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**A PROCEDURE FOR USING MULTIVARIATE
NORMAL TOLERANCE REGIONS FOR
THE CERTIFICATION OF RELATED
CHARACTERISTICS**

C. W. Holland

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A PROCEDURE FOR USING MULTIVARIATE NORMAL TOLERANCE REGIONS FOR THE CERTIFICATION OF RELATED CHARACTERISTICS

C. W. Holland

Statistical Services Department
Y-12 Technical Division

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Adapted from a dissertation that was submitted to The University of Tennessee in partial fulfillment of the requirements for the Degree Doctor of Philosophy.

Oak Ridge Y-12 Plant

P.O. Box Y, Oak Ridge, Tennessee 37830

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ABSTRACT

The purpose of this investigation was to develop a complete procedure for the use of multivariate normal tolerance regions for the simultaneous certification of up to 30 related characteristics. A Monte Carlo integration technique was developed to investigate the accuracy of an approximation to the multivariate normal tolerance region. The results indicated that the approximation is sufficiently accurate for relatively small sample sizes.

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SUMMARY

The first step in the development of the procedure involved the determination of a method for setting specification regions. The method proposed involves using the univariate specification intervals to define a rectangular solid and using this rectangular solid in conjunction with an estimate of the variance-covariance matrix to inscribe a maximum-volume ellipsoid within the rectangular solid. This inscribed ellipsoid is the specification ellipsoid.

The second step in the development of the procedure involved generating a complete set of tables for determining the equation for the calculated tolerance ellipsoid that contains, with a certain γ degree of confidence, a proportion, P , of the population of items. In generating these tables, the most complete set of noncentral Chi-Square tables in existence was generated.

The final step involved the development of a method for testing to see if the calculated tolerance ellipsoid is within the specification ellipsoid. If this test is positive, it can be said, with $100\gamma\%$ degree of confidence, that at least $100P\%$ of the items meet the specification.

INTRODUCTION

Manufactured items often have several continuous characteristics which must be measured to determine if the items conform to customer specifications. For example, a certain alloy part might have several metallurgical characteristics (such as tensile strength, elongation, and modulus of elasticity) and several chemical characteristics (such as major constituent content and various elemental impurity contents each of which must meet some specification). The procedure used to determine if the items conform to customer specification is often referred to as a "certification procedure". Sometimes the certification procedure involves 100% sampling or the measurement of each required characteristic on every item produced. Frequently, however, the quantities produced are so large that the time or cost involved prohibits measuring all characteristics on 100% of the items. In these situations, certification must be accomplished by using some form of sampling.

Methods commonly used to accomplish the certification in sampling situations involve the use of "univariate tolerance intervals" or "attribute acceptance plans". Univariate tolerance intervals are used to make confidence statements about a proportion of items from a population based on a random sample of items from the population; attribute acceptance sampling plans are used to make probability statements about the long-run average quality levels of the population of items based on random samples of items from fixed periods of production.

Both the univariate tolerance intervals and the attribute acceptance sampling plans offer advantages in that they are well documented, easy to use, and their properties have been extensively investigated. However, they also have some serious disadvantages: First, the confidence statements cannot be made about related characteristics simultaneously. Confidence statements must be made about each characteristic separately unless the characteristics are independent. (If the characteristics are statistically independent, the product rule of probability may be used to make statements about the probability that all characteristics meet the requirements simultaneously.) Also, the buyer cannot control the relationships between characteristics with the univariate tolerance intervals or the attribute acceptance sampling plans. Only the levels and variability of the individual characteristics can be controlled.

STATEMENT OF THE PROBLEM

Just as the univariate normal distribution is frequently assumed in industrial situations involving one variable, the multivariate normal distribution is frequently assumed in situations involving more than one variable. An approximation to the tolerance region for a multivariate normal distribution has been developed by John.⁽¹⁾

The purpose of this study was to develop a complete systematic procedure for the use of multivariate normal tolerance regions in situations involving up to 30 variables. However, it was recognized that, for these regions to be of much practical value in industry, they must be reasonably accurate for relatively small sample sizes. Consequently, before developing a procedure for their use, it was necessary to investigate the accuracy of John's approximation. Fortunately, this investigation indicated that the approximation is reasonably good even for small sample sizes. Based on this, a procedure for using these regions was developed.

Facilities at the Oak Ridge National Laboratory,^(a) Oak Ridge, Tennessee, were used in the pursuit of this study.

IMPORTANCE OF THE STUDY

Use of multivariate tolerance regions offers a solution to the problem of simultaneously certifying all characteristics of a production item. Through the use of these regions, legitimate confidence statements can be made about the entire set of characteristics rather than about each characteristic separately. A typical statement is: "We are 100% confident that at least 100% of the items meet the requirements for all characteristics." Also, use of multivariate tolerance regions allows a buyer to set his specifications so he can control the relationships between characteristics in addition to the levels and variabilities of the individual characteristics.

Despite the fact that multivariate tolerance regions offer some very worthwhile advantages, they are not used in industry. There are two principal reasons for this failure: First, very few people in industry are aware of the advantages of using the multivariate tolerance regions for certification purposes; further, even those who are aware of multivariate tolerance regions and their advantages cannot use them because a procedure for their use has not been developed.

Since the objective of this study is to produce a procedure for the use of multivariate normal tolerance regions for situations involving up to 30 variables; and, since the multivariate normal distribution can be assumed in a large proportion of cases, the results of this study should be extremely useful for certification purposes in industry and in other situations where it is desired to make confidence statements about multiple characteristics of a population of items based on a sample of the items.

(a) Operated for the USAEC by the Union Carbide Corporation's Nuclear Division.

USING MULTIVARIATE NORMAL TOLERANCE REGIONS

REVIEW OF THE LITERATURE

Tolerance regions make it possible to make confidence statements about proportions of individuals of a population that are included within a given region. Tolerance regions differ from confidence regions in that confidence regions are confidence statements about certain parameters of a distribution. The earliest work on tolerance regions was by Wilks.(2) He considered the univariate case in which nothing was known about the probability density function except that it was continuous. He found a relatively simple method for determining a nonparametric tolerance interval for this situation. Wald(3) expanded Wilks' method to the multivariate case, assuming that nothing was known about the joint probability density function except that it was continuous. Several authors have published later works on nonparametric tolerance regions. Nonparametric tolerance regions are very useful in situations where nothing is assumed to be known about the density function except that it is continuous. However, the nonparametric regions have a disadvantage in that large sample sizes are required to make the desired confidence statements. Usually, if a specific density function can be assumed, a smaller sample size is required. In particular, Wald and Wolfowitz(4) have worked out a tolerance-interval solution for the univariate normal distribution which requires a much smaller sample size than the Wilks nonparametric interval solution. John(1) has extended their results to the multivariate normal distribution case. The tolerance region for the multivariate normal distribution developed by John is an ellipsoid.

If the sample mean, $\bar{m}$, of an n number of observations is an unbiased estimate, S , of Σ (the population variance-covariance matrix) distributed independently of $\bar{m}$ as a Wishart variable with f degrees of freedom, then the tolerance ellipsoid that has approximately 100 γ % probability of containing at least 100 P % of the population has for its equation:

$$H = (X - \bar{m})' S^{-1} (X - \bar{m}) = \frac{\chi'^2 (1 - P, c, c/n)}{[\chi^2 (\gamma, fc)] / fc}$$

where:

- c represents the number of characteristics;
- n the number of items measured;
- $X = [X_1, X_2, \dots, X_c]'$ a c -dimensional random vector, where X_i represents the i th characteristic;
- $\bar{m} = (m_1, m_2, \dots, m_c)'$ the sample mean vector, where each m_i term = $(X_{i1} + \dots + X_{in})/n$, the sample mean of the i th characteristic;
- S^{-1} the inverse of the sample covariance matrix of order, c ;

$\chi'^2(a, f, \lambda)$	the upper 100 α % point of $\chi'^2(f, \lambda)$, the noncentral Chi-Square distribution with f degrees of freedom and noncentrality parameter, λ , defined by $P[\chi'^2(f, \lambda) > \chi'^2(a, f, \lambda)] = \alpha$;
$\chi^2(a, f)$	the upper 100 α % point of the Chi-Square distribution with f degrees of freedom;
P	the proportion of population to be included in tolerance region, and
γ	the confidence level of probability statement.

The main problem with John's work is that no tables of H are given, and nothing is mentioned about the accuracy of the approximation.

Two very complete surveys of the tolerance region literature have been published in recent years. Chew⁽⁵⁾ has surveyed the work done prior to 1965 and Guttman⁽⁶⁾ has surveyed all work done prior to 1970, including a considerable amount of recent work in the area of Bayesian tolerance regions. Guttman also derives an approximation for the multivariate normal tolerance region using a different procedure from that of John. Both Guttman and Chew present tables for the multivariate normal case. However, these tables are very incomplete. Chew presents a table for $c = 2$, with values of n from 8 to 100; Guttman presents tables for $c = 2, 3$, and 4, with n greater than 100. Unfortunately, these tables are insufficient for most industrial applications.

Several additional articles have appeared in recent years concerning specific tolerance region problem areas, including such things as tolerance regions for censored distributions and the robustness of univariate normal tolerance regions. However, none of these documents consider the multivariate case.

DISCUSSION OF THE STUDY

Accuracy of John's Approximation

As mentioned earlier, the multivariate normal tolerance regions must be reasonably accurate for small sample sizes if they are to be of much practical value in industry. That is, if $g_X(\mu, \Sigma)$ denotes the multivariate normal density, the proportion of the multivariate normal population covered by regions, R_H , defined as all points satisfying the inequality:

$$(X - m)' S^{-1} (X - m) \leq H,$$

must be such that:

$$\text{prob} \left[\int_{R_H} \dots \int g_X(\mu, \Sigma) dX_1 \dots dX_c \geq P \right] \approx \gamma.$$

Unfortunately, the integration of the multivariate normal distribution is extremely complicated when several variables are involved. Speck⁽⁷⁾ has developed tables that can be used to integrate the trivariate normal distribution. Gupta⁽⁸⁾ has summarized all

multivariate normal integration work done prior to 1963, but all of it is confined to situations involving four or less variables. Milton(9) has developed a computer program that obtains very accurate results, but his work indicates that the computer execution times would be prohibitive for situations involving up to 30 variables. Consequently, it appears that a practical method for integrating the multivariate normal distribution has not been described in the literature.

Kahn(10) suggested that Monte Carlo integration techniques were the most practical way to integrate multiple integrals involving a large number of variables. He stated that the major problem in performing the Monte Carlo multiple integration was that it required the generation of a large number of random points to achieve accurate results. However, there have been tremendous improvements in the computer execution speeds and in the quality of random-number generating techniques since 1954. Consequently, it was felt that the Monte Carlo technique might provide a solution to the problem of integrating the multivariate normal distribution for a large number of variables.

Based on Kahn's reasoning, a Monte Carlo procedure for performing the required integration was developed. The procedure consists of considering a given multivariate normal distribution, generating a very large sample of random points (eg, 25,000) from this distribution, and counting the number of points that fall on or inside the region of integration. The ratio of the number of points falling on or inside the region to the total number of points generated is an estimate of the value of the integral. The procedure can be repeated several times to obtain an estimate of the mean value of the integral and an estimate of the error associated with this mean value.

The success of the Monte Carlo integration procedure depends on the quality of the multivariate normal random number generator. The International Mathematical and Statistical Libraries (IMSL) statistical program library(11) contains a multivariate normal random number generator program that generates 247 random points before repeating itself. The quality of the numbers generated by this program was checked by comparing the integration results obtained using the program with some exact results. Twenty-five random samples of Size 1000 each were used to obtain the average value of each integral. The comparison is given in Table 1 for the case where $\mu = 0$, $\Sigma = I$, and the region of integration is the positive c-tant. The exact value of the integral is 2^{-c} .

This comparison indicated that the Monte Carlo integration technique gives reasonably accurate results using the original IMSL multivariate normal random number generator. Better accuracy could be obtained by increasing the number of random points used in the integration. However, the IBM 360-91 CPU execution time required to obtain the results in Table 1 was 24.97 seconds and it was felt that, for situations involving up to 30 variables, the execution time might be excessive. Consequently, ways to cut down on the execution time were investigated. It was felt that the

Table 1
COMPARISON OF MONTE CARLO INTEGRATION
RESULTS WITH EXACT RESULTS
FOR POSITIVE C-TANT

Number of Variables	Monte Carlo Result	Exact Result	Absolute Difference
2	0.25308	0.250	0.00308
3	0.12736	0.125	0.00236
4	0.06412	0.0625	0.00162
5	0.03224	0.03125	0.00099
6	0.01552	0.015625	0.000105

accuracy could be improved by improving the IMSL multivariate normal random number generator program. This program calculates m multivariate normal random points by first calling a subroutine that calculates $m \times c$ univariate uniform random numbers and then calling another subroutine that transforms each of these uniform random numbers to a univariate normal random number. The univariate normal random numbers are transformed to an $m \times c$ matrix of multivariate normal deviates. The procedures used by the univariate normal and multivariate normal programs are described by Strecok(12) and Hurst and Knop,(13) respectively. These procedures appear to be the best available. However, the Oak Ridge National Laboratory Computer Library has a uniform random number subroutine developed by Coveyou(14) that is acknowledged by many numerical analysts to be the best in existence. Consequently, it was inserted into the IMSL multivariate normal random point generator program to see if the integration accuracy could be improved. The resulting program will be referred to hereafter as the "modified IMSL program".

In order to check out the modified IMSL program, it was felt that several known multivariate normal integrals should be evaluated. Anis and Lloyd(15) show that when the inverse of the covariance matrix has twos on the main diagonal, ones on the two adjacent diagonals, and zeros elsewhere, the value of the integral over the positive c -tant is $1/(1+c)$. A comparison between the Monte Carlo results obtained using the modified IMSL program and the exact results is given in Table 2 for $\mu = 0$, $R = \{X: 0 \leq X_1 < \infty\}$, and two types of covariance matrices. The first type is $\Sigma = I$ and the second is the special matrix of Anis and Lloyd.

Table 2
COMPARISON OF MONTE CARLO INTEGRATION RESULTS OBTAINED USING THE
MODIFIED IMSL PROGRAM WITH EXACT RESULTS FOR POSITIVE C-TANT

Number of Variables	Type Variance-Covariance Matrix	Exact Result	Monte Carlo Result	Absolute Difference
2	$\Sigma = I$	0.2500	0.25066	0.00066
	Anis and Lloyd	0.3333	0.33378	0.00048
3	$\Sigma = I$	0.1250	0.12573	0.00073
	Anis and Lloyd	0.2500	0.25048	0.00048
4	$\Sigma = I$	0.0625	0.06147	0.00103
	Anis and Lloyd	0.20000	0.19972	0.00028
5	$\Sigma = I$	0.03125	0.03146	0.00021
	Anis and Lloyd	0.16667	0.16592	0.00075
6	$\Sigma = I$	0.01562	0.01559	0.00003
	Anis and Lloyd	0.14286	0.14181	0.00105
30	Anis and Lloyd	0.03226	0.03291	0.00065

As indicated in the table, the approximation is considerably better than that obtained using the original program and is good enough for the purposes of this study. Also, the IBM 360-91 CPU execution time required to obtain the result for $c = 30$ was only 4.7 seconds. This reasonable time lapse indicates that Monte Carlo integration is a practical and reasonably accurate technique for integrating the multivariate normal distribution on high-speed computers.

The accuracy of John's approximation for a given number of characteristics was investigated by the following procedure:

1. A random sample of n points was generated from the c -variate normal distribution with density $g_X(\mu, \Sigma)$.
2. A tolerance ellipsoid was calculated from the sample data.
3. The c -variate normal density was integrated over the calculated tolerance ellipsoid region by the previously described Monte Carlo integration technique using 25 random samples of size 1000 each.
4. Steps 1 - 3 were repeated one hundred times to obtain a frequency distribution of the proportion of the population covered.

A computer program that performs the four steps just enumerated is given in Appendix A. The program also calculates the standard error for each integral value obtained in Step 3. The standard errors obtained were approximately equal to $\sqrt{P(1-P)/25,000}$. All 1000 random points in a sample are obtained by a single call of the modified IMSL program in order to cut down on computer execution time.

Perhaps the procedure can best be understood by considering the analogous one-variable problem. Given a univariate normal distribution with mean, μ , and a variance, σ^2 (as noted in Figure 1), a sample of size, n , is generated from this distribution and the univariate tolerance limits, S_1 and S_2 , are calculated using these sample data. Then, the normal

Figure 1. COVERAGE OF $N(\mu, \sigma)$ DISTRIBUTION BY INTERNAL (S_1, S_2) .
(Shaded Area Represents the Coverage)

distribution with mean, μ , and variance, σ^2 , is integrated from S_1 to S_2 by generating 1000 random numbers from this distribution and counting the ones that fall within the calculated region. The ratio of the ones that fall within the region to the total number generated yields an estimate of the proportion of the population covered by the region. This procedure is repeated 25 times and the estimated value of the integral is the average of the 25 proportions. A frequency distribution of the coverage is obtained by generating one hundred estimated values of the integral.

The procedure yields an estimate of the distribution of the proportion of the population covered by tolerance ellipsoids obtained using John's approximation for a fixed sample size, a fixed number of characteristics, a given variance-covariance matrix, and fixed γ and P . For example, Figure 2 shows for n values of 5, 10, 20, and 40, frequency distributions of the proportion of the population covered by the calculated tolerance ellipsoids for $c = 2$, $\gamma = P = 0.95$ and the variance-covariance matrix obtained for the first two characteristics of Table A-1, Appendix A. An estimate of the sample size required for the approximation to be accurate for any number of characteristics can be obtained by repeating the procedure for varying sample sizes and a varying number of characteristics. Figures A-1 through A-6 in Appendix A show the average proportion of the population covered for varying sample sizes with $c = 5, 10, 20$, and 30 characteristics, respectively, where the characteristics involved are the first characteristics of Tables A-1 through A-6 of Appendix A.

As indicated in Figure 2 and in Figures 3 through 6, John's approximation is valid for small sample sizes and appears to be conservative for large sample sizes. That is, for large samples the probability that the proportion of the population covered will be at least P is larger than γ . For example, when $n = 40$, $c = 2$, and $\gamma = P = 0.95$, the estimate of the probability is 0.97 that the proportion of the population covered is at least 0.95; for $c = 5$ and $n = 100$, the estimated probability is 0.99 that the proportion of the population covered is at least 0.95. When $c = 30$, a sample size of 30 is not nearly sufficient. Based on the results shown in Figures 2 through 6, it appears that a good rule of thumb is that the approximation is sufficiently accurate for $n \geq c^2$ for c less than 10, for $n \geq c^2/2$ for c between 10 and 20, and for $n \geq c^2/3$ for $c \geq 20$.

Similar studies could be performed for other variance-covariance matrices, and it is possible that different sample sizes would be required for vastly different variance-covariance matrices. However, it is felt that the matrix, which is based on actual data obtained from a preproduction process described more fully in the section that follows, entitled EXAMPLES, is fairly representative of most situations involving multiple characteristics. Also, similar studies could be performed for other values of P and γ .

Procedure for Using Multivariate Normal Tolerance Regions

Introduction - The procedure for using the multivariate normal tolerance regions is as follows:

1. A specification region is established. This specification region is in the form of an ellipsoid.
2. A random sample of n items is taken over a given time period of production. Each specified characteristic is measured on each sampled item.

Figure 2. FREQUENCY DISTRIBUTIONS OF THE PROPORTION COVERED FOR $c = 2$, $\gamma = 0.95$, AND $P = 0.95$.

Figure 3. FREQUENCY DISTRIBUTIONS OF THE PROPORTION COVERED FOR $c = 5$, $\gamma = 0.95$, AND $P = 0.95$.

*One value of 88.7 not shown.

Figure 4. FREQUENCY DISTRIBUTIONS OF THE PROPORTION COVERED FOR $c = 10$, $\gamma = 0.95$, AND $P = 0.95$.

Figure 5. FREQUENCY DISTRIBUTIONS OF THE PROPORTION COVERED FOR $c = 20$, $\gamma = 0.95$, AND $P = 0.95$.

3. The data generated in Step 2 are used to calculate a tolerance ellipsoid within which it can be said, with γ degree of confidence, that at least a proportion, P , of the population of items fall.
4. If the calculated tolerance ellipsoid is completely within the specification ellipsoid, the items produced during the given production period are acceptable. Otherwise, they are not acceptable.

Method for Constructing the Specification Ellipsoid - Using ellipsoids for specification regions allows the buyer to control the relationship among characteristics as well as the level and variability of each characteristic. However, the question arises of what method should be used to construct the specification ellipsoid. One possible method suggested by Bradley and Jackson(16) is to start with a rectangular solid defined by the univariate specification limits, use an estimate of the variance-covariance matrix, and inscribe a maximum-area ellipsoid within the rectangular solid. Of course, this method results in a smaller specification region than the rectangular region, and offers the intuitive appeal to the buyer that the resulting acceptance region will not contain any points outside the rectangular solid.

Bradley and Jackson's procedure for inscribing a maximum ellipsoid within a rectangular solid is as follows for this situation: Suppose the upper and lower specification limits for characteristic, i , is S_{iu} and S_{il} , then shift the center of the specification to the origin by the transformation $L_{iu} = S_{iu} - (S_{iu} + S_{il})/2$, $i = 1, 2, \dots, c$. First, consider the side L_{iu} , the transformed upper specification limit for the first characteristic. It is desirable to find the equation for an ellipsoid that is symmetric about the center of the solid which will just touch this side. To do this, first form the equation:

Figure 6. FREQUENCY DISTRIBUTIONS OF THE PROPORTION COVERED FOR $c = 30$, $\gamma = 0.95$, AND $P = 0.95$.

$$\lambda^2 = [L_{lu}, X_2, X_3, \dots, X_c] \hat{\Sigma}^{-1} [L_{lu}, X_2, X_3, \dots, X_c]' ,$$

in which $\hat{\Sigma}$ represents a sufficiently good estimate of the variance-covariance matrix. The desired ellipsoid is then obtained by solving the following system of equations for $X_2, X_3, \dots, X_c$:

$$\frac{\partial \lambda^2}{\partial X_2} = \frac{\partial}{\partial X_2} [L_{lu}, X_2, X_3, \dots, X_c] \hat{\Sigma}^{-1} [L_{lu}, X_2, X_3, \dots, X_c]' = 0 ,$$

$$\frac{\partial \lambda^2}{\partial X_3} = \frac{\partial}{\partial X_3} [L_{lu}, X_2, X_3, \dots, X_c] \hat{\Sigma}^{-1} [L_{lu}, X_2, X_3, \dots, X_c]' = 0, \dots,$$

$$\frac{\partial \lambda^2}{\partial X_c} = \frac{\partial}{\partial X_c} [L_{lu}, X_2, X_3, \dots, X_c] \hat{\Sigma}^{-1} [L_{lu}, X_2, X_3, \dots, X_c]' = 0 .$$

The values of $X_2, X_3, \dots, X_c$ can be substituted back into the original expression for λ^2 . This procedure can then be repeated for the remaining surfaces: $X_2 = L_{2u}, X_3 = L_{3u}, \dots, X_c = L_{cu}$. The minimum value of λ^2 obtained from these c determinates will define the largest ellipsoid that can be inscribed in the rectangular solid.

The procedure can be illustrated by an example. Suppose a two-characteristic situation is present with the following specification limits:

Characteristic	Specification Limits		Transformed Specification Limits	
	Upper	Lower	Upper	Lower
X_1	+ 12	+ 8	+ 2	- 2
X_2	+ 15	+ 9	+ 3	- 3

Further, suppose that the following estimate of the variance-covariance matrix has been obtained:

$$\hat{\Sigma} = \begin{bmatrix} \frac{2}{3} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$$

Then:

$$\lambda^2 = [X_1, X_2] \begin{bmatrix} \frac{12}{5} & -\frac{6}{5} \\ -\frac{6}{5} & \frac{8}{5} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, \text{ or}$$

$$\lambda^2 = \frac{12}{5} X_1^2 - \frac{12}{5} X_1 X_2 + \frac{8}{5} X_2^2 .$$

Hence, if $X_1 = +2$:

$$\lambda^2 = \frac{8}{5} X_2^2 - \frac{24}{5} X_2 + \frac{48}{5} , \text{ and}$$

$$\frac{\partial \lambda^2}{\partial X_2} = \frac{16}{5} X_2 - \frac{24}{5} = 0 \quad X_2 = 1.5 .$$

Substituting $X_2 = 1.5$ into this equation, and solving for λ^2 , yields:

$$\lambda^2 = \frac{8}{5} \left[\frac{3}{2} \right]^2 - \frac{24}{5} \left[\frac{3}{2} \right] + \frac{48}{5} = 6 .$$

Repeating the procedure for the $X_2 = +3$ side yields $\lambda^2 = 9$. Hence, the equation for the maximum area ellipse that can be inscribed within the rectangle defined by $X_1 = \pm 2$ and $X_2 = \pm 3$ is:

$$\frac{12}{5} X_1^2 - \frac{12}{5} X_1 X_2 + \frac{8}{5} X_2^2 = 6 .$$

The rectangle and the inscribed ellipse are illustrated in Figure 7.

The method can be used for a problem involving any number of characteristics. A computer program that uses this procedure to find the equation for the specification ellipsoid is given in Appendix B.

Determination of H Values - To evaluate H requires the percentage points of the noncentral Chi-Square distribution. Haynam and Leone⁽¹⁷⁾ have developed the most complete set of tables available. Also, Fix,⁽¹⁸⁾ Patnaik, ⁽¹⁹⁾ and Siotani⁽²⁰⁾ have published limited tables and graphs. Unfortunately, not one of these tables is nearly complete enough to satisfy the needs of this study.

Consequently, a computer program was developed to calculate the percentage points of the noncentral Chi-Square distribution. The program makes use of the following equation from the National Bureau of Standards Handbook of Mathematical Functions (1964)⁽²¹⁾:

$$\text{prob } [X \leq \chi'^2(a, f, \lambda)] = \sum_{j=0}^{\infty} \exp(-\lambda/2) \frac{\left[\frac{\lambda}{2}\right]^j}{j!} \text{prob } [X \leq \chi^2(a, f+2j)].$$

The program utilizes the IMSL statistical subroutine to obtain the probabilities for the central Chi-Square distribution. The indicated summation is then performed until the summation converges. The result is the probability that a noncentral Chi-Square distributed

Figure 7. SPECIFICATION ELLIPSE INSCRIBED WITHIN THE RECTANGLE GENERATED BY UNIVARIATE SPECIFICATION LIMITS.

random variable is less than some particular value. Therefore, if $f_z(f, \lambda)$ is the density function of the noncentral Chi-Square variable with f degrees of freedom and noncentrality parameter, λ , the percentage points are then obtained by solving for $\chi'^2(a, f, \lambda)$ in the equation:

$$\text{prob } [X \geq \chi'^2(a, f, \lambda)] = 1 - \int_0^{\chi'^2(a, f, \lambda)} f_z(f, \lambda) dz = a.$$

This equation was solved for $\chi'^2(a, f, \lambda)$ by using the guaranteed convergence technique given in Brent⁽²²⁾. The complete program is given in Appendix C. The results obtained using the program are compared in Table 3 with some of the exact published results given by Patnaik.

Table 3
COMPARISON OF EXACT WITH COMPUTER-
PROGRAM PERCENTAGE POINTS OF
NONCENTRAL CHI-SQUARE

Sample Size	λ	α	Patnaik Exact, $\chi'^2(a, n-1, \lambda)$	Computer Program, $\chi'^2(a, n-1, \lambda)$
5	1	0.05	1.765	1.76491
5	1	0.95	17.309	17.30923
8	1	0.05	2.910	2.90833
8	1	0.95	16.004	16.00398

Based on these results it appears that this computer program yields very accurate percentage points of the noncentral Chi-Square distribution. Tabulated values of the noncentral Chi-Square distribution for various values of λ and α are given in Tables C-1 through C-6 of Appendix C. Another more complete set of tables has been prepared and will be published in a subsequent report. This noncentral Chi-Square table is more complete than any of those known to have been previously published. The IBM 360-91 time required to produce the entire table was less than 30 seconds.

The program for the percentage points of the noncentral Chi-Square distribution was used in conjunction with the IMSL inverse-central Chi-Square subroutine to produce a set of tables for H. These values with $\lambda = 0.90, 0.95$, and 0.99 ; $P = 0.90, 0.95$, and 0.99 , and $c = 2$ to 30 for selected values of n are given in Tables D-1 through D-29 of Appendix D. The tables for $c = 2$ agree closely with those given by Chew and should be more accurate than Chew's values since more accurate noncentral Chi-Square values have been used.

Method to Determine if the Tolerance Ellipsoid is Within the Specification Ellipsoid - The equation for the tolerance ellipsoid is:

$$(X - m)' S^{-1} (X - m) = H$$

and the equation for the specification ellipsoid is:

$$X' \Sigma^{-1} X = \lambda^2$$

These equations can be normalized to obtain:

$$(X - m)' B (X - m) = 1 \text{ where } B = S^{-1}/H, \text{ and}$$

$$X' A X = 1 \text{ where } A = \Sigma^{-1}/\lambda^2$$

The following procedure can be used to determine if the tolerance ellipsoid is within the specification ellipsoid:

1. Maximize:

$$f(X) = (X - m)' B (X - m),$$

subject to:

$$g(X) = X' A X - 1 = 0.$$

2. Substitute the solution of Step 1 (say X^*) into $f(X)$ to obtain the maximum value of f .
3. If $f(X^*) < 1$, the tolerance ellipsoid is partially outside the specification ellipsoid; if $f(X^*) > 1$, the tolerance ellipsoid is completely within or outside of the specification ellipsoid; if $f(X^*) = 1$, the tolerance ellipsoid just touches the specification ellipsoid.
4. If $f(X^*) \geq 1$, compute $m'Am$. If this quadratic form is > 1 , the tolerance ellipsoid is outside the specification ellipsoid, otherwise it is within the specification ellipsoid.

The procedure can be illustrated by an example. Consider the problem of determining if the specification ellipse illustrated in Figure 7 contains the tolerance ellipse:

$$[(X_1 - 1, X_2 - 1)] \begin{bmatrix} 2.0 & -1.0 \\ -1.0 & 2.0 \end{bmatrix} \begin{bmatrix} X_1 - 1 \\ X_2 - 1 \end{bmatrix} = 4$$

The problem is to maximize: $f(X) = \frac{1}{2}(X_1^2 + X_2^2 - X_1X_2 - X_1 - X_2 + 1)$, subject to:

$$\frac{12}{5}X_1^2 - \frac{12}{5}X_1X_2 + \frac{8}{5}X_2^2 - 6 = 0$$

The Lagrange multiplier technique can be used to solve this problem when c is small. However, use of this technique results in a problem of solving $c + 1$ nonlinear equations for $c + 1$ unknowns, and this is extremely difficult for large c . The Oak Ridge National Laboratory Computer Library has a subroutine called GRADIENT that can be used to solve an optimization problem involving a nonlinear objective function and nonlinear constraints.(23) This subroutine uses a gradient technique to find the maximum of the function. Applications of this subroutine to this problem yielded a solution of $X_1^* = 1.925$ and $X_2^* = 2.000$ for which $f(X^*) = 0.461$. Since $f(X^*)$ is < 1 , the tolerance ellipse has to be reduced before it fits within the specification ellipse, as noted in Figure 8.

Figure 8. CONTRACTION OF A TOLERANCE ELLIPSOID [$f(X^*) = 0.461$].

The procedure will work in all instances except for the unlikely case where $m = 0$, and $A = B$. A computer program that checks to see if the tolerance ellipsoid is within the specification ellipsoid is given in Appendix E. This program utilizes the subroutine GRADIENT. The program first checks to determine if $m = 0$ and $A = B$. If both of these conditions exist, the tolerance ellipsoid is the same as the specification ellipsoid. Otherwise, the steps of the procedure are performed.

EXAMPLES

Introduction

This report section provides illustrations of the use of the multivariate normal tolerance regions through three examples. The items involved are machined uranium alloy parts. There are 30 characteristics that must be measured and certified: four metallurgical characteristics, three major chemical characteristics, and 23 chemical impurity characteristics. The metallurgical and chemical samples for each measured part are taken from the excess material that is removed during the machining process.

These examples utilize data from 106 preproduction parts used to construct the 30 characteristic variance-covariance matrix given in Tables A-1 through A-6. The first example illustrates each step of the procedure for using multivariate tolerance regions to certify product characteristics for a three-characteristic situation, the second example illustrates the case where the relationship between characteristics changes for a two-characteristic situation, the third example illustrates the use of the procedure for 13 of the characteristics.

Example 1

This example illustrates each step of the procedure for using the multivariate normal tolerance regions. The characteristics involved are the first three in Table A-1. The assumed population variance-covariance matrix and its inverse are:

$$\hat{\Sigma} = \begin{bmatrix} 0.0563 & -0.0194 & -0.0057 \\ -0.0194 & 0.0167 & 0.0038 \\ -0.0057 & 0.0038 & 0.0014 \end{bmatrix}, \text{ and}$$

$$\hat{\Sigma}^{-1} = \begin{bmatrix} 32.59 & 20.05 & 78.28 \\ 20.05 & 168.93 & -376.89 \\ 78.28 & -376.89 & 2056.02 \end{bmatrix}$$

Hence, the equation for the specification ellipsoid is:

$$(X_1, X_2, X_3) \begin{bmatrix} 32.59 & 20.05 & 78.28 \\ 20.05 & 168.93 & -376.89 \\ 78.28 & -376.89 & 2056.02 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \lambda^2.$$

The original and transformed univariate specification intervals are:

Variable	Original Univariate Specification Limits		Transformed Univariate Specification Limits	
	Upper	Lower	Upper	Lower
X_1	91.0	89.0	1.0	-1.0
X_2	8.20	6.80	0.7	-0.7
X_3	2.75	2.25	0.25	-0.25

An application of the computer program for inscribing an ellipsoid within a rectangular solid, given in Appendix B, yields the specification ellipsoid:

$$(X_1, X_2, X_3) \begin{bmatrix} 32.59 & 20.05 & 78.28 \\ 20.05 & 168.93 & -376.89 \\ 78.28 & -376.89 & 2056.02 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = 17.76.$$

The data in Table 4 were obtained from a recent month's production of parts. Estimates of the mean vector, variance-covariance matrix, and its inverse calculated from these data are:

$$m = \begin{bmatrix} 90.06 \\ 7.50 \\ 2.51 \end{bmatrix} \quad S = \begin{bmatrix} 0.0892 & -0.0271 & -0.0066 \\ -0.0271 & 0.0139 & 0.0028 \\ -0.0066 & 0.0028 & 0.0010 \end{bmatrix}, \text{ and}$$

$$S^{-1} = \begin{bmatrix} 29.404 & 43.119 & 69.987 \\ 43.119 & 227.043 & -350.750 \\ 68.987 & -350.750 & 2379.120 \end{bmatrix}$$

The H value for $n = 20$, $c = 3$, and $\gamma = P = 0.95$, obtained from Table D-2 is 11.50. Hence, the tolerance ellipsoid is:

$$\begin{bmatrix} X_1 - 90.05 \\ X_2 - 7.50 \\ X_3 - 2.51 \end{bmatrix} \begin{bmatrix} 29.404 & 43.119 & 68.987 \\ 43.119 & 227.043 & -350.750 \\ 68.987 & -350.750 & 2379.120 \end{bmatrix} \begin{bmatrix} X_1 - 90.05 \\ X_2 - 7.50 \\ X_3 - 2.51 \end{bmatrix} = 11.50.$$

Application of the program for determining if the tolerance ellipsoid is within the specification ellipsoid given in Appendix E yields $f(X^*) = 1.312$, indicating that the tolerance ellipsoid is completely within the specification ellipsoid. Consequently, the given month's production can be certified with 95% confidence that at least 95% of the parts met the specification for the three characteristics.

Example 2

A plot of the tensile strength versus elongation data for the 106 previously produced parts is given in Figure 9. As indicated in the figure, the two characteristics are highly correlated. The univariate specification region is represented by the rectangular region. Application of the method for constructing the specification ellipsoid discussed earlier (page 19) yields the solid-line elliptical specification region shown in the figure.

Table 4
DATA FROM A SAMPLE OF 20 PARTS
REPRESENTING A RECENT
ONE-MONTH PRODUCTION

Sample Number	Characteristic Number		
	1	2	3
1	89.9184	7.4638	2.4873
2	90.0035	7.5150	2.4895
3	89.6112	7.6345	2.5248
4	90.1172	7.4023	2.5245
5	89.8161	7.5153	2.4825
6	90.8547	7.2611	2.4122
7	89.9658	7.5355	2.5108
8	90.1151	7.4772	2.5093
9	90.4272	7.4088	2.5035
10	90.3279	7.3822	2.4770
11	90.3406	7.3129	2.4840
12	90.0317	7.5559	2.5486
13	90.3154	7.4735	2.5040
14	89.8750	7.5185	2.5353
15	89.5985	7.5914	2.5379
16	89.8737	7.5840	2.5026
17	90.1442	7.6998	2.5283
18	89.7869	7.6548	2.5406
19	90.1513	7.4117	2.4654
20	89.8524	7.6687	2.5289

Figure 10 contains a plot of the data from 19 randomly sampled routine production parts measured during a recent month. The ellipsoid represented by the connected dots represents the tolerance region within which it can be stated with 95% confidence that the characteristic values of at least 95% of the items produced during the month are contained. Application of the computer program given in Appendix E confirms that the certification ellipse is completely contained within the specification. Consequently, it can be stated with 95% confidence that at least 95% of the items produced during the month met the elongation and tensile strength specification.

To illustrate how the relationship among characteristics can be controlled by using the multivariate normal tolerance regions, suppose that the correlation between tensile strength and elongation changed from -0.82 to -0.40 during the month, but that the levels and variability remained the same. The resulting data are given in Figure 11.

Figure 9. TENSILE STRENGTH AS A FUNCTION OF ELONGATION WITH AN INSCRIBED SPECIFICATION ELLIPSOID.

Figure 10. CALCULATED TOLERANCE ELLIPSE FOR A RECENT ONE-MONTH SAMPLE DATA WITH A CORRELATION COEFFICIENT OF -0.82.

The calculated tolerance ellipsoid, within which it can be stated with 95% confidence that the elongation and tensile-strength values of at least 95% of the items produced during the month are contained, is represented by the connected-dot ellipsoid in the figure. Portions of this calculated ellipsoid fall outside the specification ellipsoid. This fact was confirmed by applying the program in Appendix E. An $f(X^*)$ value of 0.893 was obtained. Consequently, it cannot be stated with 95% confidence that 95% of the items produced during the month met the elongation and tensile-strength specification.

Figure 11. CALCULATED TOLERANCE ELLIPSE FROM HYPOTHETICAL DATA WITH A CORRELATION COEFFICIENT OF -0.40 .

Example 3

The procedure for using the multivariate normal tolerance regions was used to see if it could be stated with 95% confidence that at least 95% of the parts produced during a recent month met a given 13-characteristic specification. Data representing all 13 characteristics from 28 routine production parts measured during the month are reported, along with the univariate specification limits in Table 5. The covariance matrix calculated from these data are listed in Table 6. This covariance matrix was used to construct a tolerance ellipsoid representing the entire month's production.

A 13-characteristic specification ellipsoid was constructed from the univariate specification limits of Table 5 and from the 106 data points used to obtain the variance-covariance matrix given in Table A-1 through A-6. Application of the program given in Appendix E indicated that the calculated tolerance ellipsoid was completely within the specification ellipsoid, since an $f(X^*)$ value of 1.153 was obtained. Consequently, it was possible to make the desired certification statement. (This example is given to demonstrate the feasibility of using the multivariate normal tolerance regions for a large number of characteristics. A sample size of $n \geq c^2/2$ would normally be necessary in order to assure adequate coverage.)

Table 5
OBSERVATIONS FROM VARIATE NORMAL DISTRIBUTION AND UNIVARIATE SPECIFICATION LIMITS

Observation	Variate Normal Distribution and Univariate Specification Limits												
	1	2	3	4	5	6	7	8	9	10	11	12	13
1	89.99	7.35	2.43	40	11	2	63	78	35	139.3	81.6	16.59	16.45
2	89.64	7.68	2.52	56	12	5	60	93	39	145.5	80.1	15.79	16.30
3	89.87	7.58	2.47	51	8	2	66	46	28	134.7	79.3	17.53	16.38
4	89.57	7.66	2.49	54	9	5	71	56	45	138.1	81.1	17.35	16.39
5	90.30	7.42	2.43	38	13	0	34	59	20	138.2	76.4	16.49	16.25
6	90.14	7.45	2.44	45	12	2	61	74	42	139.7	82.1	17.07	16.34
7	90.20	7.31	2.39	33	12	3	71	82	41	146.0	84.1	15.25	16.39
8	89.89	7.59	2.47	62	14	1	54	64	27	140.3	77.0	17.05	16.31
9	90.01	7.43	2.46	50	11	3	65	66	27	141.2	82.4	17.09	16.37
10	89.69	7.47	2.46	55	8	5	60	102	42	147.8	83.8	14.00	16.30
11	89.66	7.64	2.52	62	11	2	54	44	18	136.6	78.2	18.24	16.28
12	89.85	7.54	2.46	62	8	2	51	53	24	136.7	77.9	17.83	16.32
13	90.34	7.27	2.41	48	11	1	62	76	21	142.5	81.8	17.43	16.36
14	89.63	7.57	2.44	36	12	8	76	67	46	150.7	82.9	13.26	16.28
15	90.01	7.40	2.45	56	8	2	77	66	33	139.3	83.5	18.21	16.39
16	90.24	7.33	2.40	39	8	4	58	98	37	146.5	82.8	15.28	16.28
17	90.03	7.56	2.45	45	9	2	58	39	24	138.6	77.6	17.60	16.38
18	89.74	7.50	2.43	40	8	6	84	59	41	146.4	81.1	15.57	16.35
19	90.27	7.42	2.41	42	10	3	53	60	30	139.7	80.1	16.89	16.27
20	89.77	7.54	2.50	62	16	1	74	54	24	143.9	82.6	15.36	16.36
21	90.23	7.45	2.41	47	6	3	80	66	35	143.1	81.0	18.09	16.39
22	90.13	7.40	2.45	53	10	0	68	61	17	139.2	77.2	17.57	16.34
23	89.55	7.57	2.49	58	16	3	71	100	35	149.8	82.1	14.21	16.31
24	89.79	7.44	2.44	50	9	5	70	84	30	143.9	83.2	16.36	16.37
25	89.53	7.47	2.50	67	11	4	73	114	38	141.9	83.4	16.79	16.27
26	89.79	7.77	2.49	63	12	2	57	31	33	141.2	75.3	16.97	16.25
27	89.75	7.51	2.49	51	12	5	58	67	25	143.4	80.3	15.75	16.27
28	90.17	7.30	2.43	44	13	3	75	72	20	140.4	80.8	16.97	16.36
Lower Specification Limit	89.00	6.80	2.25	0	0	0	0	0	0	120.0	70.0	11.0	16.10
Upper Specification Limit	91.00	8.20	2.75	200	100	20	200	200	100	160.0	90.0	22.00	16.50

Table 6
COVARIANCE MATRIX CALCULATED FROM THE DATA OF TABLE 5

Characteristic Number	Variate Normal Distribution and Univariate Specification Limits												
	1	2	3	4	5	6	7	8	9	10	11	12	13
1	0.087												
2	-0.026	0.018											
3	-0.001	0.001	0.001										
4	-3.342	0.958	0.075	4,123.00									
5	-0.879	0.415	0.027	312.77	59.830								
6	0.208	-0.141	-0.008	-205.15	-30.200	25.295							
7	4.943	-3.118	-0.200	582.91	-47.519	3.993	11,244.00						
8	-4.626	6.120	0.244	11,032.00	689.470	-452.010	-14,466.00	149,680.0					
9	4.335	1.649	-0.023	1,290.40	57.663	-69.820	-175.94	25,031.0	9,520.2				
10	0.439	0.596	-0.021	479.06	43.931	-21.830	-175.94	10,735.0	3,130.2	1,248.7			
11	0.718	0.239	0.000	457.70	30.162	-29.278	182.42	7,129.8	2,003.4	744.3	511.4		
12	-0.234	-0.246	-0.007	-223.18	-22.318	-14.758	405.25	-4,582.9	-1,184.1	-462.9	-282.8	189.4	
13	-0.015	-0.007	0.000	-14.48	-1.034	0.563	19.03	-147.1	-26.3	-11.2	-6.9	4.9	0.162

CONCLUSIONS AND RECOMMENDATIONS

Conclusions

A complete procedure for the use of multivariate normal tolerance regions for the certification of related characteristics has been developed. The procedure is as follows:

1. A specification region is established. This specification region is in the form of an ellipsoid.
2. A random sample of n items is taken over a given time period of production. Each specified characteristic is measured on each sampled item.
3. The data generated in Step 2 are used to calculate a tolerance ellipsoid within which it can be said with γ confidence that at least a proportion, P , of the population of items fall. The equation for this ellipsoid is:

$$(X - m)' S^{-1} (X - m) = H$$

4. If the calculated tolerance ellipsoid is completely within the specification ellipsoid, the items produced during the given production period are acceptable. Otherwise, they are not acceptable.

This procedure should be extremely useful in industry. It is the only method available in which controlled probability statements can be made about several characteristics simultaneously. Also, it is the only method that allows the buyer to control the relationship between the characteristics. It can be used in any situation where an underlying multivariate normal distribution can be assumed.

In mass production situations, the procedure can be used to make probability statements about production for a day or a week. In low-production situations, the certification period might be a month or longer. The procedure can also be used as a lot-acceptance criterion for large lots of items.

The procedure is quick and easy to use. All that is required is the use of the H values given in Appendix D and the two computer programs given in Appendixes B and E. A large computer is not required. However, a small computer would be a necessity when a large number of characteristics is involved.

In obtaining the procedure, the most complete set of noncentral Chi-Square tables known to be in existence have been produced. These noncentral Chi-Square values have been used to produce a set of tables of H values that are required for use in constructing the tolerance ellipsoid. Also, methods for constructing specification ellipsoids and for testing to see if the calculated tolerance ellipsoid is completely within the specification ellipsoid have been developed.

The procedure utilizes an approximation to the multivariate normal tolerance region developed by John. It has been shown that John's approximation is sufficiently accurate for use with small sample sizes.

In investigating the accuracy of John's approximation, a procedure for integrating the multivariate normal distribution was developed. This procedure appears to be the only practical method in existence for integrating the multivariate normal distribution for as many as 30 variables.

Recommendations

The study has revealed several promising areas for possible further work. Guttman(6) has developed an approximation to the multivariate normal tolerance region that differs from John's. The accuracy of his approximation could be investigated in a manner similar to the investigation of the accuracy of John's approximation performed in this study. The two approximations could then be compared to see if one is superior to the other. Also, the investigation of the accuracy of John's approximation could be extended to several types of variance-covariance matrices not considered in this study. Also, the accuracy for other values of γ and P not considered in this study could be investigated.

Another possible future research topic would be the study of the robustness of multivariate normal tolerance regions. Skewed and censored distributions are common in industry, and a study of the effect of this lack of normality on γ and P would be worthwhile.

A considerable amount of work has been done by Guttman on the power of tolerance regions. A study of the power of John's multivariate normal tolerance regions would also be beneficial.

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APPENDIX A

PROGRAM FOR CHECKING THE ACCURACY OF JOHN'S APPROXIMATION

```

        DIMENSION SIGMA(50)
        DIMENSION BSIG(50,50)
        DIMENSION SINV(50,50)
        DIMENSION TEMP(100)
        DIMENSION SSIG(500),SIG(500)
        DIMENSION XBAR(50),S(50,50)
        DIMENSION ASIG(50,50)
        DIMENSION AMU(50)
        DIMENSION RVEC(1000,13), WKVEC(1000)
        DIMENSION SD(50,50)
        REAL*8 SD,WKAREA,SINV,BSIG,SEED
        INTEGER C
        DSQRT(TTT)=SQRT(TTT)
        C=30

C
C      READ SIGMA MATRIX
C
        WRITE(6,7)
        NB=1
        DO 23 I=1,C
        READ(5,6) (ASIG(J,I),J=1,30)
        SIGMA(I)=DSQRT(ASIG(I,I))
23      CONTINUE
        DO 14 I=1,C
        L=NB
        DO 11 J=1,C
        ASIG(I,J)=ASIG(I,J)/(SIGMA(I)*SIGMA(J))
        BSIG(I,J)=ASIG(I,J)
        IF(J.GT.I) GO TO 15
        SIG(L)=ASIG(I,J)
        SSIG(L)=SIG(L)
        L=L+1
11      CONTINUE
15      CONTINUE
        KEND=NB+I-1
        FORMAT(8E10.4)
        WRITE(6,117) (SIG(L),L=NB,KEND)
117      FORMAT(1X,1P10E11.3)
14      NB=KEND+1
        READ(5,6) (AMU(L),L=1,C)
        FORMAT(13F5.0)
        WRITE(6,1)
        FORMAT(1H0,'CO-VARIANCE MATRIX'/1X)
999      WRITE(6,1)
1        FORMAT(1H1,20X,'MONTE CARLO INTEGRATION OF MULTIVARIATE NORMAL')
        IR=1000
        READ(5,2) NSTRT,NBATS,ISIG,C,NST,HHH
2        FORMAT(5I5,E10.5)
        WRITE(6,3) NSTRT,NBATS,ISIG,C,NST,HHH
3        FORMAT(1H0,10X,'NHIST = NUMBER OF HISTORIES PER SAMPLE',T55,I5/
1          11X,'NSAMP = NUMBER OF SAMPLES',T55,I5/
2          11X,'ISIG = 0/1 DO NOT READ SIGMA MATRIX/READ',T55,I5/
3          11X,'KNORM = NUMBER OF RANDOM DEVIATES',T55,I5,/
3          11X,'NST = NUMBER OF SAMPLES',T55,I5,/
*        11X,'HHH =',1PE15.5)
        NUMB=C*(C+1)/2
        N=NST

```

```

WRITE(6,3017)
IDGT=0
CALL LINV1F(BSIG,C,50,SINV,IDGT,WKAREA,IER)
3017  FORMAT(1H0,'SIGMA INVERSE'/1X)
DO 29 I=1,C
WRITE(6,3003) (SINV(I,J),J=1,C)
29  CONTINUE
    IF (ISIG.EQ.0) GO TO 21
    DO 17 I=1,NUMB
17  SIG(I)=SSIG(I)
    SEED=0.01368854963220356D0
    CALL GGNRM(SEED,N,C,SIG,IR,RVEC,WKVEC,IER)
21  CONTINUE
    CALL GGNRM1(SEED,NST,C,SIG,IR,RVEC,WKVEC,IER)
    WRITE(6,77)
77  FORMAT(1H0,45X,'S CO-VARIANCE'/1X)
    DO 19 II=1,N
    DO 19 JJ=1,C
19  RVEC(II,JJ)=RVEC(II,JJ) + AMU(JJ)
    NO=C
    XN=N
    DO 10 J=1,NO
    SUM=0.0
    DO 5 I=1,N
    SUM=SUM+ RVEC(I,J)
22  RVEC(II,JJ)=RVEC(II,JJ) + AMU(JJ)
    IVAL=0
C
C    SUBTRACT SAMPLE MEAN
C
    DO 45 J=1,NSTRT
    DO 45 MM=1,C
45  RVEC(J,MM)=RVEC(J,MM) -XBAR(MM)
C
C
C    CALCUCATE (X-M) *S'*(X-M) '
C
    DO 65 J=1,NSTRT
    DO 53 II=1,C
    SUM=0
    DO 51 JJ=1,C
51  SUM=SUM+SINV(II,JJ)*RVEC(J,JJ)
53  TEMP(II)=SUM
C
C    CALCULATE (X-X) '*S'*(X-X) '
TOTS=0
    DO 55 II=1,C
55  TOTS=TOTS+RVEC(J,II)*TEMP(II)
    IF (TOTS.LE.HHH) IVAL=IVAL+1
3000  FORMAT(1X,'X1 =',1P2E15.4,5X,'SUM =',E15.4,5X,I5)
65  CONTINUE
    AA=FLOAT(IVAL)/FLOAT(NSTRT)
    AVGE=AVGE + AA**2
    AVG=AVG+AA
    WRITE(6,60) I,AA
60  FORMAT(1X,I5,1PE15.5)
100  CONTINUE
    AVG=AVG/FLOAT(NBATS)
    AVGE=AVGE/FLOAT(NBATS)
    AVGE=DSQRT(DABS(AVGE-AVG**2)/DFLOAT(NBATS-1))
    WRITE(6,105) AVG,AVGE
105  FORMAT(1H0,'AVERAGE AREA =',1PE15.5,5X,'STD. DEV. =',E15.5)
101  CONTINUE

```

```

      GO TO 999
400  WRITE(6,303) IER,IDGT
303  FORMAT(1H0,'*** IER ***',2I6)
      5 CONTINUE
      XBAR(J)=SUM/XN
10  CONTINUE
      XN1=XN-1.0
      DO 50 I=1,N0
      DO 40 K=I,N0
      SUM=0.0
      SUB=XBAR(I)*XBAR(K)
      DO 30 J=1,N
      SUM=SUM+(RVEC(J,I)*RVEC(J,K)-SUB)
30  CONTINUE
      S(I,K)=SUM/XN1
      S(K,I)=S(I,K)
40  CONTINUE
      WRITE(6,117) (S(I,K),K=1,I)
50  CONTINUE
      WRITE(6,307) (XBAR(MM),MM=1,C)
307  FORMAT(1X,'XBAR ='/1X,1P11E12.4/1X,4E11.4)
103  FORMAT(1H0,'RVEC',1P10E12.5)
      AVG=0
      DO 47 I=1,C
      DO 47 J=1,C
      SINV(I,J)=0
47  SD(I,J)=S(I,J)
      AVGE=0
      IDGT=0

C
C      INVERT S
C
      CALL LINV1F(SD,C,50,SINV,IDGT,WKAREA,IER)
      WRITE(6,3004)
3004  FORMAT(1H0,45X,'S INVERSE'/1X)
      DO 49 I=1,C
      WRITE(6,3003) (SINV(I,L),L=1,I)
3003  FORMAT(1X,1P10E12.4)
49  CONTINUE
      IF (IER.NE.0) GO TO 400
      DO 100 I=1,NBATS
      CALL GGNRM1(SEED,NSTRT,C,SIG,IR,RVEC,WKVEC,IER)
      DO 22 II=1,NSTRT
      DO 22 JJ=1,C
      GO TO 999
      END
      SUBROUTINE GGU1(SEE,N,R)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION R(1)
      DO 1 I=1,N
1  R(I)=PLTRNF(D)
      RETURN
      END

```

Table A-1
COLUMNS 1 - 5 OF THE 30-CHARACTERISTIC
VARIANCE-COVARIANCE MATRIX

C	1	2	3	4	5
1	0.0563	-0.0194	-0.0057	- 1.5633	- 0.1126
2	-0.0194	0.0167	0.0038	0.6233	0.0469
3	-0.0057	0.0038	0.0014	0.2391	0.0309
4	-1.5633	0.6233	0.2391	94.2513	5.2866
5	-0.1126	0.0469	0.0309	5.2866	7.1637
6	-0.1222	0.0390	-0.0028	- 3.8947	- 1.3684
7	0.1017	-0.2554	-0.0455	-12.4999	3.1111
8	-1.3725	-0.5300	0.0416	29.2076	- 3.5322
9	-0.8492	0.2328	0.0038	- 5.2368	- 7.1433
10	-0.2286	-0.0182	-0.0082	- 3.0178	2.2477
11	0.0101	-0.1488	-0.0268	- 4.9661	- 1.3206
12	0.1206	0.0085	0.0011	0.2753	- 1.0277
13	0.0036	-0.0015	-0.0005	- 0.1707	- 0.0222
14	0.4527	-0.2850	-0.0649	- 7.7310	1.7807
15	0.0952	0.0829	0.0011	3.6959	0.6199
16	1.6187	0.3591	-0.2041	-37.7864	-11.7193
17	0.0120	-0.0228	-0.0026	0.2330	0.2351
18	0.1812	-0.5817	-0.2363	-34.4590	6.5234
19	0.2266	-0.0524	0.0153	0.0468	- 0.4971
20	-3.0205	1.5214	0.3904	100.7250	19.4619
21	0.2892	-0.3579	-0.0814	- 7.4824	- 2.2281
22	0.1141	-0.1010	-0.0268	1.5234	0.8070
23	1.0809	-1.0112	-0.0558	7.0469	-11.3304
24	-0.1239	-0.0489	-0.0411	0.6813	0.9210
25	-1.4268	0.5144	0.1433	1.5059	-12.2427
26	-0.2881	0.2538	0.0628	32.5175	- 3.1725
27	-0.8911	0.5536	0.1358	54.1052	- 0.9795
28	0.7435	-0.4788	-0.1711	-45.8946	- 3.3684
29	2.3570	-0.1544	-0.0632	-57.0730	-11.2163
30	-0.1102	-0.2462	-0.1033	8.7193	0.2047

Table A-2
COLUMNS 6 - 10 OF THE 30-CHARACTERISTIC
VARIANCE-COVARIANCE MATRIX

C	6	7	8	9	10
1	- 0.1222	0.1017	- 1.3725	- 0.8492	- 0.2286
2	0.0390	- 0.2554	- 0.5300	0.2328	- 0.0182
3	- 0.0028	- 0.0455	0.0416	0.0038	- 0.0082
4	- 3.8947	- 12.4999	29.2076	- 5.2368	- 3.0178
5	- 1.3684	- 3.1111	- 3.5322	- 7.1433	2.2477
6	2.9123	5.3333	8.8918	8.8918	3.0570
7	5.3333	115.3331	19.9444	19.5554	15.0933
8	8.8918	19.9444	401.2559	87.6459	38.5981
9	8.8918	19.5554	87.6459	71.7016	14.1433
10	3.0570	15.0933	38.5981	14.1433	13.7445
11	1.3837	9.6583	24.9185	8.0214	3.5094
12	- 0.8106	- 1.2350	- 14.2991	- 4.9502	- 3.7263
13	- 0.0314	0.0252	- 0.3492	- 0.0417	- 0.0647
14	- 2.6316	- 1.8889	- 21.8011	-24.3011	0.7621
15	- 1.7836	5.6667	- 6.7368	1.7076	2.1211
16	1.2573	73.7776	-257.6338	-18.4122	- 3.9367
17	0.0855	1.2833	2.1693	- 0.2196	0.3368
18	-10.9415	- 22.2221	- 95.8537	-41.3538	-14.5548
19	- 2.7982	- 6.0555	- 9.2456	-10.5234	- 3.9063
20	0.5439	33.8333	- 60.5293	19.2485	- 9.0851
21	0.1520	- 1.1111	13.6024	- 4.0643	2.1120
22	- 2.0936	- 0.0555	- 6.5117	- 6.2339	- 1.2834
23	10.1462	141.3888	52.0321	-54.6343	25.0464
24	0.7193	3.8889	20.0759	11.9649	7.7175
25	14.8655	- 24.7778	39.1636	78.4968	5.2897
26	- 3.0146	- 14.6111	- 13.2310	- 1.3977	- 8.0515
27	- 2.9211	- 30.5555	0.7807	-10.2193	- 9.2657
28	5.1901	25.2777	57.1694	11.8917	12.2821
29	- 4.9854	-222.8331	-155.1024	-46.0466	-65.8439
30	- 2.8216	- 20.9443	- 16.4152	0.9181	- 4.7857

Table A-3
COLUMNS 11 - 15 OF THE 30-CHARACTERISTIC
VARIANCE-COVARIANCE MATRIX

C	11	12	13	14	15
1	0.0101	0.1206	0.0036	0.4527	0.0952
2	- 0.1488	0.0085	-0.0015	- 0.2850	0.0829
3	- 0.0268	0.0011	-0.0005	- 0.0649	0.0011
4	- 4.9661	0.2753	-0.1707	- 7.7310	3.6959
5	- 1.3206	- 1.0277	-0.0222	1.7807	0.6199
6	1.3837	- 0.8106	-0.0314	- 2.6316	- 1.7836
7	9.6583	- 1.2350	0.0252	- 1.8889	5.6667
8	24.9185	-14.2991	-0.3492	-21.8011	- 6.7368
9	8.0214	- 4.9502	-0.0417	-24.3011	1.7076
10	3.5094	- 3.7263	-0.0647	0.7621	2.1211
11	4.5139	- 0.9746	0.0155	0.2214	- 1.1189
12	- 0.9746	1.5152	0.0269	1.0122	0.8497
13	0.0155	0.0269	0.0034	0.0404	0.0502
14	0.2214	1.0122	0.0404	25.6082	3.1023
15	- 1.1189	0.8497	0.0502	3.1023	11.0292
16	-17.3488	1.9188	-0.4012	11.9416	6.2135
17	0.3998	- 0.0804	-0.0072	0.7649	- 0.1775
18	- 5.2298	- 2.0962	0.3488	18.8654	- 7.8479
19	1.4192	1.3632	0.1206	1.1637	1.4357
20	- 8.8026	- 1.4094	0.0283	-53.5730	- 4.9971
21	5.1219	- 0.5743	0.0327	12.1170	- 2.4269
22	- 0.4257	0.5025	0.0663	3.8596	2.4678
23	47.3624	8.3177	-0.2186	146.8858	19.2690
24	0.7670	- 3.1471	0.0893	- 4.3099	2.7924
25	9.1550	- 3.2228	-0.2611	-27.4795	-22.0497
26	- 2.7370	2.1424	0.0477	-12.4386	2.4064
27	- 9.5641	4.9982	-0.0783	1.5907	9.7719
28	- 2.2667	- 6.0110	-0.1149	14.4240	1.6608
29	- 4.9286	12.1925	0.2773	-37.1170	-42.0175
30	- 5.2569	- 0.0844	-0.0007	15.1842	1.2193

Table A-4
COLUMNS 16 - 20 OF THE 30-CHARACTERISTIC
VARIANCE-COVARIANCE MATRIX

C	16	17	18	19	20
1	1.6187	0.0120	0.1812	0.2266	3.0206
2	0.3591	-0.0226	- 0.5817	- 0.0524	1.5214
3	- 0.2041	-0.0026	- 0.2363	0.0153	0.3904
4	- 37.7864	-0.2330	- 34.4590	0.0468	100.7250
5	- 11.7193	0.2351	6.5234	- 0.4971	19.4619
6	- 1.2573	0.0655	- 10.9415	- 2.7982	0.5439
7	73.7776	1.2833	- 22.2221	- 6.0555	- 33.8333
8	-257.6339	2.1693	- 95.8537	- 9.2456	- 60.5293
9	- 18.4122	-0.2196	- 41.3538	-10.5234	19.2485
10	- 3.9367	0.3368	- 14.5548	- 3.9063	- 9.0851
11	- 17.3488	0.3998	- 5.2298	1.4192	- 8.8026
12	1.9188	-0.0844	- 2.0962	1.3632	- 1.4094
13	- 0.4012	-0.0072	0.3488	0.1206	0.0283
14	11.9416	0.7649	18.8654	1.1637	- 53.5730
15	6.2135	-0.1775	- 7.8479	1.4357	- 4.9971
16	785.7173	-2.3184	- 19.5234	-31.4473	-107.2397
17	- 2.3184	0.1845	0.8082	0.0316	- 2.7328
18	10.5204	0.0002	420.5094	13.8988	38.1374
19	- 31.1173	0.0316	13.6988	15.6082	4.3713
20	-107.2397	-2.7328	38.1374	4.3713	368.2271
21	- 12.2719	1.0368	5.7134	4.6170	- 41.2426
22	- 13.1959	-0.0342	21.9327	1.8596	- 0.7866
23	- 73.9470	17.6593	- 2.6901	63.2747	-336.7393
24	12.5789	0.5029	26.4648	- 1.6988	- 3.9153
25	- 65.8683	1.9289	-141.6695	-15.0906	21.1227
26	15.0615	-1.5965	- 26.7309	6.5058	58.3127
27	- 87.4093	-0.8567	- 33.8859	- 9.9649	46.3216
28	- 8.4093	1.8433	34.1140	- 7.6316	-102.2338
29	- 18.6725	-2.1702	-135.9356	63.2162	45.9095
30	13.9619	-0.1395	40.2514	-21.4824	- 31.2280

Table A-5
COLUMNS 21 - 25 OF THE 30-CHARACTERISTIC
VARIANCE-COVARIANCE MATRIX

C	21	22	23	24	25
1	0.2002	0.1141	1.0009	- 0.1239	- 1.4268
2	- 0.3579	- 0.1010	- 1.0112	- 0.0489	0.5144
3	- 0.0814	- 0.0268	- 0.0558	- 0.0411	0.1433
4	- 7.4824	1.5234	7.0469	0.6813	1.5059
5	- 2.2281	0.8070	- 11.3304	0.9210	- 12.2427
6	0.1520	- 2.0936	10.1462	0.7193	14.8655
7	- 1.1111	- 0.0555	141.3888	3.8889	- 24.7778
8	13.6024	- 6.5117	52.0321	20.0759	39.1636
9	- 4.0643	- 6.2339	- 54.6343	11.9649	78.4968
10	2.1120	- 1.2834	25.0464	7.7175	5.2897
11	5.1219	- 0.4257	47.3624	0.7670	9.1550
12	- 0.5743	0.5025	8.3177	- 3.1471	- 3.2228
13	0.0327	0.0663	- 0.2186	- 0.0893	- 0.2611
14	12.1170	3.8596	146.8858	- 4.3099	- 27.4795
15	- 2.4269	2.4678	19.2690	2.7924	- 22.0497
16	-12.2719	-13.1959	- 73.9470	12.5789	- 65.8683
17	1.0368	- 0.0342	17.6593	0.5029	1.9289
18	5.7134	21.9327	- 2.0901	26.4648	-141.6695
19	4.6170	1.8596	63.2747	- 1.6908	- 15.0906
20	-41.2426	- 0.7866	-336.7393	- 3.9153	21.1227
21	18.6549	1.5585	136.7835	2.5643	16.5702
22	1.5585	4.2632	- 14.1959	0.6784	- 23.4619
23	136.7835	-14.1959	3065.4939	-10.3654	197.6313
24	2.5643	0.6784	- 10.3654	20.4794	1.1696
25	16.5702	-23.4619	197.6313	1.1696	404.2444
26	- 6.2895	1.6140	-129.3273	- 4.4912	- 13.5965
27	-14.7924	1.8509	- 19.0204	-11.5585	1.1988
28	6.8743	0.0176	183.7572	10.1082	- 41.4678
29	37.6783	-20.3362	-189.5614	-26.5087	161.8741
30	4.2983	9.4532	-161.5932	5.1374	- 4.5673

Table A-6
COLUMNS 26 - 30 OF THE 30-CHARACTERISTIC
VARIANCE-COVARIANCE MATRIX

C	26	27	28	29	30
1	- 0.2881	- 0.8911	0.7435	2.3570	- 0.1102
2	0.2538	0.5536	- 0.4788	- 0.1544	- 0.2462
3	0.0628	0.1358	- 0.1711	- 0.0632	- 0.1033
4	32.5175	54.1052	- 45.8946	- 57.0730	8.7193
5	- 3.1725	- 0.9795	- 3.3684	- 11.2163	0.2047
6	- 3.0146	- 2.9211	5.1901	- 4.9854	- 2.8216
7	- 14.6111	-30.5555	25.2777	-222.8331	- 20.9443
8	- 13.2310	0.7807	57.1694	-155.1024	- 16.4152
9	- 1.3977	-10.2193	11.8917	- 46.0466	0.9181
10	- 8.0515	- 9.2657	12.2821	- 65.8439	- 4.7857
11	- 2.7370	- 9.5641	6.0110	- 4.9286	- 5.2569
12	2.1424	4.9982	- 2.2667	12.1925	- 0.0844
13	0.0477	- 0.0783	- 0.1149	0.2773	- 0.0007
14	- 12.4386	1.5907	14.4240	- 37.1170	15.1842
15	2.4064	9.7719	1.6608	- 42.0175	1.2193
16	15.0615	-87.4093	- 8.4093	- 18.6725	13.9619
17	- 1.5965	- 0.8567	1.8433	- 2.1702	- 0.1395
18	- 26.7309	-33.8859	34.1140	-135.9356	40.2514
19	6.5058	- 9.9649	- 7.6316	63.2162	- 21.4824
20	58.3127	46.3216	-102.2338	45.9095	- 31.2280
21	- 6.2895	-14.7924	6.8743	37.6782	4.2983
22	1.6140	1.8509	0.0176	- 20.3362	9.4532
23	-129.3273	-19.0204	183.7572	-189.5614	161.5932
24	- 4.4912	-11.5585	10.1082	- 26.5087	5.1374
25	- 13.5965	1.1988	- 41.4678	151.8741	- 4.5673
26	35.0993	16.4853	- 38.2368	- 65.1783	- 7.3129
27	16.4853	116.4677	- 12.4210	- 68.0408	14.6784
28	- 38.2368	-12.4210	86.8011	-111.9297	4.1228
29	65.1783	-68.0408	-111.9297	1366.3191	- 99.2426
30	- 7.3129	14.6784	4.1228	- 99.2426	103.5613

APPENDIX B

PROGRAM FOR SETTING SPECIFICATION ELLIPSOIDS

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      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION SI(50,50),A(50,50),WKAREA(3000),Y(50),B(50),C(50),
1 XLAMB(50)
      DATA XMINL/1.0D60/
C
C      READ PROBLEM PARAMETERS
C
      5 READ(5,10,END=99) IORDER
      10 FORMAT(I2)
      WRITE(6,11) IORDER
      11 FORMAT('1 SIGMA-INVERSE IS OF ORDER ',I5/'0')
      IF(IORDER.GT.2.AND.IORDER.LE.50) GO TO 25
      WRITE(6,20)
      20 FORMAT(' INVALID ORDER FOR SIGMA-INVERSE')
      STOP
C      READ SIGMA-INVERSE IN LOWER-TRIANGULAR FORM
      25 DO 30 I=1,IORDER
      30 READ(5,35) (SI(I,J),J=1,I)
      35 FORMAT(8F10.0)
C      COMPLETE FULL STORAGE SIGMA-INVERSE
      DO 40 I=1,IORDER
      DO 40 J=I,IORDER
      40 SI(I,J)=SI(J,I)
      WRITE(6,41)
      41 FORMAT(' SIGMA-INVERSE =')
      DO 42 I=1,IORDER
      42 WRITE(6,43) (SI(I,J),J=1,IORDER)
      43 FORMAT('0',5D25.15/(' ',25X,4D25.15))
C      READ BOUNDARY POINTS FOR RECTANGLE
      READ(5,35) (Y(I),I=1,IORDER)
      WRITE(6,36)
      36 FORMAT('0 RECTANGLE BOUNDARY POINTS')
      WRITE(6,43) (Y(I),I=1,IORDER)
C
C      INPUT COMPLETE, PROCESS EACH RECTANGLE BOUNDARY POINT
C
      DO 75 K=1,IORDER
      LOAD RIGHT-HAND SIDE
      DO 45 I=1,IORDER
      B(I)=-Y(K)*SI(I,K)*2.0
C      CONSTRUCT COEFFICIENT MATRIX
      DO 45 J=1,IORDER
      45 A(I,J)=SI(I,J)*2.0
      DO 50 I=1,IORDER
      50 A(I,K)=0.0D0
      A(K,K)=1.0D0
C      GET SOLUTION TO AX = B. RESULT IN B
      IDGT=0
      CALL LEQT2F(A,1,IORDER,50,B,IDGT,WKAREA,IER)
      B(K)=Y(K)
      WRITE(6,55) K,B(K)
      55 FORMAT('0 FOR Y(',I2,') ='D25.15/' LAMBDA-SQUARED ATTAINS A MINIMUM
1 AT')
      DO 60 I=1,IORDER
      60 WRITE(6,65) I,B(I)
      65 FORMAT(7X,'Y(',I2,') ='D25.15)

```

```

      CALL VMULFF(SI,B,IORDER,IORDER,1,50,50,C,50,IERV)
      CALL VIPRFF(B,C,IORDER,1,1,SQLAMB)
      WRITE(6,70) SQLAMB
70  FORMAT(8X,'LAMBDA-SQUARED = ',D25.15)
      XLAMB(K)=SQLAMB
      IF(XMINL .LT. SQLAMB) GO TO 75
      XMINL=SQLAMB
      INDEX=K
75  CONTINUE
      WRITE(6,80) (I,Y(I),XLAMB(I),I=1,IORDER)
80  FORMAT('1SUMMARY OF CALCULATIONS'/'0INDEX',T30,'BOUND',T55,
      1'LAMBDA-SQUARED'/(1X,I4,2D25.15))
      WRITE(6,85) INDEX,IDGT,IER,IERV
85  FORMAT('0MINIMUM LAMBDA-SQUARED OCCURS AT INDEX = ',I3/
      1' SIGNIFICANT DIGITS IN RESULTS = ',I3/
      2' LEQT2F ERROR CODE = ',I3/
      3' VMULFF ERROR CODE = ',I3)
      GO TO 5
99  STOP
      END

```

APPENDIX C

THE NONCENTRAL CHI-SQUARE DISTRIBUTION

```

      IMPLICIT REAL*8(A-H,O-Z)
      REAL*8 MACHEP
      DATA MACHEP/3.55E-15/
      COMMON/PAREM/N,NUP,S1,UP,BOT,EPS,ALAM
      DIMENSION ALP(10),AL(10)
      DIMENSION GAM(3),H(9)
      EXTERNAL P
      ALP(1)=.01
      ALP(2)=.05
      ALP(3)=.10
      ALP(4)=.90
      ALP(5)=.95
      ALP(6)=.99
      AL(1)=0
      AL(2)=.05
      AL(3)=.25
      AL(4)=.50
      AL(5)=1.0
      AL(6)=5.0
      AL(7)=10.0
      AL(8)=25.0
      EPS=1.D-6
102  FORMAT(1H1/////////69X,'TABLE '///11X,T45,
      *'PERCENTAGE POINTS OF THE NON-CENTRAL CHI-SQUARE DISTRIBUTION:'
      * //)
      DO 300 II=1,6
      WRITE(6,102)
      N=1
      NDEL=1
1    N=N+NDEL
      P1=ALP(II)
      S1=P1
      L=0
      NU=N-1
C
C      GET NON-CENYAL CHISQ
C
      DO 200 JJ=1,8
      ALAM=AL(JJ)
      NUP=N
      ALAM=ALAM/2.0D0
      XLO=.01
      XHI=300.
      T=.5D-7
      ANS=ZERO(XLO,XHI,MACHEP,T,P)
      IF (N.GT.500) DENOM=1.0
      H(JJ)=ANS
100. CONTINUE
200  CONTINUE
110  CONTINUE
      IF (N .GT. 90) GO TO 129
      WRITE(6,103) N, (H(L),L=1,8)
103  FORMAT(20X,I3, 8F13.4)
      GO TO 131
104  FORMAT(20X,I3, 8F13.4)
129  WRITE(6,104) N, (H(L),L=1,8)
131. CONTINUE

```

```

      IF (N.EQ.30) NDEL=5
      IF (N.EQ.50) NDEL=10
      IF (N.EQ.100) NDEL=100
      IF (N.EQ.100) GO TO 120
      IF (N.EQ.200) NDEL=300
      IF (N.EQ.500) NDEL=2147483000
      IF (N.GT.1000) GO TO 120
      GO TO 1
120  CONTINUE
300  CONTINUE
      WRITE(6,102)
      STOP
      END
      SUBROUTINE NCHISQ(XSQ,NU,EPS,ALAM,P)
      IMPLICIT REAL*8 (A-H,O-Z)
      DATA NUM/100/
      CALL MDCDFI(XSQ,NU,PROB,IER)
      PROB=1.0D0-PROB
      SUM=PROB
      VAL=1.0D0
      DO 10 K=2,NUM
      J=K-1
      AJ=J
      KK=NU+ 2*J
      CALL MDCDFI(XSQ,KK,PROB,IER)
      PROB=1.0D0-PROB
      VAL=(ALAM/AJ) *VAL
      TERM=VAL*PROB
      IF ( ((SUM+TERM)-SUM) .LE.EPS) GO TO 20
      SUM=SUM + TERM
10   CONTINUE
      WRITE(6,100) NUM,NU,XSQ,VAL,TERM,SUM
100  FORMAT(1H0,'ERROR IN CONVERGENCE'/1X,2I5,1P4E15.5)
20   P=DEXP(-ALAM) *SUM
      RETURN
      END
      FUNCTION ZERO(A,B,MACHEP,T,P)
      IMPLICIT REAL*8 (A-H,O-Z)
      REAL*8 MACHEP,M
      ABS(X)=DABS(X)
      SA = A
      SB = B
      PA = P(SA)
      PB = P(SB)
10   C = SA
      PC = PA
      E = SB - SA
      D = E
20   IF (ABS(PC) .GE.ABS(PB)) GO TO 30
      SA = SB
      SB = C
      C = SA
      PA = PB
      PB = PC
      PC = PA
30   TOL = 2.0*MACHEP*ABS(SB) + T
      M = 0.5*(C - SB)
      IF ((ABS(M) .LE.TOL) .OR. (PB.EQ.0.0)) GO TO 140
      IF ((ABS(E) .GE.TOL) .AND. (ABS(PA) .GT.ABS(PB))) GO TO 40
      E = M
      D = E
      GO TO 100
40   S = PB/PA

```

```

      IF (SA.NE.C) GO TO 50
      P = 2.0*M*S
      Q = 1.0 - S
      GO TO 60
50  Q = PA/PC
      R = PB/PC
      P = S*(2.0*M*Q*(Q - R) - (SB - SA)*(R - 1.0))
      Q = (Q - 1.0)*(R - 1.0)*(S - 1.0)
60  IF (P.LE.0.0) GO TO 70
      Q = -Q
      GO TO 80
70  P = -P
80  S = E
      E = D
      IF ((2.0*P.GE.3.0*M*Q-ABS(TOL*Q)).OR.(P.GE.ABS(0.5*S*Q))) GO TO 90
      D = P/Q
      GO TO 100
90  E = M
      D = E
100 SA = SB
      PA = PB
      IF (ABS(D).LE.TOL) GO TO 110
      SB = SB + D
      GO TO 130
110 IF (M.LE.0.0) GO TO 120
      SB = SB + TOL
      GO TO 130
120 SB = SB - TOL
130 PB = P(SB)
      IF ((PB.GT.0.0).AND.(PC.GT.0.0)) GO TO 10
      IF ((PB.LE.0.0).AND.(PC.LE.0.0)) GO TO 10
      GO TO 20
140 ZERO = SB
      RETURN
      END
      FUNCTION P(X)
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/PAREM/N,NUP,S1,UP,BOT,EPS,ALAM
      A=0
      CALL NCHISQ(X,NUP,EPS,ALAM,PP)
      P=S1-PP
      RETURN
      END

```

Table C-1
 PERCENTAGE POINTS OF THE NONCENTRAL CHI-SQUARE DISTRIBUTION:
 $P [X \geq \chi^2 (0.99, f, \lambda)] = 0.99$

f	λ							
	0.0	0.5	0.25	0.50	1.0	5.0	10.0	25.0
2	0.0201	0.0206	0.0228	0.0258	0.0331	0.2251	1.2268	7.8793
3	0.1148	0.1168	0.1248	0.1356	0.1600	0.5574	1.7764	8.6143
4	0.2971	0.3008	0.3162	0.3365	0.3808	0.9609	2.3467	9.3532
5	0.5543	0.5599	0.5827	0.6123	0.6758	1.4120	2.9348	10.0959
6	0.8721	0.8794	0.9091	0.9475	1.0286	1.8989	3.5383	10.8422
7	1.2390	1.2479	1.2840	1.3303	1.4272	2.4141	4.1554	11.5919
8	1.6465	1.6568	1.6986	1.7521	1.8632	2.9529	4.7845	12.3448
9	2.0879	2.0995	2.1466	2.2065	2.3304	3.5115	5.4244	13.1009
10	2.5582	2.5710	2.6228	2.6887	2.8241	4.0875	6.0740	13.8599
11	3.0535	3.0674	3.1235	3.1947	3.3407	4.6785	6.7326	14.6219
12	3.5706	3.5855	3.6455	3.7217	3.8773	5.2831	7.3994	15.3866
13	4.1069	4.1227	4.1864	4.2671	4.4316	5.8998	8.0737	16.1539
14	4.6604	4.6771	4.7442	4.8290	5.0017	6.5275	8.7550	16.9239
15	5.2293	5.2468	5.3170	5.4057	5.5860	7.1651	9.4429	17.6963
16	5.8122	5.8304	5.9035	5.9958	6.1831	7.8120	10.1368	18.4711
17	6.4078	6.4267	6.5025	6.5981	6.7920	8.4673	10.8365	19.2482
18	7.0149	7.0345	7.1128	7.2116	7.4116	9.1305	11.5416	20.0276
19	7.6327	7.6529	7.7336	7.8353	8.0411	9.8010	12.2517	20.8091
20	8.2604	8.2811	8.3641	8.4685	8.6798	10.4784	12.9666	21.5927
21	8.8972	8.9185	9.0035	9.1106	9.3270	11.1621	13.6861	22.3784
22	9.5425	9.5643	9.6513	9.7608	9.9822	11.8518	14.4100	23.1660
23	10.1957	10.2180	10.3069	10.4188	10.6447	12.5472	15.1379	23.9555
24	10.8564	10.8791	10.9698	11.0839	11.3142	13.2480	15.8698	24.7469
25	11.5240	11.5471	11.6396	11.7558	11.9902	13.9538	16.6054	25.5401
26	12.1981	12.2217	12.3158	12.4340	12.6724	14.6644	17.3446	26.3350
27	12.8785	12.9025	12.9981	13.1182	13.3604	15.3796	18.0872	27.1317
28	13.5647	13.5890	13.6862	13.8081	14.0540	16.0991	18.8332	27.9300
29	14.2565	14.2811	14.3797	14.5034	14.7526	16.8228	19.5823	28.7299
30	14.9535	14.9785	15.0784	15.2038	15.4563	17.5505	20.3344	29.5314
35	18.5089	18.5355	18.6414	18.7743	19.0415	21.2426	24.1370	33.5613
40	22.1643	22.1921	22.3031	22.4421	22.7126	25.0125	28.0012	37.6255
45	25.9013	25.9303	26.0454	26.1898	26.4797	28.8476	31.9181	41.7202
50	29.7067	29.7366	29.8555	30.0044	30.3034	32.7382	35.8806	45.8426
60	37.4849	37.5161	37.6413	37.7978	38.1117	40.6579	43.9218	54.1608
70	45.4417	45.4742	45.6043	45.7668	46.0926	48.7287	52.0920	62.5640
80	53.5401	53.5735	53.7077	53.8751	54.2108	56.9214	60.3680	71.0400
90	61.7541	61.7884	61.9260	62.0976	62.4416	65.2152	68.7331	79.5795
100	70.0649	70.0999	70.2405	70.4156	70.7667	73.5948	77.1747	88.1749

Table C-2
 PERCENTAGE POINTS OF THE NONCENTRAL CHI-SQUARE DISTRIBUTION:
 $P[X \geq \chi^2(0.95, f, \lambda)] = 0.95$

f	λ							
	0.0	0.5	0.25	0.50	1.0	5.0	10.0	25.0
2	0.1026	0.1052	0.1162	0.1316	0.1684	0.9262	3.0114	12.0802
3	0.3518	0.3578	0.3823	0.4152	0.4887	1.5164	3.7290	12.9055
4	0.7107	0.7197	0.7564	0.8046	0.9087	2.1352	4.4551	13.7329
5	1.1455	1.1570	1.2040	1.2649	1.3941	2.7766	5.1891	14.5623
6	1.6354	1.6491	1.7047	1.7762	1.9261	3.4365	5.9305	15.3936
7	2.1673	2.1829	2.2458	2.3264	2.4937	4.1119	6.6786	16.2268
8	2.7326	2.7498	2.8190	2.9073	3.0896	4.8008	7.4330	17.0617
9	3.3251	3.3436	3.4184	3.5134	3.7087	5.5013	8.1931	17.8984
10	3.9403	3.9600	4.0396	4.1406	4.3474	6.2123	8.9587	18.7368
11	4.5748	4.5956	4.6796	4.7858	5.0028	6.9325	9.7292	19.5768
12	5.2260	5.2478	5.3356	5.4466	5.6727	7.6612	10.5044	20.4183
13	5.8919	5.9146	6.0058	6.1211	6.3555	8.3974	11.2841	21.2615
14	6.5706	6.5941	6.6886	6.8077	7.0497	9.1406	12.0678	22.1061
15	7.2609	7.2852	7.3826	7.5053	7.7541	9.8903	12.8554	22.9521
16	7.9616	7.9865	8.0866	8.2126	8.4678	10.6458	13.6468	23.7996
17	8.6718	8.6973	8.7998	8.9288	9.1899	11.4069	14.4416	24.6484
18	9.3905	9.4166	9.5214	9.6532	9.9197	12.1731	15.2397	25.4985
19	10.1170	10.1437	10.2506	10.3851	10.6566	12.9442	16.0410	26.3500
20	10.8508	10.8780	10.9869	11.1238	11.4001	13.7197	16.8453	27.2027
21	11.5913	11.6189	11.7297	11.8689	12.1497	14.4995	17.6524	28.0566
22	12.3380	12.3661	12.4786	12.6200	12.9049	15.2832	18.4622	28.9117
23	13.0905	13.1190	13.2332	13.3766	13.6655	16.0708	19.2747	29.7680
24	13.8484	13.8773	13.9930	14.1383	14.4310	16.8619	20.0897	30.6254
25	14.6114	14.6406	14.7579	14.9050	15.2011	17.6564	20.9071	31.4839
26	15.3792	15.4087	15.5274	15.6762	15.9757	18.4542	21.7267	32.3435
27	16.1514	16.1813	16.3013	16.4517	16.7544	19.2550	22.5486	33.2042
28	16.9279	16.9581	17.0793	17.2313	17.5371	20.0588	23.3727	34.0659
29	17.7084	17.7389	17.8613	18.0148	18.3235	20.8654	24.1988	34.9286
30	18.4927	18.5235	18.6470	18.8020	19.1134	21.6747	25.0269	35.7923
35	22.4650	22.4971	22.6257	22.7869	23.1104	25.7578	29.1948	40.1248
40	26.5093	26.5424	26.6752	26.8415	27.1751	29.8942	33.4034	44.4791
45	30.6123	30.6463	30.7825	30.9531	31.2951	34.0754	37.6473	48.8531
50	34.7643	34.7990	34.9382	35.1125	35.4618	38.2949	41.9219	53.2449
60	43.1880	43.2240	43.3680	43.5483	43.9096	46.8301	50.5499	62.0767
70	51.7393	51.7762	51.9242	52.1092	52.4799	55.4703	59.2659	70.9646
80	60.3915	60.4292	60.5803	60.7692	61.1476	64.1957	68.0546	79.9012
90	69.1260	69.1644	69.3181	69.5103	69.8952	72.9919	76.9049	88.8804
100	77.9295	77.9684	78.1244	78.3193	78.7097	81.8481	85.8080	97.8974

Table C-3
 PERCENTAGE POINTS OF THE NONCENTRAL CHI-SQUARE DISTRIBUTION:
 $P[X \geq \chi^2(0.90, f, \lambda)] = 0.90$

f	λ							
	0.0	0.5	0.25	0.50	1.0	5.0	10.0	25.0
2	0.2107	0.2161	0.2387	0.2701	0.3443	1.6160	4.3260	14.6947
3	0.5844	0.5942	0.6349	0.6891	0.8093	2.3252	5.1199	15.5640
4	1.0636	1.0770	1.1318	1.2035	1.3573	3.0484	5.9187	16.4346
5	1.6103	1.6265	1.6924	1.7775	1.9571	3.7844	6.7222	17.3065
6	2.2041	2.2226	2.2974	2.3933	2.5934	4.5316	7.5302	18.1797
7	2.8331	2.8534	2.9355	3.0404	3.2573	5.2888	8.3425	19.0542
8	3.4895	3.5114	3.5997	3.7120	3.9431	6.0548	9.1587	19.9298
9	4.1682	4.1914	4.2849	4.4037	4.6469	6.8287	9.9788	20.8066
10	4.8652	4.8895	4.9877	5.1120	5.3658	7.6097	10.8024	21.6846
11	5.5778	5.6032	5.7054	5.8346	6.0977	8.3972	11.6294	22.5637
12	6.3038	6.3301	6.4359	6.5694	6.8408	9.1905	12.4595	23.4438
13	7.0415	7.0686	7.1776	7.3151	7.5939	9.9893	13.2927	24.3250
14	7.7895	7.8174	7.9293	8.0703	8.3559	10.7931	14.1287	25.2073
15	8.5468	8.5753	8.6898	8.8340	9.1257	11.6014	14.9674	26.0905
16	9.3122	9.3414	9.4583	9.6054	9.9028	12.4141	15.8087	26.9747
17	10.0852	10.1149	10.2340	10.3838	10.6864	13.2309	16.6525	27.8599
18	10.8649	10.8951	11.0163	11.1686	11.4760	14.0513	17.4986	28.7460
19	11.6509	11.6816	11.8047	11.9593	12.2711	14.8753	18.3469	29.6330
20	12.4426	12.4737	12.5986	12.7554	13.0713	15.7027	19.1974	30.5209
21	13.2396	13.2711	13.3976	13.5564	13.8763	16.5331	20.0500	31.4096
22	14.0415	14.0734	14.2014	14.3621	14.6856	17.3666	20.9045	32.2992
23	14.8480	14.8802	15.0097	15.1721	15.4991	18.2028	21.7609	33.1897
24	15.6587	15.6913	15.8221	15.9862	16.3164	19.0417	22.6191	34.0809
25	16.4734	16.5064	16.6385	16.8041	17.1374	19.8832	23.4791	34.9729
26	17.2919	17.3252	17.4585	17.6256	17.9617	20.7271	24.3407	35.8657
27	18.1139	18.1475	18.2819	18.4505	18.7893	21.5733	25.2040	36.7593
28	18.9392	18.9731	19.1086	19.2785	19.6200	22.4217	26.0688	37.6536
29	19.7677	19.8018	19.9384	20.1096	20.4536	23.2722	26.9352	38.5486
30	20.5992	20.6336	20.7712	20.9436	21.2899	24.1248	27.8030	39.4443
35	24.7967	24.8321	24.9740	25.1517	25.5084	28.4153	32.1624	43.9332
40	29.0505	29.0868	29.2323	29.4143	29.7795	32.7463	36.5522	48.4379
45	33.3504	33.3874	33.5358	33.7215	34.0939	37.1114	40.9683	52.9570
50	37.6886	37.7263	37.8772	38.0661	38.4445	41.5059	45.4076	57.4893
60	46.4589	46.4976	46.6526	46.8465	47.2348	50.3685	54.3458	66.5894
70	55.3289	55.3685	55.5266	55.7245	56.1207	59.3120	63.3508	75.7315
80	64.2778	64.3180	64.4788	64.6799	65.0824	68.3211	72.4113	84.9100
90	73.2911	73.3318	73.4948	73.6985	74.1064	77.3848	81.5190	94.1207
100	82.3581	82.3993	82.5641	82.7701	83.1825	86.4951	90.6672	103.3600

Table C-4
 PERCENTAGE POINTS OF THE NONCENTRAL CHI-SQUARE DISTRIBUTION:
 $P [X \geq \chi^2(0.10, f, \lambda)] = 0.10$

f	λ							
	0.0	0.5	0.25	0.50	1.0	5.0	10.0	25.0
2	4.6052	4.7201	5.1739	5.7251	6.7701	13.6402	20.9399	40.5818
3	6.2514	6.3554	6.7665	7.2692	8.2371	14.8964	22.1281	41.7046
4	7.7794	7.8765	8.2610	8.7333	9.6504	16.1433	23.3126	42.8263
5	9.2364	9.3286	9.6945	10.1452	11.0254	17.3817	24.4935	43.9470
6	10.6446	10.7332	11.0851	11.5196	12.3714	18.6125	25.6710	45.0666
7	12.0170	12.1028	12.4437	12.8653	13.6943	19.8362	26.8453	46.1853
8	13.3616	13.4450	13.7769	14.1881	14.9982	21.0536	28.0165	47.3030
9	14.6837	14.7652	15.0897	15.4920	16.2862	22.2650	29.1848	48.4197
10	15.9872	16.0670	16.3852	16.7801	17.5606	23.4709	30.3504	49.5355
11	17.2750	17.3535	17.6662	18.0545	18.8231	24.6718	31.5133	50.6503
12	18.5493	18.6266	18.9345	19.3171	20.0752	25.8680	32.6737	51.7643
13	19.8119	19.8881	20.1918	20.5693	21.3180	27.0597	33.8317	52.8774
14	21.0641	21.1393	21.4392	21.8123	22.5525	28.2474	34.9874	53.9897
15	22.3071	22.3814	22.6780	23.0469	23.7795	29.4311	36.1409	55.1011
16	23.5418	23.6154	23.9088	24.2740	24.9996	30.6112	37.2922	56.2117
17	24.7690	24.8419	25.1325	25.4944	26.2136	31.7879	38.4415	57.3215
18	25.9894	26.0616	26.3496	26.7084	27.4218	32.9613	39.5889	58.4306
19	27.2036	27.2751	27.5608	27.9167	28.6246	34.1316	40.7343	59.5388
20	28.4120	28.4830	28.7665	29.1197	29.8226	35.2990	41.8780	60.6464
21	29.6151	29.6856	29.9671	30.3178	31.0160	36.4636	43.0199	61.7531
22	30.8133	30.8833	31.1629	31.5114	32.2052	37.6256	44.1601	62.8592
23	32.0069	32.0765	32.3543	32.7006	33.3903	38.7851	45.2986	63.9645
24	33.1962	33.2654	33.5415	33.8859	34.5717	39.9422	46.4356	65.0692
25	34.3816	34.4503	34.7249	35.0674	35.7496	41.0969	47.5711	66.1732
26	35.5632	35.6315	35.9047	36.2453	36.9241	42.2495	48.7051	67.2765
27	36.7412	36.8092	37.0810	37.4200	38.0955	43.3999	49.8376	68.3791
28	37.9159	37.9836	38.2541	38.5914	39.2639	44.5483	50.9688	69.4811
29	39.0875	39.1548	39.4241	39.7599	40.4294	45.6948	52.0986	70.5825
30	40.2560	40.3231	40.5911	40.9255	41.5923	46.8394	53.2271	71.6833
35	46.0588	46.1246	46.3875	46.7156	47.3703	52.5366	58.8515	77.1782
40	51.8051	51.8698	52.1286	52.4517	53.0965	58.1959	64.4482	82.6591
45	57.5053	57.5692	57.8246	58.1435	58.7801	63.8229	70.0208	88.1271
50	63.1671	63.2303	63.4828	63.7981	64.4278	69.4218	75.5719	93.5832
60	74.3970	74.4590	74.7069	75.0165	75.6350	80.5489	86.6181	104.4631
70	85.5270	85.5881	85.8324	86.1376	86.7473	91.5978	97.6010	115.3045
80	96.5782	96.6386	96.8799	97.1815	97.7842	102.5827	108.5309	126.1120
90	107.5650	107.6248	107.8637	108.1623	108.7591	113.5142	119.4154	136.8894
100	118.4980	118.5572	118.7942	119.0903	119.6820	124.4001	130.2606	147.6398

Table C-5
 PERCENTAGE POINTS OF THE NONCENTRAL CHI-SQUARE DISTRIBUTION:
 $P [X \geq \chi'^2 (0.05, f, \lambda)] = 0.05$

f	λ							
	0.0	0.5	0.25	0.50	1.0	5.0	10.0	25.0
2	5.9915	6.1404	6.7167	7.3962	8.6422	16.3835	24.3441	45.3082
3	7.8147	7.9444	8.4509	9.0588	10.2025	17.6946	25.5754	46.4611
4	9.4877	9.6059	10.0702	10.6329	11.7072	18.9959	26.8026	47.6128
5	11.0705	11.1809	11.6162	12.1471	13.1702	20.2882	28.0258	48.7632
6	12.5916	12.6963	13.1101	13.6171	14.6004	21.5722	29.2452	49.9125
7	14.0671	14.1674	14.5646	15.0527	16.0039	22.8484	30.4611	51.0606
8	15.5073	15.6041	15.9879	16.4606	17.3852	24.1175	31.6735	52.2076
9	16.9190	17.0128	17.3855	17.8455	18.7476	25.3799	32.8826	53.3535
10	18.3070	18.3985	18.7618	19.2109	20.0936	26.6362	34.0886	54.4982
11	19.6751	19.7645	20.1197	20.5594	21.4253	27.8866	35.2916	55.6420
12	21.0261	21.1136	21.4619	21.8933	22.7443	29.1316	36.4917	56.7847
13	22.3620	22.4480	22.7901	23.2143	24.0520	30.3714	37.6890	57.9264
14	23.6848	23.7693	24.1060	24.5237	25.3495	31.6065	38.8836	59.0670
15	24.9958	25.0790	25.4108	25.8227	26.6377	32.8370	40.0756	60.2068
16	26.2962	26.3783	26.7057	27.1123	27.9175	34.0632	41.2652	61.3455
17	27.5871	27.6682	27.9915	28.3933	29.1896	35.2854	42.4524	62.4833
18	28.8693	28.9494	29.2691	29.6664	30.4545	36.5036	43.6373	63.6202
19	30.1435	30.2228	30.5390	30.9324	31.7128	37.7183	44.8199	64.7562
20	31.4104	31.4889	31.8020	32.1916	32.9650	38.9294	46.0004	65.8913
21	32.6706	32.7483	33.0586	33.4447	34.2115	40.1371	47.1788	67.0255
22	33.9244	34.0015	34.3091	34.6919	35.4527	41.3417	48.3553	68.1589
23	35.1725	35.2489	35.5539	35.9338	36.6888	42.5433	49.5297	69.2914
24	36.4150	36.4909	36.7936	37.1706	37.9202	43.7419	50.7023	70.4231
25	37.6525	37.7278	38.0283	38.4026	39.1472	44.9377	51.8731	71.5540
26	38.8851	38.9599	39.2584	39.6302	40.3700	46.1309	53.0421	72.6841
27	40.1133	40.1875	40.4841	40.8535	41.5889	47.3215	54.2094	73.8134
28	41.3371	41.4109	41.7056	42.0729	42.8039	48.5096	55.3750	74.9420
29	42.5570	42.6303	42.9233	43.2884	44.0154	49.6953	56.5390	76.0698
30	43.7730	43.8459	44.1372	44.5003	45.2235	50.8787	57.7014	77.1968
35	49.8018	49.8730	50.1572	50.5116	51.2180	56.7641	63.4914	82.8214
40	55.7585	55.8282	56.1066	56.4541	57.1470	62.6027	69.2478	88.4291
45	61.6562	61.7247	61.9985	62.3402	63.0217	68.4011	74.9745	94.0212
50	67.5048	67.5723	67.8421	68.1789	68.8510	74.1646	80.6747	99.5989
60	79.0819	79.1478	79.4113	79.7402	80.3970	85.6028	92.0060	110.7147
70	90.5312	90.5959	90.8544	91.1773	91.8222	96.9427	103.2585	121.7832
80	101.8795	101.9431	102.1978	102.5158	103.1511	108.2018	114.4446	132.8099
90	113.1453	113.2081	113.4595	113.7735	114.4009	119.3933	125.5738	143.7992
100	124.3421	124.4043	124.6529	124.9635	125.5842	130.5270	136.6536	154.7550

Table C-6
 PERCENTAGE POINTS OF THE NONCENTRAL CHI-SQUARE DISTRIBUTION:
 $P[X \geq \chi'^2(0.01, f, \lambda)] = 0.01$

f	λ							
	0.0	0.5	0.25	0.50	1.0	5.0	10.0	25.0
2	9.2103	9.4369	10.2784	11.2165	12.8487	22.2324	31.4363	54.8833
3	11.3449	11.5320	12.2448	13.0694	14.5609	23.6380	32.7427	56.0897
4	13.2767	13.4414	14.0774	14.8282	16.2179	25.0333	34.0446	57.2947
5	15.0863	15.2363	15.8202	16.5183	17.8303	26.4188	35.3420	58.4982
6	16.8119	16.9513	17.4972	18.1554	19.4058	27.7952	36.6351	59.7004
7	18.4753	18.6068	19.1234	19.7500	20.9501	29.1629	37.9241	60.9011
8	20.0902	20.2154	20.7086	21.3096	22.4676	30.5226	39.2092	62.1005
9	21.6660	21.7860	22.2601	22.8397	23.9619	31.8746	40.4904	63.2985
10	23.2092	23.3250	23.7830	24.3445	25.4359	33.2194	41.7680	64.4952
11	24.7250	24.8371	25.2814	25.8273	26.8916	34.5574	43.0420	65.6907
12	26.2170	26.3260	26.7584	27.2907	28.3310	35.8889	44.3125	66.8849
13	27.6882	27.7946	28.2165	28.7368	29.7559	37.2143	45.5798	68.0778
14	29.1412	29.2452	29.6579	30.1676	31.1675	38.5339	46.8438	69.2695
15	30.5779	30.6797	31.0843	31.5843	32.5670	39.8479	48.1047	70.4600
16	31.9999	32.0998	32.4970	32.9884	33.9554	41.1567	49.3626	71.6494
17	33.4086	33.5068	33.8973	34.3808	35.3335	42.4604	50.6176	72.8375
18	34.8053	34.9019	35.2863	35.7626	36.7022	43.7593	51.8697	74.0246
19	36.1909	36.2860	36.6649	37.1346	38.0620	45.0536	53.1192	75.2105
20	37.5662	37.6601	38.0338	38.4974	39.4136	46.3434	54.3659	76.3953
21	38.9322	39.0248	39.3938	39.8518	40.7575	47.6291	55.6101	77.5790
22	40.2893	40.3808	40.7455	41.1983	42.0942	48.9107	56.8518	78.7617
23	41.6384	41.7288	42.0894	42.5373	43.4241	50.1885	58.0910	79.9433
24	42.9798	43.0693	43.4261	43.8694	44.7477	51.4625	59.3279	81.1239
25	44.3141	44.4027	44.7559	45.1949	46.0652	52.7329	60.5625	82.3035
26	45.6417	45.7294	46.0793	46.5143	47.3770	53.9998	61.7948	83.4820
27	46.9629	47.0498	47.3966	47.8278	48.6834	55.2634	63.0250	84.6596
28	48.2782	48.3644	48.7082	49.1358	49.9846	56.5238	64.2530	85.8362
29	49.5879	49.6733	50.0143	50.4386	51.2810	57.7811	65.4790	87.0119
30	50.8922	50.9769	51.3153	51.7364	52.5727	59.0354	66.7029	88.1866
35	57.3421	57.4239	57.7509	58.1582	58.9681	65.2651	72.7942	94.0465
40	63.6907	63.7703	64.0882	64.4844	65.2731	71.4327	78.8419	99.8849
45	69.9568	70.0345	70.3450	70.7321	71.5032	77.5466	84.8507	105.7032
50	76.1539	76.2300	76.5343	76.9138	77.6701	83.6134	90.8246	111.5028
60	88.3794	88.4530	88.7474	89.1147	89.8473	95.6269	102.6803	123.0503
70	100.4252	100.4969	100.7837	101.1416	101.8559	107.5060	114.4301	134.5356
80	112.3288	112.3990	112.6797	113.0302	113.7297	119.2743	126.0901	145.9654
90	124.1163	124.1852	124.4610	124.8053	125.4928	130.9493	137.6727	157.3454
100	135.8067	135.8746	136.1462	136.4853	137.1626	142.5442	149.1876	168.6802

APPENDIX D

H VALUES

```

      IMPLICIT REAL*8 (A-H,O-Z)
      REAL*8 MACHEP
      DATA MACHEP/3.55E-15/
      COMMON/PAREN/N,NUP,S1,UP,BOT,EPS,ALAM
      DIMENSION GAM(3),H(9)
      EXTERNAL F
      EPS=1.D-6
102   FORMAT(1H1,T20,'GAM=.90',T61,'GAM=.95',T91,'GAM=.99'/
      *1X,T7,'P=',T14,'.90',T24,'.95',T35,'.99',T55,'.90',T65,'.95',
      X T75,'.99',T85,'.90',T95,'.95',T105,'.99')
      GAM(1)=0.90
      GAM(2)=0.95
      GAM(3)=0.99
      DO 120 JJ=2,30
      WRITE(6,125)JJ
125   FORMAT(1H1//////////1X,T43,'TABLE '//1X,T40,'H VALUES(C = ',I2,') '
      *//1X,T30,'=.90',T54,'=.95',T78,'=.99'//1X,T17,'N',T21,'P=.90',T29,
      *,'P=.95',T37,'P=.99',T45,'P=.90',T53,'P=.95',T61,'P=.99',T69,
      * 'P=.90',T77,'P=.95',T85,'P=.99'/)
      N=1
      NDEL=1
1    N=N+NDEL
      NUP=JJ
      ALAM=DFLOAT(NUP)/DFLOAT(N)
      ALAM=ALAM/2.0D0
      L=0
      DO 110 I=1,3
      DO 100 J=1,3
      P1=GAM(J)
      L=L+1
      S1=P1
      NU=N-1

C
C   GET CENTRAL CHISQ
C
      DF=DFLOAT(NU)*DFLOAT(NUP)
      P2=GAM(I)
      P2=1.0D0-P2
      CALL MDCHI(P2,DF,X,IER)
      DENOM=X/DF

C
C   GET NON-CENYAL CHISQ
C
      XLO=.01
      XHI=75.0
      T=.5D-7
      ANS=ZERO(XLO,XHI,MACHEP,T,F)
      IF(N.GT.500)DENOM=1.0
      H(L)=ANS/DENOM
100   CONTINUE
110   CONTINUE
      IF(N.GT.90) GO TO 129
      WRITE(6,103)N,(H(L),L=1,9)
103   FORMAT(15X,I2,1X,9F8.2)
      GO TO 131
129   IF(N.GT.500)GO TO 137
      WRITE(6,104)N,(H(L),L=1,9)
      GO TO 131
137   WRITE(6,107)(H(L),L=1,9)
107   FORMAT(18X,9F8.2)
139   CONTINUE

```

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104  FORMAT(14X,I3,1X,9F8.2)
131  CONTINUE
      IF(N.EQ.30) NDEL=5
      IF(N.EQ.50) NDEL=10
      IF(N.EQ.100) NDEL=100
      IF(N.EQ.200) NDEL=300
      IF(N.EQ.500) NDEL=2147483000
      IF(N.GT.1000) GO TO 120
      GO TO 1
120  CONTINUE
      WRITE(6,102)
      STOP
      END
      SUBROUTINE NCHISQ(XSQ,NU,EPS,ALAM,P)
      IMPLICIT REAL*8(A-H,O-Z)
      DATA NUM/100/
      CALL MDCDFI(XSQ,NU,PROB,IER)
      PROB=1.0D0-PROB
      SUM=PROB
      VAL=1.0D0
      GO TO 100
40   S = PB/PA
      IF (SA.NE.C) GO TO 50
      P = 2.0*M*S
      Q = 1.0 - S
      GO TO 60
50   Q = PA/PC
      R = PB/PC
      P = S*(2.0*M*Q*(Q - R) - (SB - SA)*(R - 1.0))
      Q = (Q - 1.0)*(R - 1.0)*(S - 1.0)
60   IF (P.LE.0.0) GO TO 70
      Q = -Q
      GO TO 80
70   P = -P
80   S = E
      E = D
      IF ((2.0*P.GE.3.0*M*Q-ABS(TOL*Q)).OR.(P.GE.ABS(0.5*S*Q))) GO TO 90
      D = P/Q
      GO TO 100
90   E = M
      D = E
100  SA = SB
      PA = PB
      IF (ABS(D).LE.TOL) GO TO 110
      SB = SB + D
      GO TO 130
110  IF (M.LE.0.0) GO TO 120
      SB = SB + TOL
      GO TO 130
120  SB = SB - TOL
130  PB = P(SB)
      IF ((PB.GT.0.0).AND.(PC.GT.0.0)) GO TO 10
      IF ((PB.LE.0.0).AND.(PC.LE.0.0)) GO TO 10
      GO TO 20
140  ZERO = SB
      RETURN
      END
      FUNCTION F(X)
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/PAREM/N,NUP,S1,UP,BOT,EPS,ALAM
      A=0
      CALL NCHISQ(X,NUP,EPS,ALAM,PP)
      F=S1-PP
      RETURN
      END

```

Table D-1
H VALUES (C = 2)

N	$\gamma = 0.90$			$\gamma = 0.95$			$\gamma = 0.99$		
	P = 0.90	P = 0.95	P = 0.99	P = 0.90	P = 0.95	P = 0.99	P = 0.90	P = 0.95	P = 0.99
2	64.27	82.04	121.97	132.11	168.65	250.73	674.56	861.08	1280.20
3	22.87	29.43	44.34	34.23	44.05	66.36	81.89	105.39	158.76
4	15.59	20.14	30.54	21.02	27.15	41.17	39.42	50.92	77.22
5	12.63	16.35	24.89	16.13	20.88	31.78	26.83	34.73	52.87
6	11.02	14.28	21.80	13.61	17.64	26.92	20.98	27.20	41.51
7	10.00	12.98	19.84	12.07	15.66	23.93	17.67	22.93	35.05
8	9.30	12.07	18.48	11.03	14.32	21.91	15.57	20.21	30.92
9	8.78	11.41	17.47	10.27	13.34	20.44	14.08	18.29	28.01
10	8.39	10.89	16.70	9.70	12.61	19.32	13.01	16.90	25.90
11	8.07	10.49	16.08	9.25	12.03	18.44	12.17	15.81	24.24
12	7.81	10.15	15.57	8.89	11.56	17.73	11.51	14.96	22.95
13	7.60	9.88	15.15	8.59	11.17	17.14	10.97	14.26	21.88
14	7.42	9.64	14.80	8.34	10.84	16.64	10.52	13.68	20.99
15	7.26	9.44	14.49	8.12	10.56	16.21	10.15	13.19	20.25
16	7.12	9.26	14.22	7.94	10.32	15.84	9.82	12.77	19.60
17	7.01	9.11	13.99	7.77	10.11	15.52	9.54	12.41	19.05
18	6.90	8.97	13.78	7.63	9.92	15.23	9.30	12.09	18.56
19	6.80	8.85	13.59	7.50	9.75	14.98	9.08	11.81	18.14
20	6.72	8.74	13.42	7.38	9.60	14.75	8.88	11.55	17.75
21	6.64	8.64	13.27	7.28	9.47	14.54	8.71	11.33	17.41
22	6.57	8.55	13.13	7.18	9.34	14.35	8.56	11.13	17.10
23	6.51	8.46	13.00	7.10	9.23	14.18	8.41	10.94	16.81
24	6.45	8.39	12.88	7.02	9.13	14.02	8.28	10.77	16.55
25	6.39	8.32	12.78	6.95	9.03	13.88	8.16	10.62	16.31
26	6.34	8.25	12.68	6.88	8.95	13.74	8.05	10.48	16.10
27	6.30	8.19	12.58	6.81	8.86	13.62	7.95	10.34	15.89
28	6.25	8.13	12.50	6.76	8.79	13.50	7.86	10.22	15.71
29	6.21	8.08	12.42	6.70	8.72	13.40	7.77	10.11	15.53
30	6.17	8.03	12.34	6.65	8.65	13.30	7.69	10.00	15.37
35	6.01	7.82	12.02	6.44	8.38	12.87	7.35	9.56	14.69
40	5.89	7.66	11.78	6.28	8.17	12.55	7.09	9.23	14.18
45	5.79	7.54	11.59	6.15	8.00	12.29	6.89	8.97	13.78
50	5.72	7.44	11.43	6.04	7.86	12.09	6.73	8.76	13.46
60	5.59	7.28	11.18	5.88	7.65	11.76	6.48	8.43	12.96
70	5.50	7.16	11.00	5.76	7.50	11.52	6.30	8.20	12.60
80	5.43	7.06	10.86	5.67	7.38	11.34	6.16	8.01	12.32
90	5.37	6.99	10.74	5.59	7.28	11.19	6.05	7.87	12.09
100	5.32	6.93	10.65	5.53	7.20	11.07	5.95	7.75	11.91
200	5.08	6.61	10.17	5.22	6.79	10.44	5.49	7.15	10.99
500	4.89	6.37	9.79	4.97	6.47	9.95	5.13	6.68	10.27
∞	4.61	5.99	9.21	4.61	5.99	9.21	4.61	5.99	9.21

Table D-2
H VALUES (C = 3)

N	$\gamma = 0.90$			$\gamma = 0.95$			$\gamma = 0.99$		
	P = 0.90	P = 0.95	P = 0.99	P = 0.90	P = 0.95	P = 0.99	P = 0.90	P = 0.95	P = 0.99
2	47.03	57.87	81.69	78.12	96.12	135.68	239.47	294.64	415.90
3	22.43	27.78	39.65	30.24	37.45	53.45	56.71	70.24	100.25
4	16.76	20.82	29.88	21.01	26.11	37.47	33.48	41.60	59.70
5	14.21	17.69	25.48	17.15	21.35	30.74	25.11	31.26	45.01
6	12.76	15.90	22.94	15.02	18.72	27.00	20.87	26.00	37.52
7	11.81	14.73	21.27	13.66	17.04	24.62	18.21	22.84	33.00
8	11.13	13.89	20.09	12.72	15.87	22.95	16.59	20.70	29.93
9	10.63	13.27	19.20	12.02	15.01	21.72	15.35	19.16	27.72
10	10.24	12.78	18.51	11.48	14.34	20.76	14.41	17.99	26.05
11	9.92	12.39	17.95	11.05	13.80	20.00	13.68	17.08	24.74
12	9.66	12.07	17.49	10.70	13.37	19.37	13.09	16.35	23.69
13	9.44	11.80	17.10	10.41	13.00	18.85	12.60	15.74	22.82
14	9.26	11.57	16.77	10.16	12.69	18.40	12.19	15.23	22.08
15	9.10	11.36	16.48	9.95	12.43	18.02	11.85	14.80	21.46
16	8.96	11.19	16.23	9.76	12.19	17.68	11.54	14.42	20.92
17	8.83	11.04	16.01	9.60	11.99	17.39	11.28	14.09	20.44
18	8.72	10.90	15.81	9.45	11.81	17.13	11.05	13.80	20.02
19	8.62	10.78	15.63	9.32	11.65	16.89	10.84	13.54	19.65
20	8.54	10.67	15.47	9.20	11.50	16.68	10.65	13.31	19.31
21	8.45	10.57	15.33	9.10	11.37	16.49	10.49	13.10	19.01
22	8.38	10.47	15.20	9.00	11.25	16.32	10.33	12.91	18.74
23	8.31	10.39	15.07	8.91	11.14	16.16	10.19	12.74	18.49
24	8.25	10.31	14.96	8.83	11.03	16.01	10.07	12.58	18.26
25	8.19	10.24	14.86	8.75	10.94	15.88	9.95	12.44	18.05
26	8.14	10.17	14.76	8.68	10.85	15.75	9.84	12.30	17.85
27	8.09	10.11	14.67	8.62	10.77	15.63	9.74	12.18	17.67
28	8.04	10.05	14.59	8.56	10.70	15.52	9.65	12.06	17.50
29	8.00	10.00	14.51	8.50	10.63	15.42	9.56	11.95	17.35
30	7.96	9.95	14.44	8.45	10.56	15.33	9.48	11.85	17.20
35	7.79	9.74	14.13	8.23	10.29	14.93	9.14	11.43	16.59
40	7.66	9.57	13.90	8.08	10.07	14.62	8.89	11.11	16.12
45	7.55	9.44	13.71	7.92	9.90	14.37	8.68	10.85	15.75
50	7.47	9.34	13.55	7.81	9.77	14.17	8.52	10.65	15.45
60	7.34	9.17	13.31	7.64	9.55	13.86	8.26	10.33	14.99
70	7.24	9.04	13.13	7.51	9.39	13.63	8.07	10.09	14.65
80	7.16	8.95	12.99	7.41	9.27	13.45	7.92	9.91	14.38
90	7.10	8.87	12.88	7.33	9.17	13.31	7.81	9.76	14.17
100	7.04	8.80	12.78	7.27	9.08	13.19	7.71	9.64	13.99
200	6.78	8.48	12.30	6.93	8.66	12.57	7.22	9.02	13.10
500	6.57	8.21	11.92	6.66	8.32	12.08	6.83	8.54	12.40
∞	6.25	7.81	11.34	6.25	7.81	11.34	6.25	7.81	11.34

Table D-3
H VALUES (C = 4)

N	$\gamma = 0.90$			$\gamma = 0.95$			$\lambda = 0.99$		
	P = 0.90	P = 0.95	P = 0.99	P = 0.90	P = 0.95	P = 0.99	P = 0.90	P = 0.95	P = 0.99
2	42.84	51.53	70.34	64.11	77.12	105.27	153.38	184.49	251.85
3	23.49	28.41	39.16	29.99	36.29	50.02	49.90	60.37	83.22
4	18.37	22.29	30.87	22.16	26.89	37.25	32.46	39.38	54.55
5	15.96	19.39	26.94	18.67	22.68	31.51	25.59	31.09	43.20
6	14.54	17.68	24.60	16.67	20.27	28.22	21.92	26.65	37.10
7	13.59	16.54	23.05	15.37	18.70	26.06	19.62	23.88	33.27
8	12.91	15.72	21.92	14.45	17.59	24.53	18.04	21.97	30.64
9	12.40	15.10	21.07	13.76	16.76	23.38	16.89	20.57	28.71
10	12.00	14.62	20.40	13.22	16.11	22.49	16.01	19.50	27.23
11	11.67	14.22	19.86	12.79	15.59	21.77	15.32	18.66	26.06
12	11.40	13.89	19.41	12.44	15.16	21.17	14.74	17.97	25.10
13	11.18	13.62	19.03	12.14	14.80	20.67	14.27	17.39	24.30
14	10.98	13.39	18.71	11.89	14.49	20.25	13.87	16.91	23.63
15	10.81	13.18	18.43	11.67	14.22	19.88	13.53	16.49	23.05
16	10.67	13.00	18.18	11.48	13.99	19.56	13.23	16.13	22.55
17	10.54	12.85	17.96	11.31	13.79	19.27	12.97	15.81	22.10
18	10.42	12.71	17.77	11.16	13.60	19.02	12.74	15.53	21.71
19	10.32	12.58	17.59	11.02	13.44	18.79	12.53	15.28	21.36
20	10.22	12.47	17.43	10.90	13.29	18.59	12.35	15.05	21.05
21	10.14	12.36	17.29	10.79	13.16	18.40	12.18	14.85	20.76
22	10.06	12.27	17.16	10.69	13.04	18.23	12.03	14.66	20.51
23	9.99	12.18	17.04	10.60	12.92	18.08	11.89	14.49	20.27
24	9.92	12.10	16.92	10.52	12.82	17.93	11.76	14.34	20.05
25	9.86	12.03	16.82	10.44	12.73	17.80	11.64	14.19	19.85
26	9.81	11.96	16.73	10.36	12.64	17.68	11.53	14.06	19.67
27	9.75	11.89	16.64	10.30	12.56	17.56	11.43	13.94	19.49
28	9.70	11.83	16.55	10.23	12.48	17.46	11.33	13.82	19.33
29	9.66	11.78	16.48	10.17	12.41	17.36	11.25	13.71	19.18
30	9.62	11.73	16.40	10.12	12.34	17.26	11.16	13.61	19.04
35	9.43	11.50	16.09	9.89	12.06	16.87	10.82	13.19	18.46
40	9.29	11.33	15.86	9.71	11.84	16.56	10.55	12.87	18.01
45	9.18	11.20	15.67	9.56	11.66	16.32	10.34	12.61	17.65
50	9.09	11.08	15.51	9.45	11.52	16.12	10.17	12.40	17.36
60	8.95	10.91	15.27	9.27	11.30	15.81	9.91	12.08	16.90
70	8.84	10.78	15.09	9.13	11.13	15.58	9.71	11.84	16.57
80	8.76	10.68	14.94	9.02	11.01	15.40	9.56	11.65	16.31
90	8.69	10.60	14.83	8.94	10.90	15.25	9.43	11.50	16.10
100	8.63	10.53	14.73	8.87	10.81	15.13	9.33	11.38	15.92
200	8.35	10.18	14.25	8.51	10.38	14.52	8.81	10.75	15.04
500	8.12	9.91	13.86	8.22	10.02	14.03	8.40	10.25	14.34
∞	7.78	9.49	13.28	7.78	9.49	13.28	7.78	9.49	13.28

Table D-4
H VALUES (C = 5)

N	$\gamma = 0.90$			$\gamma = 0.95$			$\gamma = 0.99$		
	P = 0.90	P = 0.95	P = 0.99	P = 0.90	P = 0.95	P = 0.99	P = 0.90	P = 0.95	P = 0.99
2	42.00	49.72	66.25	59.05	69.90	93.14	122.28	144.75	192.88
3	25.00	29.74	39.99	30.88	36.74	49.39	47.60	56.64	76.14
4	20.11	23.99	32.39	23.67	28.23	38.13	32.88	39.22	52.97
5	17.72	21.17	28.66	20.32	24.28	32.87	26.72	31.92	43.21
6	16.29	19.48	26.42	18.37	21.97	29.79	23.32	27.88	37.80
7	15.33	18.34	24.90	17.08	20.43	27.74	21.13	25.28	34.32
8	14.64	17.52	23.80	16.16	19.33	26.27	19.62	23.48	31.90
9	14.10	16.88	22.95	15.46	18.50	25.15	18.51	22.16	30.12
10	13.69	16.39	22.29	14.91	17.86	24.28	17.64	21.12	28.72
11	13.35	15.99	21.75	14.47	17.33	23.58	16.95	20.30	27.61
12	13.07	15.65	21.30	14.11	16.90	23.00	16.39	19.63	26.70
13	12.83	15.37	20.92	13.81	16.54	22.51	15.92	19.06	25.94
14	12.63	15.13	20.60	13.55	16.23	22.09	15.52	18.59	25.30
15	12.46	14.92	20.31	13.32	15.96	21.72	15.17	18.18	24.74
16	12.30	14.74	20.06	13.12	15.72	21.41	14.87	17.82	24.26
17	12.16	14.57	19.84	12.95	15.51	21.12	14.61	17.50	23.83
18	12.04	14.43	19.65	12.79	15.33	20.87	14.38	17.22	23.46
19	11.93	14.30	19.47	12.65	15.16	20.65	14.17	16.98	23.12
20	11.83	14.18	19.31	12.53	15.01	20.44	13.98	16.75	22.81
21	11.74	14.07	19.17	12.41	14.87	20.26	13.81	16.55	22.54
22	11.66	13.97	19.03	12.31	14.75	20.09	13.65	16.36	22.28
23	11.59	13.88	18.91	12.21	14.63	19.93	13.51	16.19	22.05
24	11.52	13.80	18.80	12.12	14.53	19.79	13.38	16.03	21.84
25	11.45	13.72	18.69	12.04	14.43	19.66	13.26	15.89	21.65
26	11.39	13.65	18.60	11.97	14.34	19.53	13.15	15.76	21.46
27	11.34	13.59	18.51	11.89	14.25	19.42	13.04	15.63	21.30
28	11.28	13.52	18.42	11.83	14.18	19.31	12.95	15.52	21.14
29	11.24	13.47	18.34	11.77	14.10	19.21	12.86	15.41	20.99
30	11.19	13.41	18.27	11.71	14.03	19.12	12.77	15.31	20.85
35	11.00	13.18	17.96	11.46	13.74	18.72	12.42	14.88	20.27
40	10.85	13.00	17.71	11.28	13.52	18.41	12.14	14.55	19.83
45	10.73	12.86	17.52	11.13	13.34	18.17	11.93	14.29	19.48
50	10.63	12.74	17.36	11.00	13.19	17.97	11.75	14.08	19.19
60	10.48	12.56	17.11	10.81	12.96	17.66	11.47	13.75	18.74
70	10.37	12.42	16.93	10.67	12.79	17.42	11.27	13.50	18.40
80	10.28	12.32	16.78	10.56	12.65	17.24	11.11	13.31	18.14
90	10.21	12.23	16.67	10.47	12.54	17.09	10.98	13.16	17.93
100	10.15	12.16	16.57	10.39	12.45	16.97	10.87	13.03	17.76
200	9.84	11.80	16.08	10.01	12.00	16.35	10.33	12.38	16.87
500	9.60	11.51	15.68	9.70	11.63	15.85	9.89	11.86	16.16
∞	9.24	11.07	15.09	9.24	11.07	15.09	9.24	11.07	15.09

Table D-5
H VALUES (C = 6)

N	$\gamma = 0.90$			$\gamma = 0.95$			$\gamma = 0.99$		
	P = 0.90	P = 0.95	P = 0.99	P = 0.90	P = 0.95	P = 0.99	P = 0.90	P = 0.95	P = 0.99
2	42.46	49.65	64.94	57.24	66.94	87.55	107.36	125.55	164.20
3	26.68	31.35	41.34	32.19	37.82	49.87	47.14	55.39	73.04
4	21.87	25.76	34.11	25.31	29.81	39.47	33.93	39.96	52.91
5	19.48	22.97	30.48	22.03	25.97	34.47	28.12	33.16	44.00
6	18.02	21.27	28.26	20.07	23.69	31.49	24.84	29.31	38.96
7	17.03	20.11	26.75	18.77	22.16	29.49	22.73	26.84	35.70
8	16.31	19.26	25.65	17.83	21.06	28.04	21.24	25.09	33.40
9	15.76	18.62	24.81	17.12	20.23	26.95	20.13	23.78	31.68
10	15.33	18.12	24.14	16.56	19.58	26.09	19.26	22.77	30.34
11	14.98	17.70	23.60	16.11	19.04	25.39	18.57	21.95	29.26
12	14.68	17.36	23.14	15.74	18.61	24.81	18.01	21.29	28.38
13	14.44	17.07	22.76	15.42	18.23	24.32	17.53	20.73	27.64
14	14.23	16.82	22.43	15.15	17.92	23.90	17.13	20.25	27.01
15	14.04	16.60	22.14	14.92	17.64	23.53	16.78	19.84	26.47
16	13.88	16.41	21.89	14.72	17.40	23.21	16.48	19.48	25.99
17	13.74	16.24	21.67	14.54	17.19	22.93	16.21	19.17	25.57
18	13.61	16.09	21.47	14.37	17.00	22.68	15.97	18.89	25.20
19	13.49	15.96	21.29	14.23	16.83	22.45	15.76	18.64	24.87
20	13.39	15.83	21.13	14.10	16.67	22.25	15.57	18.41	24.56
21	13.29	15.72	20.98	13.98	16.53	22.06	15.39	18.20	24.29
22	13.21	15.62	20.84	13.87	16.40	21.89	15.23	18.02	24.04
23	13.13	15.53	20.72	13.77	16.28	21.73	15.09	17.84	23.81
24	13.05	15.44	20.60	13.68	16.17	21.59	14.95	17.69	23.60
25	12.99	15.36	20.50	13.59	16.07	21.45	14.83	17.54	23.41
26	12.92	15.28	20.40	13.51	15.98	21.33	14.71	17.40	23.23
27	12.86	15.21	20.31	13.44	15.89	21.21	14.61	17.28	23.06
28	12.81	15.15	20.22	13.37	15.81	21.10	14.51	17.16	22.90
29	12.76	15.09	20.14	13.30	15.74	21.00	14.42	17.05	22.76
30	12.71	15.03	20.06	13.24	15.66	20.91	14.33	16.95	22.62
35	12.51	14.79	19.75	12.99	15.36	20.51	13.96	16.51	22.04
40	12.35	14.61	19.50	12.79	15.13	20.20	13.68	16.18	21.60
45	12.22	14.46	19.30	12.63	14.94	19.95	13.45	15.91	21.25
50	12.12	14.33	19.14	12.50	14.79	19.74	13.27	15.70	20.96
60	11.96	14.14	18.88	12.30	14.55	19.43	12.98	15.36	20.50
70	11.84	14.00	18.69	12.15	14.37	19.19	12.77	15.11	20.17
80	11.74	13.89	18.55	12.03	14.23	19.00	12.60	14.91	19.90
90	11.67	13.80	18.43	11.94	14.12	18.85	12.47	14.75	19.69
100	11.61	13.73	18.33	11.86	14.03	18.73	12.36	14.62	19.52
200	11.29	13.35	17.82	11.46	13.55	18.10	11.79	13.95	18.62
500	11.03	13.05	17.42	11.14	13.17	17.59	11.34	13.41	17.90
∞	10.64	12.59	16.81	10.64	12.59	16.81	10.64	12.59	16.81

Table D-6
H VALUES (C = 7)

N	$\gamma = 0.90$			$\gamma = 0.95$			$\gamma = 0.99$		
	P = 0.90	P = 0.95	P = 0.99	P = 0.90	P = 0.95	P = 0.99	P = 0.90	P = 0.95	P = 0.99
2	43.52	50.41	64.94	56.89	65.90	84.89	99.66	115.42	148.70
3	28.44	33.09	42.96	33.73	39.24	50.94	47.61	55.39	71.91
4	23.64	27.57	35.92	27.01	31.49	41.03	35.22	41.07	53.52
5	21.21	24.76	32.33	23.73	27.70	36.18	29.64	34.60	45.18
6	19.72	23.03	30.12	21.76	25.42	33.25	26.43	30.88	40.38
7	18.69	21.84	28.60	20.44	23.88	31.27	24.34	28.45	37.24
8	17.95	20.98	27.49	19.48	22.77	29.83	22.86	26.72	35.00
9	17.38	20.32	26.64	18.75	21.93	28.74	21.74	25.42	33.32
10	16.93	19.80	25.96	18.18	21.26	27.87	20.87	24.41	32.01
11	16.56	19.37	25.41	17.71	20.72	27.17	20.18	23.60	30.95
12	16.26	19.02	24.95	17.33	20.27	26.59	19.60	22.93	30.08
13	16.00	18.72	24.56	17.00	19.89	26.10	19.12	22.37	29.35
14	15.78	18.46	24.22	16.72	19.57	25.67	18.71	21.89	28.73
15	15.59	18.24	23.93	16.48	19.29	25.31	18.36	21.48	28.19
16	15.42	18.04	23.67	16.27	19.04	24.98	18.05	21.12	27.72
17	15.27	17.86	23.45	16.08	18.82	24.70	17.77	20.80	27.30
18	15.13	17.71	23.24	15.91	18.62	24.44	17.53	20.52	26.93
19	15.01	17.57	23.06	15.76	18.45	24.21	17.31	20.26	26.60
20	14.90	17.44	22.89	15.63	18.29	24.00	17.12	20.03	26.29
21	14.80	17.32	22.74	15.50	18.14	23.82	16.94	19.82	26.02
22	14.71	17.22	22.60	15.39	18.01	23.64	16.77	19.63	25.77
23	14.63	17.12	22.47	15.28	17.89	23.48	16.62	19.46	25.54
24	14.55	17.03	22.36	15.19	17.77	23.34	16.49	19.30	25.33
25	14.48	16.95	22.25	15.10	17.67	23.20	16.36	19.15	25.14
26	14.41	16.87	22.15	15.02	17.57	23.07	16.24	19.01	24.96
27	14.35	16.80	22.05	14.94	17.48	22.96	16.13	18.88	24.79
28	14.29	16.73	21.96	14.87	17.40	22.85	16.03	18.76	24.63
29	14.24	16.67	21.88	14.80	17.32	22.74	15.93	18.65	24.49
30	14.19	16.61	21.80	14.74	17.25	22.65	15.84	18.54	24.35
35	13.97	16.36	21.48	14.47	16.94	22.24	15.47	18.10	23.77
40	13.81	16.16	21.22	14.26	16.70	21.92	15.17	17.76	23.32
45	13.67	16.01	21.02	14.10	16.50	21.67	14.94	17.49	22.97
50	13.57	15.88	20.85	13.96	16.34	21.46	14.75	17.27	22.67
60	13.40	15.68	20.59	13.75	16.10	21.14	14.45	16.92	22.22
70	13.27	15.53	20.40	13.59	15.91	20.90	14.23	16.66	21.88
80	13.17	15.42	20.25	13.47	15.77	20.71	14.06	16.46	21.61
90	13.09	15.32	20.13	13.37	15.65	20.56	13.92	16.29	21.40
100	13.03	15.25	20.02	13.29	15.56	20.43	13.80	16.16	21.11
200	12.69	14.85	19.51	12.87	15.06	19.78	13.21	15.47	20.32
500	12.42	14.54	19.10	12.53	14.67	19.26	12.74	14.91	19.59
∞	12.02	14.07	18.48	12.02	14.07	18.48	12.02	14.07	18.48

Table D-7
H VALUES (C = 8)

N	$\gamma = 0.90$			$\gamma = 0.95$			$\gamma = 0.99$		
	P = 0.90	P = 0.95	P = 0.99	P = 0.90	P = 0.95	P = 0.99	P = 0.90	P = 0.95	P = 0.99
2	44.93	51.62	65.67	57.38	65.93	83.88	95.46	109.68	139.54
3	30.24	34.90	44.72	35.37	40.82	52.31	48.49	55.95	71.71
4	25.41	29.38	37.78	28.73	33.22	42.72	36.68	42.41	54.54
5	22.92	26.53	34.19	25.43	29.44	37.94	31.23	36.14	46.58
6	21.38	24.76	31.96	23.43	27.14	35.03	28.06	32.50	41.94
7	20.33	23.56	30.42	22.08	25.59	33.05	25.96	30.08	38.85
8	19.56	22.67	29.30	21.10	24.46	31.61	24.47	28.36	36.65
9	18.97	21.99	28.43	20.36	23.60	30.51	23.34	27.07	34.99
10	18.50	21.45	27.75	19.76	22.92	29.64	22.47	26.05	33.70
11	18.12	21.01	27.18	19.29	22.37	28.93	21.76	25.24	32.65
12	17.80	20.65	26.71	18.89	21.91	28.35	21.18	24.56	31.78
13	17.53	20.33	26.32	18.55	21.52	27.85	20.69	24.00	31.06
14	17.30	20.07	25.97	18.26	21.18	27.42	20.27	23.51	30.43
15	17.10	19.83	25.67	18.01	20.89	27.05	19.91	23.09	29.89
16	16.92	19.63	25.41	17.79	20.64	26.72	19.59	22.73	29.42
17	16.76	19.45	25.18	17.59	20.41	26.43	19.31	22.40	29.01
18	16.62	19.29	24.97	17.42	20.21	26.17	19.06	22.12	28.63
19	16.50	19.14	24.78	17.26	20.03	25.94	18.84	21.86	28.30
20	16.38	19.01	24.61	17.12	19.87	25.72	18.64	21.62	28.00
21	16.28	18.89	24.46	16.99	19.72	25.53	18.45	21.41	27.73
22	16.18	18.78	24.32	16.87	19.58	25.36	18.28	21.22	27.48
23	16.09	18.67	24.19	16.77	19.45	25.19	18.13	21.04	27.25
24	16.01	18.58	24.06	16.67	19.34	25.05	17.99	20.87	27.03
25	15.94	18.49	23.95	16.57	19.23	24.91	17.86	20.72	26.84
26	15.87	18.41	23.85	16.49	19.13	24.78	17.74	20.58	26.66
27	15.80	18.34	23.75	16.41	19.04	24.66	17.62	20.45	26.49
28	15.74	18.27	23.66	16.33	18.95	24.55	17.52	20.33	26.33
29	15.69	18.20	23.58	16.26	18.87	24.44	17.42	20.21	26.18
30	15.63	18.14	23.50	16.20	18.79	24.34	17.33	20.11	26.04
35	15.41	17.88	23.16	15.92	18.47	23.93	16.94	19.65	25.46
40	15.23	17.68	22.90	15.70	18.22	23.61	16.64	19.31	25.01
45	15.10	17.52	22.69	15.53	18.02	23.35	16.40	19.03	24.65
50	14.98	17.39	22.52	15.39	17.86	23.14	16.20	18.80	24.35
60	14.80	17.18	22.26	15.17	17.61	22.81	15.89	18.44	23.89
70	14.67	17.03	22.06	15.01	17.42	22.56	15.66	18.18	23.55
80	14.57	16.91	21.90	14.88	17.27	22.37	15.48	17.97	23.28
90	14.48	16.81	21.78	14.77	17.15	22.21	15.34	17.80	23.06
100	14.42	16.73	21.67	14.69	17.05	22.08	15.22	17.66	22.88
200	14.06	16.32	21.15	14.25	16.54	21.42	14.61	16.95	21.96
500	13.78	16.00	20.72	13.90	16.13	20.89	14.11	16.38	21.22
∞	13.36	15.51	20.09	13.36	15.51	20.09	13.36	15.51	20.09

Table D-8
H VALUES (C = 9)

N	$\gamma = 0.90$			$\gamma = 0.95$			$\gamma = 0.99$		
	P = 0.90	P = 0.95	P = 0.99	P = 0.90	P = 0.95	P = 0.99	P = 0.90	P = 0.95	P = 0.99
2	46.53	53.11	66.85	58.34	66.59	83.81	92.96	106.10	133.55
3	32.05	36.73	46.55	37.09	42.50	53.87	49.72	56.97	72.21
4	27.16	31.18	39.65	30.46	34.97	44.47	38.23	43.89	55.81
5	24.61	28.29	36.04	27.13	31.18	39.73	32.85	37.75	48.10
6	23.03	26.48	33.78	25.09	28.85	36.81	29.68	34.13	43.55
7	21.94	25.24	32.23	23.71	27.28	34.83	27.57	31.72	40.51
8	21.14	24.33	31.09	22.70	26.13	33.38	26.07	30.00	38.33
9	20.53	23.63	30.21	21.93	25.25	32.27	24.93	28.70	36.68
10	20.04	23.08	29.50	21.33	24.55	31.39	24.04	27.68	35.39
11	19.65	22.62	28.93	20.83	23.99	30.67	23.32	26.86	34.34
12	19.31	22.24	28.45	20.42	23.51	30.08	22.73	26.17	33.48
13	19.03	21.92	28.04	20.07	23.11	29.57	22.23	25.60	32.75
14	18.79	21.64	27.69	19.77	22.77	29.14	21.80	25.11	32.13
15	18.58	21.40	27.39	19.51	22.47	28.76	21.43	24.69	31.59
16	18.40	21.19	27.12	19.28	22.21	28.42	21.11	24.31	31.11
17	18.23	21.00	26.88	19.08	21.98	28.13	20.82	23.98	30.69
18	18.09	20.83	26.66	18.90	21.77	27.86	20.57	23.69	30.32
19	17.95	20.68	26.47	18.74	21.59	27.63	20.34	23.43	29.98
20	17.83	20.54	26.30	18.59	21.42	27.41	20.13	23.19	29.68
21	17.73	20.42	26.14	18.46	21.26	27.22	19.94	22.97	29.40
22	17.63	20.31	25.99	18.33	21.12	27.04	19.77	22.77	29.15
23	17.53	20.20	25.86	18.22	20.99	26.87	19.61	22.59	28.92
24	17.45	20.10	25.74	18.12	20.87	26.72	19.47	22.42	28.71
25	17.37	20.01	25.62	18.02	20.76	26.58	19.33	22.27	28.51
26	17.30	19.93	25.51	17.93	20.66	26.45	19.21	22.13	28.33
27	17.23	19.85	25.42	17.85	20.56	26.33	19.09	21.99	28.16
28	17.17	19.78	25.32	17.77	20.47	26.21	18.98	21.87	28.00
29	17.11	19.71	25.24	17.70	20.39	26.11	18.88	21.75	27.85
30	17.05	19.65	25.15	17.63	20.31	26.00	18.78	21.64	27.71
35	16.82	19.38	24.81	17.34	19.98	25.58	18.38	21.18	27.12
40	16.64	19.17	24.54	17.12	19.72	25.25	18.07	20.82	26.66
45	16.49	19.00	24.33	16.94	19.52	24.99	17.82	20.53	26.29
50	16.37	18.86	24.15	16.79	19.35	24.77	17.62	20.30	25.99
60	16.19	18.65	23.88	16.56	19.08	24.44	17.30	19.94	25.53
70	16.05	18.49	23.68	16.39	18.89	24.19	17.06	19.66	25.18
80	15.94	18.37	23.52	16.26	18.73	23.99	16.88	19.45	24.90
90	15.85	18.27	23.39	16.15	18.61	23.83	16.73	19.28	24.68
100	15.78	18.18	23.28	16.06	18.51	23.70	16.61	19.13	24.50
200	15.41	17.76	22.74	15.60	17.98	23.02	15.97	18.40	23.57
500	15.12	17.42	22.31	15.24	17.56	22.48	15.46	17.82	22.81
∞	14.68	16.92	21.67	14.68	16.92	21.67	14.68	16.92	21.67

Table D-9
H VALUES (C = 10)

N	$\gamma = 0.90$			$\gamma = 0.95$			$\gamma = 0.99$		
	P = 0.90	P = 0.95	P = 0.99	P = 0.90	P = 0.95	P = 0.99	P = 0.90	P = 0.95	P = 0.99
2	48.25	54.76	68.29	59.60	67.63	84.35	91.88	104.27	130.04
3	33.87	38.58	48.43	38.84	44.24	55.54	51.06	58.17	73.02
4	28.89	32.97	41.53	32.19	36.73	46.27	39.83	45.45	57.25
5	26.28	30.02	37.88	28.80	32.90	41.52	34.49	39.40	49.71
6	24.65	28.18	35.60	26.73	30.55	38.60	31.30	35.78	45.20
7	23.53	26.91	34.02	25.31	28.95	36.60	29.18	33.37	42.19
8	22.70	25.97	32.85	24.28	27.77	35.13	27.66	31.63	40.02
9	22.07	25.25	31.95	23.49	26.87	34.01	26.50	30.32	38.37
10	21.56	24.67	31.23	22.86	26.16	33.12	25.60	29.29	37.08
11	21.15	24.20	30.65	22.35	25.58	32.39	24.87	28.46	36.03
12	20.81	23.81	30.15	21.93	25.10	31.78	24.26	27.77	35.16
13	20.51	23.48	29.74	21.57	24.69	31.27	23.75	27.19	34.43
14	20.26	23.19	29.38	21.26	24.33	30.82	23.31	26.69	33.80
15	20.04	22.94	29.07	20.99	24.03	30.44	22.94	26.25	33.26
16	19.85	22.72	28.79	20.75	23.76	30.10	22.60	25.88	32.78
17	19.68	22.53	28.55	20.55	23.52	29.80	22.31	25.54	32.36
18	19.53	22.36	28.33	20.36	23.31	29.53	22.05	25.24	31.98
19	19.39	22.20	28.13	20.19	23.11	29.29	21.81	24.97	31.64
20	19.27	22.06	27.95	20.04	22.94	29.07	21.60	24.73	31.34
21	19.15	21.93	27.79	19.90	22.78	28.87	21.41	24.51	31.06
22	19.05	21.81	27.64	19.77	22.64	28.69	21.23	24.31	30.80
23	18.95	21.70	27.50	19.65	22.50	28.52	21.07	24.12	30.57
24	18.86	21.60	27.37	19.55	22.38	28.36	20.92	23.95	30.35
25	18.78	21.51	27.26	19.45	22.27	28.22	20.78	23.79	30.15
26	18.71	21.42	27.15	19.36	22.16	28.09	20.65	23.65	29.97
27	18.64	21.34	27.05	19.27	22.06	27.96	20.53	23.51	29.80
28	18.57	21.26	26.95	19.19	21.97	27.85	20.42	23.38	29.64
29	18.51	21.19	26.86	19.11	21.88	27.74	20.32	23.26	29.48
30	18.45	21.13	26.78	19.04	21.80	27.64	20.22	23.15	29.34
35	18.21	20.85	26.43	18.74	21.46	27.20	19.80	22.68	28.74
40	18.02	20.63	26.15	18.51	21.20	26.87	19.48	22.31	28.28
45	17.87	20.46	25.93	18.32	20.98	26.60	19.23	22.02	27.91
50	17.74	20.32	25.75	18.17	20.81	26.38	19.02	21.78	27.61
60	17.55	20.09	25.47	17.94	20.54	26.04	18.69	21.40	27.13
70	17.40	19.93	25.27	17.76	20.33	25.78	18.45	21.12	26.78
80	17.29	19.80	25.10	17.62	20.18	25.58	18.26	20.90	26.50
90	17.20	19.70	24.97	17.51	20.05	25.41	18.10	20.73	26.28
100	17.13	19.61	24.86	17.41	19.94	25.28	17.97	20.58	26.09
200	16.74	19.17	24.31	16.94	19.40	24.59	17.32	19.83	25.14
500	16.44	18.82	23.87	16.56	18.96	24.04	16.79	19.23	24.38
∞	15.99	18.31	23.21	15.99	18.31	23.21	15.99	18.31	23.21

Table D-10
H VALUES (C = 11)

N	$\gamma = 0.90$			$\gamma = 0.95$			$\gamma = 0.99$		
	P = 0.90	P = 0.95	P = 0.99	P = 0.90	P = 0.95	P = 0.99	P = 0.90	P = 0.95	P = 0.99
2	50.05	56.52	69.92	61.02	68.91	85.24	91.60	103.43	127.95
3	35.68	40.44	50.34	40.61	46.02	57.30	52.58	59.58	74.18
4	30.62	34.76	43.41	33.91	38.50	48.07	41.48	47.09	58.81
5	27.93	31.74	39.71	30.47	34.62	43.32	36.13	41.05	51.36
6	26.26	29.85	37.39	28.35	32.24	40.37	32.92	37.43	46.88
7	25.10	28.55	35.78	26.90	30.60	38.35	30.78	35.01	43.88
8	24.25	27.58	34.59	25.84	29.40	36.87	29.23	33.26	41.71
9	23.59	26.84	33.67	25.03	28.48	35.73	28.06	31.93	40.06
10	23.06	26.25	32.94	24.38	27.75	34.82	27.14	30.89	38.76
11	22.64	25.76	32.34	23.86	27.15	34.08	26.40	30.04	37.71
12	22.28	25.36	31.84	23.42	26.66	33.47	25.78	29.34	36.84
13	21.97	25.02	31.41	23.05	26.24	32.94	25.26	28.75	36.10
14	21.71	24.72	31.04	22.73	25.88	32.49	24.81	28.24	35.47
15	21.49	24.46	30.72	22.45	25.56	32.10	24.42	27.80	34.92
16	21.29	24.24	30.44	22.21	25.28	31.75	24.08	27.42	34.43
17	21.11	24.04	30.19	21.99	25.04	31.45	23.78	27.08	34.01
18	20.95	23.85	29.96	21.80	24.82	31.17	23.51	26.77	33.63
19	20.81	23.69	29.76	21.62	24.62	30.93	23.27	26.50	33.28
20	20.68	23.55	29.58	21.46	24.44	30.70	23.05	26.25	32.97
21	20.56	23.41	29.41	21.32	24.28	30.50	22.85	26.02	32.69
22	20.45	23.29	29.26	21.19	24.13	30.31	22.67	25.82	32.43
23	20.35	23.18	29.12	21.07	23.99	30.14	22.51	25.63	32.20
24	20.26	23.07	28.99	20.96	23.87	29.98	22.35	25.45	31.98
25	20.18	22.98	28.87	20.85	23.75	29.84	22.21	25.29	31.78
26	20.10	22.89	28.75	20.76	23.64	29.70	22.08	25.14	31.59
27	20.02	22.80	28.65	20.67	23.54	29.57	21.96	25.00	31.41
28	19.96	22.73	28.55	20.59	23.44	29.45	21.84	24.87	31.25
29	19.89	22.65	28.46	20.51	23.35	29.34	21.73	24.75	31.10
30	19.83	22.58	28.38	20.43	23.27	29.24	21.63	24.64	30.95
35	19.58	22.30	28.01	20.12	22.92	28.80	21.21	24.15	30.35
40	19.38	22.07	27.73	19.88	22.65	28.46	20.88	23.78	29.88
45	19.22	21.89	27.51	19.69	22.43	28.18	20.61	23.48	29.50
50	19.09	21.75	27.33	19.53	22.25	27.96	20.40	23.23	29.19
60	18.89	21.52	27.04	19.29	21.97	27.61	20.06	22.85	28.71
70	18.74	21.35	26.82	19.11	21.76	27.34	19.81	22.56	28.35
80	18.63	21.21	26.66	18.96	21.60	27.14	19.61	22.34	28.07
90	18.53	21.11	26.52	18.85	21.46	26.97	19.45	22.16	27.84
100	18.45	21.02	26.41	18.75	21.35	26.83	19.32	22.01	27.65
200	18.06	20.57	25.84	18.26	20.80	26.13	18.65	21.24	26.69
500	17.74	20.21	25.39	17.87	20.35	25.57	18.10	20.62	25.91
∞	17.28	19.68	24.72	17.28	19.68	24.72	17.28	19.68	24.72

Table D-11
H VALUES (C = 12)

N	$\gamma = 0.90$			$\gamma = 0.95$			$\gamma = 0.99$		
	P = 0.90	P = 0.95	P = 0.99	P = 0.90	P = 0.95	P = 0.99	P = 0.90	P = 0.95	P = 0.99
2	51.89	58.34	71.65	62.61	70.38	86.45	91.69	103.08	126.60
3	37.49	42.29	52.26	42.40	47.83	59.10	51.12	61.05	75.45
4	32.32	36.53	45.27	35.63	40.26	49.90	43.14	48.74	60.42
5	29.57	33.45	41.53	32.13	36.34	45.11	37.77	42.71	53.03
6	27.85	31.51	39.17	29.96	33.90	42.14	34.54	39.08	48.58
7	26.65	30.17	37.53	28.47	32.23	40.09	32.37	36.64	45.57
8	25.77	29.18	36.31	27.38	31.01	38.59	30.80	34.88	43.40
9	25.09	28.42	35.37	26.55	30.07	37.43	29.61	33.53	41.74
10	24.55	27.81	34.62	25.88	29.32	36.51	28.67	32.47	40.44
11	24.10	27.31	34.01	25.34	28.71	35.76	27.91	31.62	39.37
12	23.73	26.89	33.49	24.89	28.20	35.13	27.28	30.90	38.49
13	23.42	26.53	33.06	24.51	27.77	34.60	26.74	30.30	37.75
14	23.15	26.23	32.68	24.18	27.40	34.14	26.29	29.79	37.11
15	22.91	25.96	32.35	23.89	27.07	33.74	25.89	29.34	36.56
16	22.71	25.73	32.06	23.64	26.79	33.38	25.54	28.94	36.07
17	22.52	25.52	31.80	23.42	26.54	33.07	25.23	28.60	35.64
18	22.36	25.34	31.57	23.22	26.31	32.79	24.96	28.29	35.25
19	22.21	25.17	31.37	23.04	26.11	32.54	24.71	28.01	34.90
20	22.07	25.02	31.18	22.88	25.93	32.31	24.49	27.75	34.59
21	21.95	24.88	31.01	22.73	25.76	32.11	24.29	27.52	34.31
22	21.84	24.75	30.85	22.59	25.60	31.91	24.10	27.31	34.04
23	21.74	24.64	30.71	22.47	25.46	31.74	23.93	27.12	33.80
24	21.64	24.53	30.57	22.35	25.33	31.58	23.77	26.94	33.58
25	21.55	24.43	30.45	22.25	25.21	31.43	23.62	26.78	33.38
26	21.47	24.34	30.34	22.15	25.10	31.29	23.49	26.62	33.19
27	21.40	24.25	30.23	22.05	25.00	31.16	23.36	26.48	33.01
28	21.32	24.17	30.13	21.97	24.90	31.04	23.25	26.35	32.84
29	21.26	24.09	30.04	21.89	24.81	30.93	23.13	26.22	32.69
30	21.20	24.02	29.95	21.81	24.72	30.82	23.03	26.10	32.54
35	20.93	23.72	29.58	21.49	24.36	30.37	22.59	25.61	31.93
40	20.73	23.49	29.29	21.24	24.08	30.02	22.26	25.23	31.45
45	20.56	23.31	29.06	21.04	23.85	29.74	21.99	24.92	31.07
50	20.43	23.16	29.87	20.88	23.67	29.51	21.76	24.67	30.76
60	20.22	22.92	29.58	20.63	23.38	29.15	21.42	24.28	30.27
70	20.07	22.75	28.36	20.44	23.17	28.88	21.16	23.98	29.90
80	19.95	22.61	28.19	20.29	23.00	28.67	20.95	23.75	29.62
90	19.85	22.50	28.05	20.17	22.86	28.51	20.79	23.57	29.38
100	19.77	22.41	27.94	20.07	22.75	28.36	20.65	23.41	29.19
200	19.36	21.94	27.36	19.56	22.18	27.65	19.96	22.63	28.21
500	19.03	21.57	26.90	19.16	21.72	27.08	19.40	21.99	27.42
∞	18.55	21.03	26.22	18.55	21.03	26.22	18.55	21.03	26.22

Table D-12
H VALUES (C = 13)

N	$\gamma = 0.90$			$\gamma = 0.95$			$\gamma = 0.99$		
	P = 0.90	P = 0.95	P = 0.99	P = 0.90	P = 0.95	P = 0.99	P = 0.90	P = 0.95	P = 0.99
2	53.78	60.22	73.49	64.29	71.99	87.85	92.31	103.37	126.14
3	39.29	44.14	54.18	44.18	49.63	60.93	55.73	62.61	76.86
4	34.02	38.28	47.13	37.33	42.01	51.72	44.80	50.42	62.07
5	31.20	35.14	43.33	33.77	38.03	46.90	39.41	44.38	54.73
6	29.42	33.16	40.93	31.55	35.56	43.89	36.14	40.73	50.28
7	28.19	31.78	39.25	30.03	33.86	41.82	33.94	38.27	47.27
8	27.28	30.76	38.01	28.91	32.60	40.29	32.35	36.48	45.08
9	26.58	29.97	37.05	28.05	31.64	39.11	31.14	35.12	43.42
10	26.02	29.34	36.29	27.37	30.87	38.18	30.18	34.05	42.10
11	25.56	28.83	35.66	26.81	30.25	37.41	29.41	33.17	41.03
12	25.17	28.40	35.13	26.35	29.73	36.77	28.76	32.45	40.14
13	24.85	28.03	34.68	25.96	29.28	36.23	28.22	31.84	39.39
14	24.57	27.72	34.30	25.62	28.90	35.76	27.75	31.31	38.74
15	24.32	27.45	33.96	25.32	28.57	35.35	27.34	30.85	38.18
16	24.11	27.20	33.67	25.06	28.28	35.00	26.99	30.46	37.69
17	23.92	26.99	33.40	24.83	28.02	34.68	26.67	30.10	37.25
18	23.75	26.80	33.17	24.63	27.79	34.39	26.39	29.78	36.86
19	23.59	26.63	32.95	24.44	27.58	34.14	26.14	29.50	36.51
20	23.46	26.47	32.76	24.27	27.39	33.90	25.91	29.24	36.19
21	23.33	26.33	32.59	24.12	27.22	33.69	25.70	29.00	35.90
22	23.21	26.20	32.43	23.98	27.06	33.50	25.51	28.79	35.64
23	23.11	26.08	32.28	23.85	26.92	33.32	25.34	28.59	35.39
24	23.01	25.97	32.14	23.73	26.78	33.15	25.17	28.41	35.17
25	22.92	25.86	32.02	23.62	26.66	33.00	25.02	28.24	34.96
26	22.83	25.77	31.90	23.52	26.54	32.86	24.89	28.08	34.77
27	22.75	25.68	31.79	23.42	26.44	32.73	24.76	27.94	34.59
28	22.68	25.60	31.69	23.34	26.34	32.60	24.63	27.80	34.42
29	22.61	25.52	31.59	23.25	26.24	32.49	24.52	27.68	34.26
30	22.55	25.45	31.50	23.17	26.15	32.38	24.41	27.56	34.11
35	22.27	25.14	31.12	22.84	25.78	31.92	23.97	27.05	33.40
40	22.06	24.90	30.83	22.59	25.49	31.56	23.62	26.66	33.00
45	21.89	24.71	30.59	22.38	25.26	31.28	23.34	26.34	32.62
50	21.75	24.55	30.40	22.21	25.07	31.04	23.11	26.09	32.30
60	21.54	24.31	30.10	21.95	24.78	30.68	22.76	25.69	31.80
70	21.38	24.13	29.87	21.76	24.56	30.40	22.49	25.39	31.43
80	21.25	23.99	29.70	21.60	24.38	30.19	22.28	25.15	31.14
90	21.15	23.87	29.56	21.48	24.24	30.02	22.11	24.96	30.91
100	21.07	23.78	29.44	21.38	24.13	29.87	21.97	24.80	30.71
200	20.64	23.30	28.85	20.86	23.54	29.15	21.26	24.00	29.71
500	20.31	22.92	28.38	20.44	23.07	28.56	20.69	23.35	28.91
∞	19.81	22.36	27.69	19.81	22.36	27.69	19.81	22.36	27.69

Table D-13
H VALUES (C = 14)

N	$\gamma = 0.90$			$\gamma = 0.95$			$\gamma = 0.99$		
	P = 0.90	P = 0.95	P = 0.99	P = 0.90	P = 0.95	P = 0.99	P = 0.90	P = 0.95	P = 0.99
2	55.69	62.14	75.39	66.04	73.69	89.40	93.22	104.01	126.19
3	41.08	45.99	56.11	45.97	51.46	62.79	57.41	64.27	78.42
4	35.70	40.02	48.97	39.03	43.76	53.54	46.49	52.12	63.78
5	32.81	36.82	45.12	35.40	39.72	48.68	41.04	46.05	56.44
6	30.98	34.78	42.67	33.14	37.20	45.64	37.74	42.37	51.98
7	29.71	33.37	40.96	31.58	35.46	43.53	35.51	39.88	48.96
8	28.77	32.32	39.70	30.43	34.18	41.98	33.89	38.07	46.76
9	28.05	31.51	38.72	29.55	33.20	40.78	32.66	36.69	45.08
10	27.47	30.87	37.93	28.85	32.41	39.83	31.68	35.60	43.75
11	27.00	30.34	37.29	28.27	31.77	39.05	30.89	34.72	42.67
12	26.60	29.90	36.75	27.79	31.24	38.40	30.23	33.98	41.77
13	26.26	29.52	36.29	27.39	30.78	37.85	29.68	33.36	41.01
14	25.97	29.20	35.90	27.04	30.39	37.37	29.20	32.82	40.36
15	25.72	28.91	35.55	26.74	30.05	36.95	28.79	32.36	39.79
16	25.50	28.67	35.25	26.47	29.75	36.59	28.42	31.95	39.29
17	25.30	28.44	34.98	26.23	29.49	36.26	28.10	31.59	38.85
18	25.13	28.25	34.74	26.02	29.25	35.97	27.81	31.26	38.45
19	24.97	28.07	34.52	25.83	29.04	35.71	27.55	30.97	38.09
20	24.82	27.91	34.32	25.66	28.84	35.47	27.32	30.71	37.77
21	24.69	27.76	34.15	25.50	28.67	35.26	27.10	30.47	37.48
22	24.57	27.63	33.98	25.35	28.50	35.06	26.91	30.25	37.21
23	24.46	27.50	33.83	25.22	28.36	34.88	26.73	30.05	36.96
24	24.36	27.39	33.69	25.10	28.22	34.71	26.56	29.86	36.73
25	24.27	27.28	33.56	24.99	28.09	34.55	26.41	29.69	36.52
26	24.18	27.19	33.44	24.88	27.97	34.41	26.27	29.53	36.33
27	24.10	27.09	33.33	24.78	27.86	34.27	26.13	29.38	36.15
28	24.02	27.01	33.22	24.69	27.76	34.15	26.01	29.24	35.97
29	23.95	26.93	33.13	24.60	27.66	34.03	25.89	29.11	35.81
30	23.88	26.85	33.03	24.52	27.57	33.92	25.79	28.99	35.66
35	23.60	26.54	32.65	24.18	27.19	33.45	25.33	28.47	35.03
40	23.38	26.29	32.34	23.92	26.89	33.09	24.97	28.07	34.54
45	23.21	26.09	32.10	23.71	26.66	32.80	24.68	27.75	34.15
50	23.06	25.93	31.91	23.53	26.46	32.56	24.45	27.49	33.82
60	22.84	25.68	31.60	23.26	26.16	32.18	24.09	27.08	33.32
70	22.68	25.50	31.37	23.06	25.93	31.90	23.81	26.77	32.94
80	22.55	25.35	31.19	22.91	25.75	31.69	23.60	26.53	32.65
90	22.44	25.23	31.05	22.78	25.61	31.51	23.43	26.34	32.41
100	22.36	25.14	30.93	22.67	25.49	31.37	23.28	26.18	32.21
200	21.92	24.65	30.32	22.14	24.89	30.62	22.55	25.36	31.20
500	21.57	24.26	29.84	21.71	24.41	30.03	21.96	24.69	30.38
∞	21.06	23.68	29.14	21.06	23.68	29.14	21.06	23.68	29.14

Table D-14
H VALUES (C = 15)

N	$\gamma = 0.90$			$\gamma = 0.95$			$\gamma = 0.99$		
	P = 0.90	P = 0.95	P = 0.99	P = 0.90	P = 0.95	P = 0.99	P = 0.90	P = 0.95	P = 0.99
2	57.61	64.08	77.34	67.82	75.43	91.03	94.22	104.80	126.47
3	42.87	47.83	58.04	47.76	53.28	64.66	59.09	65.92	80.00
4	37.37	41.76	50.81	40.72	45.50	55.36	48.17	53.82	65.49
5	34.41	38.48	46.89	37.02	41.40	50.45	42.67	47.72	58.16
6	32.53	36.40	44.40	34.71	38.83	47.37	39.33	44.01	53.68
7	31.22	34.95	42.66	33.11	37.06	45.23	37.07	41.49	50.64
8	30.26	33.87	41.36	31.93	35.75	43.65	35.42	39.66	48.43
9	29.51	33.04	40.36	31.03	34.74	42.44	34.17	38.26	46.73
10	28.91	32.38	39.56	30.31	33.94	41.47	33.17	37.15	45.39
11	28.42	31.83	38.90	29.72	33.28	40.67	32.37	36.25	44.30
12	28.02	31.38	38.35	29.23	32.74	40.01	31.69	35.50	43.39
13	27.67	30.99	37.88	28.81	32.27	39.45	31.13	34.86	42.62
14	27.37	30.66	37.48	28.45	31.87	38.96	30.64	34.32	41.96
15	27.11	30.37	37.13	28.14	31.52	38.54	30.21	33.85	41.38
16	26.88	30.11	36.82	27.86	31.21	38.16	29.84	33.43	40.88
17	26.68	29.89	36.54	27.62	30.94	37.83	29.51	33.06	40.43
18	26.49	29.68	36.30	27.40	30.70	37.54	29.22	32.73	40.03
19	26.33	29.50	36.07	27.21	30.48	37.27	28.95	32.44	39.66
20	26.18	29.33	35.87	27.03	30.28	37.03	28.71	32.17	39.34
21	26.05	29.18	35.69	26.87	30.10	36.81	28.49	31.92	39.04
22	25.92	29.04	35.52	26.72	29.93	36.61	28.29	31.70	38.77
23	25.81	28.92	35.36	26.58	29.78	36.42	28.11	31.49	38.52
24	25.70	28.80	35.22	26.45	29.64	36.25	27.94	31.30	38.29
25	25.61	28.69	35.09	26.34	29.51	36.09	27.78	31.13	38.07
26	25.52	28.59	34.97	26.23	29.39	35.94	27.64	30.97	37.87
27	25.43	28.50	34.85	26.13	29.27	35.81	27.50	30.81	37.69
28	25.35	28.41	34.75	26.03	29.17	35.68	27.38	30.67	37.52
29	25.28	28.33	34.64	25.95	29.07	35.56	27.26	30.54	37.35
30	25.21	28.25	34.55	25.86	28.98	35.44	27.14	30.41	37.20
35	24.92	27.92	34.15	25.51	28.58	34.96	26.67	29.89	36.56
40	24.69	27.67	33.84	25.24	28.28	34.59	26.31	29.48	36.06
45	24.51	27.47	33.60	25.02	28.04	34.30	26.02	29.15	35.66
50	24.37	27.30	33.40	24.84	27.84	34.05	25.78	28.88	35.33
60	24.14	27.04	33.08	24.57	27.53	33.67	25.40	28.46	34.82
70	23.96	26.85	32.85	24.36	27.29	33.39	25.12	28.15	34.44
80	23.83	26.70	32.67	24.20	27.11	33.17	24.90	27.90	34.14
90	23.72	26.58	32.52	24.07	26.97	32.99	24.73	27.71	33.89
100	23.63	26.48	32.40	23.96	26.84	32.84	24.58	27.54	33.69
200	23.18	25.98	31.78	23.41	26.23	32.08	23.83	26.70	32.67
500	22.83	25.58	31.29	22.97	25.73	31.48	23.23	26.03	31.84
∞	22.31	25.00	30.58	22.31	25.00	30.58	22.31	25.00	30.58

Table D-15
H VALUES (C = 16)

N	$\gamma = 0.90$			$\gamma = 0.95$			$\gamma = 0.99$		
	P = 0.90	P = 0.95	P = 0.99	P = 0.90	P = 0.95	P = 0.99	P = 0.90	P = 0.95	P = 0.99
2	59.55	66.04	79.31	69.65	77.25	92.77	95.49	105.89	127.18
3	44.64	49.66	59.96	49.53	55.10	66.53	60.82	67.65	81.68
4	39.03	43.48	52.64	42.40	47.23	57.18	49.84	55.52	67.22
5	36.00	40.13	48.66	38.63	43.07	52.21	44.30	49.39	59.88
6	34.07	38.00	46.12	36.26	40.45	49.09	40.92	45.64	55.39
7	32.73	36.51	44.34	34.63	38.64	46.92	38.62	43.09	52.32
8	31.73	35.41	43.02	33.42	37.30	45.32	36.94	41.23	50.09
9	30.96	34.56	41.99	32.60	36.27	44.08	35.67	40.81	48.38
10	30.34	33.87	41.17	31.76	35.45	43.09	34.65	38.68	47.02
11	29.84	33.32	40.50	31.15	34.78	42.28	33.83	37.77	45.91
12	29.42	32.85	39.94	30.65	34.22	41.61	33.14	37.01	44.99
13	29.06	32.45	39.46	30.22	33.74	41.03	32.56	36.36	44.21
14	28.75	32.11	39.05	29.85	33.33	40.54	32.07	35.81	43.54
15	28.49	31.81	38.69	29.53	32.98	40.11	31.63	35.32	42.96
16	28.25	31.55	38.37	29.25	32.66	39.73	31.25	34.90	42.45
17	28.04	31.31	38.09	29.00	32.38	39.39	30.92	34.52	41.99
18	27.85	31.10	37.84	28.77	32.13	39.09	30.61	34.19	41.59
19	27.68	30.92	37.61	28.57	31.91	38.82	30.34	33.89	41.22
20	27.53	30.75	37.40	28.39	31.70	38.57	30.10	33.61	40.89
21	27.39	30.59	37.21	28.22	31.52	38.34	29.87	33.36	40.59
22	27.26	30.45	37.04	28.07	31.35	38.14	29.67	33.14	40.31
23	27.15	30.32	36.88	27.93	31.19	37.95	29.48	32.93	40.06
24	27.04	30.20	36.74	27.80	31.05	37.77	29.31	32.73	39.82
25	26.94	30.09	36.60	27.68	30.91	37.61	29.16	32.56	39.61
26	26.84	29.98	36.48	27.57	30.79	37.46	29.00	32.39	39.41
27	26.76	29.88	36.36	27.46	30.67	37.32	28.86	32.23	39.22
28	26.68	29.79	36.25	27.37	30.57	37.19	28.73	32.09	39.04
29	26.60	29.71	36.15	27.28	30.47	37.07	28.61	31.95	38.88
30	26.53	29.63	36.05	27.19	30.37	36.95	28.49	31.82	38.72
35	26.23	29.29	35.64	26.83	29.97	36.46	28.01	31.29	38.07
40	26.00	29.04	35.33	26.55	29.66	36.09	27.64	30.87	37.56
45	25.81	28.83	35.08	26.33	29.41	35.78	27.34	30.54	37.16
50	25.66	28.66	34.87	26.14	29.20	35.54	27.09	30.26	36.82
60	25.42	28.39	34.55	25.86	28.88	35.15	26.71	29.83	36.30
70	25.24	28.20	34.31	25.65	28.65	34.86	26.42	29.51	35.91
80	25.11	28.04	34.13	25.48	28.46	34.63	26.20	29.26	35.61
90	25.00	27.92	33.98	25.34	28.31	34.45	26.02	29.06	35.36
100	24.90	27.82	33.85	25.23	28.19	34.30	25.87	28.89	35.16
200	24.44	27.30	33.22	24.67	27.55	33.53	25.10	28.04	34.12
500	24.08	26.89	32.73	24.22	27.05	32.92	24.48	27.35	33.28
∞	23.54	26.30	32.00	23.54	26.30	32.00	23.54	26.30	32.00

Table D-16
H VALUES (C = 17)

N	$\gamma = 0.90$			$\gamma = 0.95$			$\gamma = 0.99$		
	P = 0.90	P = 0.95	P = 0.99	P = 0.90	P = 0.95	P = 0.99	P = 0.90	P = 0.95	P = 0.99
2	61.49	68.01	81.32	71.52	79.10	94.57	96.88	107.16	128.12
3	46.41	51.48	61.88	51.31	56.92	68.41	62.53	69.36	83.36
4	40.68	45.19	54.45	44.07	48.96	58.99	51.52	57.23	68.96
5	37.58	41.77	50.41	40.23	44.72	53.97	45.92	51.05	61.60
6	35.60	39.59	47.82	37.81	42.06	50.79	42.49	47.26	57.08
7	34.22	38.07	46.00	36.14	40.21	48.59	40.16	44.68	54.00
8	33.19	36.94	44.66	34.91	38.84	46.96	38.46	42.80	51.74
9	32.40	36.06	43.61	33.96	37.79	45.71	37.15	41.35	50.01
10	31.77	35.36	42.77	33.20	36.95	44.70	36.12	40.21	48.64
11	31.25	34.79	42.09	32.58	36.27	43.88	35.28	39.28	47.52
12	30.82	34.31	41.51	32.06	35.69	43.19	34.58	38.50	46.59
13	30.45	33.90	41.02	31.62	35.21	42.60	33.99	37.85	45.80
14	30.13	33.55	40.60	31.24	34.79	42.10	33.48	37.28	45.12
15	29.85	33.24	40.23	30.91	34.42	41.66	33.04	36.79	44.53
16	29.61	32.97	39.91	30.62	34.10	41.27	32.65	36.36	44.01
17	29.39	32.73	39.62	30.37	33.81	40.93	32.31	35.98	43.55
18	29.20	32.52	39.36	30.14	33.56	40.62	32.00	35.63	43.14
19	29.03	32.32	39.13	29.93	33.33	40.35	31.72	35.33	42.77
20	28.87	32.15	38.92	29.74	33.12	40.09	31.47	35.05	42.43
21	28.72	31.99	38.73	29.57	32.93	39.87	31.24	34.79	42.12
22	28.59	31.84	38.55	29.41	32.75	39.66	31.04	34.56	41.84
23	28.47	31.71	38.39	29.27	32.59	39.46	30.84	34.35	41.59
24	28.36	31.58	38.24	29.14	32.45	39.28	30.67	34.15	41.35
25	28.26	31.47	38.10	29.01	32.31	39.12	30.50	33.97	41.13
26	28.16	31.36	37.97	28.90	32.18	38.97	30.35	33.80	40.92
27	28.07	31.26	37.85	28.79	32.06	38.82	30.21	33.64	40.73
28	27.99	31.17	37.74	28.69	31.95	38.69	30.07	33.49	40.55
29	27.91	31.08	37.64	28.60	31.85	38.56	29.95	33.35	40.39
30	27.84	31.00	37.54	28.51	31.75	38.45	29.83	33.22	40.23
35	27.53	30.66	37.12	28.14	31.34	37.95	29.34	32.68	39.57
40	27.29	30.39	36.80	27.85	31.02	37.56	28.96	32.25	39.05
45	27.10	30.18	36.54	27.62	30.77	37.26	28.65	31.91	38.64
50	26.94	30.00	36.33	27.44	30.56	37.00	28.40	31.63	38.30
60	26.70	29.73	36.01	27.14	30.23	36.61	28.01	31.19	37.78
70	26.52	29.53	35.76	26.92	29.99	36.31	27.71	30.87	37.38
80	26.37	29.37	35.57	26.75	29.80	36.08	27.49	30.61	37.07
90	26.26	29.25	35.42	26.62	29.64	35.90	27.30	30.41	36.82
100	26.17	29.14	35.29	26.50	29.52	35.74	27.15	30.23	36.61
200	25.69	28.61	34.65	25.92	28.87	34.96	26.36	29.36	35.56
500	25.32	28.20	34.15	25.46	28.35	34.34	25.73	28.66	34.70
∞	24.77	27.59	33.41	24.77	27.59	33.41	24.77	27.59	33.41

Table D-17
H VALUES (C = 18)

N	$\gamma = 0.90$			$\gamma = 0.95$			$\gamma = 0.99$		
	P = 0.90	P = 0.95	P = 0.99	P = 0.90	P = 0.95	P = 0.99	P = 0.90	P = 0.95	P = 0.99
2	63.43	69.98	83.32	73.40	80.99	96.43	98.38	108.55	129.24
3	48.17	53.30	63.78	53.09	58.74	70.30	64.28	71.12	85.12
4	42.32	46.89	56.26	45.74	50.67	60.80	53.19	58.94	70.71
5	39.14	43.40	52.14	41.82	46.37	55.71	47.54	52.71	63.32
6	37.11	41.17	49.51	39.35	43.65	52.49	44.06	48.88	58.77
7	35.70	39.61	47.66	37.64	41.77	50.26	41.69	46.26	55.66
8	34.64	38.45	46.28	36.38	40.38	48.60	39.96	44.35	53.38
9	33.83	37.55	45.22	35.41	39.30	47.32	38.63	42.89	51.63
10	33.18	36.84	44.36	34.63	38.44	46.30	37.58	41.72	50.25
11	32.65	36.25	43.66	33.99	37.74	45.46	36.73	40.78	49.11
12	32.20	35.75	43.07	33.46	37.16	44.76	36.01	39.99	48.17
13	31.82	35.34	42.57	33.01	36.66	44.16	35.41	39.32	47.37
14	31.50	34.98	42.14	32.62	36.23	43.65	34.89	38.75	46.68
15	31.21	34.66	41.76	32.29	35.85	43.20	34.44	38.25	46.09
16	30.96	34.38	41.43	31.99	35.53	42.81	34.04	37.81	45.56
17	30.74	34.14	41.14	31.73	35.23	42.46	33.69	37.42	45.09
18	30.54	33.92	40.88	31.49	34.97	42.15	33.38	37.07	44.67
19	30.36	33.72	40.64	31.28	34.74	41.86	33.09	36.76	44.30
20	30.20	33.54	40.42	31.08	34.52	41.61	32.84	36.47	43.96
21	30.05	33.38	40.23	30.91	34.33	41.37	32.61	36.21	43.65
22	29.92	33.23	40.05	30.75	34.15	41.16	32.39	35.98	43.36
23	29.79	33.09	39.88	30.60	33.99	40.96	32.20	35.76	43.10
24	29.68	32.96	39.73	30.46	33.83	40.78	32.01	35.56	42.86
25	29.57	32.84	39.59	30.34	33.69	40.61	31.85	35.37	42.64
26	29.47	32.73	39.46	30.22	33.56	40.46	31.69	35.20	42.43
27	29.38	32.63	39.33	30.11	33.44	40.31	31.55	35.04	42.23
28	29.29	32.54	39.22	30.01	33.33	40.18	31.41	34.89	42.05
29	29.21	32.45	39.11	29.91	33.22	40.05	31.28	34.75	41.88
30	29.14	32.36	39.01	29.82	33.12	39.93	31.16	34.61	41.72
35	28.82	32.01	38.59	29.44	32.70	39.42	30.66	34.05	41.05
40	28.57	31.74	38.26	29.15	32.38	39.03	30.27	33.62	40.53
45	28.38	31.52	38.00	28.91	32.12	38.72	29.96	33.27	40.11
50	28.22	31.34	37.78	28.72	31.90	38.46	29.70	32.99	39.77
60	27.97	31.06	37.45	28.42	31.57	38.06	29.30	32.54	39.23
70	27.78	30.86	37.20	28.19	31.32	37.76	29.00	32.21	38.83
80	27.63	30.69	37.01	28.02	31.12	37.52	28.76	31.95	38.52
90	27.52	30.56	36.85	27.88	30.97	37.33	28.57	31.74	38.27
100	27.42	30.46	36.72	27.76	30.84	37.18	28.42	31.57	38.06
200	26.93	29.92	36.07	27.17	30.18	36.38	27.61	30.67	36.98
500	26.55	29.49	35.55	26.69	29.65	35.75	26.97	29.96	36.12
∞	25.99	28.87	34.81	25.99	28.87	34.81	25.99	28.87	34.81

Table D-18
H VALUES (C = 19)

N	$\gamma = 0.90$			$\gamma = 0.95$			$\gamma = 0.99$		
	P = 0.90	P = 0.95	P = 0.99	P = 0.90	P = 0.95	P = 0.99	P = 0.90	P = 0.95	P = 0.99
2	65.37	71.96	85.35	75.30	82.89	98.32	99.86	109.92	130.38
3	49.92	55.10	65.69	54.86	60.56	72.19	66.01	72.87	86.86
4	43.96	48.59	58.06	47.39	52.38	62.59	54.86	60.64	72.46
5	40.70	45.02	53.87	43.40	48.01	57.44	49.14	54.36	65.04
6	38.62	42.74	51.18	40.88	45.24	54.18	45.62	50.49	60.48
7	37.17	41.14	49.30	39.13	43.32	51.91	43.21	47.84	57.32
8	36.09	39.96	47.90	37.84	41.90	50.23	41.45	45.90	55.02
9	35.25	39.04	46.81	36.85	40.80	48.92	40.10	44.41	53.25
10	34.58	38.30	45.94	36.05	39.93	47.88	39.03	43.23	51.85
11	34.04	37.70	45.22	35.40	39.21	47.03	38.16	42.27	50.70
12	33.58	37.19	44.62	34.86	38.61	46.32	37.44	41.46	49.74
13	33.19	36.76	44.11	34.40	38.10	45.71	36.82	40.79	48.93
14	32.85	36.39	43.67	34.00	37.66	45.19	36.29	40.20	48.24
15	32.56	36.07	43.29	33.65	37.28	44.73	35.83	39.69	47.63
16	32.30	35.79	42.95	33.35	36.94	44.33	35.43	39.25	47.10
17	32.08	35.54	42.65	33.08	36.64	43.98	35.07	38.85	46.62
18	31.87	35.31	42.38	32.83	36.38	43.66	34.75	38.49	46.20
19	31.69	35.11	42.13	32.62	36.13	43.37	34.46	38.18	45.82
20	31.52	34.92	41.92	32.42	35.92	43.11	34.20	37.89	45.47
21	31.37	34.75	41.71	32.24	35.72	42.87	33.96	37.62	45.16
22	31.23	34.60	41.53	32.07	35.54	42.65	33.74	37.38	44.87
23	31.10	34.46	41.36	31.92	35.37	42.45	33.54	37.16	44.60
24	30.98	34.33	41.21	31.78	35.21	42.27	33.35	36.96	44.36
25	30.88	34.21	41.06	31.65	35.07	42.10	33.18	36.77	44.13
26	30.77	34.10	40.93	31.53	34.94	41.94	33.02	36.59	43.92
27	30.68	33.99	40.80	31.42	34.81	41.79	32.88	36.43	43.73
28	30.59	33.89	40.69	31.32	34.70	41.65	32.74	36.27	43.54
29	30.51	33.80	40.58	31.22	34.59	41.52	32.61	36.13	43.37
30	30.43	33.72	40.48	31.12	34.49	41.40	32.49	35.99	43.21
35	30.10	33.35	40.04	30.74	34.06	40.88	31.97	35.42	42.53
40	29.85	33.08	39.71	30.43	33.72	40.48	31.57	34.98	42.00
45	29.65	32.85	39.44	30.19	33.46	40.16	31.25	34.63	41.57
50	29.48	32.67	39.22	30.00	33.24	39.90	30.99	34.34	41.22
60	29.23	32.38	38.88	29.69	32.90	39.49	30.58	33.88	40.68
70	29.03	32.17	38.63	29.46	32.64	39.19	30.27	33.55	40.27
80	28.89	32.01	38.43	29.28	32.44	38.95	30.03	33.28	39.96
90	28.77	31.87	38.27	29.13	32.28	38.76	29.84	33.07	39.70
100	28.67	31.76	38.14	29.01	32.15	38.60	29.68	32.89	39.48
200	28.17	31.21	37.47	28.41	31.48	37.79	28.86	31.98	38.40
500	27.77	30.78	36.95	27.92	30.94	37.15	28.20	31.25	37.52
∞	27.20	30.14	36.19	27.20	30.14	36.19	27.20	30.14	36.19

Table D-19
H VALUES (C = 20)

N	$\gamma = 0.90$			$\gamma = 0.95$			$\gamma = 0.99$		
	P = 0.90	P = 0.95	P = 0.99	P = 0.90	P = 0.95	P = 0.99	P = 0.90	P = 0.95	P = 0.99
2	67.32	73.95	87.39	77.20	84.80	100.22	101.49	111.48	131.76
3	51.65	56.89	67.57	56.62	62.36	74.06	67.79	74.66	88.67
4	45.58	50.27	59.85	49.04	54.08	64.38	56.53	62.35	74.22
5	42.25	46.63	55.59	44.98	49.64	59.17	50.75	56.01	66.76
6	40.12	44.30	52.85	42.40	46.82	55.86	47.17	52.09	62.14
7	38.63	42.67	50.93	40.62	44.86	53.55	44.73	49.40	58.97
8	37.52	41.45	49.50	39.30	43.41	51.84	42.94	47.43	56.64
9	36.67	40.51	48.39	38.28	42.29	50.52	41.57	45.92	54.86
10	35.98	39.76	47.50	37.47	41.40	49.46	40.48	44.73	53.43
11	35.42	39.14	46.77	36.80	40.66	48.59	39.59	43.75	52.27
12	34.95	38.62	46.16	36.24	40.05	47.87	38.85	42.93	51.31
13	34.55	38.18	45.64	35.77	39.53	47.25	38.22	42.24	50.49
14	34.20	37.80	45.19	35.36	39.08	46.72	37.68	41.65	49.78
15	33.90	37.47	44.79	35.01	38.69	46.25	37.21	41.13	49.17
16	33.64	37.18	44.45	34.70	38.35	45.85	36.80	40.68	48.63
17	33.41	36.92	44.14	34.42	38.04	45.48	36.43	40.27	48.14
18	33.20	36.69	43.87	34.17	37.77	45.16	36.11	39.91	47.72
19	33.01	36.48	43.62	33.95	37.52	44.86	35.81	39.59	47.33
20	32.84	36.30	43.40	33.75	37.30	44.60	35.55	39.29	46.98
21	32.68	36.12	43.19	33.56	37.10	44.36	35.30	39.02	46.66
22	32.54	35.97	43.00	33.39	36.91	44.14	35.08	38.78	46.37
23	32.41	35.82	42.83	33.24	36.74	43.93	34.88	38.55	46.10
24	32.29	35.69	42.67	33.09	36.58	43.74	34.69	38.34	45.85
25	32.17	35.56	42.53	32.96	36.44	43.57	34.51	38.15	45.62
26	32.07	35.45	42.39	32.84	36.30	43.41	34.35	37.97	45.41
27	31.97	35.34	42.26	32.72	36.17	43.26	34.20	37.80	45.21
28	31.88	35.24	42.14	32.62	36.06	43.12	34.06	37.65	45.02
29	31.80	35.15	42.03	32.52	35.94	42.98	33.92	37.50	44.85
30	31.72	35.06	41.93	32.42	35.84	42.86	33.80	37.36	44.68
35	31.38	34.69	41.49	32.02	35.40	42.33	33.27	36.78	43.99
40	31.12	34.40	41.14	31.71	35.06	41.93	32.87	36.33	43.45
45	30.91	34.18	40.87	31.47	34.79	41.60	32.54	35.97	43.02
50	30.74	33.99	40.65	31.26	34.56	41.34	32.27	35.68	42.67
60	30.40	33.70	40.30	30.95	34.22	40.92	31.86	35.22	42.12
70	30.28	33.48	40.04	30.71	33.96	40.61	31.54	34.87	41.70
80	30.13	33.31	39.84	30.53	33.75	40.37	31.30	34.60	41.38
90	30.01	33.18	39.68	30.38	33.59	40.17	31.10	34.38	41.12
100	29.91	33.06	39.54	30.26	33.45	40.01	30.94	34.20	40.90
200	29.40	32.50	38.87	29.64	32.77	39.19	30.10	33.28	39.80
500	28.99	32.05	38.34	29.14	32.22	38.53	29.43	32.54	38.91
∞	28.41	31.41	37.57	28.41	31.41	37.57	28.41	31.41	37.57

Table D-20
H VALUES (C = 21)

N	$\gamma = 0.90$			$\gamma = 0.95$			$\gamma = 0.99$		
	P = 0.90	P = 0.95	P = 0.99	P = 0.90	P = 0.95	P = 0.99	P = 0.90	P = 0.95	P = 0.99
2	69.26	75.93	89.44	79.12	86.73	102.16	103.18	113.12	133.24
3	53.39	58.68	69.45	58.37	64.16	75.94	69.53	76.43	90.46
4	47.20	51.95	61.62	50.68	55.78	66.17	58.20	64.05	75.98
5	43.80	48.23	57.29	46.54	51.26	60.88	52.34	57.65	68.47
6	41.62	45.85	54.51	43.92	48.39	57.53	48.72	53.68	63.81
7	40.09	44.18	52.55	42.10	46.39	55.18	46.24	50.96	60.61
8	38.95	42.94	51.09	40.75	44.92	53.44	44.42	48.96	58.26
9	38.07	41.98	49.96	39.70	43.77	52.10	43.02	47.43	56.45
10	37.37	41.21	49.05	38.87	42.86	51.02	41.91	46.21	55.02
11	36.79	40.57	48.31	38.19	42.11	50.14	41.01	45.22	53.84
12	36.31	40.04	47.68	37.62	41.49	49.40	40.26	44.39	52.86
13	35.90	39.59	47.15	37.14	40.96	48.78	39.62	43.69	52.03
14	35.55	39.21	46.69	36.72	40.50	48.23	39.07	43.09	51.31
15	35.24	38.87	46.29	36.36	40.10	47.76	38.59	42.56	50.69
16	34.97	38.57	45.94	36.04	39.75	47.35	38.17	42.10	50.14
17	34.73	38.30	45.63	35.76	39.44	46.98	37.80	41.69	49.66
18	34.51	38.07	45.35	35.50	39.16	46.65	37.46	41.32	49.22
19	34.32	37.85	45.10	35.27	38.91	46.35	37.16	40.99	48.83
20	34.14	37.66	44.87	35.07	38.68	46.08	36.89	40.69	48.47
21	33.98	37.49	44.66	34.88	38.47	45.83	36.64	40.42	48.15
22	33.84	37.32	44.47	34.71	38.28	45.61	36.41	40.17	47.85
23	33.70	37.18	44.29	34.55	38.11	45.40	36.21	39.94	47.58
24	33.58	37.04	44.13	34.40	37.95	45.21	36.01	39.72	47.33
25	33.46	36.91	43.98	34.26	37.80	45.03	35.84	39.53	47.10
26	33.36	36.80	43.84	34.14	37.66	44.87	35.67	39.35	46.88
27	33.26	36.69	43.71	34.02	37.53	44.71	35.52	39.18	46.68
28	33.17	36.58	43.59	33.91	37.41	44.57	35.37	39.02	46.49
29	33.08	36.49	43.48	33.81	37.29	44.43	35.24	38.87	46.31
30	33.00	36.40	43.37	33.71	37.19	44.31	35.11	38.73	46.14
35	32.65	36.02	42.92	33.30	36.74	43.77	34.57	38.14	45.44
40	32.39	35.73	42.57	32.99	36.39	43.36	34.16	37.68	44.90
45	32.17	35.49	42.29	32.74	36.11	43.03	33.82	37.31	44.46
50	32.00	35.30	42.06	32.53	35.88	42.76	33.55	37.01	44.10
60	31.73	35.00	41.71	32.21	35.53	42.34	33.12	36.54	43.54
70	31.53	34.78	41.45	31.97	35.26	42.02	32.81	36.19	43.13
80	31.37	34.61	41.24	31.78	35.06	41.77	32.56	35.92	42.80
90	31.25	34.47	41.08	31.63	34.89	41.57	32.35	35.69	42.53
100	31.14	34.35	40.94	31.50	34.75	41.41	32.19	35.51	42.31
200	30.62	33.78	40.25	30.87	34.05	40.58	31.34	34.57	41.19
500	30.21	33.32	39.71	30.36	33.49	39.91	30.65	33.81	40.29
∞	29.62	32.67	38.93	29.62	32.67	38.93	29.62	32.67	38.93

Table D-21
H VALUES (C = 22)

N	$\gamma = 0.90$			$\gamma = 0.95$			$\gamma = 0.99$		
	P = 0.90	P = 0.95	P = 0.99	P = 0.90	P = 0.95	P = 0.99	P = 0.90	P = 0.95	P = 0.99
2	71.21	77.92	91.49	81.04	88.68	104.13	104.92	114.81	134.81
3	55.12	60.47	71.34	60.12	65.96	77.82	71.28	78.20	92.25
4	48.81	53.61	63.39	52.32	57.46	67.95	59.85	65.74	77.74
5	45.33	49.83	58.99	48.10	52.87	62.59	53.93	59.28	70.18
6	43.10	47.39	56.15	45.43	49.95	59.18	50.26	55.27	65.48
7	41.54	45.69	54.16	43.56	47.92	56.80	47.74	52.51	62.25
8	40.38	44.42	52.68	42.19	46.41	55.04	45.89	50.49	59.87
9	39.47	43.43	51.52	41.12	45.25	53.67	44.47	48.93	58.04
10	38.75	42.65	50.60	40.27	44.32	52.58	43.34	47.69	56.59
11	38.16	42.00	49.84	39.58	43.55	51.68	42.42	46.69	55.40
12	37.67	41.46	49.20	38.99	42.92	50.93	41.65	45.84	54.41
13	37.25	41.00	48.66	38.50	42.37	50.29	41.00	45.13	53.56
14	36.89	40.60	48.19	38.07	41.91	49.74	40.44	44.52	52.84
15	36.57	40.25	47.78	37.70	41.50	49.26	39.96	43.98	52.21
16	36.29	39.95	47.42	37.38	41.14	48.84	39.53	43.51	51.65
17	36.05	39.68	47.10	37.09	40.82	48.46	39.15	43.09	51.16
18	35.83	39.44	46.82	36.83	40.54	48.13	38.81	42.72	50.72
19	35.63	39.22	46.56	36.59	40.28	47.82	38.50	42.38	50.32
20	35.45	39.02	46.33	36.38	40.05	47.55	38.23	42.08	49.96
21	35.28	38.84	46.12	36.19	39.84	47.30	37.97	41.80	49.63
22	35.13	38.68	45.92	36.01	39.64	47.07	37.74	41.55	49.33
23	35.00	38.52	45.74	35.85	39.46	46.86	37.53	41.31	49.05
24	34.87	38.38	45.58	35.70	39.30	46.66	37.33	41.10	48.80
25	34.75	38.26	45.42	35.56	39.15	46.48	37.15	40.90	48.56
26	34.64	38.13	45.28	35.43	39.01	46.32	36.98	40.71	48.34
27	34.54	38.02	45.15	35.31	38.87	46.16	36.82	40.54	48.14
28	34.44	37.92	45.03	35.20	38.75	46.01	36.68	40.38	47.95
29	34.35	37.82	44.91	35.09	38.63	45.88	36.54	40.23	47.77
30	34.27	37.73	44.80	34.99	38.53	45.75	36.41	40.08	47.60
35	33.92	37.34	44.34	34.58	38.07	45.20	35.86	39.48	46.89
40	33.65	37.04	43.99	34.26	37.71	44.78	35.44	39.02	46.33
45	33.43	36.80	43.70	34.00	37.43	44.45	35.10	38.64	45.89
50	33.25	36.61	43.47	33.79	37.20	44.17	34.82	38.34	45.53
60	32.97	36.30	43.11	33.46	36.83	43.74	34.39	37.86	44.96
70	32.77	36.08	42.84	33.21	36.56	43.42	34.06	37.50	44.54
80	32.61	35.90	42.63	33.02	36.35	43.17	33.81	37.22	44.20
90	32.48	35.76	42.47	32.86	36.18	42.97	33.60	37.00	43.94
100	32.37	35.64	42.33	32.74	36.04	42.80	33.43	36.81	43.71
200	31.84	35.05	41.63	32.09	35.33	41.95	32.56	35.85	42.58
500	31.42	34.59	41.08	31.57	34.76	41.28	31.87	35.08	41.67
∞	30.81	33.92	40.29	30.81	33.92	40.29	30.81	33.92	40.29

Table D-22
H VALUES (C = 23)

N	$\gamma = 0.90$			$\gamma = 0.95$			$\gamma = 0.99$		
	P = 0.90	P = 0.95	P = 0.99	P = 0.90	P = 0.95	P = 0.99	P = 0.90	P = 0.95	P = 0.99
2	73.15	79.90	93.54	82.98	90.64	106.11	106.60	116.44	136.32
3	56.84	62.25	73.21	61.87	67.75	79.69	73.03	79.98	94.06
4	50.41	55.27	65.15	53.94	59.14	69.71	61.51	67.44	79.49
5	46.86	51.41	60.68	49.65	54.18	64.29	55.52	60.91	71.89
6	44.58	48.93	57.79	46.93	51.51	60.83	51.79	56.85	67.14
7	42.98	47.19	55.76	45.03	49.44	58.42	49.23	54.05	63.88
8	41.79	45.89	54.25	43.62	47.90	56.63	47.36	52.00	61.47
9	40.87	44.88	53.07	42.53	46.71	55.24	45.91	50.42	59.62
10	40.13	44.08	52.13	41.67	45.76	54.12	44.76	49.17	58.15
11	39.53	43.42	51.36	40.95	44.99	53.21	43.83	48.14	56.95
12	39.02	42.86	50.71	40.36	44.34	52.45	43.05	47.29	55.94
13	38.59	42.39	50.16	39.86	43.78	51.80	42.38	46.56	55.09
14	38.22	41.99	49.68	39.42	43.31	51.24	41.82	45.94	54.35
15	37.90	41.63	49.26	39.04	42.89	50.75	41.32	45.39	53.71
16	37.61	41.32	48.90	38.71	42.53	50.32	40.88	44.92	53.15
17	37.36	41.04	48.57	38.41	42.20	49.94	40.50	44.49	52.65
18	37.13	40.80	48.28	38.15	41.91	49.60	40.15	44.11	52.20
19	36.93	40.57	48.02	37.91	41.65	49.29	39.84	43.77	51.80
20	36.74	40.37	47.78	37.69	41.41	49.01	39.55	43.46	51.44
21	36.58	40.19	47.56	37.49	41.20	48.76	39.30	43.18	51.10
22	36.42	40.02	47.37	37.31	41.00	48.52	39.06	42.92	50.80
23	36.28	39.86	47.18	37.15	40.82	48.31	38.85	42.68	50.52
24	36.15	39.72	47.02	36.99	40.65	48.11	38.65	42.46	50.26
25	36.03	39.59	46.86	36.85	40.49	47.93	38.46	42.26	50.02
26	35.92	39.47	46.71	36.72	40.35	47.76	38.29	42.07	49.80
27	35.81	39.35	46.58	36.60	40.21	47.60	38.13	41.90	49.59
28	35.72	39.25	46.45	36.48	40.09	47.45	37.98	41.73	49.40
29	35.62	39.15	46.34	36.37	39.97	47.31	37.84	41.58	49.21
30	35.54	39.05	46.22	36.27	39.86	47.18	37.71	41.43	49.04
35	35.18	38.65	45.76	35.85	39.39	46.63	37.15	40.82	48.32
40	34.90	38.35	45.40	35.52	39.03	46.20	36.72	40.35	47.76
45	34.68	38.10	45.11	35.25	38.74	45.86	36.37	39.97	47.31
50	34.49	37.90	44.87	35.04	38.50	45.58	36.09	39.66	46.94
60	34.21	37.59	44.50	34.70	38.13	45.14	35.65	39.17	46.37
70	34.00	37.36	44.23	34.45	37.86	44.82	35.31	38.81	45.94
80	33.84	37.18	44.02	34.25	37.64	44.56	35.06	38.52	45.60
90	33.70	37.04	43.85	34.10	37.47	44.35	34.85	38.29	45.33
100	33.60	36.92	43.70	33.97	37.32	44.19	34.67	38.10	45.10
200	33.05	36.32	42.99	33.30	36.60	43.33	33.79	37.13	43.96
500	32.62	35.85	42.44	32.78	36.02	42.64	33.08	36.35	43.03
∞	32.01	35.17	41.64	32.01	35.17	41.64	32.01	35.17	41.64

Table D-23
H VALUES (C = 24)

N	$\gamma = 0.90$			$\gamma = 0.95$			$\gamma = 0.99$		
	P = 0.90	P = 0.95	P = 0.99	P = 0.90	P = 0.95	P = 0.99	P = 0.90	P = 0.95	P = 0.99
2	75.09	81.88	95.60	84.91	92.60	108.11	108.39	118.20	138.00
3	58.56	64.02	75.08	63.61	69.54	81.56	74.78	81.75	95.87
4	52.01	56.92	66.90	55.56	60.82	71.48	63.16	69.13	81.25
5	48.38	52.99	62.36	51.20	56.07	65.98	57.10	62.53	73.59
6	46.05	50.46	59.42	48.42	53.05	62.48	53.32	58.42	68.80
7	44.42	48.68	57.35	46.48	50.95	60.02	50.72	55.59	65.50
8	43.20	47.35	55.81	45.05	49.38	58.20	48.81	53.51	63.07
9	42.26	46.33	54.62	43.94	48.17	56.79	47.35	51.91	61.20
10	41.50	45.50	53.66	43.05	47.21	55.66	46.18	50.63	59.71
11	40.88	44.83	52.87	42.33	46.41	54.74	45.23	49.59	58.49
12	40.36	44.26	52.21	41.72	45.75	53.96	44.43	48.73	57.47
13	39.92	43.78	51.64	41.20	45.19	53.30	43.76	47.99	56.61
14	39.54	43.37	51.16	40.76	44.70	52.73	43.18	47.35	55.86
15	39.21	43.01	50.74	40.37	44.28	52.24	42.68	46.80	55.21
16	38.92	42.69	50.36	40.03	43.90	51.80	42.23	46.32	54.64
17	38.66	42.40	50.03	39.73	43.57	51.41	41.84	45.88	54.14
18	38.43	42.15	49.73	39.46	43.28	51.06	41.48	45.50	53.68
19	38.22	41.92	49.47	39.21	43.01	50.75	41.17	45.15	53.28
20	38.04	41.72	49.22	38.99	42.77	50.46	40.88	44.84	52.91
21	37.88	41.53	49.00	38.79	42.55	50.21	40.62	44.55	52.57
22	37.71	41.36	48.80	38.61	42.35	49.97	40.38	44.29	52.26
23	37.56	41.20	48.62	38.44	42.16	49.75	40.16	44.05	51.98
24	37.43	41.05	48.44	38.28	41.99	49.55	39.95	43.82	51.71
25	37.30	40.92	48.29	38.14	41.83	49.36	39.76	43.62	51.47
26	37.19	40.79	48.14	38.00	41.68	49.19	39.59	43.42	51.24
27	37.08	40.68	48.00	37.88	41.55	49.03	39.43	43.25	51.03
28	36.98	40.57	47.87	37.76	41.42	48.88	39.27	43.08	50.84
29	36.89	40.46	47.75	37.65	41.30	48.74	39.13	42.92	50.65
30	36.80	40.37	47.64	37.55	41.18	48.60	39.00	42.77	50.48
35	36.43	39.96	47.16	37.11	40.71	48.04	38.43	42.15	49.75
40	36.15	39.65	46.80	36.77	40.34	47.61	37.99	41.67	49.18
45	35.92	39.40	46.50	36.50	40.04	47.26	37.64	41.29	48.73
50	35.73	39.20	46.26	36.28	39.80	46.97	37.35	40.97	48.35
60	35.44	38.88	45.89	35.94	39.43	46.53	36.90	40.47	47.77
70	35.23	38.64	45.61	35.68	39.14	46.20	36.56	40.10	47.33
80	35.06	38.46	45.39	35.48	38.92	45.94	36.30	39.82	46.99
90	34.93	38.31	45.22	35.32	38.75	45.73	36.08	39.58	46.72
100	34.82	38.19	45.08	35.19	38.60	45.56	35.91	39.39	46.49
200	34.26	37.58	44.35	34.52	37.86	44.69	35.01	38.40	45.32
500	33.82	37.10	43.79	33.98	37.28	44.00	34.28	37.61	44.39
∞	33.20	36.42	42.98	33.20	36.42	42.98	33.20	36.42	42.98

Table D-24
H VALUES (C = 25)

N	$\gamma = 0.90$			$\gamma = 0.95$			$\gamma = 0.99$		
	P = 0.90	P = 0.95	P = 0.99	P = 0.90	P = 0.95	P = 0.99	P = 0.90	P = 0.95	P = 0.99
2	77.02	83.86	97.65	86.85	94.56	110.11	110.21	120.00	139.73
3	60.27	65.78	76.94	65.35	71.33	83.42	76.53	83.53	97.69
4	53.60	58.57	68.65	57.18	62.48	73.23	64.81	70.82	83.00
5	49.90	54.56	64.03	52.74	57.66	67.67	58.67	64.15	75.28
6	47.52	51.98	61.04	49.91	54.59	64.11	54.84	59.99	70.45
7	45.85	50.16	58.94	47.93	52.45	61.62	52.20	57.12	67.11
8	44.60	48.81	57.37	46.47	50.86	59.77	50.27	55.01	64.66
9	43.64	47.76	56.15	45.34	49.62	58.34	48.78	53.39	62.76
10	42.87	46.92	55.17	44.44	48.64	57.19	47.59	52.09	61.25
11	42.23	46.23	54.37	43.69	47.83	56.25	46.62	51.04	60.02
12	41.70	45.66	53.70	43.08	47.16	55.46	45.81	50.16	58.99
13	41.25	45.16	53.12	42.55	46.58	54.79	45.13	49.41	58.11
14	40.87	44.74	52.63	42.10	46.09	54.21	44.54	48.76	57.36
15	40.53	44.37	52.20	41.70	45.66	53.71	44.03	48.20	56.71
16	40.23	44.05	51.82	41.35	45.28	53.27	43.57	47.71	56.13
17	39.96	43.76	51.48	41.04	44.94	52.87	43.17	47.27	55.61
18	39.73	43.50	51.18	40.77	44.64	52.52	42.81	46.88	55.15
19	39.51	43.27	50.91	40.52	44.36	52.20	42.49	46.53	54.74
20	39.32	43.06	50.66	40.29	44.12	51.91	42.20	46.21	54.37
21	39.15	42.86	50.44	40.09	43.89	51.65	41.93	45.91	54.02
22	38.99	42.69	50.23	39.90	43.69	51.41	41.69	45.65	53.71
23	38.84	42.53	50.04	39.72	43.50	51.18	41.46	45.40	53.42
24	38.70	42.38	49.87	39.56	43.32	50.98	41.26	45.18	53.16
25	38.57	42.24	49.71	39.42	43.16	50.79	41.06	44.97	52.91
26	38.46	42.11	49.56	39.28	43.01	50.61	40.88	44.77	52.68
27	38.35	41.99	49.42	39.15	42.87	50.45	40.72	44.59	52.47
28	38.25	41.88	49.28	39.03	42.74	50.30	40.56	44.42	52.27
29	38.15	41.78	49.16	38.92	42.62	50.15	40.42	44.26	52.08
30	38.06	41.68	49.05	38.81	42.50	50.02	40.28	44.11	51.91
35	37.68	41.27	48.56	38.37	42.02	49.45	39.70	43.48	51.17
40	37.39	40.95	48.19	38.03	41.64	49.01	39.26	42.99	50.59
45	37.16	40.69	47.89	37.75	41.34	48.65	38.90	42.60	50.13
50	36.97	40.48	47.64	37.53	41.10	48.36	38.60	42.27	49.75
60	36.67	40.16	47.27	37.18	40.71	47.91	38.14	41.77	49.16
70	36.45	39.92	46.98	36.91	40.43	47.58	37.80	41.40	48.72
80	36.28	39.73	46.76	36.71	40.20	47.32	37.53	41.10	48.38
90	36.14	39.58	46.59	36.55	40.02	47.10	37.32	40.87	48.10
100	36.03	39.46	46.44	36.41	39.87	46.93	37.14	40.67	47.86
200	35.46	38.84	45.71	35.72	39.12	46.04	36.22	39.67	46.69
500	35.02	38.35	45.13	35.18	38.53	45.34	35.49	38.86	45.74
∞	34.38	37.65	44.31	34.38	37.65	44.31	34.38	37.65	44.31

Table D-25
H VALUES (C = 26)

N	$\gamma = 0.90$			$\gamma = 0.95$			$\gamma = 0.99$		
	P = 0.90	P = 0.95	P = 0.99	P = 0.90	P = 0.95	P = 0.99	P = 0.90	P = 0.95	P = 0.99
2	78.94	85.83	99.69	88.77	96.51	112.10	111.98	121.75	141.41
3	61.97	67.54	78.79	67.08	73.10	85.28	78.28	85.31	99.52
4	55.18	60.21	70.38	58.79	64.14	74.98	66.45	72.50	84.75
5	51.41	56.13	65.69	54.27	59.25	69.34	60.24	65.76	76.97
6	48.98	53.49	62.65	51.39	56.13	65.74	56.36	61.55	72.09
7	47.27	51.64	60.51	49.38	53.94	63.21	53.68	58.65	68.72
8	46.00	50.26	58.92	47.89	52.33	61.34	51.71	56.51	66.24
9	45.02	49.19	57.68	46.73	51.07	59.88	50.20	54.86	64.32
10	44.23	48.34	56.68	45.81	50.07	58.71	49.00	53.55	62.79
11	43.58	47.63	55.87	45.06	49.25	57.76	48.01	52.48	61.55
12	43.04	47.04	55.18	44.42	48.56	56.96	47.19	51.58	60.50
13	42.58	46.54	54.60	43.89	47.97	56.28	46.49	50.82	59.62
14	42.18	46.11	54.09	43.42	47.47	55.69	45.89	50.17	58.85
15	41.84	45.73	53.66	43.02	47.03	55.18	45.37	49.60	58.19
16	41.53	45.40	53.27	42.67	46.64	54.73	44.91	49.10	57.60
17	41.26	45.11	52.93	42.35	46.30	54.32	44.50	48.65	57.08
18	41.02	44.84	52.62	42.07	45.99	53.97	44.14	48.25	56.62
19	40.80	44.61	52.34	41.81	45.71	53.64	43.81	47.89	56.20
20	40.60	44.39	52.09	41.58	45.46	53.35	43.51	47.57	55.82
21	40.42	44.19	51.86	41.37	45.23	53.08	43.24	47.27	55.47
22	40.26	44.02	51.65	41.18	45.02	52.84	42.99	47.00	55.16
23	40.11	43.85	51.46	41.01	44.83	52.61	42.76	46.75	54.87
24	39.97	43.70	51.28	40.84	44.65	52.40	42.55	46.52	54.60
25	39.84	43.56	51.12	40.69	44.49	52.21	42.36	46.31	54.35
26	39.72	43.43	50.96	40.55	44.34	52.03	42.18	46.11	54.11
27	39.61	43.31	50.82	40.42	44.19	51.87	42.01	45.93	53.90
28	39.50	43.19	50.69	40.30	44.06	51.71	41.85	45.75	53.70
29	39.41	43.08	50.56	40.18	43.94	51.56	41.70	45.59	53.51
30	39.31	42.98	50.45	40.08	43.82	51.43	41.56	45.44	53.33
35	38.93	42.56	49.95	39.62	43.32	50.85	40.97	44.80	52.58
40	38.63	42.24	49.57	39.27	42.94	50.40	40.52	44.30	51.99
45	39.39	41.98	49.27	38.99	42.63	50.04	40.15	43.90	51.53
50	38.20	41.77	49.02	38.76	42.38	49.75	39.85	43.58	51.15
60	37.90	41.44	48.63	38.41	41.99	49.29	39.39	43.07	50.55
70	37.67	41.19	48.35	38.14	41.70	48.95	39.04	42.68	50.10
80	37.50	41.00	48.12	37.93	41.48	48.68	38.77	42.39	49.75
90	37.36	40.85	47.94	37.77	41.29	48.47	38.54	42.15	49.47
100	37.24	40.72	47.80	37.63	41.14	48.29	38.36	41.95	49.23
200	36.66	40.09	47.05	36.93	40.38	47.39	37.43	40.93	48.04
500	36.21	39.59	46.47	36.37	39.77	46.68	36.69	40.11	47.08
∞	35.56	38.89	45.64	35.56	38.89	45.64	35.56	38.89	45.64

Table D-26
H VALUES (C = 27)

N	$\gamma = 0.90$			$\gamma = 0.95$			$\gamma = 0.99$		
	P = 0.90	P = 0.95	P = 0.99	P = 0.90	P = 0.95	P = 0.99	P = 0.90	P = 0.95	P = 0.99
2	80.87	87.80	101.74	90.70	98.48	114.11	113.83	123.59	143.21
3	63.67	69.29	80.64	68.80	74.88	87.14	80.02	87.09	101.34
4	56.76	61.84	72.11	60.39	65.80	76.72	68.08	74.18	86.50
5	52.91	57.68	67.34	55.80	60.83	71.01	61.80	67.37	78.65
6	50.43	55.00	64.26	52.87	57.65	67.36	57.87	63.11	73.73
7	48.69	53.12	62.08	50.82	55.44	64.79	55.15	60.16	70.32
8	47.39	51.71	60.46	49.30	53.79	62.89	53.16	58.00	67.81
9	46.39	50.62	59.20	48.12	52.51	61.41	51.62	56.33	65.87
10	45.58	49.74	58.19	47.18	51.49	60.23	50.39	54.99	64.33
11	44.92	49.03	57.35	46.41	50.65	59.26	49.39	53.91	63.06
12	44.37	48.42	56.66	45.77	49.95	58.45	48.56	53.00	62.01
13	43.90	47.91	56.06	45.22	49.36	57.75	47.85	52.23	61.11
14	43.49	47.47	55.55	44.75	48.84	57.16	47.24	51.56	60.34
15	43.14	47.09	55.10	44.34	48.40	56.64	46.71	50.99	59.67
16	42.83	46.75	54.71	43.98	48.00	56.18	46.24	50.48	59.07
17	42.55	46.45	54.36	43.65	47.65	55.77	45.83	50.03	58.55
18	42.30	46.18	54.05	43.37	47.34	55.41	45.46	49.62	58.08
19	42.08	45.94	53.77	43.11	47.06	55.08	45.12	49.26	57.65
20	41.88	45.72	53.51	42.87	46.80	54.78	44.82	48.93	57.27
21	41.70	45.52	53.28	42.66	46.57	54.51	44.54	48.63	56.92
22	41.53	45.34	53.07	42.46	46.36	54.26	44.29	48.35	56.59
23	41.38	45.17	52.87	42.28	46.16	54.03	44.06	48.10	56.30
24	41.23	45.01	52.69	42.12	45.98	53.82	43.84	47.86	56.03
25	41.10	44.87	52.52	41.96	45.81	53.62	43.65	47.65	55.77
26	40.98	44.74	52.37	41.82	45.65	53.44	43.46	47.45	55.54
27	40.86	44.61	52.22	41.69	45.51	53.27	43.29	47.26	55.32
28	40.76	44.50	52.09	41.56	45.37	53.11	43.13	47.08	55.12
29	40.66	44.39	51.96	41.45	45.25	52.97	42.98	46.92	54.92
30	40.56	44.29	51.84	41.34	45.13	52.83	42.84	46.76	54.74
35	40.17	43.86	51.34	40.87	44.62	52.24	42.24	46.11	53.98
40	39.87	43.52	50.95	40.52	44.23	51.78	41.78	45.61	53.39
45	39.62	43.28	50.64	40.23	43.92	51.42	41.40	45.20	52.92
50	39.43	43.04	50.39	40.00	43.67	51.12	41.10	44.87	52.53
60	39.12	42.71	50.00	39.63	43.27	50.66	40.63	44.35	51.93
70	38.89	42.46	49.71	39.36	42.97	50.31	40.27	43.97	51.47
80	38.71	42.26	49.48	39.15	42.74	50.04	39.99	43.66	51.12
90	38.57	42.11	49.30	38.98	42.56	49.82	39.77	43.42	50.83
100	38.45	41.98	49.15	38.84	42.40	49.64	39.58	43.22	50.60
200	37.86	41.33	48.39	38.13	41.63	48.73	38.64	42.18	49.39
500	37.40	40.83	47.80	37.56	41.01	48.01	37.88	41.36	48.42
∞	36.74	40.11	46.96	36.74	40.11	46.96	36.74	40.11	46.96

Table D-27
H VALUES (C = 28)

N	$\gamma = 0.90$			$\gamma = 0.95$			$\gamma = 0.99$		
	P = 0.90	P = 0.95	P = 0.99	P = 0.90	P = 0.95	P = 0.99	P = 0.90	P = 0.95	P = 0.99
2	82.79	89.77	103.79	92.64	100.45	116.13	115.70	125.45	145.04
3	65.36	71.04	82.48	70.52	76.64	88.99	81.76	88.86	103.17
4	58.33	63.46	73.83	61.99	67.44	78.46	69.72	75.85	88.24
5	54.41	59.24	68.99	57.32	62.40	72.68	63.36	68.97	80.33
6	51.88	56.51	65.86	54.34	59.18	68.97	59.37	64.66	75.36
7	50.11	54.58	63.65	52.25	56.92	66.37	56.62	61.68	71.92
8	48.78	53.15	61.99	50.71	55.24	64.44	54.59	59.48	69.38
9	47.76	52.04	60.71	49.51	53.95	62.94	53.03	57.79	67.42
10	46.93	51.15	59.68	48.55	52.91	61.74	51.79	56.44	65.86
11	46.26	50.41	58.83	47.76	52.06	60.75	50.77	55.33	64.58
12	45.69	49.80	58.13	47.11	51.34	59.93	49.92	54.41	63.51
13	45.21	49.28	57.52	46.55	50.74	59.22	49.20	53.63	62.60
14	44.80	48.83	57.00	46.07	50.21	58.62	48.58	52.96	61.82
15	44.44	48.44	56.55	45.65	49.76	58.09	48.04	52.37	61.14
16	44.12	48.09	56.15	45.28	49.36	57.62	47.57	51.85	60.54
17	43.84	47.79	55.79	44.95	49.00	57.21	47.15	51.39	60.00
18	43.59	47.51	55.47	44.66	48.68	56.84	46.77	50.98	59.53
19	43.36	47.27	55.19	44.40	48.40	56.51	46.43	50.61	59.10
20	43.15	47.04	54.93	44.16	48.13	56.20	46.12	50.28	58.71
21	42.97	46.84	54.69	43.94	47.90	55.93	45.84	49.97	58.35
22	42.80	46.65	54.47	43.74	47.68	55.68	45.69	49.69	58.03
23	42.64	46.48	54.27	43.56	47.48	55.44	45.35	49.44	57.73
24	42.49	46.32	54.09	43.39	47.30	55.23	45.13	49.20	57.45
25	42.30	46.18	53.92	43.23	47.13	55.03	44.93	48.90	57.20
26	42.23	46.04	53.76	43.08	46.97	54.85	44.74	48.78	56.96
27	42.12	45.91	53.62	42.95	46.82	54.67	44.57	48.59	56.74
28	42.01	45.80	53.48	42.82	46.68	54.51	44.40	48.41	56.53
29	41.91	45.68	53.35	42.70	46.55	54.36	44.25	48.24	56.33
30	41.81	45.58	53.23	42.59	46.43	54.22	44.11	48.08	56.15
35	41.41	45.14	52.72	42.12	45.92	53.62	43.50	47.42	55.38
40	41.10	44.80	52.32	41.76	45.52	53.16	43.03	46.91	54.78
45	40.85	44.54	52.01	41.47	45.21	52.79	42.65	46.50	54.30
50	40.65	44.31	51.75	41.23	44.95	52.49	42.34	46.16	53.91
60	40.33	43.97	51.36	40.86	44.54	52.02	41.86	45.64	53.30
70	40.10	43.72	51.06	40.58	44.24	51.67	41.50	45.24	52.84
80	39.92	43.52	50.83	40.36	44.01	51.39	41.22	44.94	52.48
90	39.77	43.36	50.64	40.19	43.82	51.17	40.99	44.69	52.19
100	39.65	43.23	50.49	40.05	43.66	50.99	40.80	44.48	51.95
200	39.05	42.57	49.72	39.32	42.87	50.07	39.84	43.44	50.73
500	38.58	42.06	49.13	38.75	42.25	49.34	39.07	42.60	49.75
∞	37.92	41.34	48.28	37.92	41.34	48.28	37.92	41.34	48.28

Table D-28
H VALUES (C = 29)

N	$\gamma = 0.90$			$\gamma = 0.95$			$\gamma = 0.99$		
	P = 0.90	P = 0.95	P = 0.99	P = 0.90	P = 0.95	P = 0.99	P = 0.90	P = 0.95	P = 0.99
2	84.71	91.74	105.83	94.57	102.42	118.15	117.58	127.34	146.90
3	67.05	72.78	84.31	72.24	78.41	90.83	83.51	90.64	105.00
4	59.90	65.08	75.55	63.58	69.08	80.19	71.34	77.52	89.98
5	55.91	60.78	70.63	58.84	63.97	74.33	64.91	70.57	82.00
6	53.33	58.00	67.45	55.80	60.69	70.58	60.87	66.20	76.99
7	51.52	56.04	65.20	53.68	58.40	67.94	58.08	63.18	73.51
8	50.17	54.58	63.52	52.11	56.70	65.98	56.02	60.96	70.94
9	49.12	53.45	62.22	50.89	55.37	64.46	54.44	59.24	68.96
10	48.28	52.54	61.17	49.91	54.32	63.24	53.18	57.87	67.38
11	47.59	51.80	60.31	49.11	53.45	62.24	52.15	56.75	66.08
12	47.01	51.17	59.59	48.44	52.73	61.40	51.28	55.82	65.00
13	46.52	50.64	58.97	47.88	52.11	60.69	50.55	55.03	64.08
14	46.10	50.18	58.45	47.39	51.58	60.07	49.92	54.34	63.29
15	45.73	49.78	57.98	46.96	51.12	59.54	49.38	53.75	62.60
16	45.41	49.43	57.58	46.58	50.71	59.06	48.89	53.22	61.99
17	45.12	49.12	57.22	46.25	50.35	58.64	48.46	52.76	61.45
18	44.86	48.84	56.89	45.95	50.02	58.27	48.08	52.34	60.97
19	44.63	48.59	56.60	45.68	49.73	57.93	47.74	51.97	60.53
20	44.42	48.36	56.34	45.44	49.46	57.62	47.42	51.63	60.14
21	44.23	48.15	56.10	45.21	49.22	57.34	47.14	51.32	59.78
22	44.06	47.96	55.88	45.01	49.00	57.09	46.88	51.03	59.45
23	43.90	47.79	55.67	44.82	48.80	56.85	46.64	50.77	59.15
24	43.75	47.63	55.49	44.65	48.61	56.63	46.42	50.53	58.87
25	43.61	47.48	55.31	44.49	48.44	56.43	46.21	50.31	58.61
26	43.48	47.34	55.15	44.34	48.28	56.24	46.02	50.10	58.37
27	43.36	47.21	55.00	44.21	48.13	56.07	45.84	49.91	58.14
28	43.25	47.09	54.86	44.08	47.99	55.91	45.68	49.73	57.93
29	43.15	46.98	54.73	43.95	47.85	55.75	45.52	49.56	57.74
30	43.05	46.87	54.61	43.84	47.73	55.61	45.37	49.40	57.55
35	42.64	46.43	54.09	43.36	47.21	55.00	44.76	48.73	56.77
40	42.33	46.08	53.69	42.99	46.81	54.54	44.28	48.21	56.17
45	42.07	45.81	53.37	42.70	46.48	54.18	43.90	47.79	55.68
50	41.87	45.58	53.11	42.45	46.22	53.85	43.58	47.45	55.28
60	41.55	45.23	52.71	42.08	45.81	53.38	43.09	46.91	54.66
70	41.31	44.98	52.41	41.79	45.50	53.02	42.72	46.52	54.20
80	41.12	44.77	52.17	41.57	45.26	52.74	42.44	46.20	53.84
90	40.98	44.61	51.98	41.40	45.07	52.52	41.21	45.95	53.54
100	40.85	44.48	51.83	41.25	44.91	52.33	42.01	45.74	53.30
200	40.24	43.81	51.05	40.52	44.11	51.40	41.04	44.68	52.06
500	39.76	43.29	50.45	39.93	43.48	50.66	40.26	43.83	51.07
∞	39.09	42.56	49.59	39.09	42.56	49.59	39.09	42.56	49.59

Table D-29
H VALUES (C = 30)

N	$\gamma = 0.90$			$\gamma = 0.95$			$\gamma = 0.99$		
	P = 0.90	P = 0.95	P = 0.99	P = 0.90	P = 0.95	P = 0.99	P = 0.90	P = 0.95	P = 0.99
2	86.63	93.70	107.87	96.51	104.39	120.18	119.41	129.15	148.70
3	68.73	74.51	86.14	73.95	80.16	92.67	85.24	92.41	106.83
4	61.46	66.70	77.25	65.17	70.72	81.91	72.97	79.18	91.72
5	57.40	62.32	72.27	60.35	65.53	75.98	66.46	72.16	83.67
6	54.77	59.49	69.03	57.26	62.20	72.18	62.37	67.74	78.61
7	52.92	57.50	66.75	55.11	59.87	69.50	59.54	64.69	75.09
8	51.55	56.01	65.04	53.50	58.14	67.51	57.45	62.43	72.49
9	50.48	54.86	63.72	52.26	56.80	65.97	55.84	60.69	70.49
10	49.62	53.93	62.65	51.27	55.72	64.73	54.56	59.30	68.89
11	48.92	53.17	61.78	50.46	54.84	63.72	53.52	58.17	67.58
12	48.33	52.54	61.05	49.77	54.11	62.87	52.64	57.22	66.49
13	47.83	52.00	60.42	49.20	53.48	62.14	51.90	56.42	65.56
14	47.40	51.53	59.88	48.70	52.94	61.52	51.26	55.72	64.76
15	47.03	51.12	59.41	48.26	52.47	60.98	50.70	55.12	64.06
16	46.69	50.77	59.00	47.88	52.05	60.50	50.21	54.59	63.44
17	46.40	50.45	58.63	47.54	51.68	60.07	49.78	54.12	62.90
18	46.14	50.16	58.30	47.24	51.35	59.69	49.39	53.69	62.41
19	45.90	49.91	58.01	46.96	51.06	59.35	49.04	53.31	61.97
20	45.69	49.67	57.74	46.71	50.79	59.03	48.72	52.97	61.57
21	45.49	49.46	57.49	46.49	50.54	58.75	48.43	52.65	61.20
22	45.32	49.27	57.27	46.28	50.32	58.49	48.16	52.37	60.87
23	45.15	49.09	57.07	46.09	50.11	58.25	47.92	52.10	60.56
24	45.00	48.93	56.88	45.91	49.92	58.03	47.69	51.86	60.28
25	44.86	48.78	56.70	45.75	49.74	57.83	47.49	51.63	60.02
26	44.73	48.63	56.54	45.60	49.58	57.64	47.29	51.42	59.77
27	44.61	48.50	56.38	45.46	49.43	57.46	47.11	51.22	59.55
28	44.50	48.38	56.24	45.33	49.28	57.29	46.94	51.04	59.33
29	44.39	48.27	56.11	45.20	49.15	57.14	46.78	50.87	59.14
30	44.29	48.16	55.98	45.09	49.02	56.99	46.64	50.71	58.95
35	43.87	47.70	55.46	44.60	48.49	56.38	46.01	50.03	58.16
40	43.55	47.35	55.05	44.22	48.09	55.90	45.52	49.50	57.55
45	43.29	47.07	54.73	43.92	47.76	55.52	45.13	49.08	57.06
50	43.08	46.85	54.46	43.67	47.49	55.21	44.81	48.73	56.65
60	42.76	46.49	54.05	43.29	47.07	54.73	44.32	48.19	56.02
70	42.51	46.23	53.75	43.00	46.76	54.37	43.95	47.78	55.55
80	42.33	46.02	53.51	42.78	46.52	54.08	43.65	47.47	55.19
90	42.18	45.86	53.32	42.60	46.32	53.86	43.42	47.21	54.89
100	42.05	45.72	53.16	42.45	46.16	53.67	43.22	47.00	54.64
200	41.43	45.04	52.37	41.70	45.35	52.72	42.23	45.92	53.39
500	40.94	44.52	51.76	41.12	44.71	51.98	41.44	45.06	52.39
∞	40.26	43.77	50.89	40.26	43.77	50.89	40.26	43.77	50.89

APPENDIX E

PROGRAM FOR DETERMINING IF THE TOLERANCE ELLIPSOID IS WITHIN THE SPECIFICATION ELLIPSOID

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      IMPLICIT REAL*8 (A-H,O-Z)
      COMMON/INPDAT/A (30,30), B(30,30), AMU (30), POPT
      COMMON/LINCOM/XL%50<, XB%50<, XH%50<, CMH, TOLCON, KNT%50<, KON%50<,
1      N, NLINEQ, NNOTEQ, IOUT, ITER, ITERMX
      CALL INPUT
C
C      CHECK TO SEE IF A=B
C
      SUM=0
      DO 2 I=1,N
      DO 1 J=1,N
1      SUM=SUM + (A(I,J)-B(I,J)).**2
      IF (DABS(SUM).GT.1.0D-5) GO TO 10
2      CONTINUE
C
C      A=B
C
C      CHECK TO SEE IF SAMPLE MEAN IS 0
C
      DO 5 I=1,N
      IF (AMU(I).NE.0) GO TO 10
5      CONTINUE
C
C      SAMPLE MEAN IS 0
C
      WRITE(6,1001)
1001  FORMAT(1H1,'MATRICES A AND B ARE EQUAL .  SAMPLE MEAN= 0 STOP')
      STOP
10      CALL OPTIM
      IF (POPT.GE.1.0D0) GO TO 20
      WRITE(6,1010)
      WRITE(6,1011) POPT
1011  FORMAT(1H0,'P (X*) =',1PE15.4)
1010  FORMAT(1H0,'*****  CANNOT CERTIFY *****'/1H0)
      STOP
20      SUM=0
      DO 25 I=1,N
      DO 25 J=1,N
25      SUM=SUM + B(I,J)*AMU(I)*AMU(J)
      IF (SUM.LT.1.0D0) GO TO 35
      WRITE(6,1010)
      STOP
35      WRITE(6,1015)
1015  FORMAT(1H0,45X,' CAN CERTIFY  '/1X)
      WRITE(6,1011) POPT
      STOP
      END
      SUBROUTINE CSUM%X<
      IMPLICIT REAL*8 A-H,O-Z<
      COMMON/INPDAT/A (30,30), B(30,30), AMU (30), POPT
      COMMON/LINCOM/XL%50<, XB%50<, XH%50<, CMH, TOLCON, KNT%50<, KON%50<,
1      N, NLINEQ, NNOTEQ, IOUT, ITER, ITERMX
      COMMON/USER/BSUM%50<, BSUMNT%50<, PPC%50,50<
      DIMENSION X%50<
      SUM=0.0
      DO 1 I=1,N

```

```

DO 1 J=1,N
1  SUM=SUM + B(I,J)*X(I)*X(J)
  BSUM(1)=SUM-1.0
  RETURN
  END
  SUBROUTINE PARTL%X<
  IMPLICIT REAL*8%A-H,O-Z<
  COMMON/USER/BSUM%50<,BSUMNT%50<,FPC%50,50<
  COMMON/INPDAT/A(30,30),B(30,30),AMU(30),FOPT
  DIMENSION X%50<
  DO 5 K=1,N
  SUM=0
    DO 1 I=1,N
1    SUM=SUM + B(I,K)*X(I)
5    FPC(K,1)=2.0*SUM
    RETURN
    END
    SUBROUTINE EVAL%X,Y,IOUT<
    IMPLICIT REAL*8%A-H,O-Z<
    DIMENSION X%50<,D%4<,XD%4<
    COMMON/LINCOM/XL%50<,XB%50<,XH%50<,CHM,TOLCON,KNT%50<,KON%50<,
1      N,NLINEQ,NNOTEQ,IOTT,ITER,ITERMX
    COMMON/INPDAT/A(30,30),B(30,30),AMU(30),FOPT
    Y=0
    DO 5 I=1,N
952  FORMAT(1X,I3,1P10E12.4)
20   CONTINUE
    DO 30 I=1,N
    READ(5,910) (B(I,J),J=1,N)
    DO 25 J=1,N
25   B(I,J)=B(I,J)/BNORM
    WRITE(6,952) I,(B(I,J),J=1,N)
30   CONTINUE
    READ(5,910) (AMU(I),I=1,N)
    WRITE(6,953) (AMU(I),I=1,N)
953  FORMAT(1H0,' MEANS=',1P10E12.4)
    RETURN
    END

```

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