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Informal Report

BEAM-BEAM RESONANCES IN HEAD-ON COLLISIONS
OF ELECTRONS AND POSITRONS

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ABSTRACT

Observations of electron-positron collisions demonstrate the existence of a limiting strength to the beam-beam force. The tune dependence of this beam-beam limit suggests the influence of nonlinear resonances. However, beam-beam resonances differ considerably from the normal multipole resonances induced by magnetic fields. In particular, the resonant widths from the beam-beam interaction have a rather slow variation with amplitude for large amplitudes, in marked contrast to the resonant widths caused by multipole fields, which rise with amplitude with a power corresponding to the order of the nonlinear resonance. This peculiar resonance behavior of the beam-beam interaction has resulted in an inability to explain the tune dependence of the beam-beam limit in terms of conventional resonances. However, it is found here that a closer analysis of beam-beam resonances indicates that the lock-in process can indeed occur. This mechanism of particle trapping is exploited to arrive at a picture of beam-beam resonances, one which contains within it the basic ingredients permitting beam size growth and particle loss to external apertures. Our results suggest that an isolated resonance mechanism may be the source of the observed beam-beam limit in electron-positron collisions.

I. INTRODUCTION

Observations on the collisions of stored electrons and positrons have demonstrated the existence of a limiting strength to the force between the two beams.¹⁻³ It has become evident that this beam-beam limit is strongly tune dependent,² suggesting that the nonlinear resonances induced by the interaction of the two beams play an important role in determining the limit.

However, beam-beam resonances differ considerably from the normal multipole resonances induced by magnetic fields.^{4,5} The two most striking differences are: (1) The nonlinear detuning remains much larger than the resonant

widths even for very large amplitudes. This is in marked contrast to the case of strong conventional resonances, where one expects the resonant width function to dominate at physically relevant amplitudes (i.e. within the vacuum chamber). This dominance threshold in amplitude denotes the start of an unstable region in amplitude. Such an unstable region is beyond physical relevance in the case of the beam-beam interaction. (2) The resonant widths from the beam-beam interaction have a rather slow variation with amplitude for large amplitudes (increasing linearly with amplitude for all resonances), again contrasting with the behavior of the resonant widths for multipole fields, where they rise with increasing amplitude with a power corresponding to the order of the nonlinear resonance.

This peculiar resonance behavior of the beam-beam force has resulted in an inability to explain the tune dependence of the beam-beam limit in terms of conventional resonances. There have evolved two non-conventional treatments. Both involve superimposing on the resonances a stochastic process, differing only in the source of the stochasticity. One theory presumes that the stochasticity arises from external sources,⁶ for example, from intra beam scattering or quantum fluctuations, while the other theory supposes that the stochasticity is simulated by the interaction of many overlapping resonances, the source of the stochasticity thus being from within the Hamiltonian system itself.⁷ The mathematical difficulties that arise from attempting to include resonant streaming in a diffusion equation tend to obscure the fine details of the resonance behavior. When the source of the stochasticity is presumed intrinsic, the primary means of study is through numerical experiments.⁸ The numerical accuracy problems associated with strongly nonlinear equations⁹ again tends to obscure the finer resonance details. The result in both cases is that conclusions derived from the studies are simply rough limiting strengths. However, it should be emphasized that the estimates of the strengths have an indeterminate uncertainty. In other words, although not inconsistent with either of these approaches, it is not clear whether the observed beam-beam limits are a reflection of either of the assumed processes.

It is our contention here that conventional resonance^{10,11} behavior may play a substantial role in determining the beam-beam limits. This does not contradict the diffusion-stochastic treatments, but rather supercedes them. Our aim here is to provide a careful analysis of beam-beam nonlinear resonances

in electron-positron collisions and to propose a mechanism for beam growth and particle loss to the aperture limit.

In Part 2, we give the details of one dimensional beam-beam resonances, emphasizing the distinctive amplitude dependence of both the nonlinear detuning and the resonant widths. We show how lock-in¹⁰ and particle transport can occur as a consequence of a modulating beam-beam strength or modulating tune. We include in our analysis the head-on collisions of both flat and round bunches, as well as the case when the bunch centers are displaced during collisions.

In Part 3, we describe a picture of the beam-beam limit resulting from beam-beam nonlinear resonances. The model appears consistent with the fundamental observations^{1,2} of (1) a limiting beam-beam strength, (2) a transverse weak beam growth near the limit, and (3) a strong tune dependence of the limiting strength.

II. BEAM-BEAM RESONANCES

Equation of Motion

We consider the beam-beam interaction occurring in a lattice defined by the β -function,¹² $\beta(s)$, where s measures the azimuthal distance along the equilibrium orbit from some fixed reference point. With z representing the transverse vertical displacement, we use normal coordinates $[\eta, \eta']$ and the independent variable, θ (essentially, the physical azimuth):

$$\theta = \frac{1}{v} \int^s \frac{ds'}{\beta(s')} , \quad (2.1)$$

$$z = \sqrt{\beta} \eta , \quad (2.2)$$

where v is the lattice tune.

For Gaussian beams of rms horizontal and vertical sizes at the collision point, $a/\sqrt{2}$ and $b/\sqrt{2}$, and for M interactions of equal strength, the equation of motion for a particle of the weak beam in the force field of the strong beam can be written¹³

$$\eta'' + v^2 \eta = - \sum_{i=0}^{M-1} 2\pi v \, ab \, \xi_i \delta(\theta - \theta_i) (\eta - v_i) \int_0^{\infty} \frac{dt}{(a^2+t)^{\frac{1}{2}} (b^2+t)^{3/2}} e^{-\beta(\eta-v_i)^2/(b^2+t)}, \quad (2.3)$$

where ξ_i is the beam-beam strength parameter at interaction i , with its value for a weak beam particle at the center of the strong beam given by,¹⁴

$$\xi = \frac{N r_e \beta^*}{\pi a b \gamma}, \quad (2.4)$$

β^* is the value of β at the collision point,

$\delta(\theta - \theta_i)$ is the Dirac Delta function,

θ_i is the azimuth of the i^{th} interaction,

v_i is the vertical displacement in normal coordinates of the central orbit of the weak particle from the center of the strong beam, for interaction i ,

r_e is the classical electron radius,

and γ is the energy in electron mass units.

We have chosen to write the equation of motion in the vertical plane. There is no loss of generality in this respect. The horizontal motion can be obtained by simply reinterpreting the various parameters involved. We have included explicitly a vertical orbit shift, v ; that is, a weak particle has an equilibrium orbit which is shifted by an amount v with respect to the strong beam center. The primary effect of such an orbit shift is the excitation of odd-ordered resonances. On the other hand, a horizontal orbit shift only modifies the even-ordered beam-beam resonances and we will include its effect approximately as a variation of the strength parameter.

We can simplify (2.3) for the two cases we will treat, (1) $a \gg b$, and (2) $a = b$. We obtain

$$\eta'' + v^2 \eta = - \sum_{i=0}^{M-1} 2\pi v \, \xi_i \delta(\theta - \theta_i) (\eta - v_i) \left\{ \begin{array}{l} 2 \int_0^1 dt \, e^{-\frac{\beta(\eta-v_i)^2}{b^2} t^2}, \quad a \gg b \\ \int_0^1 dt \, e^{-\frac{\beta(\eta-v_i)^2}{b^2} t}, \quad a = b. \end{array} \right. \quad (2.5)$$

The equations of motion in phase-amplitude variables (I, ϕ) defined by

$$\begin{aligned}\eta &= \sqrt{I} \cos \phi, \\ \eta' &= -\nu \sqrt{I} \sin \phi,\end{aligned}\quad (2.6)$$

are

$$\begin{aligned}I' &= \frac{2\eta'}{\nu} (\eta'' + \nu^2 \eta), \\ \phi' &= \nu - \frac{\cos \phi}{\sqrt{I}} (\eta'' + \nu^2 \eta).\end{aligned}\quad (2.7)$$

Resonance Approximation

The approximate equation for the resonant phase, Ψ , for a resonance of order p induced by the m^{th} azimuthal harmonic of the beam-beam force has the general form¹⁵

$$\begin{aligned}\Psi' &= \left(\nu - \frac{m}{p}\right) + \xi_s F(\alpha) + \\ &+ \left[\bar{\xi}_s G_p(\alpha) + \bar{\omega}_s \bar{G}_p(\alpha)\right] \cos p\Psi,\end{aligned}\quad (2.8)$$

where

$$\Psi = \phi - \frac{m}{p} \theta, \quad (2.9)$$

$$\xi_s = \sum_{i=0}^{M-1} \xi_i \begin{cases} 1 & \text{if } a \gg b \\ \frac{1}{2} & \text{if } a = b \end{cases} \quad (2.10)$$

$$\bar{\xi}_s = \sum_{i=0}^{M-1} \xi_i \cos m\theta_i \begin{cases} 1 & \text{if } a \gg b \\ \frac{1}{2} & \text{if } a = b \end{cases} \quad (2.11)$$

$$\bar{\omega}_s = \sum_{i=0}^{M-1} \xi_i \nu_i \cos m\theta_i / \sqrt{I_{\text{rms}}} \begin{cases} 1 & \text{if } a \gg b \\ \frac{1}{2} & \text{if } a = b \end{cases} \quad (2.12)$$

$$\alpha = I/I_{\text{rms}}, \quad (2.13)$$

and

$$I_{\text{rms}} = b^2/\beta^* \text{ is the rms squared amplitude.} \quad (2.14)$$

The expressions for the resonant strengths must be evaluated at an azimuth where the sin components of the sum are zero. For example, with two evenly spaced collision points ($M=2$), simply choose $\theta_0 = 0$. The functions of α , F , G_p , and $\bar{G}_p$ in the two cases $a \gg b$, $a = b$, are given by

$$F(\alpha) = \frac{1}{\pi} \int_0^{2\pi} d\phi \cos^2 \phi \left\{ \begin{array}{l} \int_0^1 dt e^{-\alpha \cos^2 \phi t^2}, a \gg b \\ \int_0^1 dt e^{-\alpha \cos^2 \phi t}, a = b \end{array} \right. \quad (2.15)$$

$$G_p(\alpha) = \frac{2}{\pi} \int_0^{2\pi} d\phi \cos p\phi \cos^2 \phi \left\{ \begin{array}{l} \int_0^1 dt e^{-\alpha \cos^2 \phi t^2}, a \gg b \\ \int_0^1 dt e^{-\alpha \cos^2 \phi t}, a = b \end{array} \right. \quad (2.16)$$

and

$$\bar{G}_p(\alpha) = \frac{2}{\pi} \int_0^{2\pi} d\phi \cos p\phi \cos^2 \phi \left\{ \begin{array}{l} \int_0^1 dt e^{-\alpha \cos^2 \phi t^2} [\sqrt{\alpha \cos \phi t^2} - \frac{1}{\sqrt{\alpha \cos \phi}}], a \gg b \\ \int_0^1 dt e^{-\alpha \cos^2 \phi t} [\sqrt{\alpha \cos \phi t} - \frac{1}{\sqrt{\alpha \cos \phi}}], a = b. \end{array} \right. \quad (2.17)$$

Note that in general,

$$\begin{aligned} G_p(\alpha) &\neq 0 \text{ only if } p \text{ even,} \\ \bar{G}_p(\alpha) &\neq 0 \text{ only if } p \text{ odd,} \end{aligned} \quad (2.18)$$

which simply means that for weak particle orbits that are centered, only even ordered resonances are excited, while the odd ones are introduced by orbit shifts with respect to the strong beam center.

In terms of the Bessel functions, I_n , defined by

$$I_n(y) = \frac{1}{\pi} \int_0^\pi e^{y \cos \theta} \cos n\theta d\theta, \quad (2.19)$$

we have for $a \gg b$

$$F(\alpha) = e^{-\alpha/2} \left[I_0(\alpha/2) + I_1(\alpha/2) \right] , \quad (2.20)$$

$$G_p(\alpha) = -2(-1)^{\frac{p}{2}} \frac{e^{-\alpha/2}}{p^2-1} \left\{ \left(1 + \frac{p^2}{\alpha}\right) I_{p/2}(\alpha/2) + I'_{p/2}(\alpha/2) \right\} , \quad (2.21)$$

$$\bar{G}_p(\alpha) = (-1)^{\frac{p-1}{2}} \frac{e^{-\alpha/2}}{\sqrt{\alpha}} \left\{ \left(1 - \frac{1}{p}\right) I_{\frac{p+1}{2}} - \left(1 + \frac{1}{p}\right) I_{\frac{p-1}{2}} \right\} , \quad (2.22)$$

where

$$I'_n = \frac{1}{2} (I_{n+1} + I_{n-1}), \quad n \geq 1 , \quad (2.23)$$

while for $a = b$,

$$F(\alpha) = \frac{2}{\alpha} \left[1 - e^{-\alpha/2} I_0(\alpha/2) \right] , \quad (2.24)$$

$$G_p(\alpha) = (-1)^{\frac{p}{2}+1} \frac{4}{\alpha} e^{-\alpha/2} I_{p/2}(\alpha/2) , \quad (2.25)$$

$$\bar{G}_p(\alpha) = (-1)^{\frac{p-1}{2}} \frac{2}{\sqrt{\alpha}} e^{-\alpha/2} \left[I_{\frac{p+1}{2}}\left(\frac{\alpha}{2}\right) - I_{\frac{p-1}{2}}\left(\frac{\alpha}{2}\right) \right] . \quad (2.26)$$

The resonant invariant can be written

$$C = (v - \frac{m}{p})\alpha + \xi_s U(\alpha) + \left[\bar{\xi}_s V_p(\alpha) + \bar{w}_s \bar{V}_p(\alpha) \right] \cos p\psi , \quad (2.27)$$

from which the equations for phase and amplitude are obtained,

$$\begin{aligned} \psi' &= \frac{\partial C}{\partial \alpha} , \\ \alpha' &= -\frac{\partial C}{\partial \psi} . \end{aligned} \quad (2.28)$$

The functions U , V_p and $\bar{V}_p$ are obtained from F , G_p and $\bar{G}_p$. They are:
for $a \gg b$,

$$U(\alpha) = 2 \left\{ e^{-\alpha/2} \left[(1+\alpha) I_0(\alpha/2) + \alpha I_1(\alpha/2) \right] - 1 \right\} \quad (2.29)$$

$$v_p(\alpha) = 4(-1)^{\frac{p}{2} + 1} \frac{e^{-\alpha/2}}{p^2 - 1} \left[(1+\alpha) I_{p/2}(\alpha/2) + \alpha I'_{p/2}(\alpha/2) \right], \quad (2.30)$$

$$\tilde{v}_p(\alpha) = (-1)^{\frac{p-1}{2}} \sqrt{2} \int_0^{\alpha/2} dx \frac{e^{-x}}{\sqrt{x}} \left\{ \left(1 - \frac{1}{p}\right) I_{\frac{p+1}{2}}(x) - \left(1 + \frac{1}{p}\right) I_{\frac{p-1}{2}}(x) \right\}, \quad (2.31)$$

where we have taken $U, v_p, \tilde{v}_p \rightarrow 0$ as $\alpha \rightarrow 0$;

while for $a = b$,

$$U(\alpha) = 2 \int_0^{\alpha/2} dx \frac{1}{x} \left[1 - e^{-x} I_0(x) \right], \quad (2.32)$$

$$v_p(\alpha) = 4(-1)^{\frac{p}{2} + 1} \int_0^{\alpha/2} dx \frac{1}{x} e^{-x} I_{p/2}(x), \quad (2.33)$$

$$\tilde{v}_p(\alpha) = 4(-1)^{\frac{p-1}{2}} \frac{\sqrt{\alpha}}{p} e^{-\alpha/2} \left[I_{\frac{p+1}{2}}\left(\frac{\alpha}{2}\right) + I_{\frac{p-1}{2}}\left(\frac{\alpha}{2}\right) \right]. \quad (2.34)$$

Note that in both cases, $U(\alpha) \approx \alpha$, as $\alpha \rightarrow 0$. This term corresponds to the linear tune shift from the beam-beam interaction.

Fixed Points and Lock-In Islands

Fixed points are obtained from the invariant by applying the condition

$$\frac{\partial G}{\partial \alpha} = \frac{\partial G}{\partial \Psi} = 0. \quad (2.35)$$

The fixed points are therefore determined by the functions F, G_p and $\bar{G}_p$. However, since $F \gg G_p$ and $F \gg \bar{G}_p$ for all relevant α while $\xi_s \geq \bar{\xi}_s$ and $\xi_s \geq \bar{\omega}_s$, then the fixed point amplitude, α_F , can be found from

$$\left(v - \frac{m}{p}\right) + \xi_s F(\alpha_F) = 0. \quad (2.36)$$

We have a sequence of equally spaced, alternating stable and unstable fixed points at the amplitude α_F , forming islands capable of trapping or locking in particles. The extent in amplitude of the islands can be expressed in terms of the maximum and minimum amplitudes of the separatrix (the trajectory forming

the island boundary). We can show that these are, for even ordered resonances,

$$\alpha_{F\pm} = \alpha_F \pm 2 \sqrt{-\frac{\bar{\xi}_s |\bar{V}_p|}{F_s F'}}, \quad (2.37a)$$

while for odd-ordered resonances,

$$\alpha_{F\pm} = \alpha_F \pm 2 \sqrt{-\frac{\bar{\omega}_s |\bar{V}_p|}{\bar{\xi}_s F'}}, \quad (2.37b)$$

$$\begin{aligned} & -\frac{1}{\alpha} e^{-\alpha/2} I_1(\alpha/2), \quad a \gg b \\ \text{where } F' = \frac{dF}{d\alpha} = & \left\{ \begin{aligned} & -\frac{1}{\alpha} e^{-\alpha/2} I_1(\alpha/2), \quad a \gg b \\ & \frac{2}{\alpha^2} \left\{ e^{-\alpha/2} \left[\left(1 + \frac{\alpha}{2}\right) I_0 - \frac{\alpha}{2} I_1 \right] - 1 \right\}, \quad a = b. \end{aligned} \right. \end{aligned} \quad (2.38)$$

The Trapping Criterion

Since the detuning functions $[U(\alpha)$ and its derivatives] are much larger than the resonant width functions $[V_p, \bar{V}_p$ and their derivatives], even for large α beam-beam resonances have a characteristic form quite different from conventional resonances. The primary effect is that the large amplitude phase space region as well as the small amplitude region is dominated by the detuning function. This is in contrast to conventional resonances, where a physically relevant large α may allow the resonant width function to become dominant, thus creating a large amplitude unstable region in phase space. However, the beam-beam resonance still has an impact on the phase space topology. It allows for a phase space region where amplitude modulation can occur. The center of this amplitude region is defined by the fixed point amplitude, α_F , Eq. (2.36), while the amplitude modulation can be obtained from (2.37). Again, since the detuning function is much larger than the resonant width function, this amplitude modulation is small. Thus, from the point of view of particle stability, the static phase space topology appears innocuous.

On the other hand, if the parameters v and ξ_s have an adiabatic variation with time, particles can lock into the resonance and be transported to larger amplitudes. The amplitude through which transport can occur is determined by

(2.36). The picture is that as the parameters v and ξ_s change slowly, particles can be trapped in the islands (regions of oscillation about stable centers) and be dragged to different amplitudes.¹⁶

The mathematical criterion for adiabaticity is found by comparing the speed with which the island center, α_F , is changing with amplitude to the rate of particle amplitude change due to the resonance.¹⁰ The question we wish to answer is, how slowly do the parameters v and ξ_s have to be changing in order for trapping to occur? We expect the result to depend on the amplitude at which the particles meet the islands.

The adiabaticity condition is obtained by relating the fixed point speed from (2.36),

$$\alpha_F' = - \frac{\delta_{\text{per}} \left(\frac{\bar{v}}{\xi_s} \right)}{2\pi F'(\alpha_F)} , \quad (2.39)$$

where

$$\bar{v} = v - m/p , \quad (2.40)$$

and

δ_{per} means the rate of change per period,

to the resonant amplitude change from (2.27) and (2.28),

$$\alpha' = p [\bar{\xi}_s V_p(\alpha_F) + \bar{\omega}_s \bar{V}_p(\alpha_F)] , \quad (2.41)$$

for $\alpha \approx \alpha_F$, and where we have taken the phase term, which changes rapidly, to be one. We thus obtain a necessary condition for particle trapping,

$$\alpha' \geq \alpha_F' . \quad (2.42)$$

For even-ordered resonances, we have

$$\frac{1}{\xi_s} \delta_{\text{per}} \left(\frac{\bar{v}}{\xi_s} \right) \leq \left| 2\pi p V_p(\alpha_F) F'(\alpha_F) \right| , \quad (2.43a)$$

while for odd-ordered resonances,

$$\frac{1}{\xi_s} \delta_{\text{per}} \left(\frac{\bar{v}}{\xi_s} \right) \leq \left| 2\pi p \bar{V}_p(\alpha_F) F'(\alpha_F) \right| . \quad (2.43b)$$

Time Varying Parameters

In the one-dimensional analysis we have presented, we have included the variation of the force with vertical (or horizontal if we wish to consider horizontal resonances) displacement about the mean orbit, z , as well as a shift in the orbit from the center of the force, i.e. the strong beam. The first is of course the basic displacement dependent force which gives the resonant behavior. The latter is an external parameter embodied in the quantity $\bar{\omega}_s$. The other external parameters are $\bar{\nu}$, $\bar{\epsilon}_s$ and $\bar{\xi}_s$. We wish now to consider the effect on the resonance characteristics of modulations in the external parameters.

The most significant source of variation of both the tune $\bar{\nu}$ and the force $\bar{\epsilon}_s$ is the synchrotron motion. For a particle with non-zero synchrotron amplitude, each revolution finds it in a different azimuthal position with respect to the center of the strong bunch and with a different momentum. For example, if the β -function variation is large at the collision point (as in low- β interaction regions), the force experienced by a particle will modulate with azimuthal position in the bunch. The momentum variation causes the tune to modulate via the chromaticity of the machine. Another example results if there is a non-zero dispersion function in the collision region; then, the central orbit of the weak beam particle will shift with respect to the strong beam as the momentum changes, thus providing a modulation of the force.

Now, these changes all are periodic, with the period of the synchrotron oscillations. It is therefore tempting to treat a total synchrotron period as a "machine period". Doing this reduces the modulation of the external parameters to zero. This, however, leads us to an apparent contradiction. By treating a single revolution as the basic period, external parameters are modulated with the synchrotron period, thus allowing, in principal, the lock-in process to occur. By treating the synchrotron period as the basic machine period, there is no modulation of external parameters and no lock-in can result from this description. Yet, it is the same physical system. Which version is correct?

We expect that the answer lies in an understanding of the adiabaticity criterion (2.43). Consider an amplitude region not close to the boundary implied in (2.43). Suppose we are well into the adiabatic region. From the one revolution picture, a periodic modulation of parameters would produce a periodic amplitude modulation via particle trapping. Treating the

whole period as a basic machine, no such amplitude modulation results. In the latter case, we are simply observing the mean change over a longer time scale. In other words, the one revolution analysis simply brings out the shorter time scale fine structure, the latter giving only the mean amplitude change. The point is, if short time effects are negligible, the two descriptions are equivalent. However, they would not be the same if we had an amplitude barrier nearby. Then, particles whose amplitude reached a certain value would be lost to the sink. This would not be observable in the long time description. In this case, which period we choose to describe the particle motion would be a crucial factor.

In the same way, we can see that the adiabaticity boundary itself can act like a barrier. When the boundary is crossed from inside to outside the adiabaticity region, the islands become leaky and particles can escape. By such a mechanism particles can be given a net amplitude shift, thus changing the beam shape.

Thus, even in the presence of forces tending to create an equilibrium distribution in amplitude (i.e. radiation damping and quantum fluctuations), the combination of amplitude boundaries and resonant lock-in allows (1) the loss of particles in the case of physical boundaries and (2) a change in the transverse equilibrium beam size when the adiabaticity boundary is appropriately crossed. To describe these effects, we look for the influence of modulating parameters, and so study the resonant system in detail within the synchrotron period.

We will treat here the case of even-ordered resonances with two symmetrically placed crossings per turn. Thus $M = 2$ and since $\theta_0 = 0$, we see from (2.11) and (2.10)

$$\xi_s^m = \begin{cases} \xi_s = 2\xi_s & , m \text{ even} \\ 0 & , m \text{ odd} \end{cases} \quad (2.44)$$

The physically significant external parameters are

$$f = \bar{v}/\xi_s \quad (2.45)$$

and

$$r = \frac{1}{\xi_s} \delta_{\text{rev}} \left(\frac{\bar{v}}{\xi_s} \right) \quad , \quad (2.46)$$

where we have taken the period of the system as a revolution. The modulation of f during the synchrotron period will, through (2.36), define the limits of particle transport in amplitude if they are trapped. The value of r determines from (2.43) whether or not particles can be trapped in the moving islands at each amplitude.

There are other sources of time variation, in particular, diffusion phenomena arising from, say, quantum fluctuations or intra beam scattering. We presume these effects to be negligible in comparison to the stronger and faster effects resulting from the synchrotron motion.

Beam-Beam Resonance Functions

The resonance functions are universal in the sense that they are strength independent, and functions of amplitude only. In Fig. 1, we have the detuning functions for the cases $a \gg b$ and $a = b$. We plot

$$D(\alpha) = -F(\alpha) \quad , \quad (2.47)$$

which then can be used to compare with the value of f , defined in (2.44). In Figs. 2 ($a \gg b$) and 3 ($a = b$), we give plots, for various even-ordered resonances, of the functions

$$R_p(\alpha) = 2\pi p |V_p(\alpha)F'(\alpha)| \quad , \quad (2.48)$$

which can be used in conjunction with the modulating r , given in (2.45). In Figs. 4 and 5 the corresponding odd-ordered resonance functions

$$\tilde{R}_p(\alpha) = 2\pi p |\tilde{V}_p(\alpha)F'(\alpha)| \quad (2.49)$$

are shown for completeness.

For the independent variable, instead of the squared amplitude α , we use the displacement in units of rms displacement, σ , which is related to α by

$$\sigma = \sqrt{2\alpha} \quad . \quad (2.50)$$

III. BEAM-BEAM LIMIT

Parameter Modulation Due to Synchrotron Oscillations

(a) $\bar{\nu}$ Modulation (Chromaticity Effect)

Let $\bar{\nu}_0$ be the tune of the synchronous particle; then the modulation of $\bar{\nu}$ about $\bar{\nu}_0$ due to the chromaticity can be written

$$\bar{\nu} = \bar{\nu}_0 + \Delta_{ch} u \quad , \quad (3.1)$$

where

$$u = k \sin (2\pi \nu_s n + \phi_s) \quad , \quad (3.2)$$

$$\Delta_{ch} = \xi_{ch} \left(\frac{\delta p}{p} \right)_{rms} \quad , \quad (3.3)$$

$\left(\frac{\delta p}{p} \right)_{rms}$ is the rms momentum spread,

ξ_{ch} is the absolute chromaticity,

Δ_{ch} is the maximum tune modulation
for a peak momentum excursion of
one rms,

k is a parameter denoting the synchro-
tron amplitude; specifically, the
peak momentum excursion in units
of rms momentum spread of the bunch,

ν_s is the synchrotron wave number,

$$N_s = \frac{1}{\nu_s} \quad , \quad (3.4)$$

N_s is the number of particle revolutions
in a synchrotron period,

n is a variable denoting the revolution
number during a synchrotron period,

and ϕ_s is a phase; if $\phi_s = 0$, the initial momentum ($n = 0$) is the synchronous momentum, while if $\phi_s = \pi/2$, the initial momentum is its peak value, $k(\delta p/p)_{rms}$.

(b) ξ Modulation Due to Low β^*

In an interaction region design with low β^* , meaning essentially

$$\beta^* \lesssim \sigma_\ell, \quad (3.5)$$

where σ_ℓ is one rms bunch length, the beam-beam strength parameter of the strong beam, ξ , depends on the azimuthal position of the weak beam particle relative to the synchronous particle.¹⁷ This is a result of the significant β change along the bunch length coupled with the changing density distribution as the weak particle passes through the strong beam bunch. For an ideal collision, where the centers of the two bunches coincide at the minimum β , a particle at the azimuthal center of one bunch sees the maximum density of the other bunch at the minimum β , β^* . However, particles at azimuths different from the synchronous particle see the peak density at higher values of β , and so the force over the collision region is larger. As can be seen from the equation of motion (2.5), the azimuthal dependence of the beam-beam force term separates out. Thus, the force can still be approximated by a delta function, but we must modify the strength. If we let ξ_0 be the strength parameter for the synchronous particle, then, for a particle a distance v from the center of the bunch, in units of rms bunch length, σ_ℓ , the strength ξ is given by

$$\xi = \xi_0 F_\beta(v, w), \quad (3.6)$$

where

$$w = \beta^*/\sigma_\ell, \quad (3.7)$$

and

$$F_\beta(v, w) = \frac{\int_{-\infty}^{\infty} dq \sqrt{1 + \frac{q^2}{2w^2}} e^{-(q-v/\sqrt{2})^2}}{\int_{-\infty}^{\infty} dq \sqrt{1 + \frac{q^2}{2w^2}} e^{-q^2}}. \quad (3.8)$$

The quantity v can be expressed in terms of the synchrotron oscillation parameters, k , v_s , ϕ_s and n , previously introduced:

$$v = k \sin (2\pi v_s n + \phi_s + \pi/2) \quad . \quad (3.9)$$

Note that F_β is positive and that further,

$$F_\beta \geq 1 \quad ,$$

which simply represents the fact that non-synchronous particles experience a larger force, as we have pointed out above. Also note that F_β has an oscillation frequency which is twice the synchrotron frequency.

(c) § Modulation Due to a Non-Zero Dispersion

We are concerned here with the effect of a horizontal dispersion function on the magnitude of the vertical resonance strength parameter, ξ . The effect of such a non-zero dispersion can be approximated as a modulating beam-beam strength. An estimate of the modulating strength can be made by computing the changing force field that a weak beam particle will encounter as its central orbit moves across the strong beam in the median plane. If $\langle x \rangle$ is the distance from the strong beam center, the strength changes, relative to its value at the center, by the factor

$$F_\eta = \frac{2\sigma_h^{*2}}{\langle x \rangle^2} \left[1 - e^{-\langle x \rangle^2 / 2\sigma_h^{*2}} \right] \quad ,$$

where σ_h^* is the rms horizontal size. If the dispersion function at the collision point, η^* , is small, then we can neglect the density overlap between betatron and momentum size, and thus we can interpret σ_h^* as the horizontal betatron size of the strong beam. Defining

$$t = \frac{\eta^* (\delta p/p)_{rms}}{\sigma_h^*} \quad , \quad (3.10)$$

we have for the strength

$$\tau = \xi_0 F_\eta (tu) \quad , \quad (3.11)$$

with

$$F_{\eta}(tu) = \frac{2}{t^2 u^2} \left[1 - e^{-t^2 u^2 / 2} \right] \quad (3.12)$$

Note that our assumption concerning the size of the dispersion function can be stated

$$t < 1 \quad (3.13)$$

Models of the Beam-Beam Limit For SPEAR and ACO

We have presented a theory for beam growth and beam loss due to resonances excited by the beam-beam interaction. If we now take the parameters for existing electron-positron colliding beam facilities, we should be able to describe the two-beam behavior at the beam-beam limit, and in particular the dependence on tune and beam-beam strength. We will test our theory by applying it to the cases of SPEAR¹ and ACO.²

SPEAR:

In SPEAR, the dominant tune modulating source is the low β^* effect previously discussed. The modulating function F_{β} can be considered a function of k and

$$h = 2\pi\nu_s n \quad (3.14)$$

Then, $F_{\beta}(k, h)$ goes from 1 to its maximum value as h goes from 0 to $\pi/2$; that is, over 1/4 of a synchrotron period. From (2.45), (2.46), (3.6), (3.8) and (3.9), we can thus write for f and r ,

$$f = \frac{\bar{\nu}}{\xi_{so} F_{\beta}(k, h)} \quad (3.15)$$

$$r = \left| \frac{2\pi\nu_s \bar{\nu}}{\xi_{so}^2} \frac{F'_{\beta}(k, h)}{[F_{\beta}(k, h)]^3} \right| \quad (3.16)$$

The procedure is to solve

$$D(\sigma) = f(h, k) \quad (3.17)$$

for σ as a function of h (given k), using Fig. 1. The solution is then used to plot $r(\sigma)$ and compare this function with the resonance curves of Fig. 2. For SPEAR, we take $w = \beta^*/\sigma_L = 1$ and $v_s = 3.9 \times 10^{-3}$ (the low voltage value at $E = 1.5$ GeV).

For beam loss, particles must be transported to a physical aperture in one synchrotron period. For beam growth, particles need only be transported across the adiabaticity boundary, while moving from smaller to larger values of σ .

Our theory requires the knowledge of the physical aperture boundary in units of rms beam size. Let this be σ_{AP} . In order to get particle loss to this boundary, the islands must reach this point during a synchrotron period. Since the maximum value of f is

$$f_{\max} = \bar{v}/\xi_{so} F_{\max}(k) \quad (3.18)$$

we have the necessary condition,

$$f_{\max} = \frac{\bar{v}}{\xi_{so} F_{\max}(k)} > D(\sigma_{AP}) \quad (3.19)$$

For the purposes of our qualitative discussion, let us assume a value for $\sigma_{AP} \approx 8$. This choice is not critical to the theory, but some choice is necessary to give a coherent picture of the mechanism. Even though the natural single beam size might place the aperture at larger values of σ , the "two-beam" size is known to be appreciably larger³ than the single beam size, thus bringing in the value of σ_{AP} . In fact, as we will see, near the beam-beam limit, the theory presented here contains a mechanism for this growth. With $\sigma_{AP} = 8$, we obtain from Fig. 1 ($a \gg b$),

$$\frac{\bar{v}}{\xi_{so} F_{\max}(k)} > 0.2 \quad (3.20)$$

Now, since

$$F_{\min}(k) = 1 \text{ for all } k, \quad (3.21)$$

then for each value of $\bar{v}/\xi_{so}$, there is a minimum of k , $k_{\min}$, for which the physical aperture will be reached during the synchrotron oscillation. We plot $k_{\min}$ versus $\bar{v}/\xi_{so}$ in Fig. 6. We also plot on this figure the minimum σ , which is

independent of k . It is the solution of

$$D(\sigma_{\min}) = \bar{v}/\xi_{so} \quad . \quad (3.22)$$

The meaning of these curves is that for each value of $\bar{v}/\xi_{so}$, islands move from $\sigma_{\min}$ and reach the aperture limit, σ_{AP} , as long as $k \geq k_{\min}$.

However, the motion of the islands is not sufficient. Do the particles lock-in? Which ones? To answer these questions we must relate the value of r to the resonance function, R_p (Fig. 2). For each set of values of the pair $[\bar{v}/\xi_{so}, k]$, we can solve (3.17) and plot $r(\sigma)$ along the path from $\sigma_{\min}$ to $\sigma_{\max}$ (e.g. taking $k = k_{\min}$, $\sigma_{\max} = \sigma_{AP}$). Note that aside from the known parameter values, r is proportional to $1/\xi_{so}$.

To see the significance of this, let us consider the 8th order resonance, $p = 8$. Take the point on the $k_{\min}$ curve $[-.33, 3]$. We plot in Fig. 7, $R_8(\sigma)$ (for $a \gg b$) along with $r(\sigma)$ for various values of ξ_{so} . As is evident, the ability of the moving islands to retain and transport particles is sensitive to ξ_{so} . In particular, there is a threshold value of ξ_{so} which corresponds to the condition that the transport can occur from $\sigma_{\min}$ (defined by $\bar{v}/\xi_{so}$ alone) to the aperture, σ_{AP} . For $\xi_{so} < \xi_{so}(\text{threshold})$, the adiabaticity boundary intercedes during the island motion. Close to the threshold value, particles will be transported from $\sigma_{\min}$ to some larger amplitude where they will leak out of the island. Other particles will be picked up at an amplitude larger than $\sigma_{\min}$ and be transported to the aperture boundary. However, as ξ_{so} is reduced, both of these processes become less and less significant. For $\xi_{so} > \xi_{so}(\text{threshold})$, particles can continue to lock-in from $\sigma_{\min}$ to σ_{AP} . However, the significance of being above the strength threshold becomes clear when we recall that we have been dealing with a select group of particles, i.e. those with $k = k_{\min}$ (in this case, $k_{\min} = 3$). Thus, if we are at the threshold for $k = k_{\min}$, then increasing k does not allow particles to be transported the full distance from $\sigma_{\min}$ to σ_{AP} . On the other hand, above the strength threshold, there is a band of k which does allow the full transport. We can see this in Fig. 7, where we have plotted for ξ_{so} above the threshold, $r(\sigma)$ for two values of k above $k_{\min}$.

It is thus clear that the resonance reduces the beam lifetime by providing what may be termed a "resonance aperture". For example, the betatron aperture in the example of Fig. 7 was reduced from the physical limit, $\sigma_{AP} = 8$ to the resonance aperture, $\sigma_{RES} = 5$. In electron beams, because of quantum feeding,

this is a significant effect.

To get a rough estimate of what may be termed a beam-beam limit, we make the assumption that the creation of a resonance aperture at $\sigma = 5$ (i.e. the loss boundary being moved from $\sigma_{AP} = 8$ to $\sigma_{RES} = 5$) reduces the beam lifetime to an unacceptable value. We can then say that this is the beam-beam limit and ask what the nature of this limit is. In particular we can ask whether a range of $\bar{v}$ will allow such a condition to prevail as a function of the beam-beam ξ parameter. We note that for two beam collision regions

$$\xi = \frac{1}{2} \xi_{so} \quad . \quad (3.23)$$

This then is analogous to a resonance stop band. As we have already seen, particles in a small band around $k = 3$ will be transported to $\sigma_{AP} = 8$ from $\sigma_{RES} = 5$ if $\xi = .033$ and $\bar{v} = -.022$ (see Fig. 7). If we assume this can cause sufficient deterioration in the beam lifetime, then for $\bar{v} = -.022$, $\xi = .033$ is the beam-beam limit. Changing $\bar{v}$ alters the conditions prescribed above so that either the resonance aperture is at a value of σ greater than 5 or the islands do not reach the physical aperture at $\sigma = 8$. (Note the lifetime is not determined by the betatron aperture alone; one must also include the constraint that only particles of the specified synchrotron amplitudes are involved). Since it is clear that the beam-beam limit increases as $\bar{v}$ is changed, we consider what values of $\bar{v}$ cause beam deterioration for different ξ . Suppose we double ξ from 0.033. Now, it is clear that for $\bar{v}$ doubled, beam deterioration will result. However, for $\bar{v}$ larger than this, loss will result as well. The reason is that for larger $\bar{v}$, particles with $k > 3$ can be transported. These particles will have betatron amplitudes $\sigma < 5$ and still be transported to the physical aperture at $\sigma = 8$. Thus, we expect a stopband to develop below the resonance and furthermore below a line,

$$\bar{v} = -c\xi \quad , \quad (3.24)$$

where in the example above, $c = 0.67$. We sketch the stopband in Fig. 8.

Thus the resonance model has all the ingredients for a qualitative explanation of the observations. First, there is a limiting beam-beam strength which is tune sensitive; and second, beam growth can result if islands transport particles across the adiabaticity boundary while moving outward in σ .

ACO:

In ACO, where the β^* is large enough that the low β^* effect so pronounced in SPEAR is absent, and where the dispersion function at the collision point is small, the dominant modulating source is the tune modulation through the chromaticity. Our model for a beam-beam limit for ACO is therefore contained in the expressions for f and r ,

$$f = \frac{\bar{\nu}_o + \Delta_{ch} k \sin h}{\xi_{so}}, \quad (3.25)$$

$$\text{and} \quad r = \left| \frac{2\pi\nu_s \Delta_{ch} k \cos h}{\xi_{so}^2} \right|. \quad (3.26)$$

We take for ACO the following parameters²: $V_{rf} = 17.5$ kV, $E = 510$ MeV, h (rf harmonic number) = 2, $\nu_h = 2.845$, ξ_{ch} (vertical) = -3.1, and $(\delta p/p)_{rms} = 7.8 \times 10^{-4}$. This gives

$$\nu_s = \frac{1}{\nu_h} \left(\frac{h \text{ eV}}{2\pi E} \right)^{\frac{1}{2}} = 1.16 \times 10^{-3}, \quad (3.27)$$

$$\text{and} \quad \Delta_{ch} = \xi_{ch} \left(\frac{\delta p}{p} \right)_{rms} = -2.43 \times 10^{-3}. \quad (3.28)$$

For two crossings per turn, we have again $\xi = \frac{1}{2} \xi_{so}$. The two essential observations on ACO near the beam-beam limit are (1) the increase in beam size and (2) the opening up of disallowed regions of $\bar{\nu}$ near high order resonances. As more and more resonances become harmful, the region of disallowed $\bar{\nu}$ is extended until no operating point is available.

Our model of the beam-beam limit can describe these observations in the following sense. First, we can show how beam growth results as a consequence of particles being transported to increasing σ and crossing the adiabaticity boundary and thus being left at larger amplitudes. They cannot be transported back since the islands are leaky and cannot hold the particles. Beam loss occurs as a second stage. The increased beam size results in the physical aperture moving to smaller values of σ (the displacement in rms beam size units). That is, a larger beam size means that the physical aperture occurs at a smaller value of σ_{AP} than would be the case for a single beam. Thus, as a consequence

of resonance transport to σ_{AP} , we will show how a deteriorating beam lifetime results over a widening range of $\bar{v}$ as ξ increases.

Using (3.27) and (3.28), r and f can be written

$$f = \frac{\bar{v}_0}{\xi_{so}} + 2.43 \times 10^{-3} \frac{k \sin h}{\xi_{so}} , \quad (3.29)$$

$$r = 1.77 \times 10^{-5} \left| \frac{k \cos h}{\xi_{so}^2} \right| . \quad (3.30)$$

With the aid of Fig. 1 ($a = b$), we solve $D(\sigma) = f(h)$ and substitute into (3.30) to find $r(\sigma)$. We superimpose the $r(\sigma)$ curves on the resonance curves $R_p(\sigma)$, as shown in Fig. 9. A few examples are shown to illustrate the basic effects.

For small ξ (e.g. $\xi = 0.015$), transport to the aperture for even the 6th order resonance does not occur. However, beam size growth can occur, with substantial effect seen near the 6th and 8th order resonances. For $\xi = 0.03$, beam loss can result from transport to the aperture near the 6th, 8th and 10th order resonances. At this strength, the stopband for the 8th order resonance is large and actually extends from $\bar{v}_0 = 0$ to $\bar{v}_0 \approx -0.018$. The 10th order stopband is smaller and extends from $\bar{v}_0 \approx -0.006$ to $\bar{v}_0 \approx -0.01$.

Thus, using the theory of Section 2, together with ACO parameters, a qualitative picture of ACO near the beam-beam limit emerges, one which contains a tune sensitive limiting beam-beam strength (stopbands), and one which predicts increasing beam size as the limit is approached.

IV. CONCLUSIONS

We have proposed a model to describe the deteriorating effect of the beam-beam interaction on beam quality in electron-positron collisions. The extension to e-e and proton-proton collisions is straightforward. At the core of the theory is the excitation by the beam-beam interaction of nonlinear resonances. Their characteristics are quite different from conventional resonances and are here properly treated. With increasing beam-beam strength and decreasing resonance order, the resonances will influence ever smaller betatron amplitudes, thus creating resonance barriers at amplitudes within the physical amplitude

boundary. The picture is then of particles being fed to the resonance amplitude by quantum fluctuations. The transport between the resonance and physical boundaries is mediated by time modulation of the beam-beam strength parameter, ξ , and the tune, $\bar{\nu}$.

Lifetime estimates are complicated by the fact that the resonance mechanism we are proposing will influence only a select region of all three dimensions, horizontal and vertical transverse as well as longitudinal (synchrotron). Thus, the lifetime must reflect a feeding into an intersection of three separate spaces. Our model has been simplified and in its present form is not capable of providing accurate lifetime estimates.

We have considered only one-dimensional resonances, neglecting the coupling resonances, even though the operation of electron storage rings is essentially on the coupling line. Furthermore, we have neglected the dependence of the tune on the amplitude in the non-resonating dimension. This has a relevant impact on the tune parameter $\bar{\nu}$ even in the case of one-dimensional resonances, implying that only a particular region of amplitude in the non-resonant dimension can be affected by the resonance.

Another simplification is related to the sources of time modulation of ξ and $\bar{\nu}$. We have chosen one dominant source to describe the observations in each of the two cases dealt with, SPEAR and ACO. Although adequate for our purposes in establishing the mechanism, to determine a more complete picture will require the simultaneous inclusion of other time varying sources. An example of another source in SPEAR is the time modulation of the beam-beam force due to the linear betatron coupling. That is, because of the changing beam shape due to the transfer of energy between horizontal and vertical planes, the force between the beams fluctuates with the linear coupling period.

In conclusion, however, we contend that by choosing the dominant source of parameter modulation and by treating one-dimensional beam-beam resonances, we have established that these beam-beam resonances can cause growth in size and beam loss to physical apertures. Our quantitative estimates of the resonance effects suggest that a single resonance mechanism may indeed be the source of the observed beam-beam limiting strength in electron-positron collisions.

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FIGURE CAPTIONS

- Fig. 1 Detuning functions, $D(\sigma)$, for the cases $a \gg b$ and $a = b$. σ is the displacement from the strong beam center in units of the strong beam rms size.
- Fig. 2 Resonance functions $R_p(\sigma)$ for even-ordered resonances ($a \gg b$).
- Fig. 3 Resonance functions $R_p(\sigma)$ for even-ordered resonances ($a = b$).
- Fig. 4 Resonance functions $\bar{R}_p(\sigma)$ for odd-ordered resonances ($a \gg b$).
- Fig. 5 Resonance functions $\bar{R}_p(\sigma)$ for odd-ordered resonances ($a = b$).
- Fig. 6 Range of betatron amplitude (σ) for lock-in in SPEAR: Dependence on the distance of the tune from the resonance, $\bar{\nu}$, and on the synchrotron amplitude, k . For each value of $\bar{\nu}/\xi_{so}$, k_{min} is the minimum synchrotron amplitude required to reach $\sigma_{AP} = 8$. The minimum betatron amplitude, σ_{min} , during the excursion is also shown. σ_{min} is independent of k .
- Fig. 7 Trapping for the 8th order resonance ($a \gg b$). Dependence on synchrotron amplitude, k , distance of tune from the resonance, $\bar{\nu}$, and beam-beam strength, ξ_{so} . For two interaction regions per revolution, the usual strength parameter, $\xi = \frac{1}{2} \xi_{so}$.
- Fig. 8 Sketch of stopband for 8th order resonance ($a \gg b$). $\bar{\nu}$ is the distance of the tune from the resonance value, ξ is the beam-beam strength parameter.
- Fig. 9 Trapping for even-ordered resonances ($a = b$). Dependence on $\bar{\nu}_o$, the distance of the synchronous tune from the resonant tune, and on the beam-beam strength, ξ . The synchrotron amplitude, k , is taken to be $k = 3$ in these examples.

Fig. 1

Fig. 2

$\max \approx R_6(3) \approx 0.15$

Fig. 3

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Fig. 5

Fig. 6

Fig. 7

Fig. 8

Fig. 9