$\Delta U_n^{s+1} = \dot{U}_n^s \Delta t + \frac{(\Delta t)^2}{6} [4\ddot{U}_n^s - \ddot{U}_n^{s-1}], \quad (8c)$$

where Δt is the discrete time increment, such that $t^s = s\Delta t$ and s is the number of time increments. The total displacement at the $(s+1)$ th instant are then obtained from

$$U_i^{s+1} = U_i^s + \Delta U_i^{s+1}.$$

Assuming the initial velocity and acceleration are given, the first displacement increment is given as

$$\Delta U_n^1 = U_n^1 - U_n^0 = \Delta t \dot{U}_n^0 + \frac{(\Delta t)^2}{2} \ddot{U}_n^0.$$

The velocity at t^1 is then calculated as

$$\dot{U}_n^1 = \dot{U}_n^0 + \Delta t \ddot{U}_n^0.$$

The necessary time step Δt for stability of this algorithm is governed by the dilatational speed of sound C_L of the material, given as

$$C_L = \sqrt{\frac{E(1-\nu)}{\rho(1+\nu)(1-2\nu)}},$$

the space mesh, and the rate of application of the loads. It is calculated from

$$\Delta t = \frac{\Delta L}{C_L},$$

where ΔL is a characteristic length usually taken as the smallest side of the smallest element. The actual value used is taken to be

$$0.1\Delta t \leq \Delta t_{\text{act}} \leq 0.5\Delta t,$$

depending on the rise time of the load functions. The largest value of Δt that permits stable solution is used and is determined by trial and error.

Calculation of Equivalent Forces

The forces in Eqs. (8) can be interpreted as forces exerted by the elements associated with the n th mass. Conversely, each element can be shown to be associated with a number of masses. It is therefore convenient to consider a typical axisymmetric quadrilateral element, as shown in Fig. 3, and to denote the displacements and equivalent forces by the same

symbols as used above, except that the subscript n now refers to the number of quadrilateral nodes. These may be referred to as local quantities, whereas the displacements and forces in Eq. (5) are denoted as system quantities. For any given element, the sets of forces and displacements are related by simple transformation matrices, whose elements consist of zeroes and ones.

The surface integral is evaluated only if one of the sides of the element experiences surface tractions. In practice, it is usually replaced by statically equivalent concentrated loads, so that the discussion can be restricted to the volume integrals.

The quantities shown in Eq. (5) are tensor components. In cylindrical coordinates, we can express Eq. (5) in physical components as follows, per unit radian⁷:

$$R_{kn} = \int \left[F_i \frac{\partial u_i}{\partial U_{kn}} - \sigma_{ij} \frac{\partial e_i}{\partial U_{kn}} \right] X_1 dX_1 dX_2 \quad \begin{matrix} k = 1, 2 \\ n = 1, 4 \end{matrix}$$

In matrix notation, we can express the element forces as follows⁸:

$$R_{kn} = \int \left[(u')^T (F) - (e')^T (\sigma) \right] X_1 dX_1 dX_2. \quad (8)$$

⁷ Matrix notation will be adopted where convenient.

⁸ Superscript T indicates transpose.

where:

$$\{u'\} = \left\{ \frac{\partial u_1}{\partial U_{kn}} \quad \frac{\partial u_2}{\partial U_{kn}} \right\}$$

$$\{F\} = \{F_1 F_2\}$$

$$\{\sigma\} = \{\sigma_{11} \sigma_{22} \sigma_{33} \sigma_{12}\}$$

$$\{e'\} = \{e'_{11} e'_{22} e'_{33} e'_{12}\}$$

where:

$$e'_{11} = \frac{\partial u_1}{\partial X_1}$$

$$e'_{22} = \frac{\partial u_2}{\partial X_2}$$

$$e'_{33} = u_1/X_1$$

$$e'_{12} = \frac{1}{2} \left(\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1} \right).$$

The equivalent forces required for substitution into Eqs. (8) are evaluated element-by-element directly from Eq. (9) by considering the material properties, the strain state, and the assumed displacement field.

The total Almansi strains existing in the instantaneous configuration can be calculated from the relations given below, if the strain state in the previous configuration and the incremental strains are known. These relations are derived in Appendix A. Thus, in matrix form we have

$$\begin{bmatrix} \epsilon_{11}^s \\ \epsilon_{22}^s \\ \epsilon_{33}^s \\ \epsilon_{12}^s \end{bmatrix} = \begin{bmatrix} \left(\frac{\partial x_1}{\partial X_1}\right)^2 & \left(\frac{\partial x_2}{\partial X_1}\right)^2 & 0 & 2\left(\frac{\partial x_1}{\partial X_1} \frac{\partial x_2}{\partial X_1}\right) \\ \left(\frac{\partial x_1}{\partial X_2}\right)^2 & \left(\frac{\partial x_2}{\partial X_2}\right)^2 & 0 & 2\left(\frac{\partial x_1}{\partial X_2} \frac{\partial x_2}{\partial X_2}\right) \\ 0 & 0 & \left(\frac{x_1}{X_1}\right)^2 & 0 \\ \frac{\partial x_1}{\partial X_1} \frac{\partial x_1}{\partial X_2} + \frac{\partial x_2}{\partial X_1} \frac{\partial x_2}{\partial X_2} & 0 & \left(\frac{\partial x_1}{\partial X_1} \frac{\partial x_2}{\partial X_2} + \frac{\partial x_2}{\partial X_2} \frac{\partial x_1}{\partial X_1}\right) \end{bmatrix} \begin{bmatrix} \epsilon_{11}^{s-1} \\ \epsilon_{22}^{s-1} \\ \epsilon_{33}^{s-1} \\ \epsilon_{12}^{s-1} \end{bmatrix} + \begin{bmatrix} \Delta \epsilon_{11} \\ \Delta \epsilon_{22} \\ \Delta \epsilon_{33} \\ \Delta \epsilon_{12} \end{bmatrix}$$

or

$$\{\epsilon\}^s = [D] \{\epsilon\}^{s-1} + \{\Delta \epsilon\}, \quad (10)$$

where:

$$\epsilon_{11}^s = \frac{\partial u_1}{\partial X_1} - \frac{1}{2} \left[\left(\frac{\partial u_1}{\partial X_1} \right)^2 + \left(\frac{\partial u_1}{\partial X_2} \right)^2 \right]$$

$$\epsilon_{22}^s = \frac{\partial u_2}{\partial X_2} - \frac{1}{2} \left[\left(\frac{\partial u_2}{\partial X_1} \right)^2 + \left(\frac{\partial u_2}{\partial X_2} \right)^2 \right]$$

$$\epsilon_{33}^s = u_1/X_1 - \frac{1}{2} (u_1/X_1)^2$$

$$\epsilon_{12}^s = \frac{1}{2} \left[\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1} - \left(\frac{\partial u_1}{\partial X_1} \frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1} \frac{\partial u_2}{\partial X_2} \right) \right].$$

The two coordinate systems (x) and (X) are related through the displacement field, i.e.,

$$X_i = x_i + \Delta u_i(X_i, t), \quad (11)$$

where u_i are the physical components of the incremental displacement vector. Thus, the matrix $[D]$ and the incremental strains may be expressed in terms of the incremental displacement components as follows:

$$\Delta \epsilon_{11} = \frac{\partial \Delta u_1}{\partial X_1} - \frac{1}{2} \left[\left(\frac{\partial \Delta u_1}{\partial X_1} \right)^2 + \left(\frac{\partial \Delta u_2}{\partial X_1} \right)^2 \right]$$

$$\Delta \epsilon_{22} = \frac{\partial \Delta u_2}{\partial X_2} - \frac{1}{2} \left[\left(\frac{\partial \Delta u_1}{\partial X_2} \right)^2 + \left(\frac{\partial \Delta u_2}{\partial X_2} \right)^2 \right]$$

$$\Delta \epsilon_{33} = \frac{\Delta u_1}{X_1} - \frac{1}{2} \left(\frac{\Delta u_1}{X_1} \right)^2$$

$$\Delta \epsilon_{12} = \frac{1}{2} \left[\frac{\partial \Delta u_1}{\partial X_2} + \frac{\partial \Delta u_2}{\partial X_1} - \left(\frac{\partial \Delta u_1}{\partial X_1} \frac{\partial \Delta u_1}{\partial X_2} + \frac{\partial \Delta u_2}{\partial X_2} \frac{\partial \Delta u_2}{\partial X_1} \right) \right] \quad (12)$$

$$[D] = \begin{bmatrix} \left(1 - \frac{\partial \Delta u_1}{\partial X_1}\right)^2 & -\left(\frac{\partial \Delta u_2}{\partial X_1}\right)^2 & 0 & -2\left(1 - \frac{\partial \Delta u_1}{\partial X_1}\right) \frac{\partial \Delta u_2}{\partial X_1} \\ -\left(\frac{\partial \Delta u_1}{\partial X_2}\right)^2 & \left(1 - \frac{\partial \Delta u_2}{\partial X_2}\right)^2 & 0 & -2\left(1 - \frac{\partial \Delta u_2}{\partial X_2}\right) \frac{\partial \Delta u_1}{\partial X_2} \\ 0 & 0 & \left(1 - \frac{\Delta u_1}{X_1}\right)^2 & 0 \\ \left(1 - \frac{\partial \Delta u_1}{\partial X_1}\right) \frac{\partial \Delta u_1}{\partial X_2} & \left(1 - \frac{\partial \Delta u_2}{\partial X_2}\right) \frac{\partial \Delta u_2}{\partial X_1} & 0 & \left(1 - \frac{\partial \Delta u_1}{\partial X_1}\right) \left(1 - \frac{\partial \Delta u_2}{\partial X_2}\right) + \frac{\partial \Delta u_1}{\partial X_2} \frac{\partial \Delta u_2}{\partial X_1} \end{bmatrix} \quad (13)$$

In this paper we apply the concept of isoparametric elements [7,8]. Thus, with reference to Fig. 4, we assume the following coordinate transformations for each element:

$$\begin{aligned} X_1(s, t) &= h_i X_{1i} & i &= 1 \rightarrow 4 \\ X_2(s, t) &= h_i X_{2i} & i &= 1 \rightarrow 4 \end{aligned} \quad (14)$$

where the h_i are interpolation polynomials and functions of the natural coordinate (s, t) , as follows:

$$\begin{aligned} h_1 &= \frac{1}{4} (1 + s) (1 + t) \\ h_2 &= \frac{1}{4} (1 - s) (1 + t) \\ h_3 &= \frac{1}{4} (1 + s) (1 - t) \\ h_4 &= \frac{1}{4} (1 - s) (1 - t). \end{aligned} \quad (15)$$

The incremental and total displacement patterns may be expressed in the same form:

$$\begin{aligned} \Delta u_1(s, t) &= h_i \Delta U_{1i} \\ \Delta u_2(s, t) &= h_i \Delta U_{2i} \\ u_1(s, t) &= h_i U_{1i} \\ u_2(s, t) &= h_i U_{2i}. \end{aligned}$$

To express the partial derivatives in Eqs. (12) and (13), we use the chain rule. Thus,

$$\frac{\partial \Delta u_1}{\partial X_1} = \frac{1}{J} \left[\frac{\partial h_1}{\partial s} \frac{\partial h_1}{\partial t} - \frac{\partial h_1}{\partial t} \frac{\partial h_1}{\partial s} \right] \Delta U_{11} X_{2j}$$

$$\frac{\partial \Delta u_1}{\partial X_2} = \frac{1}{J} \left[\frac{\partial h_1}{\partial s} \frac{\partial h_1}{\partial t} + \frac{\partial h_1}{\partial t} \frac{\partial h_1}{\partial s} \right] \Delta U_{11} X_{1j}$$

$$\frac{\partial \Delta u_2}{\partial X_1} = \frac{1}{J} \left[\frac{\partial h_1}{\partial s} \frac{\partial h_1}{\partial t} - \frac{\partial h_1}{\partial t} \frac{\partial h_1}{\partial s} \right] \Delta U_{21} X_{2j}$$

$$\frac{\partial \Delta u_2}{\partial X_2} = \frac{1}{J} \left[\frac{\partial h_1}{\partial s} \frac{\partial h_1}{\partial t} + \frac{\partial h_1}{\partial t} \frac{\partial h_1}{\partial s} \right] \Delta U_{21} X_{1j}$$

where:

$$J = \left[\frac{\partial h_1}{\partial s} \frac{\partial h_1}{\partial t} - \frac{\partial h_1}{\partial t} \frac{\partial h_1}{\partial s} \right] X_{11} X_{2j}$$

Let

$$Y_i = \frac{1}{J} \left[\frac{\partial h_1}{\partial s} \left(\frac{\partial h_1}{\partial t} X_{2j} \right) - \frac{\partial h_1}{\partial t} \left(\frac{\partial h_1}{\partial s} X_{2j} \right) \right]$$

$$Z_i = \frac{1}{J} \left[\frac{\partial h_1}{\partial s} \left(\frac{\partial h_1}{\partial t} X_{1j} \right) + \frac{\partial h_1}{\partial t} \left(\frac{\partial h_1}{\partial s} X_{1j} \right) \right]$$

Then,

$$\frac{\partial \Delta u_1}{\partial X_1} = Y_1 \Delta U_{11}$$

$$\frac{\partial \Delta u_1}{\partial X_2} = Z_1 \Delta U_{11}$$

$$\frac{\partial \Delta u_2}{\partial X_1} = Y_1 \Delta U_{21}$$

$$\frac{\partial \Delta u_2}{\partial X_2} = Z_1 \Delta U_{21}.$$

Likewise, the linear components of the Almansi strains can be expressed in terms of the total displacements, as follows:

$$e_{11} = \frac{\partial u_1}{\partial X_1} = Y_1 U_{11}$$

$$e_{22} = \frac{\partial u_2}{\partial X_2} = Z_1 U_{21}$$

$$e_{33} = \frac{\partial u_1}{\partial X_1} = \frac{h_1 U_{11}}{X_1^s} = G_1 U_{11}$$

$$e_{12} = \frac{1}{2} \left(\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1} \right) = \frac{1}{2} (Z_1 U_{11} + Y_1 U_{21}),$$

whence,

$$\frac{\partial e_{11}}{\partial U_{11}} = Y_1 \quad \frac{\partial e_{11}}{\partial U_{21}} = 0$$

$$\frac{\partial e_{22}}{\partial U_{1i}} = 0$$

$$\frac{\partial e_{22}}{\partial U_{2i}} = Z_i$$

$$\frac{\partial e_{33}}{\partial U_{1i}} = G_i$$

$$\frac{\partial e_{33}}{\partial U_{2i}} = 0$$

$$\frac{\partial e_{12}}{\partial U_{1i}} = \frac{1}{2} Z_i$$

$$\frac{\partial e_{12}}{\partial U_{2i}} = \frac{1}{2} Y_i$$

We can now express the internal forces in the following matrix form by using Eq. (9):

$$\begin{Bmatrix} R_{11} \\ R_{21} \\ R_{12} \\ R_{22} \\ R_{13} \\ R_{23} \\ R_{14} \\ R_{24} \end{Bmatrix} = \int_V \begin{Bmatrix} Y_1 & 0 \\ 0 & Z_1 \\ Y_2 & 0 \\ 0 & Z_2 \\ Y_3 & 0 \\ 0 & Z_3 \\ Y_4 & 0 \\ 0 & Z_4 \end{Bmatrix} \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} dV - \int_V \begin{Bmatrix} Y_1 & 0 & G_1 & Z_1 \\ 0 & Z_1 & 0 & Y_1 \\ Y_2 & 0 & G_2 & Z_2 \\ 0 & Z_2 & 0 & Y_2 \\ Y_3 & 0 & G_3 & Z_3 \\ 0 & Z_3 & 0 & Y_3 \\ Y_4 & 0 & G_4 & Z_4 \\ 0 & Z_4 & 0 & Y_4 \end{Bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \end{Bmatrix} dV$$

or

$$\{R\} = \int_V [H]^T \{F\} dV - \int [B]^T \{\sigma\} dV. \quad (16)$$

These integrals can be evaluated numerically by using Gaussian quadrature.

Thus, assuming a one-radian segment, we can write

$$\begin{aligned} \{R\} &= \int_{-1}^{+1} \int_{-1}^{+1} \left([H]^T \{F\} - [B]^T \{\sigma\} \right) X_{11} \det \bar{J} \, ds \, dt \\ &= \sum_i \left([H(s, t)]_i^T \{F\}_i - [B(s, t)]_i^T \{\sigma\}_i \right) X_{11} \det J_i(s, t) W_i, \end{aligned}$$

where the W_i are Gaussian weighting factors and the summation extends over the chosen number of integration points (four in this investigation).

Incremental Stress-Strain Relations

Equation (16) can be evaluated if the instantaneous stress state is available. The instantaneous stress state can be calculated from the incremental stress-strain relations adopted for the material and the stresses in the previous configurations.

General incremental stress-strain relations were derived for metals by Hartzman and Hutchinson [3], based on the following assumptions:

- 1) The material properties remain invariant during deformation.
- 2) The strain components are composed of elastic and plastic components.

3) The plastic strain increments are given by the Prandtl-Reuss rules with the associated yield conditions of von Mises.

4) The elastic strain components are related to the stresses by Hooke's law.

5) The temperature history is prescribed.

Equation (10) can thus be written as follows:

$$\{\epsilon\}_E^s = [D] \{\epsilon\}_E^{s-1} + \{\Delta\epsilon\} - \{\Delta\epsilon\}_P - \{\Delta\epsilon\}_T, \quad (17)$$

where the subscripts E, P, T represent elastic, plastic, and thermal components, respectively.

The elastic strain components are related to the stresses by the relations

$$\{\epsilon\}_E = [B]^{-1} \{\sigma\}.$$

(The $[B]$ matrix may be isotropic or anisotropic but is assumed invariant in time.) Substituting into Eq. (17), we get

$$\{\sigma\}^s = [B] [D] [B]^{-1} \{\sigma\}^{s-1} + [B] \left\{ \{\Delta\epsilon\} - \{\Delta\epsilon\}_P - \{\Delta\epsilon\}_T \right\}. \quad (18)$$

Let

$$\{\sigma\}^* = [B] [D] [B]^{-1} \{\sigma\}^{s-1}.$$

Equation (18) may then be written as

$$\{\sigma\}^s = \{\sigma\}^* + [B] \left\{ \{\Delta\epsilon\} - \{\Delta\epsilon\}_P - \{\Delta\epsilon\}_T \right\}$$

or, in indicial notation for the case of an isotropic material, as

$$\sigma_{ij}^s = \sigma_{ij}^* + 2G(\Delta\epsilon_{ij} - \Delta\epsilon_{ijp}) + \delta_{ij} \left[\lambda(\Delta\epsilon_{kk} - \Delta\epsilon_{kkp}) - (3\lambda + 2G)\alpha\Delta T \right], \quad (19)$$

where λ, G = Lamé's constants.

In accordance with the Prandtl-Reuss theory, the stresses in a yielding body are assumed to satisfy a yield condition of the form

$$f(\sigma_{ij}) - \left[k(\epsilon_e^p, \theta) \right]^2 = 0,$$

where:

$f(\sigma_{ij})$ = yield function

$$k = \sigma_e / \sqrt{3}$$

$$\epsilon_e^p = \sum \Delta\epsilon_e^p$$

$$\Delta\epsilon_e^p = \left[\frac{2}{3} \Delta\epsilon_{ijp} \Delta\epsilon_{ijp} \right]^{1/2}$$

σ_e = instantaneous yield stress determined from a one-dimensional tensile test

ΔT = temperature increment.

If we assume

$$f(\sigma_{ij}) = J_2 = \frac{1}{2} S_{ij} S_{ij},$$

where:

$$S_{ij} = \sigma_{ij} - \frac{1}{3} \delta_{ij} \sigma_{kk},$$

we get the von Mises yield criterion:

$$\sigma_e^2 = \frac{3}{2} S_{ij} S_{ij},$$

and the corresponding flow rule:

$$\Delta \epsilon_{ij}^p = \Delta \Lambda S_{ij}, \quad (20)$$

where $\Delta \Lambda$ = a non-negative quantity to be determined.

Substituting Eq. (20) into (19), we get

$$\sigma_{ij}^s = \sigma_{ij}^{*s} + 2G(\Delta \epsilon_{ij} - \Delta \Lambda S_{ij}) + \delta_{ij} \left[\lambda \Delta \epsilon_{kk} - (3\lambda + 2G)\alpha \Delta T \right]. \quad (21)$$

Let

$$\bar{\sigma}_{ij}^s = \sigma_{ij}^{*s} + 2G\Delta \epsilon_{ij} + \delta_{ij} \left[\lambda \Delta \epsilon_{kk} - (3\lambda + 2G)\alpha \Delta T \right].$$

Equation (21) can then be rewritten as

$$\sigma_{ij}^s = \bar{\sigma}_{ij}^s - \beta \Delta \Lambda S_{ij},$$

where:

$$\beta = \begin{cases} 1 & \text{if loading} \\ 0 & \text{if unloading} \end{cases}$$

$$\Delta \Lambda = \frac{2G\Delta \Lambda}{3}.$$

If $\beta = 0$, we get the result

$$\sigma_{ij}^s = \bar{\sigma}_{ij}^s.$$

the stresses obtained by neglecting the plastic strain increments. The assumption is now made that

$$S_{ij} = \sigma_{ij}^s - \frac{1}{3} \delta_{ij} \sigma_{kk}^s,$$

so that, assuming $\beta = 1$, Eq. (21) can be rewritten as

$$\sigma_{ij}^s = \bar{\sigma}_e^s - 3\bar{\Delta\lambda} \left(\sigma_{ij}^s - \frac{1}{3} \delta_{ij} \sigma_{kk}^s \right).$$

Solving for σ_{ij}^s , we get

$$\sigma_{ij}^s = \frac{1}{(1 + 3\bar{\Delta\lambda})} \left[\sigma_{ij}^s + \bar{\Delta\lambda} \delta_{ij} \bar{\sigma}_e^s \right]. \quad (22)$$

The quantity $\bar{\Delta\lambda}$ can be shown to be given as

$$\bar{\Delta\lambda} = \frac{1}{3} \left(\frac{\bar{\sigma}_e^s}{\sigma_e^s} \pm 1 \right), \quad (23)$$

where:

$$\sigma_e^s = \frac{(\sigma_e^{s-1} + F^s \Delta T) + \left(\frac{H^s}{3G} \right) \bar{\sigma}_e}{(1 + H^s/3G)}$$

$$H^s = \left(\frac{\partial \sigma_e^s}{\partial \epsilon_e^p} \right)^{s-1}; \quad F^s = \left(\frac{\partial \sigma_e^s}{\partial T} \right)^{s-1}. \quad (24)$$

The negative sign is chosen because the equivalent stress σ_e^B based on the actual stresses is always smaller than $\bar{\sigma}_e^B$, which is calculated by neglecting the plastic strain components. Hence, if $\sigma_e = \sigma_e(\epsilon_e^P, 0)$ is available, the equivalent stress at time t^S may be calculated if all other quantities are known. For a bilinear material, the quantity H' can be shown to be given as

$$H' = \frac{EE^*}{E - E^*}$$

where $E^* =$ slope of the uniaxial stress-strain curve in the inelastic region. The algorithm applied to obtain the stress state at a given instant can thus be summarized as follows:

- 1) Calculate the stress $\bar{\sigma}_{ij}^B$, i.e., calculate the stresses by disregarding the plastic components ($\beta = 0$).
- 2) Calculate $\bar{\sigma}_e$ based on $\bar{\sigma}_{ij}^B$.
- 3) If $\bar{\sigma}_e^B$ is greater than a quantity $\sigma_{e\max}$, the material of the element has yielded. We therefore proceed to calculate the stresses that contain plastic components; if they are less than or equal to $\sigma_{e\max}$, the plastic calculation is not performed, and the stresses in the element are indeed σ_{ij}^B .
- 4) Calculate σ_e^B from Eq. (24).
- 5) Calculate $\bar{\Delta\lambda}$ from Eq. (23).
- 6) Calculate σ_{ij}^B from Eq. (22) ($\beta = 1$).
- 7) Set $\sigma_{e\max} = \sigma_e^B$. $\sigma_{e\max}$ is the greatest equivalent stress experienced by the element. At the next cycle, this is then used to check

whether yielding has occurred. Implicit in this step is the assumption that the Bauschinger effect is not taken into account. Also, if unloading and reloading occurs, σ_e^{s-1} is replaced by $\sigma_{e\max}$.

Once the instantaneous stress state has been calculated, the element forces can be calculated. This procedure is done element-by-element for all elements, from which the forces acting on the masses are then assembled.

COMPUTATIONAL PROCEDURE

For practical application, the theory presented above is implemented by a computer program, ADYN.

The general procedure on which this program is based is sequential in time and can be summarized as follows:

- A. For each quadrilateral:
 1. Assume we know
 - a. The incremental corner displacements ΔU_{in} ;
 - b. The element corner coordinates at t^{s-1} ;
 - c. For each Gaussian integration point, the stress and strain tensors σ_{ij}^{s-1} and ϵ_{ij}^{s-1} .
 2. Calculate the element corner coordinates at t^s .
 3. Calculate
 - a. The incremental strains $\Delta \epsilon_{ij}$;
 - b. The current stresses σ_{ij}^s ;
 - c. The current strains ϵ_{ij}^s , at each integration point.
 4. Calculate internal forces R_{in} , and add to the resultant internal forces of associated masses.
- B. For each mass:
 1. Calculate external force resultants;
 2. Calculate new displacement increments by applying the numerical algorithm as described above;
 3. Introduce boundary conditions;
 4. Calculate new system coordinates and total displacements.
- C. Return to Step A and repeat until the problem is over.

Because the calculation is sequential, a global stiffness matrix is never assembled; and so the practical consideration of bandwidth is avoided, i. e., the mesh nodes and elements can in general be numbered arbitrarily. In practice, these nodes and elements are numbered sequentially so that most of the mesh is generated automatically.

Figure 5 is a flow diagram that illustrates the above procedure.

ILLUSTRATIVE EXAMPLES OF SOLUTION

Dynamic Collapse of a Spherical Dome

The method of analysis is illustrated by determining the dynamic collapse pressure of a built-in spherical cap or dome with variable thickness as shown in Fig. 6. The finite-element model chosen to represent this structure is shown in Fig. 7, which also shows the assumed time history of the applied load. Figure 7 also shows the material properties and the engineering uniaxial stress-strain relation (this is transformed in the program into true stress-Almansi strain data). The material properties are assumed to be strain-independent.

A number of runs were made by varying the external pressure between 4000 and 10000 psi, with the same pressure-time history. Figure 8 shows the maximum central displacement experienced under the various pressures. The dome appears to collapse at about 8000 psi, since after this point the deflections became substantial.

Figure 9 shows a half-cross section of the dome under application of 10000 psi pressure at various instants during collapse (these figures are taken from the graphical output of the program).

Figure 10 shows the displacement history of the center of the dome (node 1).⁹ Since no damping is included in the analysis, the dome oscillates elastically, once maximum deflection is achieved. The fundamental period can thus be approximated from the graph and compared with an analytical solution. The fundamental period after collapse thus obtained is 1.4×10^{-4} s. A formula for determining the natural frequency and period of clamped shallow spherical shells is presented by Reissner [9] and is reproduced here for convenience:

$$\omega = \sqrt{\frac{E}{\rho}} \frac{h}{a^2} \left[\frac{\mu^4 + k^4 \left(\frac{1-\nu}{1+\nu} \right)}{12(1-\nu^2)} \right]^{1/2}$$

where

$$k^4 = 48 (1 + \nu^2) (H/h)^2 \quad (25)$$

H = rise of the shell above the base circle

h = shell thickness

a = base circle radius

$$\mu = \mu(k^4)$$

ρ = material density.

For this calculation the following parameters were chosen:

$$H = 1.750 \text{ in.},$$

$$h = 0.225 \text{ in. (mean thickness),}$$

$$a = 4.30 \text{ in.},$$

⁹This history was obtained by plotting the displacement every 100 cycles of computation. Total computation was 5000 cycles. Complete time for this run, including graphics, was around 8 min on a CDC 7600 computer.

$$E = 30 \times 10^6 \text{ psi}$$

$$\rho = 7.324 \times 10^{-4} \text{ lb. s}^2/\text{in.}^4.$$

Then $K^4 = 4.907 \times 10^3$, and μ is determined from Reissner as 5.717. The calculated period is thus $1.384 \times 10^{-4} \text{ s.}$

The fundamental period of the shell was also obtained from the solution of the 4000-psi pressure loading. The measured period from the numerical output was $1.939 \times 10^{-4} \text{ s.}$ The calculated value was obtained again by applying Eq. (25) with the following geometric parameters:

$$H = 1.450 \text{ in.},$$

$$h = 0.225 \text{ in.},$$

$$a = 4.70 \text{ in.}$$

The calculated period with these values was $1.904 \times 10^{-4} \text{ s.}$ which compares closely with the measured result.

Impact of a Cylinder Against a Rigid Wall

Figure 11 shows an aluminum cylinder impacting a rigid wall at 14700 ips. This problem has also been examined by Wilkins and Guinan [10] and by Raftopoulos and Al-Saliki [11], where it was applied to the determination of the dynamic yield strengths of various materials. Figure 12 compares the result obtained by the present method with that obtained by Wilkins and Guinan (a finite difference method) and with an experimental result from Raftopoulos and Al-Saliki (experimental results are also shown by Wilkins and Guinan). The material was assumed to be elastic-perfectly plastic.

The final length calculated by both methods is almost identical. The flared diameter calculated by the present method is considerably smaller than Wilkins and Guinan's but it is much closer to their experimental shape. The final shape also seems to correspond closer to Raftopoulos and Al-Saliki's experimental shape.

CONCLUSION

The limited results presented above indicate that the present formulation based on the finite element method is capable of describing the dynamic response of axisymmetric solids where large displacements and deformation occur. This is shown by the overall close agreement between calculated and experimental results for the cylinder impact problem, where considerable deformation occurred.

Experience with the dome-collapse problem indicates that it would be desirable to include in future work a more refined isoparametric element (10-node) capable of describing thick shells. It would also be desirable to include other constitutive relations, such as for soils and rubber, which, when combined with the existing relations for metals, would permit the analysis of a very wide variety of problems in the dynamics of solids.

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APPENDIX A

KINEMATICS OF INCREMENTAL DEFORMATION

The covariant components of the Almansi strain tensor are given at a generic instant as

$$\epsilon_{ij}^s = \frac{1}{2} \left[g_{ij}^s - \bar{g}_{kl} \frac{\partial \bar{x}^k}{\partial X^i} \frac{\partial \bar{x}^l}{\partial X^j} \right],$$

where the superscript s denotes current time. If we now add and subtract the expression

$$\frac{1}{2} g_{mn}^{s-1} \frac{\partial x^m}{\partial X^i} \frac{\partial x^n}{\partial X^j},$$

where the superscript $s-1$ denotes a previous instant of time, the following results are obtained after algebraic manipulation:

$$\epsilon_{ij}^s = \epsilon_{mn}^{s-1} \frac{\partial x^m}{\partial X^i} \frac{\partial x^n}{\partial X^j} + \Delta \epsilon_{ij}, \quad (A-1)$$

where $\Delta \epsilon_{ij}$ are the incremental Almansi strains experienced during the time interval $\Delta t (= t^s - t^{s-1})$. In terms of physical components, we have, in general, ^{A-1}

$$\epsilon_{(ij)} = \frac{1}{\sqrt{g_{ii}g_{jj}}} \epsilon_{ij},$$

^{A-1} Bars under the symbols suspend the summation convention.

so that Eq. (A-1) becomes

$$\epsilon_{(ij)}^s = \Delta \epsilon_{(ij)} + \sqrt{\frac{g_{mm}^{s-1} g_{nn}^{s-1}}{g_{ii}^s g_{jj}^s}} \epsilon_{(mn)}^{s-1} \frac{\partial x^m}{\partial X^i} \frac{\partial x^n}{\partial X^j}. \quad (\text{A-2})$$

In cylindrical coordinates the nonzero components of the metric tensors are

$$g_{11}^{s-1} = g_{11}^s = 1, \quad g_{33}^s = (X_1)^2,$$

$$g_{22}^{s-1} = g_{22}^s = 1, \quad g_{33}^{s-1} = (x_1)^2.$$

Substituting into Eq. (A-2) and expanding, we get Eq. (10) in the text.

FIGURE CAPTIONS

- Fig. 1. Body configurations during the deformation process.
- Fig. 2. Definition of nodal displacements for a four-node region.
- Fig. 3. Definition of local quantities.
- Fig. 4. Coordinate systems for a four-node isoparametric quadrilateral element.
- Fig. 5. Numerical algorithm for calculation of response over a generic time increment.
- Fig. 6. Dome with pressure on the convex side.
- Fig. 7. Finite-element model and material properties.
- Fig. 8. Peak central deflection versus step pressure magnitude.
- Fig. 9. Stages in dome collapse due to dynamic loading (10,000 psi, step loading).
- Fig. 10. Central displacement history of the dome under dynamic loading (10,000 psi, step loading).
- Fig. 11. Impact of a cylinder against a rigid wall.
- Fig. 12. Comparison of deformed cylinder shapes.

Hartzman - Fig. 1

Hartzman - Fig. 2

Hartzman - Fig. 3

Hartzman - Fig. 4

Hartzman - Fig. 6

Hartzman - Fig. 7

Hartzman - Fig. 8

Hartzman - Fig. 9

Hartzman - Fig. 10

Hartzman - Fig. 11

Matl: Al 6061-T6

$V = 14,700 \text{ in.}^3$

$\sigma_y = 61,000 \text{ psi}$

$\nu = \phi, \phi$

$L_0 = 0.924 \text{ in.}$

$D_0 = 0.308$

$L_f = 0.651 \text{ in.}$

$D_f = 0.591$

$L_f/L_0 = 0.703$

$D_f/D_0 = 1.93$

$L_0 = \text{initial length}$

$D_0 = \text{initial dia.}$

$L_f = \text{final length}$

$D_f = \text{flared dia.}$

(a) Finite element method

Matl: Al 6061-T6

$V = 14,700 \text{ in.}^3$

$\sigma_y = 61,000 \text{ psi}$

$L_0 = 0.929 \text{ in.}$

$D_0 = 0.301 \text{ in.}$

$L_f = 0.650 \text{ in.}$

$D_f = 0.690 \text{ in.}$

$L_f/L_0 = 0.703$

$D_f/D_0 = 2.29 \text{ (calc.)}$

$D_f/D_0 = 2.07 \text{ (exp.)}$

(b) Finite difference method
(Ref. 10)

Matl: Al 6061-T6

$V = 14,300 \text{ in.}^3/\text{sec}$

$L_0 = 1,000 \text{ in.}$

$D_0 = 0.400 \text{ in.}$

$L_f = 0.799 \text{ in.}$

$D_f = 0.746 \text{ in.}$

$L_f/L_0 = 0.799$

$D_f/D_0 = 1.86$

(c) Experimental method
(Ref. 11)

Hartzman - Fig. 12