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THE SLOWING-DOWN DISTRIBUTION OF
ENERGETIC NUCLEI IN A FULLY IONIZED
GAS

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UNIVERSITY OF CALIFORNIA
Radiation Laboratory
Livermore, California

Contract No. W-7405-eng-48

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May 6, 1958

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THE SLOWING-DOWN DISTRIBUTION OF ENERGETIC NUCLEI IN A FULLY IONIZED GAS

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ABSTRACT

The steady-state energy distribution is obtained for a group of initially monoenergetic charged particles slowing down in a fully ionized gas. The distribution function is obtained analytically by solving the Fokker-Planck equation in energy space. With reference to thermonuclear reactions, the results obtained suggest that in some situations the contribution of the slowing-down distribution to the reaction rate may be significant.

1. INTRODUCTION

When an energetic atomic nucleus is introduced into a fully ionized gas the nucleus will tend to lose energy and slow down as it experiences successive Coulomb collisions with the electrons and ions comprising the gas. The loss of energy by the nucleus is a random process rather than a deterministic one, so that we cannot expect to predict with certainty the energy the nucleus will have at any specified time after the nucleus is introduced into the system. However, we can determine the probability $P(E, t)dE$ that, at any time t after the nucleus is introduced, the energy of the nucleus will lie between the specified limits E and $E + dE$.

We shall first consider the problem of determining the time-dependent energy distribution $P(E, t)$ of a single nucleus, and then treat the problem in which monoenergetic nuclei are introduced into the fully ionized gas at a steady rate. It will be shown that the energy distribution of the energetic nuclei that have been introduced into the system consists of two parts: a time-dependent part that represents those nuclei that are in the process of slowing down; and a part that increases steadily with time

and which represents those nuclei that have become thermalized. We shall refer to the former as the slowing-down distribution and the latter as the thermal distribution.

A typical application of the results to be obtained would be the calculation of the fast reaction rate in a fully ionized deuterium or deuterium-tritium gas. The term fast reaction rate refers to those reactions in which one of the reactant nuclei is still in the act of slowing down from an initially high energy ($\sim$ Mev). These fast reactions are distinguished from the thermal reactions taking place between nuclei belonging to a Maxwellian distribution.

It will always be assumed in the treatment to follow that the number of fast nuclei in the system is small compared with the number of thermal nuclei present. This restriction permits us to neglect the collisions of the fast nuclei among themselves.

2. THE EQUATION GOVERNING THE TIME-DEPENDENT ENERGY DISTRIBUTION

We consider a homogeneous fully ionized gas that is in thermal equilibrium at a specified temperature T . At time $t=0$, an energetic nucleus with kinetic energy E_0 is introduced into the gas. We wish to determine the probability $P(E, t)dE$ that at some later time $t > 0$ the energy of the nucleus lies between the limits E and $E+dE$.

We observe that the energy of the nucleus will change, on the average, only by small fractional amounts during each Coulomb encounter, as a consequence of the long range of the Coulomb interaction. The nucleus will slow down as a result of a large number of weak encounters rather than a small number of strong ones. For this reason, we can employ the Fokker-Planck equation as an approximate description of the diffusion of the nucleus in energy-space. That is, the probability density function $P(E, t)$ will satisfy the equation¹

$$\frac{\partial p}{\partial t} = - \frac{\partial}{\partial E} (Ap) + \frac{1}{2} \frac{\partial^2}{\partial E^2} (Bp) \quad (1)$$

in which

$$A = A(E) = \left(\frac{\Delta E}{\Delta t} \right)_{Av} \quad (2)$$

¹ M. C. Wang and G. E. Uhlenbeck, Revs. Mod. Phys. 17, 333, (1945).

$$B = B(E) = \left(\frac{(\Delta E)^2}{\Delta t} \right)_{Av} \quad (3)$$

The quantity A is the rate of increase of the mean energy of the nucleus and B is the rate of increase of the dispersion of the energy of the nucleus.

The slowing down of the nucleus will be primarily due to collisions with the free electrons of the ionized gas rather than with the ions, unless the root-mean-squared electron speed is considerably greater than the initial speed of the energetic nucleus. We shall assume that the latter is not the case, so that the energy lost by the nucleus in collisions with the ions of the ionized gas can be neglected.

Approximate expressions for A and B have been obtained by S. Chandrasekhar and are given in convenient form by L. Spitzer, Jr.² Assuming that the energy loss is mainly due to collisions with electrons, and assuming further that the initial energy E_0 of the nucleus satisfies the inequality

$$E_0 < (A_N/A_e) kT = 1823 A_N kT, \quad (4)$$

the expressions for the quantities A and B assume the simple form

$$A = -(E - 3kT/2)/\tau, \quad B = 2kTE/\tau, \quad (5), (6)$$

where A_N is the mass number of the nucleus, and A_e is the mass number of the electron ($= 1/1823$). The requirement (4) simplifies the expressions for A and B , which would otherwise be more complicated functions of E involving the error function (see reference 2). If the energetic nucleus were a 4-Mev alpha particle, for example, condition (4) would require that $kT > (1/1.823) \cong 0.5$ kev.

The quantity τ appearing in the expressions for A and B is called the slowing-down time and is equal to the time required for the mean energy of the particle to be reduced to within $(1/e)$ of its final equilibrium value. It is given by the expression

$$\tau \text{ (seconds)} = \frac{(1.00 \times 10^{13}) A_N}{n_e Z_n^2 \ln \Lambda} (kT)^{3/2}, \quad (7)$$

where

$$\Lambda = (9.32 \times 10^{13}) kT / \sqrt{n_e}, \quad (8)$$

² L. Spitzer, Jr., Physics of Fully Ionized Gases, (Interscience Pub. Inc., New York, 1956).

Z_N is the atomic number of the nucleus, and n_e is the number of electrons per cm^3 of the ionized gas. (kT is in units of kev.) For example, the slowing-down time for a deuteron in 10-kev electrons with $n_e = 5 \times 10^{22} \text{ cm}^{-3}$ is equal to 1.5×10^{-9} seconds.

If we substitute the expressions (2) and (3) for A and B into the Fokker-Planck equation (1), we obtain the following equation to be satisfied by the probability density $P(E, t)$.

$$\tau \frac{\partial P}{\partial t} = \frac{\partial}{\partial E} [(E + 3kT/2)P] + kT \frac{\partial^2}{\partial E^2} (EP) \quad (9)$$

The initial condition to be satisfied by $P(E, t)$ is that the nucleus have an energy exactly equal to E_0 when $t = 0$. That is,

$$P(E, 0) = \delta(E - E_0), \quad 0 \leq E < \infty \quad (10)$$

The boundary conditions to be satisfied by $P(E, t)$ can be taken to be:

$$P(0, t) = P(\infty, t) = 0, \quad 0 \leq t < \infty \quad (11)$$

Conditions (11) are equivalent to the requirement that the nucleus cannot attain either a negative or an infinite kinetic energy.

The problem of determining the time-dependent probability density function $P(E, t)$ has thus been reduced to the mathematical problem of solving the linear partial differential equation (9) subject to the initial and boundary conditions (10) and (11), respectively. We shall not attempt to solve this equation for the time-dependent density function $P(E, t)$, but will obtain only the steady-state solution that is approached asymptotically for large t .

It will simplify the notation if we introduce the dimensionless variables

$$\theta = t/\tau, \quad x = E/kT \quad (12)$$

Equation (9) and the initial and boundary conditions (10) and (11) then become, respectively,

$$\frac{\partial P}{\partial \theta} = \frac{\partial}{\partial x} [(x + 3/2)P] + \frac{\partial^2}{\partial x^2} (xP) \quad (13)$$

$$P(x, 0) = \delta(x - x_0), \quad 0 \leq x < \infty \quad (14)$$

$$P(0, \theta) = P(\infty, \theta) = 0, \quad 0 \leq \theta < \infty \quad (15)$$

where $x_0 = E_0/kT$ and where $P(x, \theta)$ is related to $P(E, t)$ according to the relation:

$$P(x, \theta) = kT P(E, t) \quad (16)$$

The steady-state solution of (13) that is approached for large θ is easily found to be

$$P(x, \theta) \xrightarrow[\theta \rightarrow \infty]{} P(x) = 2 \sqrt{\frac{x}{\pi}} e^{-x} \quad (17)$$

or in terms of E and t we have

$$P(E, t) \xrightarrow[t \rightarrow \infty]{} \frac{2}{\sqrt{\pi}} \frac{\sqrt{E}}{(kT)^{3/2}} e^{-E/kT} \quad (18)$$

which is simply the Maxwellian energy distribution for a particle in thermal equilibrium at the temperature of the ionized gas. We see, therefore, that the equilibrium distribution predicted by the approximate theory we have employed is exactly correct.

3. SOLUTION OF THE STEADY-STATE SLOWING-DOWN PROBLEM

We now consider a somewhat different problem in which monoenergetic nuclei of initial energy E_0 are introduced into a fully ionized gas at a steady rate of N_0 nuclei per cubic centimeter per second. We assume that the nuclei have been introduced at this steady rate for a long time ($t \gg \tau$) so that the system has reached a quasi steady-state condition of equilibrium, and that the ionized gas is maintained isothermally at the temperature T . We wish to determine the energy distribution $N(E, t)$, where $N(E, t)dE$ is the number of nuclei per cubic centimeter with energies between the limits E and $E + dE$ at the time t .

The equation that $N(E, t)$ must satisfy is the same as that satisfied by $P(E, t)$ Eq. (9). It will be convenient to write Eq. (9) as the two separate equations:

$$\frac{\partial N}{\partial t} = -\frac{\partial F}{\partial E} \quad (19)$$

$$F = -\frac{1}{\tau} \left[(E - 3kT/2)N + kT \frac{\partial}{\partial E} (EN) \right] \quad (20)$$

The quantity $F(E, t)$ represents the net flux of nuclei at the point E in energy-space at the time t . Equation (19) expresses the conservation of

nuclei in energy-space. Equations (19) and (20) can be written in the dimensionless form

$$\frac{\partial y}{\partial \theta} = - \frac{\partial f}{\partial x} \quad (21)$$

$$-f = (x - 3/2)y + \frac{\partial}{\partial x}(xy) \quad (22)$$

respectively, where

$$y = (kT/N_0\tau)N, \quad f = (1/N_0)F \quad (23)$$

We shall now look for a quasi steady-state solution of Eqs. (21) and (22) of the form:

$$y(x, \theta) = y_s(x) + \theta P(x) \quad (24)$$

That is, we look for an energy distribution that consists of two parts: a time-independent part $y_s(x)$ that represents the fast nuclei still in the act of slowing down, and a part that increases linearly with time and represents the Maxwellian distribution of the nuclei that have become thermalized.

Substituting (24) into Eqs. (21) and (22) leads to the following equation to be satisfied by the slowing-down distribution $y_s(x)$:

$$- \frac{df}{dx} = \frac{d}{dx} \left[\frac{d}{dx}(xy_s) + (x - 3/2)y_s \right] = P(x) \quad (25)$$

Integrating Eq. (25) once, we obtain:

$$\begin{aligned} -f(x) &= \frac{d}{dx}(xy_s) + (x - 3/2)y_s \\ &= \text{erf } \sqrt{x} - P(x) + \text{constant.} \end{aligned} \quad (26)$$

To evaluate the constant, we observe that:

$$f(x_0^+) = \int_{x_0}^{\infty} P(x) dx = 1 - \text{erf } \sqrt{x_0} + P(x_0) \quad (27)$$

$$f(x_0^-) = - \int_0^{x_0} P(x) dx = - \text{erf } \sqrt{x_0} + P(x_0) \quad (28)$$

That is, the flux is discontinuous at $x = x_0 (= E_0/kT)$, the magnitude of the discontinuity being given by

$$f(x_0^+) - f(x_0^-) = 1 \quad (29)$$

Hence, the equation satisfied by the slowing-down distribution is:

$$\frac{d}{dx}(xy_s) + (x - 3/2)y_s = \text{erf } \sqrt{x} - P(x), \quad 0 \leq x < x_0 \quad (30)$$

$$\frac{d}{dx}(xy_s) + (x - 3/2)y_s = \text{erf } \sqrt{x} - P(x) - 1, \quad x_0 < x < \infty \quad (31)$$

Integrating Eq. (30) we find that

$$y_s^-(x) = 2P(x) \left[1 + \sqrt{\pi} \int_0^{\sqrt{x}} e^{w^2} \text{erf}(w) dw \right] - 2 \text{erf } \sqrt{x} \quad (32)$$

$$= 2P(x) \int_0^{\pi/2} \frac{\sin^2 \theta}{\cos \theta} (e^{x \cos^2 \theta} - 1) d\theta \quad (33)$$

$$= P(x) \sum_{n=1}^{\infty} \frac{(n-1)!}{(2n+1)!} (4x)^n, \quad 0 \leq x \leq x_0 \quad (34)$$

Integrating Eq. (31) we obtain

$$y_s^+(x) = 2P(x) \left[a - \sqrt{\pi} \int_{\sqrt{x_0}}^{\sqrt{x}} e^{w^2} (1 - \text{erf}(w)) dw \right] \quad (35)$$

$$+ 2(1 - \text{erf } \sqrt{x}), \quad x_0 \leq x < \infty$$

where the constant a is to be chosen so that $y_s^-(x_0) = y_s^+(x_0)$.

For large values of x , $y_s^-(x)$ and $y_s^+(x)$ are given approximately by the relations:

$$y_s^-(x) \approx 1/(x - 3/2), \quad 3/2 \ll x \leq x_0 \quad (36)$$

$$y_s^+(x) \approx 2a P(x), \quad x_0 \leq x < \infty \quad (37)$$

A graph of the slowing-down distribution $y_f^-(x)$ is shown in Fig. 1. The Maxwellian distribution of the thermal nuclei is also shown for comparison.

The asymptotic form of $y_f^-(x)$ for large x represents the exact steady-state solution that would be obtained if the dispersion coefficient B were equal to zero. This is to be expected because the dispersion of the

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Fig. 1. Graph of the slowing-down distribution y_s^- of energetic nuclei (solid curve). The normalized Maxwell distribution P and the zero dispersion approximation $1/(x - 3/2)$ are shown for comparison (dashed curves).

energy of a fast nucleus increases as its energy decreases and will become unimportant for sufficiently large values of its energy E . A point of interest, however, is the fact that the slowing-down distribution $y_f^-(x)$ is represented with considerable accuracy by its asymptotic form (36) for values of x as small as $x = 5$. (See dashed curve of Fig. 1.) In most applications it is the values of the slowing-down distribution for energies very much larger than the thermal energy that are important, so that the simple expression (36) can often be used.

The slowing-down distribution that has been obtained can also be employed when the rate $N_0(t)$ at which the fast nuclei are introduced into the system varies with the time, provided that the rate varies slowly enough that

$$\left| \tau \frac{dN_0}{dt} \right| \ll 1 \quad (38)$$

In this situation N_s is related to the dimensionless distribution y_s according to

$$N_s = \left[\tau N_0(t)/kT \right] y_s \quad (39)$$

If on the other hand the dispersion coefficient B is set equal to zero ab initio, the time-dependent slowing-down distribution $Y_s(Z, \theta)$ can readily be obtained for any arbitrary rate $N_0(t)$ of introduction of the fast nuclei, as follows.

The equation satisfied by $Y_s(Z, \theta)$ is the first-order equation

$$\frac{\partial Y_s}{\partial \theta} = \frac{\partial}{\partial Z} (ZY_s) \quad (40)$$

where

$$Y_s(Z, 0) = (E_0 - 3kT/2)N_s(E, t) \quad (41)$$

and

$$Z = (x - 3/2)/(x_0 - 3/2) \quad (42)$$

The boundary condition to be satisfied at $E = E_0$ ($Z = 1$), is

$$Y_s(1, \theta) = \tau N_0(t) = G(\theta) \quad (43)$$

The solution of Eq. (40) that satisfies the boundary condition (43) is:

$$Y_s(Z, \theta) = (1/Z) G(\theta + \ln Z) \quad , \quad 0 < Z \leq 1 \quad (44)$$

We note that, if the rate N_0 at which the fast nuclei are introduced is independent of time, the slowing-down distribution $y_s(x)$ becomes:

$$y_s(x) = Y_s / (x_0 - 3/2) N_0 \tau = 1 / (x - 3/2) , \quad (45)$$

as asserted earlier.

In cases where a knowledge of the slowing-down distribution is required only for values of $E \gg 3kT/2$ ($Z \gg 0$), approximately correct results can be obtained by using the simple solution (44). In obtaining the result (44) we have assumed that $\theta \gg 1$ so that the initial slowing-down distribution $Y_s(Z, 0)$ existing at $t = 0$ need not be taken into account.

The physical significance of the solution (44) is that the fast nuclei slow down and steadily approach the asymptotic energy limit $3kT/2$. They become thermalized to the extent that their energy approaches the mean energy $3kT/2$ of the ionized gas, but do not approach a Maxwellian distribution because the dispersion of their energies during the time they have been slowing down has been disregarded.

4. CONCLUSIONS

The ratio of the number $N_s(E)$ of slowing-down nuclei to the number $N_T(E)$ of thermal nuclei with energy E per unit energy interval is given approximately by the relation:

$$\frac{N_s(E)}{N_T(E)} = \frac{\sqrt{\pi}}{2} \left(\frac{N_0 \tau}{N_T} \right) x^{-3/2} e^x , \quad (46)$$

provided that $x (= E/kT)$ is large compared with $3/2$. N_T denotes the total number of thermal nuclei per unit volume.

Owing to the exponential factor appearing in Eq. (46) we note that N_s can become comparable with N_T if x is very large, even though the quantity $N_0 \tau$, which is a measure of the total number of slowing-down nuclei per unit volume, is extremely small compared with the total number of thermal nuclei per unit volume. With reference to thermonuclear reactions, this result suggests that in some situations the contribution of the slowing-down nuclei to the reaction rate may be significant.