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Test Problems for Nonlinear Programming Algorithms

by

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TEST PROBLEMS FOR NONLINEAR PROGRAMMING ALGORITHMS

1. INTRODUCTION

The efficiency of any new nonlinear programming algorithm should be tested using problems for which the optimal solutions are known. An approach has been suggested by Rosen and Suzuki (1)* for generating test problems with a single optimal point, but there is no widely known technique for constructing more general nonlinear problems with several relative optima. We shall consider two methods of generating test problems for nonlinear programming algorithms. The first method generates problems with a separable nonlinear objective function and linear constraints, so that there is a maximum of $(k-2)^n$ optimal points, where n represents the number of variables, and k is the degree of each of the polynomials in the separable objective function. The second method constructs problems so that both the concave objective function and the constraints are nonlinear.

*The numbers in parentheses indicate references in the Bibliography.

Neither method uses the most general form of a nonlinear equation for the objective function because of the difficulty of solving a set of simultaneous nonlinear equations, but the logic is valid for the general nonlinear objective function.

The first approach initially considers a nonlinear function as the objective function of an unconstrained problem. Optimal points for such a problem are located by solving the simultaneous equations generated by setting the first partial derivatives of the objective function equal to zero. The solutions of this set of equations are called stationary points, and it is a well-known result from classical optimization that every optimal point must be a stationary point. It should be noted that it is convenient to consider only separable objective functions, so that each partial derivative is a function (in general a nonlinear function) of only a single variable. When the stationary points are found, linear constraints are constructed from upper and lower bounds on the values of each variable over the set of stationary points. For such a constraint set, every stationary point will be a solution. It is necessary to specialize the objective function further so that for a maximization problem the value of $f(\bar{x})$ must decrease monotonically in every direction as the distance from the set of stationary points increases. We are thus assured that the value of such an objective function at the boundary of any constraint set which contains every

stationary point cannot exceed its largest value at a maximal point. Then the local maximum corresponding to the largest value of the objective function over the set of maximal points will be the global maximum for the unconstrained problem, and will also be the global maximum for any constrained problem in which every stationary point is an element of the set of feasible solutions. An analogous condition holds for minimization problems. Therefore boundary points for the constrained problem cannot be optimal points and can be safely disregarded.

The second method constructs test problems with a nonlinear objective function and nonlinear constraints. An approach by Rosen and Suzuki (1) generates problems of the type described above so that a point specified beforehand by the user will be an optimal solution to the problem constructed. We would like to modify this approach so that the global optimum and one or more additional local optima of the generated problem are known. It is possible to combine two of the Rosen-Suzuki problems so that their optima will be the optima for a problem with constraints of increased complexity. In this way the number of optimal points may be increased. It is necessary that every optimal point of the resultant problem be identified, since we pick the global optimum to be the point corresponding to the

largest (or smallest) value of the objective function over the set of maximal (minimal) points. We may use the Rosen-Suzuki approach to generate nonlinear problems which maximize a concave function over a convex set so that a given point is the optimal solution. Since such a problem has only one optimum, the point specified must be the global optimum and there are no other optimal points. If we generate several problems, each with a single global optimum (to be designated simple problems) so that each problem optimizes the same nonlinear objective function, and so that the global optima of no two problems are equal, then it is possible to combine the constraints in such a way that the constraint set is no longer convex. Each point that was a global optimum for one of the simple problems will be an optimal point of the new problem and there will be no additional optimal points. Therefore the local optima of the new problem have been located, and the global optimum will be the optimal point at which the objective function is largest (or smallest). The method for combining convex constraints so that the set of feasible solutions generated is not convex will be described in detail later.

2. FIRST METHOD

The notation used in this section corresponds to that of reference (2). The definitions and results from the first three chapters of the same text are used. Both methods of generating test problems for nonlinear programming algorithms are illustrated for maximization problems, but it is relatively simple to state analogous results for the generation of minimization problems. Corresponding changes for minimization problems are mentioned as they arise in the exposition.

2.1 OPTIMAL POINTS

In the first approach we have noted that the objective function is chosen to be a nonlinear function which is separable, that is, we can write

$$f(\bar{x}) = f_1(x_1) + f_2(x_2) + \dots + f_n(x_n). \quad (1)$$

The stationary points are defined to be the solutions of

$$\partial f(\bar{x}) / \partial x_j = 0, \quad j = 1, \dots, n. \quad (2)$$

Since $f(\bar{x})$ is separable the partial derivatives above are each functions of a single variable, therefore,

$$\partial f(\bar{x}) / \partial x_j = f'_j(x_j), \quad j = 1, \dots, n. \quad (3)$$

Then the set of stationary points can be described as the set of all possible combinations of solutions to the n

derivative equations (3), so that $\bar{x} = [\bar{x}_1, \dots, \bar{x}_n]$, where x_1 is any root of the equation corresponding to the first partial derivative, x_2 is any root of the equation corresponding to the second partial derivative, etc. Furthermore,

$$\max f(\bar{x}) = \max \sum_{j=1}^n f_j(x_j) = \sum_{j=1}^n \left[\max f_j(x_j) \right] \quad (4)$$

Let us consider an objective function of the form

$$f(\bar{x}) = \sum_{j=1}^n f_j(x_j) = \sum_{j=1}^n (a_{1j}x_j + \dots + a_{kj}x_j^k) \quad (5)$$

We want the degree k of each polynomial in $f(\bar{x})$ to be large enough so that each partial derivative has more than one root corresponding to a maximal point. The partial derivative of $f(\bar{x})$ with respect to x_j will then be a polynomial in x_j of degree less than or equal to $k - 1$ for every j . A polynomial of degree $k - 1$ has at most $k - 1$ roots. If we impose suitable conditions on the coefficients a_{ij} it is possible to obtain a partial derivative equation with $k - 1$ simple roots. We also want k to be even so that for a maximization problem when a_{kj} is chosen to be negative the objective function approaches $-\infty$ as $x_j \rightarrow \pm\infty$, $j = 1, \dots, n$. It will be recalled that this condition is necessary to insure that the objective function does not take on a maximal value at a boundary point of the constraint set to be generated. Suppose that $f(\bar{x})$ is

a function of a single variable x . Consider a polynomial in x of odd degree, say $k = 3$. Although the unconstrained minimization problem has only two stationary points, $\bar{x}^{(1)}, \bar{x}^{(2)}$, the constrained problem illustrated in Figure 1 assumes its global minimum at a boundary point, $\bar{x}^{(3)}$. Therefore we must ascertain first that the set of constraints is generated so that every stationary point of $f(\bar{x})$ is an element of the solution space, and second that in a maximization problem

$$f_j(x_j) \rightarrow -\infty \text{ as } x_j \rightarrow \pm\infty, \quad j = 1, \dots, n \quad (6)$$

or that in a minimization problem

$$f_j(x_j) \rightarrow +\infty \text{ as } x_j \rightarrow \pm\infty, \quad j = 1, \dots, n. \quad (7)$$

The sign of the coefficient of the highest power of x_j will determine the behavior of $f_j(x_j)$ "away from" the stationary points. We want a_{kj} , the coefficient of the highest power of x_j to have the same sign for every variable x_j . Then it is possible to describe $f(\bar{x})$ as either monotonically increasing or monotonically decreasing in any direction as the distance from the set of stationary points increases. If $a_{kj} < 0$ for every j , the problem will be a maximization problem, and if $a_{kj} > 0$ for every j , the problem constructed will be a minimization problem. Suppose $k = 2$. Then

$$f_j(x_j) = a_{1j}x_j + a_{2j}x_j^2, \quad (8)$$

$$\partial f(\bar{x}) / \partial x_j = f'_j(x_j) = a_{1j} + 2a_{2j}x_j = 0. \quad (9)$$

Therefore every stationary point must have

$$x_j = -a_{1j} / 2a_{2j}, \quad j = 1, \dots, n. \quad (10)$$

Figure 1. A Constrained Polynomial of Odd Degree

There will be only one optimal point for such an unrestricted problem,

$$\bar{x} = (-a_{11}/2a_{21}, -a_{12}/2a_{22}, \dots, -a_{1n}/2a_{2n}). \quad (11)$$

Since we want to generate problems with more than one optimum, it is necessary to increase the degree of $f_j(x_j)$. Let us consider the next even integer, $k = 4$.

Then

$$f_j(x_j) = a_{1j}x_j + a_{2j}x_j^2 + a_{3j}x_j^3 + a_{4j}x_j^4 \quad (12)$$

$$\partial f(\bar{x})/\partial x_j = f'_j(x_j) = a_{1j} + 2a_{2j}x_j + 3a_{3j}x_j^2 + 4a_{4j}x_j^3 = 0. \quad (13)$$

In order to solve the nonlinear derivative equation (13) more easily, let us restrict the coefficients a_{3j} , a_{1j} , so that

$$a_{3j} = 1, \quad a_{1j} = 0, \quad j = 1, \dots, n. \quad (14)$$

Then

$$\partial f(\bar{x})/\partial x_j = x_j(2a_{2j} + 3x_j + 4a_{4j}x_j^2) = 0. \quad (15)$$

The value of x_j at any stationary point must be one of the following:

$$\begin{aligned} x_j^{(1)} &= 0, \\ x_j^{(2)} &= (-3 - \sqrt{9-32a_{2j}a_{4j}})/(8a_{4j}), \\ x_j^{(3)} &= (-3 + \sqrt{9-32a_{2j}a_{4j}})/(8a_{4j}). \end{aligned} \quad (16)$$

If we specify that the product $a_{2j}a_{4j} < 0$, then the square root in (16) will be a real number. Let us consider the sign of the second derivative at values of x_j corresponding to stationary points of $f(\bar{x})$.

$$f_j''(x_j) = 2a_{2j} + 6a_{3j}x_j + 12a_{4j}x_j^2. \quad (17)$$

Evaluating (17) at the optimal points,

$$f_j''(x_j^{(1)}) = 2a_{2j}, \quad (18)$$

$$f_j''(x_j^{(2)}) = .1875(6+2\sqrt{9-32a_{2j}a_{4j}} - 21.3333a_{2j}a_{4j})/a_{4j},$$

$$f_j''(x_j^{(3)}) = .1875(6-2\sqrt{9-32a_{2j}a_{4j}} - 21.3333a_{2j}a_{4j})/a_{4j}.$$

Suppose that we are generating a maximization problem. The coefficients of the objective function will be selected so that $a_{4j} < 0$, $a_{2j} > 0$. Then $f_j(x_j)$ has a minimum at $x_j^{(1)} = 0$ and maxima at $x_j^{(2)}$ and $x_j^{(3)}$. If $a_{4j} > 0$, $a_{2j} < 0$, $f_j(x_j)$ has a maximum at $x_j = x_j^{(1)}$ and minima at the other two points.

Then in summary, if $k = 4$, $a_{3j} = 1$, $a_{1j} = 0$, and the product $a_{2j}a_{4j} < 0$, $j = 1, \dots, n$, there will be as many as 3^n stationary points. If $a_{4j} < 0$ for $j = 1, \dots, n$, there will be as many as 2^n local maxima, one local minimum (at $\bar{x} = \bar{0}$), and the other stationary points will be saddle points for the unconstrained problem. If $a_{4j} > 0$ for $j = 1, \dots, n$, we will have 2^n local minima, one local maximum (at $\bar{x} = \bar{0}$), and $3^n - 2^n - 1$ saddle points.

2.2 TRANSFORMATION

Suppose that it is necessary to generate a problem of the form

$$\max \text{ (or min) } f(\bar{x}) = \sum_{j=1}^n f_j(x_j), \quad (19)$$

subject to the restrictions,

$$x_j \geq 0, \quad j = 1, \dots, n. \quad (20)$$

Using the method of the previous section, it is obvious that stationary points which satisfy the derivative equations (15) may have negative values. Such points would not even be feasible for the restricted problem of equations (19), (20). Therefore either the degree of the polynomials $f_j(x_j)$ must be increased, or alternatively the polynomial $f_j(x_j)$ must be transformed so that every root of $f'_j = 0$ is non-negative.

Suppose that

$$f_j(x_j) = a_{2j}x_j^2 + x_j^3 + a_{4j}x_j^4, \quad (21)$$

and that $z < 0$ is the smallest root of the equation

$$f'_j(x_j) = 2a_{2j}x_j + 3x_j^2 + 4a_{4j}x_j^3. \quad (22)$$

Then a transformation which takes x_j into $x_j - z$, will map f_j into a polynomial of the same degree, say

$$f_j(x_j) \rightarrow g_j(x_j) = b_{0j} + b_{1j}x_j + b_{2j}x_j^2 + b_{3j}x_j^3 + b_{4j}x_j^4, \quad (23)$$

where

$$\begin{aligned}
 b_{4j} &= a_{4j}, \\
 b_{3j} &= 4a_{4j}z+1, \\
 b_{2j} &= 6a_{4j}z^2+3z+a_{2j}, \\
 b_{1j} &= 4a_{4j}z^3+3z^2+2a_{2j}z, \\
 b_{0j} &= a_{4j}z^4+z^3+a_{2j}z^2.
 \end{aligned} \tag{24}$$

If we substitute g_j for f_j in the objective function, then the values of x_j over the new set of stationary points will be non-negative. This procedure must be repeated for every variable which has a negative value at some stationary point.

2.3 GENERATING A CONSTRAINT SET

In the previous sections we have found the optimal points of the unconstrained nonlinear objective function $f(\bar{x})$. If a set of constraints is generated around the stationary points of $f(\bar{x})$ so that every stationary point is a solution to the constraint set, then we have chosen $f(\bar{x})$ so that the optima of $f(\bar{x})$ over any such set of constraints will be identical to those of the unconstrained optimization problem. The above assertion is not true for the general nonlinear function $f(\bar{x})$. Let us consider the counterexample in Figure 1, where the objective function is a function of a single variable. Let

$$f(\bar{x}) = x^3 - 6x^2 + 11x - 6 = (x-1)(x-2)(x-3), \tag{25}$$

$$f'(\bar{x}) = 3x^2 - 12x + 11, \quad (26)$$

$$f''(\bar{x}) = 6x - 12. \quad (27)$$

The stationary points of this problem occur at $\bar{x}^{(1)}=1.42554$, $\bar{x}^{(2)}=2.5744563$. The first point is a maximum; the second is a minimum.

If we consider the constrained problem

$$\min f(\bar{x}) = x^3 - 6x^2 + 11x - 6 \quad (28)$$

over the constraints

$$x \geq 1, \quad (29)$$

$$x \leq 3,$$

then the minimal solution will indeed be equal to the minimum of the unconstrained problem. But if we had chosen instead of (29) the following constraints,

$$x \geq 0, \quad (30)$$

$$x \leq 3,$$

then the minimal point would occur at $\bar{x} = 0$, a boundary point of the constraint set.

This difficulty can be overcome by suitable restrictions on the objective function $f(\bar{x})$. For a maximization problem the function will be required to be decreasing everywhere outside of the set of stationary points. If we consider the separable function $f(\bar{x}) = \sum_{j=1}^n f_j(x_j)$, then it is sufficient that for every j , $f_j(x_j)$ decreases as $x_j \rightarrow -\infty$, for x_j less

than the smallest of the stationary values of x_j . It is also required that $f_j(x_j)$ decrease as $x_j \rightarrow \infty$, for x_j greater than the largest of the stationary values of x_j . For the case in which the functions $f_j(x_j)$ are polynomials, then the degree of each polynomial must be even. For a maximization problem the coefficient of the term of highest degree of f_j must be negative in sign for every j . For a minimization problem the coefficient of the highest-order term must be positive for each polynomial f_j .

The method of generating constraints for the objective function described above is straightforward. For each variable x_j , we determine the upper and lower bounds of x_j over the set of stationary points. Then we construct $2n$ constraints of the form

$$\begin{array}{ll} x_1 \geq l_1 & x_1 \leq u_1 \\ x_2 \geq l_2 & x_2 \leq u_2 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ x_n \geq l_n & x_n \leq u_n \end{array} \quad (31)$$

This constraint set corresponds to the smallest set of feasible solutions which contains every stationary point of $f(\bar{x})$. A problem with any constraint set which contains this minimal

set will have the same optimal points as the unrestricted problem. It is possible to form linear combinations of inequalities which have the same sense. In general this increases the set of feasible solutions. The following four constraints

$$\begin{array}{ll} x_1 \geq l_1 & x_1 \leq u_1 \\ x_2 \geq l_2 & x_2 \leq u_2 \end{array} \quad (32)$$

could be replaced by

$$\begin{array}{l} x_1 + x_2 \geq l_1 + l_2 \\ x_1 + x_2 \leq u_1 + u_2 \end{array} \quad (33)$$

as shown in Figure 2. Any point which is feasible for the constraint set (32) will be feasible for the constraint set (33) because of the properties of vector addition.

2.4 SUMMARY OF FIRST METHOD

The method of Section 2 can be described as follows:

- a) The coefficients of the objective function are either read in as input, or generated by a random number subroutine on the library tape.
- b) The signs of the coefficients are adjusted so that the stationary points will not be imaginary numbers and so that the problem will have either multiple maxima or minima as specified by the user.
- c) The stationary points are found by computing the roots of the derivative equations (15).

Figure 2. Constraints Generated by Linear Combination of Inequalities.

- d) The sign of the second derivative, $f''(\bar{x})$, for each stationary point is stored in the array ID2.
- e) The objective function is transformed so that all components of each stationary point are nonnegative.
- f) The largest and smallest values of the objective function over the set of stationary points are computed.
- g) Upper and lower bounds are found for each variable over the set of stationary points. Linear combinations of the resulting inequalities are used to generate a set of linear constraints.

2.5 DESCRIPTION OF APPENDICES FOR THE FIRST METHOD

In order to test the ideas described in this section, a program was written in FORTRAN for the IBM 7072. The program written to implement the first method is described in detail in Appendices 7.1 through 7.5.

Appendix 7.1 provides a glossary of the FORTRAN names corresponding to each mathematical symbol used.

Appendix 7.2 gives a detailed flow chart of the program.

Appendix 7.3 shows the format for data input.

Appendix 7.4 is a listing of the FORTRAN program.

Appendix 7.5 provides a sample of typical output for a generated problem.

3. SECOND METHOD

The second approach to the problem of constructing nonlinear programming test problems is derived from a paper by J. B. Rosen and S. Suzuki [1]. I have tried to use notation which is consistent with that of the Rosen-Suzuki paper throughout this section. The method suggested by Rosen and Suzuki yields a problem modified from the problem specified by the user so that any arbitrary point originally selected as the optimum solution satisfies the Kuhn-Tucker condition and therefore is an optimum for the new problem. We would like to use this approach to construct test problems which have more than one optimal point, and for which the global optimum is known. Consider first the procedure for modifying a given nonlinear programming problem so that a specified point is optimal. Let the original problem be

$$\max_{\bar{x}} \text{ (or min) } \{ \theta(\bar{x}) \mid q_i(\bar{x}) \geq 0, \quad i = 1, \dots, k \}, \quad (34)$$

and the desired optimum $\bar{x}^{(1)} \in E^m$. We must require that $\bar{x}^{(1)}$ be a solution to the constraint set of the final problem, and that the objective function and constraints satisfy the Kuhn-Tucker condition at $\bar{x}^{(1)}$. Using the notation of the Rosen-Suzuki paper, let the modified objective function be

$$\phi(\bar{x}) = \theta(\bar{x}) + \bar{c}'\bar{x}, \quad (35)$$

and let the new constraints be

$$h_i(\bar{x}) = q_i(\bar{x}) + b_i \geq 0, \quad i = 1, \dots, k. \quad (36)$$

The Kuhn-Tucker condition for the new problem is

$$\nabla \phi(\bar{x}^{(1)}) + \sum_{i=1}^k u_i^{(1)} \nabla h_i(\bar{x}^{(1)}) = 0 \quad (37)$$

where $\bar{u}^{(1)} \in E^k$ is the desired solution for the dual of the modified problem. If we choose the coefficients of the linear terms in the objective function so that

$$\bar{c} = -\nabla \phi(\bar{x}^{(1)}) - \sum_{i=1}^n u_i^{(1)} \nabla q_i(\bar{x}^{(1)}), \quad (38)$$

equation (37) will be satisfied. Then select the b_i in (36) so that constraints designated as active by the optimal dual solution $\bar{u}^{(1)}$ are equalities, and inactive constraints are inequalities at $\bar{x}^{(1)}$. If $u_i^{(1)} > 0$, we want

$$h_i(\bar{x}^{(1)}) = q_i(\bar{x}^{(1)}) + b_i = 0, \quad (39)$$

that is,

$$b_i = -q_i(\bar{x}^{(1)}). \quad (40)$$

On the other hand, if $u_i^{(1)} = 0$, then

$$h_i(\bar{x}^{(1)}) = q_i(\bar{x}^{(1)}) + b_i > 0, \quad (41)$$

$$b_i > -q_i(\bar{x}^{(1)}). \quad (42)$$

Note that b_i is uniquely determined only if the i^{th} constraint is active at $\bar{x}^{(1)}$. The new problem is

$$\max_{\bar{x}} \text{ (or min) } \{ \phi(\bar{x}) \mid h_i(\bar{x}) \geq 0, \quad i = 1, \dots, k \}. \quad (43)$$

So much for the Rosen-Suzuki paper. If we assume that the original problem (34) stated in terms of $\phi(\bar{x})$ and $q_i(\bar{x})$ has a global optimum, then the modified problem (43) will also have a global optimum since the addition of linear terms to the objective function and constant terms to the constraints will not affect the convexity or concavity of either the objective function or the solution space. Therefore if we choose a strictly concave function $\phi(\bar{x})$ and a bounded set of concave constraints $q_i(\bar{x}) \geq 0$, both problems (34) and (43) will have a single global optimum. If we have two such problems with the same objective function is it possible to change or combine the constraints so that the optimal global solutions of the original problems are still optimal points, and the optimal point corresponding to the larger value of $\phi(\bar{x})$ is the new global optimum? It may be well to remark here that the original distinction between "old" and "new" problems represented by (34) and (43) has been discarded. We are now considering "simple" problems in the form of (43) which have only one (global) optimum and "combined" problems whose optima are those of the simple problems. What we wish to do is to modify the constraint set of a problem with one optimum so that a different point will be optimal. Alternatively, we would like to combine the constraints so that both optima will be optima for the given objective function over the new constraint set.

In order to change a typical linear programming problem with a single optimum into a problem with more than one optimum (we disregard the special case in which a constraint is parallel to the objective function and hence an infinite number of points is optimal) there are two general approaches. We can either nonlinearize the objective function so that it will have additional extreme points or we can change the constraint set. This can be done by adding one or more nonlinear constraints. Essentially this changes the set of feasible solutions into a set which is no longer convex. However it would seem to be difficult (but not impossible) to locate optimal points for such a problem.

There is a simple alternative method for constructing a non-convex set from a convex set, that is, for changing a problem with a single optimum into a problem with more than one optimum. The intersection of two or more convex sets is convex, but their union need not be convex. Consider the intersections A and B of the same circle with two perpendicular lines shown in Figure 3. We have two convex sets A and B. If the intersection of the lines is a point which satisfies the constraint represented by the circle as a strict inequality, then $A \cup B$ is a non-convex set. The only difficulty remaining is the expression of the set-theoretic concept of union in the usual nonlinear programming formulation. The concept of intersection is achieved by merely

adding a new constraint to the constraint set.) But we do not need the inclusive union. The exclusive union is also a non-convex set which is the union of two convex sets $A \cap B'$ and $A' \cap B$ corresponding to the constraint sets of two simple problems.

Let us consider the sets shown in Figure 3 which represent either an optimization problem in E^2 or the projection of a problem onto E^2 . The objective function $\phi(\bar{x})$ is the same for each problem. The circle represents the set of feasible solutions for the constraints common to each problem. Let these constraints be $h_i(\bar{x}) \geq 0$, $i=1, \dots, k-1$. Let the last constraint $h_k^1(\bar{x}) \geq 0$ correspond to the k^{th} constraint of the first problem, $h_k^2(\bar{x}) \geq 0$ the k^{th} constraint of the second problem. Then the combined problem whose constraint set corresponds to the exclusive union of the sets of feasible solutions to the simple problems has the constraint set

$$\begin{aligned} h_i(\bar{x}) &\geq 0, \quad i = 1, \dots, k-1, \\ h_k^1(\bar{x}) h_k^2(\bar{x}) &\leq 0. \end{aligned} \tag{44}$$

This is in essence the method for generating a problem with two optimal points. The requirement that the constraints $h_k^1(\bar{x}) \geq 0$ and $h_k^2(\bar{x}) \geq 0$ be orthogonal is somewhat arbitrary. We want to insure that the set of feasible solutions to the first problem neither contains nor is contained by the set of feasible solutions to the second problem. Obviously the

Figure 3. The Union of Two Convex Solution Spaces

lines $h_k^1(\bar{x}) = 0$, and $h_k^2(\bar{x}) = 0$ should not be parallel, and their intersection should satisfy the remaining $k-1$ constraints. If we have $\bar{x} \in E^m$, where $m > 2$, then we can construct a combination problem with more than two optimal points. It is reasonable to require that a third simple problem differ from the first and second only in the k^{th} constraint, and that $h_k^3(\bar{x}) = 0$ be normal to the plane determined by $h_k^1(\bar{x})=0$ and $h_k^2(\bar{x}) = 0$. For example, let

$$\begin{aligned} h_k^1(\bar{x}) &= -x_1 + b_k^1 \geq 0, \\ h_k^2(\bar{x}) &= -x_2 + b_k^2 \geq 0, \\ h_k^3(\bar{x}) &= -x_3 + b_k^3 \geq 0. \end{aligned} \tag{45}$$

Any point which satisfies these three constraints as equalities must be of the form $(b_k^1, b_k^2, b_k^3, \dots)$. Therefore it is relatively easy to determine whether such a point satisfies the first $k-1$ constraints. For the combined problem constructed from three simple problems the k^{th} constraint is of the form

$$h_k^1(\bar{x}) h_k^2(\bar{x}) h_k^3(\bar{x}) \geq 0. \tag{46}$$

Points which satisfy (46) either satisfy all three constraints (45) simultaneously, or they satisfy one and only one of the inequalities in (45). If they satisfy one and only one of

the inequalities in (45), then such points are feasible points for one of the three simple problems generated by the Rosen-Suzuki approach in which each problem has k constraints. We must consider a fourth "simple" problem which has the first $k-1$ constraints as the other problems but three additional constraints, $h_k^1 \geq 0$, $h_k^2 \geq 0$, $h_k^3 \geq 0$. Therefore the combined problem can have four optimal points. There is no easy extension of this approach to more than four optimal points.

3.1 SUMMARY OF THE SECOND METHOD

The method of Section 3 can be described as follows:

- a) The quadratic coefficients of the objective function are read in. Each simple problem has the same objective function, and this objective function will be used for the problem to be generated.
- b) The constraints common to each simple problem are read in. Coefficients of linear terms of the J^{th} constraint are read into $DUE(I,J)$. Coefficients of quadratic terms are read into the array $QUE(I,J)$.
- c) The coefficients of the last constraint are read into the array QK . There is one such constraint for each simple problem, and these constraints are the only constraints which are different for the simple problems.

- d) The optimal solution for each simple problem is specified. The optimal solution for each simple problem must not be a feasible solution for any other simple problem. Then the set of feasible solutions to the combined problem will be non-convex.
- e) The coefficients of linear terms of the objective function are chosen so that the Kuhn-Tucker conditions are satisfied at each optimum.
- f) The values of the objective function are computed at the optimal points. If any two such values are identical, the specified optima are changed, and steps d), e) and f) are repeated.
- g) The constant term of each constraint is adjusted so that points specified as optima satisfy each constraint.

3.2 DESCRIPTION OF APPENDICES FOR THE SECOND METHOD

In order to generate nonlinear test problems using the ideas described in Section 3, another FORTRAN program was written. The remaining appendices describe the second program.

Appendix 7.6 is a glossary of mathematical symbols used in the description of the method and the corresponding FORTRAN names.

Appendix 7.7 contains a flow chart of the second program.

Appendix 7.8 gives the format for data input.

Appendix 7.9 is a listing of the second program.

Appendix 7.10 is a sample of the output generated by the program. This sample is described in greater detail in Section 4.

Appendix 7.11 is the paper by Rosen and Suzuki referred to in Section 3.

4. TYPICAL PROBLEMS

- Two sample problems have been chosen to illustrate the methods described in previous sections. The procedure of Section 2 finds the stationary points of an unconstrained objective function. Then a set of linear constraints is generated so that every stationary point is contained in the solution space. The objective function can be restricted so that the values of $f(\bar{x})$ at boundary points of the constraint set will not be better than the maximum (or minimum) value of the objective function over the set of stationary points. The sample problem of Appendix 7.5 is used as an example.

- a) The coefficients of quartic and quadratic terms of the objective function are read in as input data.
- b) The program checks the signs of these coefficients so that a problem with either several maxima or several minima is constructed. The sample problem is a maximization problem.

$$\begin{aligned} \max_{\bar{x}} f(\bar{x}) = & -x_1^4 + x_1^3 + 5x_1^2 - 2x_2^4 + x_2^3 + 4x_2^2 \\ & - 3x_3^4 + x_3^3 + 3x_3^2 - 4x_4^4 + x_4^3 + 2x_4^2 \\ & - 5x_5^4 + x_5^3 + x_5^2. \end{aligned} \quad (47)$$

- c) Then the roots of the five partial derivative equations are calculated. There are three such roots for each equation:

$$\begin{array}{lll}
 x_1^{(1)} = 2 & x_1^{(2)} = 0 & x_1^{(3)} = -1.25 \\
 x_2^{(1)} = 1.2049 & x_2^{(2)} = 0 & x_2^{(3)} = -.82993 \\
 x_3^{(1)} = .84307 & x_3^{(2)} = 0 & x_3^{(3)} = -.59307 \\
 x_4^{(1)} = .60246 & x_4^{(2)} = 0 & x_4^{(3)} = -.41496 \\
 x_5^{(1)} = .4 & x_5^{(2)} = 0 & x_5^{(3)} = -.25
 \end{array} \quad (48)$$

Any point $\bar{x} = (x_1, x_2, x_3, x_4, x_5)$, where each variable is one of the three values given above, will be a stationary point of $f(\bar{x})$.

- d) The signs of the second partial derivatives, $\partial^2 f / \partial x_i^2$, $i = 1, \dots, 5$, are stored in the array ID2. By inspection of this array stationary points which are optimal can be found.
- e) The objective function (47) is transformed so that all components of stationary points are non-negative. The transformed objective function is:

$$\begin{aligned}
 \max_{\bar{x}} f(\bar{x}) = & -x_1^4 + 6x_1^3 - 8.1250x_1^2 - 2x_2^4 + 7.6394x_2^3 \\
 & - 6.7551x_2^2 - .000000581x_2 - 3x_3^4 \\
 & + 8.1168x_3^3 - 5.1104x_3^2 - .000000593x_3 \\
 & - 4x_4^4 + 7.6394x_4^3 - 3.3776x_4^2 - .000000249x_4 \\
 & - 5x_5^4 + 6x_5^3 - 1.625x_5^2.
 \end{aligned} \quad (49)$$

- f) The global maximum z^* of the transformed objective function is computed:

$$z^* = 11.761. \quad (50)$$

- g) A linear constraint set is generated from linear combinations of inequalities obtained from each variable over the set of stationary points. The original inequalities corresponding to equations (31) are:

$$\begin{aligned} x_1 &\geq 0 \\ x_1 &\leq 3.2500 \\ x_2 &\geq 0 \\ x_2 &\leq 2.0349 \\ x_3 &\geq 0 \\ x_3 &\leq 1.4361 \\ x_4 &\geq 0 \\ x_4 &\leq 1.0174 \\ x_5 &\geq 0 \\ x_5 &\leq .65000. \end{aligned} \quad (51)$$

Since the value for NCOMB is 2, the inequalities for two variables will be multiplied by coefficients specified in the array CMULT. Here $CMULT(1) = 2$, $CMULT(2) = 3$. The original set of constraints (51) is replaced by

$$\begin{aligned}
 2x_1 - 3x_2 &\geq 0 \\
 2x_1 - 3x_2 &\leq 12.605 \\
 2x_3 - 3x_4 &\geq 0 \\
 2x_3 - 3x_4 &\leq 5.9246 \\
 2x_5 &\geq 0 \\
 2x_5 &\leq 1.3000.
 \end{aligned}
 \tag{52}$$

A second type of test problem is generated by the method of Section 3. A listing of the corresponding FORTRAN program is contained in Appendix 7.9, and a sample problem in Appendix 7.10 has been generated to illustrate the second approach. The first method constructs problems with a separable polynomial objective function of at least fourth order and linear constraints. The second approach can be used to generate problems in which the objective function and constraints are very general nonlinear functions. A concave maximization problem subject to concave constraints is given as an example. The problem specified by the user is

$$\max_{\bar{x}} -x_1^2 - x_2^2 - 2x_3^2 - x_4^2 + \bar{c}'\bar{x}
 \tag{53}$$

subject to

$$-x_1^2 - x_2^2 - x_3^2 - x_4^2 - x_1 + x_2 - x_3 + x_4 + b_1 \geq 0.
 \tag{54}$$

Two maximal points are required.

- a) The quadratic terms of the objective function are read into the array THUT.

$$\theta(\bar{x}) = -x_1^2 - x_2^2 - 2x_3^2 - x_4^2. \quad (55)$$

Linear coefficients are not input quantities. They are computed by the program to satisfy the Kuhn-Tucker condition at the optimal points and then they are stored in the array C.

- b) The coefficients of quadratic and linear terms of each constraint except the last are read into the matrices QUE and DUE respectively. Constant terms of each constraint are determined by the program. The only concave constraint in this sample problem is

$$q_1(\bar{x}) = -x_1^2 - x_2^2 - x_3^2 - x_4^2 + x_1 - x_2 + x_3 - x_4 + b_1 \geq 0 \quad (56)$$

- c) Since two maximal points are desired, the variable constraints are of the form

$$q_2^1(\bar{x}) = -x_1 + b_2^1 = 0 \quad (57)$$

$$q_2^2(\bar{x}) = -x_2 + b_2^2 = 0.$$

- d) Now we may specify the optimal points of the nonlinear problem:

$$\begin{aligned} \bar{x}^{(1)} &= (0, 3, 1, 1), \\ \bar{x}^{(2)} &= (1, 2, 1, 1). \end{aligned} \quad (58)$$

- e) From the functions specified in equations (55)-(57) and the optimal solutions (58) we can set up the Kuhn-Tucker conditions at each of the optimal points.

$$\nabla \phi(\bar{x}^{(1)}) + \sum_{i=1}^2 u_i^{(1)} \nabla h_i(\bar{x}^{(1)}) = \begin{bmatrix} -2x_1^{(1)} + c_1 \\ -2x_2^{(1)} + c_2 \\ -4x_3^{(1)} + c_3 \\ -2x_4^{(1)} + c_4 \end{bmatrix} + u_3^{(1)} \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (59)$$

$$\nabla \phi(\bar{x}^{(2)}) + \sum_{i=1}^2 u_i^{(2)} \nabla h_i(\bar{x}^{(2)}) = \begin{bmatrix} -2x_1^{(2)} + c_1 \\ -2x_2^{(2)} + c_2 \\ -4x_3^{(2)} + c_3 \\ -2x_4^{(2)} + c_4 \end{bmatrix} + u_3^{(2)} \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} \quad (60)$$

Then the equations can be solved for $\bar{c}$ and $u_3^{(1)}$ and $u_3^{(2)}$.

$$\bar{c} = \begin{bmatrix} 2x_1^{(1)} \\ 2x_2^{(1)} \\ 4x_3^{(1)} \\ 2x_4^{(1)} \end{bmatrix} - u_3^{(1)} \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2x_1^{(2)} \\ 2x_2^{(2)} \\ 4x_3^{(2)} \\ 2x_4^{(2)} \end{bmatrix} - u_3^{(2)} \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 4 \\ 2 \end{bmatrix} \quad (61)$$

$$u_3^{(1)} = 2x_1^{(2)} - 2x_1^{(1)} = 2, \quad (62)$$

$$u_3^{(2)} = 2x_2^{(1)} - 2x_2^{(2)} = 2. \quad (63)$$

- f) If the global maximum is to be unique, we must specify that $\phi(\bar{x}^{(1)}) \neq \phi(\bar{x}^{(2)})$. For the initial optimal points (58), $\phi(\bar{x}^{(1)}) = \phi(\bar{x}^{(2)}) = 12$. Therefore we must change the optimal points so that

$$\begin{aligned}\bar{x}^{(1)} &= (0, 3, 1, 1), \\ \bar{x}^{(2)} &= (1, 1.9, 1, 1).\end{aligned}\tag{64}$$

Using these values in (61), we obtain the same values for $\bar{c}$ and $u_3^{(1)}$:

$$u_3^{(2)} = 2.2.\tag{65}$$

- g) In the last step of the algorithm the constant terms of the constraints are specified. Equations (57) are satisfied as equalities. Therefore $b_1^1 = 0$, $b_2^2 = 1.9$. The first constraint (56) is inactive. Hence let

$$b_1 = \left[\max_{\bar{x}^{(1)}, \bar{x}^{(2)}} -q_1(\bar{x}) \right] + 1 = \max(14, 7.51) + 1 = 15.\tag{66}$$

The problem we have constructed is

$$\max_{\bar{x}} (-x_1^2 - x_2^2 - 2x_3^2 - x_4^2 + 2x_1 + 6x_2 + 4x_3 + 2x_4)\tag{67}$$

over the constraints

$$\begin{aligned}-x_1^2 - x_2^2 - x_3^2 - x_4^2 + x_1 - x_2 + x_3 - x_4 + 15 &\geq 0, \\ -x_1(-x_2 + 1.9) &\leq 0.\end{aligned}\tag{68}$$

The objective function assumes its global optimum at $\bar{x}^{(1)} = (0, 3, 1, 1)$, where $\phi(\bar{x}^{(1)}) = 13$. It has a local optimum at $\bar{x}^{(2)} = (1, 1.9, 1, 1)$.

5. CONCLUSIONS AND SUGGESTIONS FOR FURTHER RESEARCH

We have seen that it is possible to generate nonlinear test problems by suitable restrictions on the form of the objective function and constraints. In the first method, these restrictions justified the assertion that the global optimum occurs at a stationary point. In the second approach we could assert that the global optimum could be found by evaluating the optimal solutions of two or three simpler problems. There are several extensions and refinements of the methods described above which come to mind. For the first method it is possible to generate a nonlinear constraint set so that every stationary point of the restricted objective function (12), (14), is a feasible solution to the set of constraints. A second suggestion for an extension of the method would be to program a subroutine which computes all solutions to a set of simultaneous nonlinear equations, (2). Then the objective function would not be required to be separable. Some additional analysis of the behavior of the function at the boundary points is necessary.

For the method described in Section 3 the evaluation of the gradient $\nabla \theta$ used in the computation of linear coefficients $\bar{c}$ in the objective function should not properly be contained in the main program. The evaluation of $\nabla \theta$ should be performed in a subroutine supplied by the user with every

new problem, so that it would not be necessary to recompile the main program for each new problem.

Using the second method we can generate problems with no more than four optimal points. In order to extend this approach, we could use problems which have several known optima as simple problems and use a method analogous to that of Section 3 to construct problems in which any optimum of any simple problem will be an optimum of the generated problem. The only difference between any two simple problems or between any simple problem and the problem finally generated occurs only in the k^{th} constraint. It should be possible to generate problems with additional optima by letting the last two constraints (the k^{th} and $k-1^{\text{st}}$) of a simple problem differ from the last two constraints of any other simple problem.

A final suggestion would be to generate a linear programming problem in which the objective function is identical to one of the constraints. Then that constraint can be replaced by a nonlinear constraint so that each extreme point of the optimal hyperplane of the linear programming problem satisfies the nonlinear constraint as an equality. Then the optimal points of the new problem could be located without great difficulty.

6. ACKNOWLEDGEMENT

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7. APPENDICES

APPENDIX 7.1

List of FORTRAN Names for Mathematical Symbols in the First Program

	<u>Symbolic Representation</u>	<u>FORTRAN Name</u>
number of variables	n	NV ≤ 100
input indicator:		INP
INP = 0 generates random coefficients		
INP = 1 reads in coefficients of objective function		
max or min problem indicator:		MX
MX = 2, maximization problem		
MX = 1, minimization problem		
coefficients of objective function		
$\sum_{j=1}^n a_{2j}x_j^2 + x_j^3 + a_{4j}x_j^4.$		
coefficient of x_j^4	a_{4j}	AN(100)
coefficient of x_j^3	a_{3j}	BN(100)
coefficient of x_j^2	a_{2j}	CN(100)
Coefficient of x_j	a_{1j}	DN(100)
stationary points	x_{ij}	ID2(100,3)

APPENDIX 7.1

(continued)

	<u>Symbolic Representation</u>	<u>FORTRAN Name</u>
sign of $f''_j(x_{ij})$		ID2(100,3)
term in $f(\bar{x})$ corresponding to $f_j(x_{ij})$	$a_{1j}^4 x_{ji}^4 + a_{2j}^2 x_{ji}^2$ $+ a_{3j}^3 x_{ji}^3 + a_{4j}^4 x_{ji}^4$	S(100,3)
local minimum (global if MX = 1)	$\min \{f(\bar{x})\}$ $\bar{x}$ a stationary point	ZMIN
local maximum (global if MX = 2)	$\max \{f(\bar{x})\}$ $\bar{x}$ a stationary point	ZMAX
smallest value for x_j over the set of stationary points	$\min_i \{x_{ji}\}$	XMI
largest value for x_j over the set of stationary points	$\max_i \{x_{ji}\}$	XMA
coefficients of constraints	c'_{ij}	C(50,101)
RHS of constraint set	d'_{ij}	D(50).

APPENDIX 7.2

Flow Chart of the First Program

APPENDIX 7.3

Format for Data Input for the First Program

First card: NV,INP,MX (I2,2I1)

Second card: if INP = 1 read in 2NV numbers for the
coefficients of the objective function:

AN(1),CN(1),AN(2),CN(2),...,AN(NV),CN(NV) (8F10.5)

Third type of card: NCØMB(I2)

Fourth type of card: CMULT(1),CMULT(2),...,CMULT(NCØMB) (8F10.5)

```

FLAG FORTRAN,MAIN,LIST,DECK
C  RANDOM PROBLEM GENERATOR FOR NON-LINEAR PROGRAMMING
C  PROBLEMS
C  NV IS THE NUMBER OF VARIABLES (NO MORE THAN 100)
  DIMENSION AN(100),BN(100),CN(100),DN(100),X(100,3),S(1
    X00,3),
    IC(50,101),D(50),CMULT(25),ID2(100,3)
10  READ 1000,NV,INP,MX
1000 FORMAT (I2,2I1)
    DO 20 I=1,NV
      AN(I)=0
      BN(I)=1.
      CN(I)=0
20  DN(I)=0
      IF (INP) 70,21,30
21  DO 25 I=1,NV
      AN(I)=RANF(0)
      CN(I)=RANF(0)
22  IF (MX-1) 70,23,24
23  CN(I)=-CN(I)
      GO TO 25
24  AN(I)=-AN(I)
25  CONTINUE
      GO TO 35
C  INPUT OBJECTIVE FUNCTION
30  READ 2000,(AN(I),CN(I),I=1,NV)
2000 FORMAT (8F10.5)
35  PRINT 4100,(AN(I),BN(I),CN(I),I=1,NV)
4100 FORMAT (1H1,9X,I),BN(I),CN(I),I=1,NV)
X47HUNTRANSFORMED OBJECTIVE FUNCTION IS SUM OVER I
133HOF AN*X**4+BN*X**3+CN*X**2, WHERE//10X,5HAN(I),15X,
X5HBN(I),15X,
25HCN(I)//(3E20.5))
40  DO 104 I=1,NV
45  IF (AN(I)*CN(I)) 50,60,70
50  AN8=8.*AN(I)
    ASQ=SQRTF(9.-4.*AN8*CN(I))
    X(I,1)=(-3.-ASQ)/AN8
    X(I,2)=0
    X(I,3)=(-3.+ASQ)/AN8
    DO 55 J=1,3,2
      D2FX=(6.*AN(I)*X(I,J)+3.*BN(I))*X(I,J)+CN(I)
      IF (D2FX) 55,54,55
54  AN(I)=AN(I)+.01
      GO TO 45
55  CONTINUE

```

```

        GO TO 80
60      X(I,1)=0
        X(I,2)=0
        X(I,3)=0
        GO TO 80
70      PRINT 3000
3000    FORMAT (2X,21HERROR IN COEFFICIENTS)
        GO TO 10
C   THE ARRAY X(I,J) CONTAINS ALL POSSIBLE EXTREME POINT
C   VALUES
C   FOR THE ITH VARIABLE
80      IF (AN(I)) 103,102,101
101     ID2(I,1)=1
        ID2(I,2)=-1
        ID2(I,3)=1
        GO TO 104
102     ID2(I,1)=0
        ID2(I,2)=0
        ID2(I,3)=0
        GO TO 104
103     ID2(I,1)=-1
        ID2(I,2)=1
        ID2(I,3)=-1
104     CONTINUE
C   NEW OBJECTIVE FUNCTION IS THE SUM OVER I OF
C   AN*X**4+BN*X**3+CN*X**2+
C   DN*X
        PRINT 3900
3900    FORMAT (1H0,2X,1HI,6X,6HX(I,1),6X,2HD2,6X,6HX(I,2),6X,
        X2HD2,6X,
        16HX(I,3),6X,2HD2)
        DO 116 I=1,NV
        PRINT 3905,I,(X(I,J),ID2(I,J),J=1,3)
3905    FORMAT (14,3(E16.5,I4))
        XMIN=1.E20
        DO 106 J=1,3
        IF (X(I,J)-XMIN) 105,106,106
105     XMIN=X(I,J)
106     CONTINUE
        IF (XMIN) 107,116,116
107     ANZ=AN(I)*XMIN
        BN(I)=4.*ANZ+1.
        DN(I)=((BN(I)+2.)*XMIN+2.*CN(I))*XMIN
        CN(I)=(6.*ANZ+3.)*XMIN+CN(I)
C   TRANSFORM VARIABLES SO THAT ALL OPTIMAL POINTS ARE
C   FEASIBLE

```

```

      DO 115 J=1,3
115  X(I,J)=X(I,J)-XMIN
116  CONTINUE
C   FIND GLOBAL OPTIMUM FOR 2**NV LOCAL OPTIMA OF F(X)
      DO 140 I=1,NV
      DO 120 J=1,3
120  S(I,J)=(((AN(I)*X(I,J)+BN(I))*X(I,J)+CN(I))*X(I,J)+DN(
      X(I))*X(I,J)
      IF (S(I,1)-S(I,2)) 130,130,125
125  SAVE=S(I,1)
      S(I,1)=S(I,2)
      S(I,2)=SAVE
      SAVE=X(I,1)
      X(I,1)=X(I,2)
      X(I,2)=SAVE
      SAVE=ID2(I,1)
      ID2(I,1)=ID2(I,2)
      ID2(I,2)=SAVE
130  IF (S(I,2)-S(I,3)) 140,140,131
131  SAVE=S(I,2)
      S(I,2)=S(I,3)
      S(I,3)=SAVE
      SAVE=X(I,2)
      X(I,2)=X(I,3)
      X(I,3)=SAVE
      SAVE=ID2(I,2)
      ID2(I,2)=ID2(I,3)
      ID2(I,3)=SAVE
      IF (S(I,1)-S(I,2)) 140,140,132
132  SAVE=S(I,1)
      S(I,1)=S(I,2)
      S(I,2)=SAVE
      SAVE=X(I,1)
      X(I,1)=X(I,2)
      X(I,2)=SAVE
      SAVE=ID2(I,1)
      ID2(I,1)=ID2(I,2)
      ID2(I,2)=SAVE
140  CONTINUE
      ZMIN=0
      ZMAX=0
      DO 150 I=1,NV
      ZMIN=ZMIN+S(I,1)
150  ZMAX=ZMAX+S(I,3)
C   GENERATE A SET OF CONSTRAINTS
C   XMA AND XMI ARE UPPER AND LOWER BOUNDS RESPECTIVELY USED
C   TO GENERATE

```

```

C  A CONVEX SET OF LINEAR CONSTRAINTS STORED IN C(I,J), D(I)
      IJ=1
      IJ1=2
      NCT=0
      READ 4000, NCOMB, (CMULT(I), I=1, NCOMB)
4000  EFORMAT (I2/(8F10.5))
      DO 250 I=1, NV
      XMA=-1.E20
      XMI=1.E20
      DO 230 K=1, 3
      IF (X(I,K)-XMA) 210, 210, 200
200  XMA=X(I,K)
210  IF (X(I,K)-XMI) 220, 230, 230
220  XMI=X(I,K)
230  CONTINUE
      NCT=NCT+1
      C(IJ, I)=CMULT(NCT)
      D(IJ)=D(IJ)+CMULT(NCT)*XMA
      C(IJ1, I)=CMULT(NCT)
      D(IJ1)=D(IJ1)+CMULT(NCT)*XMI
      IF (NCOMB-NCT) 240, 240, 250
240  C(IJ, 101)=-1.
      C(IJ1, 101)=0
      IJ=IJ1+1
      IJ1=IJ+1
      NCT=0
250  CONTINUE
      IF (NCT) 251, 252, 251
251  C(IJ, 101)=-1.
      C(IJ1, 101)=0
      GO TO 253
252  IJ1=IJ1-2
253  PRINT 4101, (AN(I), BN(I), CN(I), DN(I), I=1, NV)
4101  EFORMAT (1H0, 9X, I), BN(I), CN(I), DN(I), I=1, NV)
      X48HTRANSFORMED OBJECTIVE FUNCTION IS SUM OVER I OF
      135HAN*X**4+BN*X**3+CN*X**2+DN*X, WHERE//10X, 5HAN(I), 15
      XX, 5HBN(I),
      215X, 5HCN(I), 15X, 5HDN(I)/(4E20.5))
      PRINT 4300
4300  EFORMAT (1H0, 2X, 1HI, 6X, 6HX(I, 1), 6X, 2HD2, 6X, 6HS(I, 1),
      X14X, 6HX(I, 2),
      16X, 2HD2, 6X, 6HS(I, 2), 14X, 6HX(I, 3), 6X, 2HD2, 6X, 6HS(I, 3))
      DO 255 I=1, NV
255  PRINT 4301, I, (X(I, J), ID2(I, J), S(I, J), J=1, 3)
4301  FORMAT (I4, E16.5, I4, E16.5, 2(E20.5, I4, E16.5))
      PRINT 4400, ZMIN, ZMAX

```

```
4400 FORMAT (1H0,9X,5HZMIN=,E12.5,2X,5HZMAX=,E12.5)
270 DO 300 I=1,IJ1
    PRINT 4500,I,(C(I,J),J=1,NV)
4500 FORMAT (1H0,9X,11HCONSTRAINT ,I2/(4E20.5))
    IF (C(I,101)) 280,290,290
280 PRINT 4600,D(I)
4600 FORMAT (10X,10HLESS THAN ,E12.5)
    GO TO 300
290 PRINT 4700,D(I)
4700 FORMAT (10X,13HGREATER THAN ,E12.5)
300 CONTINUE
    GO TO 10
END
```

Sample Problem Generated by the First Method

UNTRANSFORMED OBJECTIVE FUNCTION IS SUM OVER I OF $AN \cdot X^{**4} + BN \cdot X^{**3} + CN \cdot X^{**2}$, WHERE

AN(I)	BN(I)	CN(I)
-C.10000E 01	0.10000E 01	0.50000E 01
-C.20000E 01	0.10000E 01	0.40000E 01
-C.30000E 01	0.10000E 01	0.30000E 01
-C.40000E 01	0.10000E 01	0.20000E 01
-C.50000E 01	0.10000E 01	0.10000E 01

I	X(I,1)	D2	X(I,2)	D2	X(I,3)	D2
1	0.20000E 01	-1	0.00000E 00	1	-0.12500E 01	-1
2	0.12049E 01	-1	0.00000E 00	1	-0.82993E 00	-1
3	0.84307E 00	-1	0.00000E 00	1	-0.59307E 00	-1
4	0.60246E 00	-1	0.00000E 00	1	-0.41496E 00	-1
5	0.40000E 00	-1	0.00000E 00	1	-0.25000E 00	-1

TRANSFORMED OBJECTIVE FUNCTION IS SUM OVER I OF $AN \cdot X^{**4} + BN \cdot X^{**3} + CN \cdot X^{**2} + DN \cdot X$, WHERE

AN(I)	BN(I)	CN(I)	DN(I)
-C.10000E 01	0.60000E 01	-0.81250E 01	0.00000E 00
-C.20000E 01	0.76394E 01	-0.67551E 01	-0.58095E-06
-C.30000E 01	0.81168E 01	-0.51104E 01	-0.59307E-06
-C.40000E 01	0.76394E 01	-0.33776E 01	-0.24898E-06
-C.50000E 01	0.60000E 01	-0.16250E 01	0.00000E 00

I	X(I,1)	D2	S(I,1)	X(I,2)	D2	S(I,2)	X(I,3)	D2	S(I,3)
1	0.12500E 01	1	-0.34180E 01	0.00000E 00	-1	0.00000E 00	0.32500E 01	-1	0.85820E 01
2	0.82993E 00	1	-0.12346E 01	0.00000E 00	-1	0.00000E 00	0.20349E 01	-1	0.21064E 01
3	0.59307E 00	1	-0.47545E 00	0.00000E 00	-1	0.00000E 00	0.14361E 01	-1	0.74051E 00
4	0.41496E 00	1	-0.15433E 00	0.00000E 00	-1	0.00000E 00	0.10174E 01	-1	0.26330E 00
5	0.25000E 00	1	-0.27344E-01	0.00000E 00	-1	0.00000E 00	0.65000E 00	-1	0.68656E-01

ZMIN=-0.53097E 01 ZMAX= 0.11761E 02

CONSTRAINT	1			
C.20000E 01	0.30000E 01	0.00000E 00	0.00000E 00	
C.00000E 00				
LESS THAN	0.12605E 02			

CONSTRAINT	2			
C.20000E 01	0.30000E 01	0.00000E 00	0.00000E 00	
C.00000E 00				
GREATER THAN	0.00000E 00			

CONSTRAINT	3			
C.00000E 00	0.00000E 00	0.20000E 01	0.30000E 01	
C.00000E 00				
LESS THAN	0.59246E 01			

CONSTRAINT	4			
C.00000E 00	0.00000E 00	0.20000E 01	0.30000E 01	
C.00000E 00				

GREATER THAN C.CCCCCOE CO

CONSTRAINT 5
C.OCCCCOE CO 0.0CCCCOE CO 0.00000E 00 0.00000E 00
C.20000E 01
LESS THAN 0.13000E 01

CONSTRAINT 6
C.OCCCCOE CO 0.0CCCCOE CO 0.00000E 00 0.00000E 00
C.20000E 01
GREATER THAN C.C0000E CO

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APPENDIX 7.6

List of FORTRAN Names for
Mathematical Symbols in the Second Program

	<u>Symbolic Representation</u> <u>(Rosen and Suzuki)</u>	<u>FORTRAN Name</u>
Number of variables	n	NV(<50)
Number of constraints -1	$k-1$	NCON
Number of optima		NØPT(<2)
Coefficients of quadratic terms of objective function	$\theta(\bar{x})$	THUT(50)
Coefficients of constraints $q_i(\bar{x})$:		
Coefficients of quadratic terms		QUE(50,50)
Coefficients of linear terms		DUE(50,50)
Coefficients of last constraint q_k		QK(2,50)
Optimal solutions to primal problems	$\bar{x}^0$	X(2,50)
Coefficients of partial derivative of $(\bar{x})$	$\frac{\delta \theta}{\delta x_i}$	D(2)
Coefficients of linear terms of objective function	$\bar{c}$	C(50)
Optimal solutions to dual problems	$\bar{u}^0$	UK(2)
Optimal values of objective function	$\Phi(\bar{x}^0)$	Z(2)
Constant terms of constraints	b_i	B(50)
Constant terms of last constraints	b_k	BK(2)

APPENDIX 7.8

Format for Data Input for the Second Program

First card: NV, NCØN, NØPT (3I2)

Second card: THUT(1), ..., THUT(NV) (8F10.5)

Third type of card: QUE(1,1), DUE(1,1), QUE(2,1), ..., DUE(NV,1),
QUE(1,2), DUE(1,2), QUE(2,2), ..., DUE(NV,2),
.
.
.
QUE(1, NCØN), DUE(1, NCØN), ..., DUE(NV, NCØN)
(8F10.5)

Fourth type of card: X(1, NØPT1), ..., X(1, NV) (8F10.5)

```

FLAG FORTRAN,MAIN,LIST,DECK
  DIMENSION THUT(50),QUE(50,50),DUE(50,50),QK(5,50),X(5,
    X50),B(50),
    IC(50),D(5),BK(5),UK(5),Z(5)
C  NCON IS THE NUMBER OF CONSTRAINTS MINUS ONE
C  NV IS THE NUMBER OF VARIABLES
C  NOPT IS THE DESIRED NUMBER OF OPTIMA IN THE COMBINED
C  PROBLEM
C  THUT(I) IS THE ARRAY OF COEFFICIENTS OF QUADRATIC TERMS
C  OF THE OB-
C  JECTIVE FUNCTION
10  READ 1000,NV,NCON,NOPT,((THUT(I),I=1,NV)
    NOPT1=NOPT+1
1000 FORMAT (3I2/(8F10.5))
C  THE ARRAY QUE CONTAINS COEFFICIENTS OF THE QUADRATIC
C  TERMS, ARRAY
C  DUE CONTAINS COEFFICIENTS OF LINEAR TERMS OF THE JTH
C  CONSTRAINT
DO 20 J=1,NCON
20  READ 1100,((QUE(I,J),DUE(I,J),I=1,NV)
1100 FORMAT (8F10.5)
DO 25 I=1,NV
DO 25 J=1,NOPT
25  QK(J,I)=0
DO 26 I=1,NOPT
26  QK(I,I)=-1.
  READ 1100,(X(1,I),I=NOPT1,NV)
  DO 30 I=NOPT1,NV
  DO 30 J=2,NOPT1
30  X(J,I)=X(1,I)
  DO 50 I=1,NV
  D(I)=-2.*THUT(I)
C  CHECK FOR D(I) GREATER THAN ZERO
  IF (D(I)) 40,40,50
40  PRINT 1200
1200 FORMAT (1H1,18HSOMETHING IS WRONG)
  GO TO 10
50  CONTINUE
C  SATISFY CONDITIONS THAT
C  X(I,I) LESS THAN X(J,I) FOR J NOT EQUAL I
  VAL=-1.
  DO 67 J=1,NOPT
  VAL=VAL+1.
  DO 67 I=1,NOPT
  IF (I-J) 64,65,66

```

```
64  X(I,J)=VAL+1.
    GO TO 67
65  X(I,J)=VAL
    BK(I)=VAL
    VAL=VAL+1.
    GO TO 67
66  X(I,J)=VAL
67  CONTINUE
C  COMPUTE THE COEFFICIENTS OF LINEAR TERMS IN THE OBJECTIVE
C  FUNCTION
C  COMPUTE U(K), THE ONLY NON-ZERO DUAL VARIABLE FOR EACH
C  PROBLEM
    DO 68 I=2,NOPT
      J=I-1
      C(J)=D(J)*X(I,J)
68  UK(J)=C(J)-D(J)*X(J,J)
      C(NOPT)=D(NOPT)*X(1,NOPT)
      UK(NOPT)=C(NOPT)-D(NOPT)*X(NOPT,NOPT)
      DO 70 I=NOPT1,NV
69  C(I)=D(I)*X(1,I)
C  COMPUTE THE OBJECTIVE FUNCTION AT THE OPTIMAL POINTS
    DO 200 J=1,NOPT
      JM1=J-1
      Z(J)=0
      DO 170 I=1,NV
170  Z(J)=Z(J)+(THUT(J)*X(J,I)+C(I))*X(J,I)
      IF (JM1) 200,200,180
180  DO 200 I=1,JM1
      IF (Z(J)-Z(I)) 200,190,200
190  X(J,J)=X(J,J)-.1
      BK(J)=X(J,J)
      GO TO 67
200  CONTINUE
      DO 71 I=1,NOPT
71  X(NOPT1,I)=X(I,I)
      DO 150 J=1,NCON
      B(J)=-1.E20
      DO 100 K=1,NOPT1
      BJ=0
      DO 80 I=1,NV
80  BJ=-(QUE(I,J)*X(K,I)+DUE(I,J))*X(K,I)+BJ
      IF (B(J)-BJ) 90,100,100
90  B(J)=BJ
100  CONTINUE
150  B(J)=B(J)+1.
```

```

      PRINT 1300,(I,THUT(I),C(I),I=1,NV)
1300 EORMAT (1H1,9X,HUT(I),C(I),I=1,NV)
      X48H OBJECTIVE FUNCTION IS SUM OVER I OF THUT(I)*X(I)
      120H**2+C(I)*X(I), WHERE//3X,1H1,6X,7HTHUT(I),13X,4HC(I
      X)//
      2U14,E16.5,E20.5))
      DO 210 J=1,NCON
      PRINT 1400,J,(QUE(I,J),I=1,NV)
1400 EORMAT (1H0,9X,11HCONSTRAINT ,I2/10X,
      X24H COEFFICIENTS OF SQUARED
      15H TERMS/(4E20.5))
      PRINT 1500,(DUE(I,J),I=1,NV)
1500 EFORMAT (10X,28H COEFFICIENTS OF LINEAR TERMS/(4E20.5))
      SAVE=-B(J)
210 PRINT 1600,SAVE
1600 FORMAT (10X,13H GREATER THAN ,E12.5)
      DO 220 J=1,NOPT
      PRINT 1700,J,(QK(J,I),I=1,NV)
1700 FORMAT (1H0,9X,28H LAST CONSTRAINT FOR PROBLEM ,I1/10X,
      128H COEFFICIENTS OF LINEAR TERMS/(4E20.5))
      SAVE=-BK(J)
220 PRINT 1900,SAVE,UK(J)
1900 EFORMAT (10X,9H EQUAL TO ,E12.5,33H,
      X CORRESPONDING DUAL VARIABLE IS
      1E12.5)
      PRINT 1950,NOPT
1950 FORMAT (1H0,9X,18H COLUMNS DESIGNATE ,I1,
      X22H OPTIMAL POINTS IN NV-
      15H SPACE)
      DO 230 I=1,NV
230 PRINT 2000,I,(X(J,I),J=1,NOPT)
2000 EFORMAT (I4,E16.5,4E20.5)
      PRINT 2100,(J,Z(J),J=1,NOPT)
2100 EFORMAT (1H0,4X,5(1HZ,I1,1H=,E12.5,5X))
      GO TO 10
      END

```


APPENDIX 7.10

Sample Problem

OBJECTIVE FUNCTION IS SUM OVER I OF $THUT(I)*X(I)**2+C(I)*X(I)$, WHERE

I	THUT(I)	C(I)
1	-0.10000E 01	0.20000E 01
2	-0.10000E 01	0.60000E 01
3	-0.20000E 01	0.40000E 01
4	-0.10000E 01	0.20000E 01

CONSTRAINT 1
 COEFFICIENTS OF SQUARED TERMS
 -0.10000E 01 -0.10000E 01 -0.10000E 01 -0.10000E 01
 COEFFICIENTS OF LINEAR TERMS
 0.10000E 01 -0.10000E 01 0.10000E 01 -0.10000E 01
 GREATER THAN -0.15000E 02

LAST CONSTRAINT FOR PROBLEM 1
 COEFFICIENTS OF LINEAR TERMS
 -0.10000E 01 0.00000E 00 0.00000E 00 0.00000E 00
 EQUAL TO -0.00000E 00, CORRESPONDING DUAL VARIABLE IS 0.20000E 01

LAST CONSTRAINT FOR PROBLEM 2
 COEFFICIENTS OF LINEAR TERMS
 0.00000E 00 -0.10000E 01 0.00000E 00 0.00000E 00
 EQUAL TO -0.19000E 01, CORRESPONDING DUAL VARIABLE IS 0.22000E 01

COLUMNS DESIGNATE 2 OPTIMAL POINTS IN NV-SPACE

1	0.00000E 00	0.10000E 01
2	0.30000E 01	0.19000E 01
3	0.10000E 01	0.10000E 01
4	0.10000E 01	0.10000E 01

Z1= 0.13000E 02 Z2= 0.12790E 02 Z

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STOP.

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APPENDIX 7.11

Rosen and Suzuki

TECHNIQUES

The Techniques Department is interested in publishing short descriptions of Techniques which improve the logistics of information processing. To quote from the policy statement, *Communications of the ACM* 1 (Jan. 1958), 5: "It is preferable that techniques contributed be factual and in successful usage, rather than speculative or theoretical. One of the major criteria for acceptance and the question one should answer before submitting any material is—Can the reader use this tomorrow?" Clear, concise statements of fairly well-known but rarely documented methods will contribute significantly to raising the general level of professional competence.—C.L.McC.

CONSTRUCTION OF NONLINEAR PROGRAMMING TEST PROBLEMS

In order to test a nonlinear programming algorithm it is very useful to be able to construct test problems with known optimum solutions. The purpose of this note is to describe a simple procedure for constructing such test problems. A concave maximization problem subject to concave constraints is used as an example.

The concave maximization problem is

$$\max_x \{\phi(x) \mid h_i(x) \geq 0, \quad i = 1, 2, \dots, k\},$$

where $x \in E^m$, and $\phi(x)$ and $h_i(x)$ are real-valued concave functions of x . The procedure will be described for $\phi(x) = \theta(x) + c'x$, and $h_i(x) = q_i(x) + b_i$, $i = 1, \dots, k$, where $\theta(x)$ and $q_i(x)$, $i = 1, \dots, k$ are any selected differentiable concave functions of x , c is a vector in E^m and the b_i are scalars.

Step I. Choose any $x^0 \in E^m$ as a desired optimum point, and any set of $u_i^0 \geq 0$, $i = 1, \dots, k$, as the corresponding optimum dual solution. That is, we first specify the primal and dual solution to the problem.

Step II. Choose b_i , $i = 1, \dots, k$, so that $h_i(x^0) = 0$ for $u_i^0 > 0$ and $h_i(x^0) \geq 0$ for $u_i^0 = 0$. Note that $u_i^0 > 0$ means that the i th constraint is active.

Step III. Let

$$c = -\nabla\theta(x^0) - \sum_{i=1}^k u_i^0 \nabla q_i(x^0).$$

This choice satisfies the Kuhn-Tucker condition

$$\nabla\phi(x^0) + \sum_{i=1}^k u_i^0 \nabla h_i(x^0) = 0,$$

and therefore ensures that x^0 is an optimum solution to the concave programming problem.

We illustrate this procedure below by applying it to the quadratic problem where $\theta(x) = x'Q_0x$, $q_i(x) = x'Q_i x + a_i'x$, and the Q_i , $i = 0, 1, \dots, k$, are negative semidefinite matrices.

Example (quadratic problem with four variables and three constraints). Let

$$Q_0 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

Prepared under NASA grant Ns G 565 at Stanford University.

$$Q_2 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}, \quad Q_3 = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$a_1 = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \quad a_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad a_3 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

(Step I)

Let

$$x^0 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ -1 \end{bmatrix}$$

and

$$u^0 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

(Step II)

$$h_1(x^0) = x^0'Q_1x^0 + a_1'x^0 + b_1 = -8 + b_1$$

Since $u_1 > 0$, $b_1 = 8$.

$$h_2(x^0) = x^0'Q_2x^0 + a_2'x^0 + b_2 = -9 + b_2$$

since $u_2 = 0$, we choose $b_2 = 10$ so that $h_2(x^0) = 1 > 0$.

$$h_3(x^0) = x^0'Q_3x^0 + a_3'x^0 + b_3 = -5 + b_3$$

Since $u_3 > 0$, $b_3 = 5$.

(Step III)

$$c = \begin{bmatrix} 2x_1^0 \\ 2x_2^0 \\ 4x_3^0 \\ 2x_4^0 \end{bmatrix} + \begin{bmatrix} 2x_1^0 + 1 \\ 2x_2^0 - 1 \\ 2x_3^0 + 1 \\ 2x_4^0 - 1 \end{bmatrix} + 2 \begin{bmatrix} 4x_1^0 + 2 \\ 2x_2^0 - 1 \\ 2x_3^0 \\ 0 \quad -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 21 \\ -7 \end{bmatrix}.$$

The constructed problem is:

minimize

$$\phi = -x_1^2 - x_2^2 - 2x_3^2 - x_4^2 + 5x_1 + 5x_2 + 21x_3 - 7x_4$$

subject to

$$\begin{aligned} -x_1^2 - x_2^2 - x_3^2 - x_4^2 - x_1 &+ x_2 - x_3 + x_4 + 8 \geq 0 \\ -x_1^2 - 2x_2^2 - x_3^2 - 2x_4^2 + x_1 &+ x_4 + 10 \geq 0 \\ -2x_1^2 - x_2^2 - x_3^2 &- 2x_1 + x_2 + x_4 + 5 \geq 0 \end{aligned}$$

and has as its optimum function value $\phi(x^0) = 44$.

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