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(A-7.390.31) AEC D-3874

CLINTON ENGINEER WORKS
CARBIDE AND CARBON CHEMICALS CORPORATION
Y-12 PLANT

Oak Ridge, Tennessee
August 25, 1947

AEC D-3874

CRITICAL DIMENSIONS OF UNTAMPED CONICAL VESSELS

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Photostat Price \$ 1.80

Microfilm Price \$ 1.80

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CRITICAL DIMENSIONS OF UNTAMPED CONICAL VESSELS

Abstract

The need often arises for determining the critical chemical concentration of uranium solution in the conical bottom of a plant reactor or storage vessel, or the dimension if the concentration is known. This report describes the mathematical analysis of Poisson's equation for a spherical sector, which approximates a right circular cone. The ratio of the critical dimension of an equivalent sphere to the height of the sector for various sector angles is derived from a comparison of first eigenvalues. No description of further relations between composition and dimensions is discussed in the report.

Analysis

The "pile equation" of thermal neutron diffusion theory is, for steady state,

$$\nabla^2 n + k^2 n = 0$$

a form of Poisson's equation appearing frequently in heat flow problems. Here k is a geometry parameter that must be equated to a function of the composition of the fissionable solution. The value of k is to be found in terms of the boundary conditions for neutron density in an untamped spherical sector, i.e.,

$$n = 0 \text{ at } r = 0, r = a, \theta = \theta_1$$

Since neutron density is independent of the polar angle, only the radial and azimuthal portion of the laplacian is needed:

$$\nabla^2 = \frac{1}{r^2 \sin \theta} \left\{ \sin \theta \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} \right\}$$

Substitute in Poisson's equation and rearrange to obtain

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial n}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 n}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial n}{\partial \theta} + k^2 n = 0$$

The usual method of separation of variables is applied, by assuming that

$$n = R \Theta$$

thus

$$\frac{1}{R} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) + k^2 r^2 = \frac{-\frac{d^2 \Theta}{d\theta^2} \cot \theta \frac{d\Theta}{d\theta}}{\Theta^2}$$

65-2

Set each side equal to a constant, $l(l+1)$, to obtain two linear differential equations of second order, each involving one variable only.

$$\frac{R''}{R} + \frac{2}{r} \frac{R'}{R} + k^2 = \frac{l(l+1)}{r^2} \quad (1)$$

$$\frac{\Theta''}{\Theta} + \cot \theta \frac{\Theta'}{\Theta} = -l(l+1) \quad (2)$$

where primes refer to differentiation with respect to the appropriate variable.

Equation (1) may be put in the form of the familiar Bessel equation by the substitutions $x = kr$ and $R = R/\sqrt{x}$, thus

$$\frac{d}{dr} = k \frac{d}{dx} \quad \text{and} \quad \frac{d^2}{dr^2} = k^2 \frac{d^2}{dx^2}$$

Differentiation leads immediately to

$$f'' + \frac{f'}{x} + \left[1 - \frac{1}{4x^2} - \frac{l(l+1)}{x^2}\right] f = 0$$

or, since $l(l+1) + \frac{1}{4} = (l + \frac{1}{2})^2$

$$f'' + \frac{f'}{x} + \left[1 - \frac{(l + \frac{1}{2})^2}{x^2}\right] f = 0$$

Its solution is the half-odd order Bessel function

$$f = J_{l + \frac{1}{2}}(x)$$

thus
$$R \sim \frac{J_{l + \frac{1}{2}}(x)}{\sqrt{x}} = \frac{J_{l + \frac{1}{2}}(kr)}{\sqrt{kr}}$$

Equation (2) may be put in the form of the familiar Legendre equation by the substitutions $x = \cos \theta$ and $\Theta = y$. Thus

$$\frac{d}{d\theta} = \frac{d}{d(\cos \theta)} \frac{d(\cos \theta)}{d\theta} = -\sin \theta \frac{d}{d(\cos \theta)} = -\sqrt{1-x^2} \frac{d}{dx}$$

and
$$\frac{d^2}{d\theta^2} = -\sqrt{1-x^2} \frac{d}{dx} \left(-\sqrt{1-x^2} \frac{d}{dx} \right) = (1-x^2) \frac{d^2}{dx^2} - x \frac{d}{dx}$$

* For analogous derivations, see Margenau and Murphy, "The Mathematics of Physics and Chemistry".

Differentiation leads to

$$(1 - x^2) \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + \ell(\ell + 1)y = 0$$

Its solution is the Legendre function

$$y = P_\ell(x) = P_\ell(\cos \theta) = \textcircled{H}$$

The neutron density, except for an arbitrary multiplying constant, is then

$$n = R \textcircled{H} = \frac{J_{\ell + \frac{1}{2}}(kr)}{\sqrt{kr}} P_\ell(\cos \theta)$$

Application

The physical significance of this solution is that the variation of density in the radial direction is independent of that in the angular sense. To satisfy the boundary condition $n = 0$ at $\theta = \theta_1$, for a given sector, $P_\ell(\cos \theta_1) = 0$. In general then, ℓ is non-integral and its determination would be difficult. A simpler device is to choose a set of values of θ_1 such that $P_1, P_2, \dots$ vanish. Once the ℓ 's are chosen, the heights of the sectors for criticality, a , are found from the roots of the half-odd Bessel function, using the boundary condition $n = 0$ at $r = a$. Thus, $J_{\ell + \frac{1}{2}}(ka) = 0$.

As an example, choose a sector of about 54° half angle, i.e., $\theta_1 = \cos^{-1} \frac{1}{\sqrt{3}}$. This satisfies the equation

$$P_2(\cos \theta_1) = \frac{1}{2}(3 \cos^2 \theta_1 - 1) = 0.$$

The value of ℓ is thus 2. Solve the equation

$$J_{5/2}(x) = 0,$$

$$\text{or } (3/x^2 - 1) \sin x - \frac{3}{x} \cos x = 0.$$

This root is $x = 5.76$, so that $ka = 5.76$. If k is given by the composition of the pile, a may be found, or the composition may be found for a given sector height.

Since the critical relation for a sphere of radius A is given by

$$k = \pi/A$$

it is convenient to determine the ratio of this "equivalent" sphere radius to

the height of the spherical sector, a , as a function of the half angle. For the example above, if

$$k = \pi/A = \pi/5.76a$$

then $A = 0.545a$ for $\theta_1 = 54^\circ 44'$.

A smooth curve, Figure 1, was drawn through a series of such calculated points, using the data on roots of Legendre and Bessel functions below:

	Root of $P_n(x)$ ($\cos \theta_1$)	Root of $J_{n-\frac{1}{2}}(x)$ (ka)	
0	1	3.1416	sphere
1	0	4.493	hemisphere
2	0.57735	5.763	
3	0.77460	6.987	
4	0.86114	8.183	
5	0.906	9.37	

Since a right circular cone is closely approximated by a spherical sector, at least for narrow cones, Figure 1 will give (conservatively) the critical relations for many plant reactors having conical bottoms.

The suggestions of Dr. F. A. Ficken, Mr. Sidney Visner and Mr. J. W. Morritt were helpful in this work.

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FIGURE 1

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Fig-1

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