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S0USA: A Root Finder for Nonlinear Equations

Keywords and Phrases: root finder, nonlinear equation solver,
bisection method, roots, zeros.

by

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SQUSA: A Root Finder for Nonlinear Equations

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ABSTRACT

A marching search and bisection procedure for solving under rather general conditions, the nonlinear equation $f(x)=0$ is presented. The algorithm design and operating characteristics are discussed. Results of applying this procedure to a set of test functions are given. Finally the Fortran coding of the procedure is included.

Keywords and Phrases: root finder, nonlinear equation solver, bisection method, roots, zeros.

INTRODUCTION

A great amount of scientific computing activity deals with solving equations, i. e. finding roots of functions. Procedures abound for solving equations. No single method is universally optimal. Here optimality must be defined in such terms as function complexity, number of times the solution must be obtained, deadlines, and computing costs. There seems to be a time and place for each method. Rice[1] has combined several methods into one procedure in an attempt to obtain a good practical root finding package. Here we present a rather simple procedure which can be used with little effort when conditions do not justify a more sophisticated procedure.

The marching search and bisection procedure is a relatively foolproof process for locating the real zeros of functions. By properly constructing the algorithm one may also use it to isolate extrema, or odd order poles of a function under rather general conditions. It can also be used to solve systems of equations. This will be discussed more below.

One might question the use of such a brute force attack when elegant procedures such as Newton's method or its relatives are available. To answer this question first note that the program described herein is not a panacea, for example one would normally not use it to solve one linear equation or even to extract square roots. Following is a list of reasons and circumstances where this procedure is applicable.

1. The function may be too difficult to differentiate.
2. The function is quite flat i.e. the derivative is nearly zero or if a system of functions is being considered the Jacobian is nearly singular.
3. The job at hand is a "one-shot" application and the time (cost) does not permit an in-depth analysis of the functions.
4. The user may be relatively ignorant of numerical methods and has no desire to devote the time and energy to learning them.
5. This procedure can be used to drive the error to the computational limits (noise level) where other methods fail.

Two highly desirable characteristics for any root finding algorithm are also present in SQUA. These are

1. An error bound can be readily obtained.
2. Roots of arbitrary order present no difficulties.

The algorithm given here is not without its shortcomings however. Most have been identified and should the user encounter such he will be dually notified. The identified pitfalls are discussed at length below.

Once a root has been bracketed the algorithm becomes a simple bisection process which is known to have a convergence factor of 1/2. That is the interval containing the root is halved with each step.

APPLICATION TO A SYSTEM OF EQUATIONS

The algorithm has been implemented as a FORTRAN subroutine named S0USA which is listed in Appendix B. The reverse communication technique has been employed to aid the user. By including two or three simple statements the user may have a root finder in place. Explicit details for calling this subprogram will be discussed later. For now consider the solution of a system of equations without much of this detail. Figure 1 indicates how the two nonlinear equations

$$f(x,y) = y-x^2+y^2$$

$$g(x,y) = x-x^2-y^2$$

can be solved using S0USA. The extension to higher dimensioned systems is straight forward.

This block has been simplified somewhat since KEY carries other information which can be useful and which has not been needed here. The first two blocks show the initialization essentials. Once the starting point (x,y); step sizes $\Delta x, \Delta y$; convergence criteria $\epsilon x, \epsilon y$; and logic switches KEYX, KEYY are given the solution loops follow. First the equation $f(x,y) = 0$ is solved for x while keeping y fixed. When x has been refined so as to pass the convergence criteria (This test is made in S0USA where KEYX is set to 4 if the test is not passed.), the solution moves on

Figure 1. Flow Chart Showing the Use of S0USA to Solve Two Equations in Two Unknowns.

to the second equation $g(x,y) = 0$. Then while keeping x fixed, y is determined in a manner similar to that used for x . If the new point (x,y) is sufficiently close to the starting point (x^*,y^*) the solution is assumed to be (x,y) . Otherwise the point (x,y) becomes the new starting point, the convergence criteria are sharpened if necessary, and the procedure repeated.

This procedure was programmed. Taking $\epsilon_0 = 10^{-3}$, $h = 0.5$, $\epsilon = 10^{-7}$ and starting from $(x_0, y_0) = (0.8, 0.4)$ the nontrivial solution $(0.77184452, 0.41964338)$ was obtained. Initially about 22 evaluations for each function was needed to solve the individual equations. At the end of the process only 4 evaluations were required for each equation. In all a total of 673 and 682 evaluations of f and g respectively were required.

DISCUSSION

Now let us turn to additional information provided by the subroutine SOUSA. What other information can be obtained? Figures 2 and 3 are representative of the possible situations which might be encountered during a search of a function for zero. Figure 2a denotes the class of functions which have zeros in an interval, the slope might be negative, and/or curvature negative etc. Similarly d. might well be the negative of the function shown.

FIGURE 2. Potential Function Shapes Which Stop the Search

The procedure is stopped when any one of the shapes shown in Figure 2 is encountered. Case a. represents the class of primary target functions for which the routine was designed. The other cases might be considered nuisances, however, S9USA will yield the locations of extrema or odd order poles too.

Figure 3 depicts situations where the procedure output is of questionable value.

FIGURE 3. Function Shapes Which Cause the Procedure to Fail

Figure 3a has no zero. The program given here permits one to double the step size at regular intervals thereby covering the range more quickly. Figure 3b has zeros but the step size might not permit detection of them. If this occurs the algorithm believes it has case 3a again. For example, suppose $f(x) = \sin x$ and $\Delta x = 2\pi$. Unless round-off and truncation error save the march it will be fruitless! Case 3b will generally yield one of the zeros after suitably changing the step size. The other zeros are not located.

Case 3c is seen by the algorithm as a special case of 3a. That is, the march simply moves away from the pole in a direction depending upon the starting point and step sign and magnitude.

The algorithm programmed here correctly identifies each case in Figure 2. Those in Figure 3 are identified as "not successful". The user may then wish to take further action. For example, a different starting point and/or step size might be successful in case 3b. Cases 3a and 3c have no zeros so obviously the user cannot expect to find one.

During the marching search the general strategy is to move toward the origin if possible. The decisions underlying this strategy are tabulated in Table 1. More information is required than shown in Table 1 to properly implement the search. For example, conditions 1 and 2 can be repeated cyclicly to produce a three step loop. Similarly, conditions 5 and 6 might produce a loop. When these conditions occur, preventative action is called for. If two consecutive sign changes of Δx occur, then Δx is halved and the search continued. This strategy leads to the determination of local extrema such as shown in Figure 2b and 2c.

Once the root has been bracketed the strategy is determined from Table 2. At each step a and b are the end points of the interval containing the root giving an estimate of the error. The eight entries of Table 2 are readily implemented in Fortran by an assigned G0 T0 statement with index constructed as follows. Assign a binary digit to each of columns 2, 3 and 4. Associate column 2 with the 4 digit, column 3 with the 2 digit and column 4 with the 1 digit. Now set each digit 1 if the corresponding key is positive and 0 if the key is negative. The resulting integer is increased by 1 to obtain the G0 T0 index ranging between 1 and 8 inclusively.

TABLE 1. Decision Table Used for Marching Search

Condition No.	$F(a)$	$F(b)$	$F(a)-F(b)$	Action
1	<	<	$\leq$	$\left\{ \begin{array}{l} a \leftarrow b \\ F(a) \leftarrow F(b) \\ \Delta x \leftarrow -\Delta x \end{array} \right\} \quad b \leftarrow a + \Delta x$
2	<	<	>	
3	<	>	---	Bracketing occurs
4	>	<	---	Bracketing occurs
5	>	>	<	$\Delta x \leftarrow -\Delta x$ $\left\{ \begin{array}{l} a \leftarrow b \\ F(a) \leftarrow F(b) \end{array} \right\} \quad b \leftarrow a + \Delta x$
6	>	>	$\geq$	

TABLE 2. Decision Table When Root Lies Between a and b

Condition No.	$(f(b)-f(a)) \cdot \Delta x$	Δx	$f(y)$	Action
1	<	<	<	$\left. \begin{array}{l} a \leftarrow y \\ b \leftarrow y \\ b \leftarrow y \\ a \leftarrow y \\ b \leftarrow y \\ a \leftarrow y \\ a \leftarrow y \\ b \leftarrow y \end{array} \right\} \quad \begin{array}{l} \text{Followed by} \\ \Delta x \leftarrow \Delta x/2 \\ y \leftarrow a + \Delta x \end{array}$
2	<	<	>	
3	<	>	<	
4	<	>	>	
5	>	<	<	
6	>	<	>	
7	>	>	<	
8	>	>	>	

In contrast to Table 1 there is only one place where failure may occur.

Namely, Δx can be reduced to the point that when added to a the result is unchanged. This is caused by the granularity of the number system used in digital computing. The only solution seems to be either accept the quantity x as the solution or use it as a starting point in a double precision pursuit of the root.

PROGRAM USE

The subroutine described here is called by the following FORTRAN statement:

CALL S0USA (X, FX, DX, KEY, MAXDBL, IMAX, EPSLN, FP, IP) where at the entry to S0USA the arguments are given in the

Entry Table: X is the approximate value of the sought root (or other characteristic) of the function $f(x)$.

FX is the value of $f(x)$ at the point x .

DX is the initial step size Δx .

KEY is the type indicator. KEY must be less than or equal to -3 initially. It is reset by S0USA and on output indicates the function's behavior according to the Return Table below.

MAXDBL is the number of consecutive steps permitted in one direction before the step size is doubled. If $\text{MAXDBL} \leq 0$ then it is replaced by 5.

IMAX is the maximum of steps to be permitted during the search.

EPSLN is the convergence criterion. The search is stopped when

(a) $f(x) = 0$ or

(b) $x \neq 0$ and $|\Delta x/x| \leq \text{EPSLN}$ or

(c) $x = 0$ and $\Delta x = 0$ or

(d) $f(x) \neq 0$ and $|1./f(x)| \leq \text{EPSLN}$.

See KEY in the Return Table below for the type of point x is when the search is terminated.

FP is a REAL array dimensioned greater than or equal to 11.

FP is used for intermediate storage. Some of the values stored in FP are discussed in the Return Table.

IP is an INTEGER array dimensioned greater than or equal to 5.

IP is used for intermediate storage. Some of the values of

IP are discussed below in the Return Table

At the time of returning from S0USA to the calling program the argument in the call list are given in the

Return Table:

X is the new value of X to be used to evaluate $f(x)$ for the next step if the search is continuing. Otherwise X is the sought root.

Key is -2 if f has an odd order pole very near X

-1 if $f(x) < 0$ and $f(x)$ is a relative maximum

0 $f(x) = 0$

1 if $f(x) > 0$ and $f(x)$ is a relative minimum

2 the step size has been reduced to the point that when added to x the value of x remain unchanged.

3 if none of the above conditions hold and IMAX steps have been taken.

FP(1) is the current step size

FP(2) } are end points of the interval containing the

FP(3) } sought value of x.

Other call list arguments are used by the subprogram but are of little use to the user. The user is free to interrogate the two arrays FP and IP but he is warned against modifying the contents. To do so could lead to confusion and ultimately to disaster for the logic behind the search.

An estimate of the error involved in x is given by the following inequality

$$\min[FP(2), FP(3)] \leq x \leq \max[FP(2), FP(3)].$$

Appendix A contains a brief performance profile for S0USA. Test functions 1 through 11 are from Rice[2]. Examination of the results shown in this profile show that S0USA may not correctly identify those roots where the function does not change sign in a small neighborhood of the root. For example the root at -157.2 in Function 4 is positive near -157.2. Using $\epsilon = 10^{-7}$ S0USA thought it had located a minimum, see Table A-2. However taking $\epsilon = 0$ produced the desired results as shown in Table A-3.

REFERENCES

1. John R. Rice, "A polyalgorithm for the automatic solution of nonlinear equations," Proc. of 24th National Conference ACM Publication P-69, (1969), pp. 179-183.
2. John R. Rice, "A set of 74 Test Functions for Nonlinear Equation Solvers," Purdue University Computer Science Department Document No. CSD TR 34, April 1969.

Appendix A

Table A-1. Test Functions

$$1. f(x) = \cos(x) - xe^x$$

$$2. f(x) = (y + 16) \log_{10}(1 + y^2) \sqrt{|y - .8|}$$

$$y = x - 1312$$

$$3. f(x) = (x - 17)^2 |x - 17.1|^{1.8} (x - 20)$$

$$4. f(x) = |x + 157.2|^{1.5} |x - 361.2|^7 (x - 10^{-6}) / |x - 10^{-11}|$$

$$5. f(x) = (1 + x^2) \operatorname{sgn}(\sin(x)) \quad |x| < 10$$

$$6. f(x) = 81 - y(108 - y(54 - y(12 - y))) = (3 - y)^4$$

$$y = x - 1.11111$$

$$7. f(x) = \begin{cases} 1 + x^4 & |x| < 10^8 \\ 1.222 \times 10^9 - |x| & |x| \geq 10^8 \end{cases}$$

$$8. f(x) = 20/y^{15} + 36/y^{25} + 40/y^{35} + (475 + 1.12/x)/y^{40}$$

$$-1.12/x - 4.5 - 6/y^4 - 3/y^8$$

$$y = 1 + x$$

$$9. f(x) = x - m - e \sin(x) \quad (a) \quad m = .8, e = .2$$

$$(b) \quad m = .8, e = 15.$$

$$10. \begin{cases} f(x,y) = y - x^2 + y^2 \\ g(x,y) = x - x^2 - y^2 \end{cases}$$

Tables A-2 and A-3 were obtained using 48-bit mantissas for floating point numbers. The step size was doubled after every 5 consecutive steps made in the same direction. Table A-2 is associated with $\epsilon = 10^{-7}$ which demands approximately half the precision available. Table A-3 has $\epsilon = 0$ which demands full precision for convergence.

Table A-2. Test Function Results with $\epsilon = 10^{-7}$

Function No.	x_0	Solution x	No. of Evaluations	Point Type Identification
1	0.5	0.51773737	21	root
	1.5	0.51773736	30	root
2	1310	1296.0001	22	root
	1340	1295.9999	22	root
	1295	1296.0000	21	root
3	15	17.099998	43	max
	18	17.099999	44	max
	22	20.000001	23	root
4	-150	-157.19998	45	max
	150	1.0000000×10^{-6}	58	root
	350	361.19996	45	root
5	3	3.1415929	21	root
	-3	-3.1415929	21	root
6	1	1.8887695	26	root
	10	1.8895874	36	root
7	-10^{10}	1.2220000×10^9	32	root
	100	0	203	min
	10^{10}	1.2220000×10^9	32	root
8	-5	-4.3196499	54	max
	-2	-2.0755178	21	root
	-1.5	-2.0755178	24	root
	-1.1	-2.0755177	28	root
	-0.5	0.097993976	34	root
	0.5	3.2985349×10^{11}	203	max
9(a)	.8	0.96433382	22	root
(b)	.8	-0.057176229	32	root
10 { x,f y,g	.8	0.77184452	673	root
	.4	0.41964338	682	root

Table A-3. Test Function Results with $\epsilon = 0$.

<u>Function No.</u>	<u>x_0</u>	<u>Solution x</u>	<u>No. of Evaluations</u>	<u>Identification</u>
3	15	17.100000	93	root
	18	17.100000	88	root
4	-150	157.20000	90	root
	350	361.20000	97	root

```

SUBROUTINE SOUSA(X,FX,DX,KEY,MAXDBL,IMAX,EPSLN,FP,IP)
DIMENSION FP(11),IP(5)
C----- THIS ROUTINE LOCATES THE POINT Y AT WHICH F IS AS
C CLOSE TO ZERO AS POSSIBLE, OR WHERE F HAS AN ODD-ORDER
C POLE. A MARCHING TECHNIQUE IS USED TO BRACKET Y.
C ONCE Y HAS BEEN BRACKETED, A BISECTION SEARCH IS
C CONDUCTED FOR ITS LOCATION.
C
C----- INPUT
C X - APPROXIMATE VALUE OF Y.
C FX - THE FUNCTION VALUE AT X
C DX - INITIAL STEP SIZE
C KEY - TYPE INDICATOR FOR Y.
C MUST BE .LE.-3 FOR INITIAL CALL
C MAXDBL - MAXIMUM NUMBER OF CONSECUTIVE STEPS IN ONE
C DIRECTION PERMITTED BEFORE STEP SIZE IS
C DOUBLED. IF MAXDBL .LE. 0 THEN IT IS
C REPLACED WITH 5.
C IMAX - MAXIMUM NUMBER OF STEPS TO BE TAKEN DURING
C THE SEARCH
C EPSLN - CONVERGENCE CRITERION
C SOLUTION IS ASSUMED WHEN
C IF Y .NE. 0 THEN ABS(DX/Y) .LE. EPSLN
C IF Y .EQ. 0 THEN DX .EQ. 0
C A POLE IS ASSUMED WHEN F(Y) .NE. 0 AND
C ABS(1./F(Y)) .LE. EPSLN
C FP - FLOATING POINT ARRAY USED FOR INTERMEDIATE
C STORAGE. MUST BE DIMENSIONED .GE. 11
C IP - INTEGER ARRAY USED FOR INTERMEDIATE STORAGE.
C MUST BE DIMENSIONED .GE. 5.
C
C **CAUTION** FP AND IP MUST NOT BE MODIFIED BY
C THE CALLING PROGRAM DURING THE SEARCH.
C
C----- OUTPUT
C X - SOLUTION VALUE FOR Y
C KEY - TYPE INDICATOR FOR Y.
C -2 IF F HAS AN ODD-ORDER POLE VERY CLOSE TO Y
C -1 IF F(Y) .LT. 0 AND F(Y) IS A RELATIVE
C MAXIMUM.
C 0 IF F(Y) .EQ. 0
C 1 IF F(Y) .GT. 0 AND F(Y) IS A RELATIVE
C MINIMUM.
C 2 IF DX IS SO SMALL THAT X .EQ. X+DX
C BUT THE CONVERGENCE CRITERION IS NOT
C SATISFIED.
C 3 IF NONE OF THE ABOVE ARE TRUE, AND IMAX
C ITERATIONS HAVE BEEN MADE

```

SOU0010
 SOU0020
 SOU0030
 SOU0040
 SOU0050
 SOU0060
 SOU0070
 SOU0080
 SOU0090
 SOU0100
 SOU0110
 SOU0120
 SOU0130
 SOU0140
 SOU0150
 SOU0160
 SOU0170
 SOU0180
 SOU0190
 SOU0200
 SOU0210
 SOU0220
 SOU0230
 SOU0240
 SOU0250
 SOU0260
 SOU0270
 SOU0280
 SOU0290
 SOU0300
 SOU0310
 SOU0320
 SOU0330
 SOU0340
 SOU0350
 SOU0360
 SOU0370
 SOU0380
 SOU0390
 SOU0400
 SOU0410
 SOU0420
 SOU0430
 SOU0440
 SOU0450
 SOU0460
 SOU0470
 SOU0480
 SOU0490

C	4 IF SEARCH IS STILL IN PROGRESS	SOU0500
C	IP(1) - NUMBER OF STEPS TAKEN	SOU0510
C	FP(1) - CURRENT STEP SIZE DX	SOU0520
C	FP(2) - END POINTS OF INTERVAL CONTAINING Y.	SOU0530
C	FP(3) - Y IS IN (MIN(FP(2),FP(3)),MAX(FP(2),FP(3)))	SOU0540
C		SOU0550
C		SOU0560
C	---- USAGE * AN EXAMPLE APPLYING SOUSA TO SOLVE THE	SOU0570
C	EQUATION F(X)=0. WHERE F IS A FUNCTION	SOU0580
C	SUBPROGRAM MIGHT BE CODED AS	SOU0590
C		SOU0600
C	X=X0	SOU0610
C	KEY=-3	SOU0620
C	XXX FX=F(X)	SOU0630
C	CALL SOUSA(X,FX,DX,KEY,MAXDBL,IMAX,EPSLN,FP,IP)	SOU0640
C	IF(KEY.EQ. 4) GO TO XXX	SOU0650
C	IF(KEY.EQ.-2) F HAS AN ODD-ORDER POLE AT X	SOU0660
C	IF(KEY.EQ.-1) F HAS A RELATIVE MAXIMUM AT X	SOU0670
C	IF(KEY.EQ.-1) F HAS A RELATIVE MAXIMUM AT X	SOU0680
C	IF(KEY.EQ. 0) THE SOLUTION IS AT X	SOU0690
C	IF(KEY.EQ. 1) F HAS A RELATIVE MINIMUM AT X	SOU0700
C	IF(KEY.EQ. 2) STEP SIZE REDUCED TO THE POINT THAT X	SOU0710
C	DOES NOT CHANGE	SOU0720
C	IF(KEY.EQ. 3) IMAX STEPS HAVE NOT PRODUCED A SOLUTION.	SOU0730
C		SOU0740
C	XXX=X	SOU0750
C	Y=X	SOU0760
C	F=FX	SOU0770
C	KKK=4	SOU0780
C	JMX=MAXDBL	SOU0790
C	IF(JMX.LE.0) JMX=5	SOU0800
C	ICYCL=0	SOU0810
C	IF(KEY.GT.-3) GO TO 1	SOU0820
C	DEL=DX	SOU0830
C	I=0	SOU0840
C	J=0	SOU0850
C	K=0	SOU0860
C	FP(6)=Y	SOU0870
C	FP(7)=Y	SOU0880
C	FP(9)=F	SOU0890
C	FP(10)=F	SOU0900
C	A=Y	SOU0910
C	FA=F	SOU0920
C	GO TO 2	SOU0930
C	1 DFL =FP(1)	SOU0940
C	A =FP(2)	SOU0950
C	R =FP(3)	SOU0960
C	FA =FP(4)	SOU0970
C	FR =FP(5)	SOU0980

```

      I  =IP(1)
      J  =IP(2)
      K  =IP(3)
      KASE=IP(4)
      JSW =IP(5)
      IF(JSW) 3,17,24
C
C----- DEFINE LIMITS OF INTERVAL CONTAINING ZERO.
      2 R=A+DEL
      Y=B
      JSW=-1
      IF(ICYCL-2) 28,31,31
      3 FB=F
      IF(FA) 5,4,12
      4 Y=A
      GO TO 27
      5 IF(FB) 7,6,14
      6 Y=B
      GO TO 27
      7 IF(FA-FB) 8,9,10
      8 FA=FB
      9 A=B
C----- COUNT STEPS IN LONG MARCHES
      J=J+1
      IF(J.NE.JMX) GO TO 11
C----- DOUBLE STEP SIZE IF BRACKETING DOES NOT OCCUR SOON
      J=0
      DEL=2.*DEL
      GO TO 11
C----- ABOUT FACE
      10 DEL=-DEL
      J=0
C----- MAKE A COUNT OF THE NUMBER OF ABOUT FACES. A LOOP
C      IS POSSIBLE IF F HAS A LOCAL MAXIMUM OR MINIMUM WITH
C      NO ROOT.
      K=K+1
      IF(K.GE.2) GO TO 33
C----- COUNT TOTAL NUMBER OF STEPS TAKEN
      11 IF(ICYCL.GT.0) GO TO 2
      I=I+1
      IF(I-IMAX)2,2,35
C
      12 IF(FB) 14,6,13
      13 IF(FA-FB) 10,9,8
C
C----- BRACKETING ACCOMPLISHED
      14 KASE=0
      IF(((FB-FA)*DEL).GT.0.) KASE=2
      IF(DEL.GT.0.) KASE=KASE+1

```

```

SOU0990
SOU1000
SOU1010
SOU1020
SOU1030
SOU1040
SOU1050
SOU1060
SOU1070
SOU1080
SOU1090
SOU1100
SOU1110
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SOU1450
SOU1460
SOU1470

```

```

      KASE=2*KASE+1
      JSW=0
      GO TO 22
C
C----- CONVERGING MARCH BEGAN
      17 KAS=KASE
         IF(F) 19,27,18
      18 KAS=KAS+1
      19 GO TO (20,21,21,20,21,20,20,21),KAS
      20 A=Y
         GO TO 22
      21 B=Y
C----- DOUBLE TIME
      22 DEL=DEL/2.
         Y=A+DEL
         IF(Y.EQ.A ) GO TO 39
         IF(Y.EQ.0.) GO TO 23
         IF(ABS(DEL/Y)-EPSLN) 27,27,31
C
C----- IF Y=0. WE REQUIRE DEL TO BE ZERO FOR A ROOT TO BE
C ASSUMED THERE.
      23 IF(DEL) 28,27,28
      24 IF(F) 25,27,26
      25 KKK=-1
         GO TO 32
      26 KKK=1
         GO TO 32
      27 KKK=0
         IF(F.EQ.0.) GO TO 32
C----- A POLE MAY HAVE BEEN FOUND
         IF(ABS(1./F).LE.EPSLN) KKK=-2
         GO TO 32
C
      28 FP(11)=FP(10)
         FP(10)=FP( 9)
         FP( 9)=F
         FP( 8)=FP( 7)
         FP( 7)=FP( 6)
         FP( 6)=XXX
         ABSE=ABS(DEL/4.)
         IF(ABS(Y-FP(8)).LE.ABSE) GO TO 29
         IF(ABS(Y-FP(7)).GT.ABSE) GO TO 31
         F=FP(10)
         GO TO 30
      29 F=FP(11)
      30 XXX=Y
C----- COUNT REPEATED POINTS
         ICYCL=ICYCL+1
         IF(JSW) 3,17,24

```

```

SOU1480
SOU1490
SOU1500
SOU1510
SOU1520
SOU1530
SOU1540
SOU1550
SOU1560
SOU1570
SOU1580
SOU1590
SOU1600
SOU1610
SOU1620
SOU1630
SOU1640
SOU1650
SOU1660
SOU1670
SOU1680
SOU1690
SOU1700
SOU1710
SOU1720
SOU1730
SOU1740
SOU1750
SOU1760
SOU1770
SOU1780
SOU1790
SOU1800
SOU1810
SOU1820
SOU1830
SOU1840
SOU1850
SOU1860
SOU1870
SOU1880
SOU1890
SOU1900
SOU1910
SOU1920
SOU1930
SOU1940
SOU1950

```

```

1  IP(1)=I
   IP(2)=J
   IP(3)=K
   IP(4)=KASE
   IP(5)=JSW
   FP(1)=DEL
   FP(2)=A
   FP(3)=B
   FP(4)=FA
   FP(5)=FB
32 X=Y
   KEY=KKK
   RETURN
C
33 K=0
   DEL=DEL/2.
   AP=AMAX1(ABS(A),ABS(B))
   IF(AB.EQ.0.) GO TO 11
   IF(ABS(DEL/AB)-EPSLN) 34,34,11
C---- LOCAL EXTREME FOUND
34 Y=(A+B)/2.
   JSW=1
   GO TO 28
35 IF(ABS(FA)-ABS(FB)) 36,34,37
36 Y=A
   GO TO 38
37 Y=B
38 KKK=3
   GO TO 32
39 KKK=2
   GO TO 32
   END

```

```

SOU1970
SOU1980
SOU2990
SOU2000
SOU2010
SOU2020
SOU2030
SOU2040
SOU2050
SOU2060
SOU2070
SOU2080
SOU2090
SOU2100
SOU2110
SOU2120
SOU2130
SOU2140
SOU2150
SOU2160
SOU2170
SOU2180
SOU2190
SOU2200
SOU2210
SOU2220
SOU2230
SOU2240
SOU2250
SOU2260
SOU2270
SOU2280-

```