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Energy Deposition of Alpha Particles in a
DT Plasma by Rutherford Scattering

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NOTE ON THE CALCULATION OF ENERGY DEPOSITION OF ALPHA PARTICLES
IN A DT PLASMA BY RUTHERFORD SCATTERING

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ABSTRACT

The approximations involved in computing range energy deposition of alpha particles in a DT plasma are discussed, and some numerical results are given.

I. INTRODUCTION

In this Note some of the formulae needed for calculation of range energy deposition of fast particles in a DT plasma are summarized for the case of Rutherford scattering only. Approximate analytic expressions for calculating range, elapsed time, and energy deposited to ions are derived, and the approximations involved are discussed. The Coulomb logarithm term is discussed in the Appendix. Numerical results for the range, elapsed time, and fractional energy deposited to ions are given for a 3.5-MeV alpha particle in a 50-50 DT plasma at various temperatures and densities.

II. ENERGY DEPOSITION RATE

Let m be the mass, v and $E = mv^2/2$ the lab speed and energy, and Ze the charge of a fast ion that moves through a hot plasma being slowed down by Rutherford scattering with particles of mass m_j , charge $Z_j e$, and number density N_j in a spherically symmetric Maxwell distribution of velocities at temperature T_j , where e is the magnitude of the electron charge. By integrating the energy exchange in a single binary collision over the Maxwell distribution and the c.m. scattering angle θ , one obtains for the average energy loss per unit length of path by particle m to particles m_j caused by Rutherford scattering

$$-\left\langle \frac{dE}{dx} \right\rangle_j = 2\pi N_j Z^2 Z_j^2 e^4 \frac{m}{m_j E} F_j(y_j) \Lambda_j, \quad (1)$$

where

$$F_j(y_j) = \phi(y_j) - \left[1 + (m_j/m) \right] y_j \phi'(y_j), \quad (2)$$

$$y_j^2 = (m_j/m) (E/kT_j), \quad (3)$$

k is Boltzmann's constant ϕ and ϕ' are the error function and its derivative given by

$$\phi(y) = \frac{2}{\pi^{1/2}} \int_0^y \exp(-x^2) dx, \quad \phi'(y) = \frac{2}{\pi^{1/2}} \exp(-y^2), \quad (4)$$

and Λ_j is the Coulomb logarithm term, which is discussed in the Appendix.

A. Rate of Energy Loss to Ions in a DT Plasma

For energy loss of a fast ion to thermal ions in a fully ionized DT plasma, let $j = D, T$ represent the species of ion, and let

$$m = m_p \Lambda, \quad m_j = m_p \Lambda_j, \quad (j = D, T) \quad (5)$$

where m_p is the proton mass. Also let N be the number density of atoms in the plasma and f_j the fraction of atoms of species j , so that

$$N_j = f_j N, \quad \sum_j f_j = 1,$$

$$N_e = \sum_j Z_j N_j = N, \quad N = (\rho/\rho_0) N_0, \quad (j = D, T),$$

$$N_0 = 5.1 \times 10^{22} \text{ cm}^{-3}, \quad \rho_0 = 0.2125 \text{ g/cm}^3, \quad (6)$$

where N_0 and ρ_0 are normal liquid number and mass densities, and ρ/ρ_0 is the compression relative to normal density.

Then Eqs. (1)-(3) may be written

$$-\langle \frac{dE}{dx} \rangle_j = 2\pi f_j N (Z_j e^2)^2 (A/A_j E) F_j(y_j) \Lambda_j, \quad (7)$$

$$F_j(y_j) = \phi(y_j) - [1 + (A_j/A)] y_j \phi'(y_j), \quad (8)$$

$$y_j^2 = (A_j/A) (E/kT_j), \quad (j = D, T; A_D = 2, A_T = 3) \quad (9)$$

where T_1 is the ion temperature, and the total energy loss rate by Rutherford scattering to the ions in the plasma may be written

$$-\langle \frac{dE}{dx} \rangle_i = - \sum_{j=D,T} \langle \frac{dE}{dx} \rangle_j = \frac{B_1}{E} \text{ MeV/cm}, \quad (10)$$

where

$$B_1 = 0.130284 A Z^2 N \left[\frac{1}{2} f_D F_D(y_D) \Lambda_D + \frac{1}{3} (1 - f_D) F_T(y_T) \Lambda_T \right] \text{ MeV}^2/\text{cm} \\ (E \text{ in MeV; } N \text{ in } 10^{24} \text{ cm}^{-3}). \quad (11)$$

The Λ_j ($j = D, T$) are given by Eqs. (A20) of the Appendix as

$$\Lambda_j = \begin{cases} \Lambda_{qj} = K_j - \frac{1}{2} \ln(\rho/\rho_0) + \frac{1}{2} \ln(E k T_e), & E > E^* \\ \Lambda_{cj} = \Lambda_{qj} + \frac{1}{2} \ln(E/E^*), & E < E^* \quad (j = D, T) \end{cases}$$

$$K_D = 16.4864 + \ln [2A^{1/2}/(A+2)],$$

$$K_T = 16.4864 + \ln [3A^{1/2}/(A+3)],$$

$$E^* = 10^{-1} Z^2 A \text{ MeV},$$

$$(E \text{ and } kT_e \text{ in MeV})$$

(12)

where T_e is the electron temperature.

B. Rate of Energy Loss to Electrons in a DT Plasma

For energy loss of a fast ion to the thermal electrons in a fully ionized DT plasma, Eq. (1) may be written ($j = e$)

$$-\langle \frac{dE}{dx} \rangle_e = \frac{B_e}{E} \text{ MeV/cm} \quad (13)$$

where

$$B_e = 239.216 A Z^2 N F_e(y_e) \Lambda_e \text{ MeV}^2/\text{cm}, \quad (14)$$

$$(E \text{ in MeV; } N \text{ in } 10^{24} \text{ cm}^{-3})$$

$$F_e(y_e) = \phi(y_e) - [1 + (m_e/m_p A)] y_e \phi'(y_e), \quad (15)$$

$$y_e^2 = (m_e/m_p A) (E/kT_e), \quad (16)$$

m_e is the electron mass, and Λ_e is given by Eqs. (A21) of the Appendix.

III. RANGE, TIME, AND ENERGY DEPOSITED TO IONS

Assuming the fast particle does not deviate significantly from a straight line path as it is slowed down from an energy E to some lower energy E' , then, using Eqs. (10) and (13), the distance it traverses is given by

$$s(E, E') = \int_{E'}^E \left[\langle \frac{dE}{dx} \rangle_i + \langle \frac{dE}{dx} \rangle_e \right]^{-1} dE \\ = \int_{E'}^E \frac{E dE}{B_1 + B_e} \text{ cm}. \quad (17)$$

Similarly, the time $t(E, E')$ that elapses and the energy $\Delta E_1(E, E')$ that is deposited to ions as the energy of the fast particle decreases from E to E' are given by

$$t(E, E') = \int_{E'}^E \left[\langle \frac{dE}{dx} \rangle_i + \langle \frac{dE}{dx} \rangle_e \right]^{-1} \frac{dE}{v} \\ = k_t \int_{E'}^E \frac{E^{1/2} dE}{B_1 + B_e} \text{ sec} \quad (18)$$

and

$$\Delta E_1(E, E') = - \int_{E'}^E \left[\left\langle \frac{dE}{dx} \right\rangle_1 + \left\langle \frac{dE}{dx} \right\rangle_e \right]^{-1} \left\langle \frac{dE}{dx} \right\rangle_1 dE$$

$$= \int_{E'}^E \frac{B_1 dE}{B_1 + B_e} \text{ MeV,} \quad (19)$$

where the constant k_t comes from the relation $v^{-1} = (Am_p/2E)^{1/2} = k_t E^{-1/2}$ sec/cm and is given by

$$k_t = 0.72248 \times 10^{-9} A^{1/2} (\text{g MeV/erg})^{1/2} (E \text{ in MeV}) \quad (20)$$

A. Approximate Analytic Expressions for Range, Etc.

It is seen from Eq. (9) that in the case $A \approx A_j$, for most of the range of the fast particle, $y_j \gg 1$. Similarly, from Eq. (16) it is seen that $y_e \ll 1$. It follows, from Eq. (8), that

$$F_j(y_j) \approx F_j(y_j)_{\text{appr.}} = 1, \quad j = D, T \quad (E \gg kT_1). \quad (21)$$

Also, if one expands Eq. (15) in powers of y_e and neglects y_e^2 compared to unity, the result is

$$F_e(y_e) \approx (4/3\pi^{1/2}) y_e^3 [1 - (3/2)(kT_e/E)]. \quad (y_e^2 \ll 1).$$

Then, if $(3/2)(kT_e/E)$ is neglected compared to unity, one obtains

$$F_e(y_e) \approx F_e(y_e)_{\text{appr.}} = (4/3\pi^{1/2}) (m_e/m_p)^{3/2} \times (E/AkT_e)^{3/2} \quad (y_e^2 \ll 1; E \gg kT_e) \quad (22)$$

The ratios of the F - functions to their approximate values $F/F_{\text{appr.}}$ are shown in Figs. 1 and 2 plotted as functions of E/kT for the case of a fast alpha particle ($A = 4$). Figures 1a and 1b ($j = D, T$; $A_j = 2, 3$, respectively) are the functions given by Eqs. (8) for E/kT ranging from 1.5 to 20. It is seen that the approximation of Eq. (21) is quite good for $E/kT \gtrsim 10$ and is too high at smaller values of E/kT . Figures 2a and 2b give the ratio of the expressions in Eqs. (15) and (22) for E/kT ranging from 1.5 to 100 and to 1000, respectively. It is seen that the approximate formula for F_e is everywhere high. So the fit can be improved by decreasing the numerical coefficient in Eq. (22) by an appropriate factor $1/C_A$ ($C_A \geq 1$). In all cases the approximate F functions are quite high at very small values of E/kT . This will result in a fairly large error in the total rate of energy loss toward the tail end of the range of the fast particle.

Fig. 1. The ratio $F_j(y_j)/F_j(y_j)_{\text{appr.}}$ for an alpha particle ($A = 4$) plotted as a function of E/kT_1 in the interval $1.5 \leq E/kT_1 \leq 20$: (a) $j = D$, $A_j = 2$; (b) $j = T$, $A_j = 3$.

Fig. 2. The ratio $F_e(y_e)/F_e(y_e)_{\text{appr.}}$ for an alpha particle ($A = 4$) plotted as a function of E/kT_e : (a) $1.5 \leq E/kT_e \leq 100$; (b) $1.5 \leq E/kT_e \leq 1000$.

After substituting the approximate expressions (21) and (22) into Eqs. (10) and (13), the rates of energy loss to ions and electrons may be written (including the numerical factor $1/C_A$ in the latter)

$$-\langle \frac{dE}{dx} \rangle_i \approx B_1' E^{-1} \text{ MeV/cm}, \quad (23)$$

where

$$B_1' = 0.130284 A Z^2 N \left[\frac{1}{2} f_D \Lambda_D + \frac{1}{3} (1 - f_D) \Lambda_T \right], \quad (24)$$

and

$$-\langle \frac{dE}{dx} \rangle_e \approx B_e' E^{1/2} \text{ MeV/cm}, \quad (25)$$

where

$$B_e' = (0.00228721/C_A) A^{-1/2} Z^2 N (kT_e)^{-3/2} \Lambda_e. \quad (26)$$

$$(E \text{ and } kT_e \text{ in MeV; } N \text{ in } 10^{24} \text{ cm}^{-3}).$$

In this approximation, the ratio of the rates of energy loss to electrons and ions is given by

$$\langle \frac{dE}{dx} \rangle_e / \langle \frac{dE}{dx} \rangle_i = (B_e' / B_1') E^{3/2} = (E/\hat{E})^{3/2}, \quad (27a)$$

where

$$\hat{E} = (B_1' / B_e')^{2/3} = 14.806 C_A^{2/3} A kT_e \left[\frac{1}{2} f_D \Lambda_D + \frac{1}{3} (1 - f_D) \Lambda_T \right]^{2/3} \Lambda_e^{-2/3} \text{ MeV} \quad (kT_e \text{ in MeV}). \quad (27b)$$

When $E = \hat{E}$, the approximate loss rates are equal, and when $E < \hat{E}$, loss to ions exceeds that to electrons.

If one substitutes Eqs. (23) and (25) into Eq. (17), the latter becomes

$$s(E, E') \approx \int_{E'}^E \left[B_1' E^{-1} + B_e' E^{1/2} \right]^{-1} dE. \quad (28)$$

In this expression, B_e' is independent of E , but B_1' depends on E through the terms Λ_D and Λ_T . [See Eqs. (12).] Since this is a slowly varying logarithmic dependence, it is not too bad an approximation to replace $\ln E$ by some average value $\langle \ln E \rangle$ appropriate for the energy range anticipated in a given problem and assume B_1' to be constant. If this is done, Eq. (28) can be integrated. With the same approximation, after substituting Eqs. (23) and (25) into Eqs. (18) and (19), the latter can also be integrated. Express E in units of $\hat{E}$ through the parameter $u = E/\hat{E}$, and the following approximate analytic expressions for range, time, and energy to ions result:

$$s(E, E') \approx (\hat{E}^2 / B_1') [\phi(u) - \phi(u')] \text{ cm}; \quad (29a)$$

$$t(E, E') \approx (k_e \hat{E}^{3/2} / B_1') [\psi(u) - \psi(u')] \text{ sec}; \quad (29b)$$

$$\Delta E_1(E, E') \approx \hat{E} [\chi(u) - \chi(u')] \text{ MeV}, \quad (29c)$$

where $\hat{E}$ is given by Eq. (27b),

$$\phi(u) = \int_0^u (1 + u^{3/2})^{-1} u du = 2u^{1/2} + f_1(u) - f_2(u),$$

$$\psi(u) = \int_0^u (1 + u^{3/2})^{-1} u^{1/2} du = (2/3) \ln(1 + u^{3/2}),$$

$$\chi(u) = \int_0^u (1 + u^{3/2})^{-1} du = f_1(u) + f_2(u),$$

$$f_1(u) = (1/3) \ln[(1 - u^{1/2} + u)/(1 + u^{1/2})^2],$$

$$f_2(u) = (2/3^{1/2}) \left\{ (\pi/3) + \tan^{-1} [3^{1/2}(u^{1/2} - 1)/(u^{1/2} + 1)] \right\}.$$

IV. 3.5-MEV ALPHA PARTICLE IN A 50-50 DT PLASMA

The above formulas apply to a fast alpha particle moving through a fully ionized plasma composed of a 50-50 mixture of deuterium and tritium if one substitutes $A = 4$, $Z = 2$, and $f_D = 0.5$. To investigate the effect of the approximations made in arriving at Eqs. (29), one could compare, for example, the integrands in Eqs. (17) and (28). Figure 3 shows such a comparison for temperatures $T_e = 5, 50, 100$ keV. The Λ_D and Λ_T in Eq. (24) have been replaced by

$$\bar{\Lambda}_D = 16.08 - \frac{1}{2} \ln(\rho/\rho_0) + \frac{1}{2} \ln(kT_e) + \frac{1}{2} \langle \ln E \rangle, \quad (30a)$$

$$\bar{\Lambda}_T = 16.33 - \frac{1}{2} \ln(\rho/\rho_0) + \frac{1}{2} \ln(kT_e) + \frac{1}{2} \langle \ln E \rangle, \quad (30b)$$

and Λ_e is given by Eqs. (A21), where, with $Z = 2$, $kT_e^* \approx 0.15$ keV, and

$$\Lambda_e = 12.93 - \frac{1}{2} \ln(\rho/\rho_0) + \ln(kT_e). \quad (31)$$

The examples are computed for the case $\rho/\rho_0 = 1$ [that is, normal liquid density; see Eq. (6)], $C_A = 1$, $\langle \ln E \rangle = \ln(3.5)$, $T_1 = T_e$, and E in the range $\frac{3}{2} T_e \leq E \leq 3.5$ MeV. Figure 3 shows that the approximate integrands are everywhere too small and, as indicated above, deviate by quite a large amount at low energies because of the approximations used for the functions F_j and F_e . [See Eqs. (21) and (22).] It should be observed that, although Figs. 1 and 2 show that at low energies the error introduced by using $F_e(y_e)_{\text{appr.}}$ is much greater than that introduced by using $F_j(y_j)_{\text{appr.}}$, it is at these energies where loss to ions dominates. This can be seen from Table I, where the fractional rate of energy deposition to ions is given by

$$\delta E_1 / \delta E \equiv \left\langle \frac{dE}{dx} \right\rangle_1 \left(\left\langle \frac{dE}{dx} \right\rangle_1 + \left\langle \frac{dE}{dx} \right\rangle_e \right)^{-1}.$$

The entries in Table I were computed using Eqs. (27).

The fit can be improved somewhat by adjusting the parameters C_A and $\langle \ln E \rangle$. Some indication as to what these adjustments might be can be obtained by observing that for the three temperatures illustrated $T_e = 5, 50, 100$ keV, the energies at which loss to ions equals that to electrons are $\hat{E} \approx 0.25, 2.2, 4.3$ MeV, respectively, or E/kT_e between 40 and 50. The effect of adjusting C_A will be most strongly felt in the region $\hat{E} < E \leq 3.5$ MeV where loss to electrons dominates, and the effect of adjusting $\langle \ln E \rangle$ is most strongly felt in the region $\frac{3}{2} T_e \leq E < \hat{E}$, where loss to ions dominates. Of course, there is no

Table I. Approximate values of the fractional rate of energy deposition to ions $\delta E_1 / \delta E$ as a function of E (MeV) and T_e (keV).

E	$T_e = 5 (\hat{E} \approx 0.25 \text{ MeV})$		$T_e = 50 (\hat{E} \approx 2.2 \text{ MeV})$		$T_e = 100 (\hat{E} \approx 4.3 \text{ MeV})$	
	E/kT_e	$\delta E_1 / \delta E$	E/kT_e	$\delta E_1 / \delta E$	E/kT_e	$\delta E_1 / \delta E$
3.5	700	0.02	70	0.33	35	0.57
2.5	500	0.03	50	0.45	25	0.69
1.5	300	0.06	30	0.64	15	0.83
1.0	200	0.11	20	0.76	10	0.90
0.5	100	0.26	10	0.90	5	0.96
0.25	50	0.50	5	0.96	2.5	0.99
0.05	10	0.92	—	—	—	—

Fig. 3. Integrands (cm/MeV) in the accurate (ACC.) and approximate expressions for computing range, with $C_A = 1$ and $\langle \ln E \rangle = \ln(3.5)$ in the latter, for $1.5 T_e \leq E \leq 3.5$ MeV: (a) $T_e = 5$ keV; (b) $T_e = 50$ keV; (c) $T_e = 100$ keV.

Table II. Values of Λ_D and $\bar{\Lambda}_D/\Lambda_D$, for two different values of $\langle \ln E \rangle$, as functions of E (MeV) and T_e (keV).

E	T_e			$\frac{\bar{\Lambda}_D}{\Lambda_D} [\langle \ln E \rangle = \ln(1.5)]$			$\frac{\bar{\Lambda}_D}{\Lambda_D} [\langle \ln E \rangle = \ln(0.8)]$		
	5	50	100	5	50	100	5	50	100
0.25	11.81	12.96	13.31	1.19	1.17	1.17	1.03	1.12	1.11
0.5	12.50	13.65	14.00	1.12	1.11	1.11	0.97	1.06	1.06
1.0	13.20	14.35	14.69	1.07	1.05	1.06	0.92	1.01	1.01
1.5	13.60	14.75	15.10	1.03	1.03	1.03	0.89	0.98	0.98
2.0	13.78	14.93	15.26	1.02	1.02	1.02	0.88	0.97	0.97
2.5	13.89	15.04	15.39	1.01	1.01	1.01	0.88	0.96	0.96
3.5	14.06	15.21	15.56	1.0	1.0	1.0	0.86	0.95	0.95

sharp division between the two energy regions, but the relative rate of energy loss varies continuously with E as indicated in Table I. (See also Fig. 5.) Thus, at $T_e = 5$ keV, loss to electrons is greater for most of the range; at $T_e = 100$ keV, loss to ions is greater for all of the range; and the case $T_e = 50$ keV is intermediate. From Table II it is seen that a choice of $\langle \ln E \rangle = \ln(0.8)$ provides a reasonably good fit of $\bar{\Lambda}_D$ to Λ_D for the relevant energy regions in the three cases. [Λ_T differs only slightly from Λ_D . See Eqs. (12) and (30).] Below $E/kT_i \approx 10$, the error in $F_j(y_j)$ appr. begins to dominate; this corresponds to $E = 0.05, 0.5, 1.0$ MeV for the three cases $T_e = 5, 50, 100$ keV, respectively. In the regions where loss to electrons is greater, it is seen from Fig. 2 that, for $T_e = 5$ keV, F_e appr. $/F_e$ ranges approximately from a maximum of 1.06 at $E = 3.5$ MeV to a minimum of 1.03 at $E = 0.5$ MeV, and rises to 1.04 at $E = 0.25$ MeV. For $T_e = 50$ keV, $F_e/\text{appr. } F_e$ lies between 1.03 at $E = 3.5$ MeV and 1.04 at $E = 2.2$ MeV. It appears from this that a value of $C_A = 1.03$ or 1.04 would be appropriate.

Figure 4 shows comparisons of the accurate and approximate integrands for range calculations with $\langle \ln E \rangle = \ln(0.8)$ and $C_A = 1.035$. Comparing with Fig. 3 shows that some improvement has been achieved. No doubt, further refinements could be made to improve the fit. In fact, one could make the parameters dependent upon T_e . The approximate and accurate integrands for calculating time lapse are qualitatively similar to those for calculating range. Figure 5 shows a comparison of the integrands for computing the energy deposition to ions. Table III compares the ranges and times computed by numerical integration of Eqs. (17) and (18) with those

Fig. 4. Integrands (cm/MeV) in the accurate (ACC.) and approximate expressions for computing range, with $C_A = 1.035$, and $\langle \ln E \rangle = \ln(0.8)$ in the latter, for $1.5 T_e \leq E \leq 3.5$ MeV: (a) $T_e = 5$ keV; (b) $T_e = 50$ keV; (c) $T_e = 100$ keV.

Fig. 5. Integrands in the accurate (ACC.) and approximate (APP.) expressions for computing energy to ions ($\delta E_1/\delta E$), with $C_A = 1.035$ and $\langle \ln E \rangle = \ln(0.8)$ in the latter, for $1.5 T_e \leq E \leq 3.5$ MeV: (a) $T_e = 5$ keV; (b) $T_e = 50$ keV; (c) $T_e = 100$ keV.

Table III. Ranges and times for various intervals in E (MeV) computed by numerical integration and the approximate analytical formulas.

	$3.5 > E > 0.5$	$0.5 > E > 1.5 T_e$	$3.5 > E > 1.5 T_e$
	<u>Range (cm)</u>		
	$T_e = 5$ keV		
Num. integ.	0.442	0.0954	0.537
Analytical	0.441	0.0884	0.529
	$T_e = 50$ keV		
Num. integ.	4.53	0.254	4.78
Analytical	4.56	0.178	4.74
	$T_e = 100$ keV		
Num. integ.	6.42	0.357	6.78
Analytical	6.44	0.168	6.61
	<u>Time (nsec)</u>		
	$T_e = 5$ keV		
Num. integ.	0.523	0.304	0.827
Analytical	0.523	0.268	0.791
	$T_e = 50$ keV		
Num. integ.	4.82	0.744	5.56
Analytical	4.83	0.470	5.30
	$T_e = 100$ keV		
Num. integ.	6.66	0.974	7.63
Analytical	6.62	0.421	7.04

computed by the approximate analytical expressions of Eqs. (29a) and (29b) with $C_A = 1.035$ and $\langle \ln E \rangle = \ln(0.8)$. The errors in range are seen to be less than about 2% and those in time less than 8%. Table III shows, further, that most of the error occurs at the end of the alpha particle's path ($E < 0.5$ MeV), which is expected because of the F function approximations. The error in the approximate expression for ΔE_1 is somewhat compensated for by the fact that $\langle dE/dx \rangle_1$ appears in both numerator and denominator [see Eq. (19)], especially at low energies where loss to ions dominates (see Fig. 5). For the three temperatures $T_e = 5, 50, 100$ keV, values of ΔE_1 computed from Eq. (29c) and from numerical integration of Eq. (19) differ by 2.6%, 1.2% and 1.4%, respectively.

Fig. 6. Range $R(\text{gm/cm}^2) = \rho s(3.5, 1.5 T_e)$ vs $T_e(\text{keV})$ for various $\rho(\text{gm/cm}^3)$: (a) num. integ.; (b) approx. analytical formula, $C_A = 1.035$, $\langle \ln \rangle = \ln(0.8)$.

Fig. 7. Density $\times$ time $T(\text{n sec gm/cm}^3) = \rho t(3.5, 1.5 T_e)$ vs $T_e(\text{keV})$ for various $\rho(\text{gm/cm}^3)$: (a) num. integ.; (b) approx. analytical formula, $C_A = 1.035$, $\langle \ln \rangle = \ln(0.8)$.

(a)

(b)

Fig. 8. Fraction of initial energy to ions $(\Delta E_i + 1.5 T_e)/E_0$ for a 3.5 MeV alpha particle in a 50-50 DT plasma plotted as a function of T_e (keV) for several plasma densities $\rho(\text{gm/cm}^3)$ calculated by: (a) numerical integration; (b) approximate analytical formula with $C_A = 1.035$ and $\langle \ln \rangle = \ln(0.8)$.

Figures 6, 7, and 8 show range, time, and fraction of initial energy to ions, respectively, computed by numerical integration of Eqs. (17), (18), and (19) and by using the approximate analytical formulas (29a), (29b), and (29c), with $C_A = 1.035$ and $\langle \ln \rangle = \ln(0.8)$ in the latter. The calculations were done for an alpha particle of initial energy $E_0 = 3.5$ MeV and final energy $E' = \frac{3}{2} T_e$ in a 50-50 DT plasma at a variety of densities ρ and temperatures T_e .

assuming $T_i = T_e$. In Fig. 6, the range is expressed in g/cm³; in Fig. 7, the time lapse is also multiplied by ρ in order to get all of the curves on one

graph. In Fig. 8, $(\Delta E_i + \frac{3}{2} T_e)/E_0$ is plotted so that the final $\frac{3}{2} T_e$ of the energy is properly accounted for as being added to the thermal ions in the plasma.

APPENDIX COULOMB LOGARITHM TERM

A variety of approaches to the computation of the logarithm term can be found in the literature. Some of these will be outlined in this Appendix.

The logarithm term arises from the integration over the scattering angle ϑ ; specifically

$$\int_{\vartheta_m}^{\pi} \sigma^{(R)}(\vartheta, V) (1 - \cos \vartheta) \sin \vartheta d\vartheta = 2L^2, \quad (A1)$$

where

$$\Lambda = \int_{\vartheta_m}^{\pi} \cot(\vartheta/2) d(\vartheta/2) = \ln(2/\vartheta_m). \quad (A2)$$

Here

$$\sigma^{(R)}(\vartheta, V) = L^2 / (1 - \cos \vartheta)^2 \quad (A3)$$

is the Rutherford differential scattering cross section in the c.m. system,

$$L = |ZZ_j| e^2 / \mu V^2, \quad (A4)$$

$$\mu = m m_j / (m + m_j), \quad (A5)$$

and V is the magnitude of the relative velocity of the interacting particles. It is seen that the integral in Eq. (A2) diverges logarithmically as ϑ_m goes to zero. The integration over ϑ can be cut off at some minimum scattering angle ϑ_m (maximum impact parameter) because the electrons in the plasma shield the scattering center so that the radius of the scattering field is limited. In the classical description, ϑ is related to the impact parameter b through the expression

$$\sin^2(\vartheta/2) = 1/[1 + (b/L)^2]. \quad (A6)$$

For small ϑ (large b), Eq. (A6) becomes

$$\vartheta = 2L/b. \quad (A7)$$

If the shielding radius is taken to be the Debye length

$$\lambda_D = (kT_e / 4\pi N_e e^2)^{1/2}, \quad (A8)$$

where T_e and N_e are the temperature and number density of the electrons, then $b \leq \lambda_D$, $\vartheta \geq 2L/\lambda_D$, and the minimum value of ϑ in the classical picture is

$$\vartheta_{mc} = 2L/\lambda_D. \quad (A9)$$

In cases where the wave properties of the particles must be considered, the classical cut-off given by Eq. (A9) is not valid.^{1,2} If $2L/\lambda < 1$, where $\lambda = h/\mu V$ is the de Broglie wavelength and $2L$ is the classical distance of closest approach, then ϑ_m should be given by the quantum-mechanical expression

$$\vartheta_{mq} = \lambda/\lambda_D. \quad (A10)$$

Thus, substituting Eqs. (A9) and (A10) into Eq. (A2), one obtains

$$\Lambda = \begin{cases} \Lambda_q = \ln(2\lambda_D/\lambda), & \lambda/2L > 1 \\ \Lambda_c = \ln(\lambda_D/L), & \lambda/2L < 1. \end{cases} \quad (A11)$$

Another approach^{3,4} to the quantum-mechanical problem is to replace the cutoff Coulomb field by a Debye field with potential

$$U = -(ZZ_j e^2 / r) \exp(-r/\lambda_D).$$

The cross section for this screened potential is, in the Born approximation,⁵

$$\sigma(\vartheta, V) = (L/2)^2 / (\sin^2 \frac{\vartheta}{2} + a^2)^2, \quad (A12)$$

where

$$a = \lambda/2\lambda_D = \vartheta_{mq}/2. \quad (A13)$$

If in Eq. (A1) one replaces $\sigma^{(R)}(\vartheta, V)$ by Eq. (A12) and integrates, the result is the same except that Λ is replaced by

$$\Lambda'_q = \frac{1}{2} \left[\ln \frac{1+a^2}{a^2} - \frac{1}{1+a^2} \right].$$

Since $a \ll 1$, this becomes

$$\Lambda'_q = \ln(1/a) - \frac{1}{2} = \ln(2\lambda_D/\lambda) - \frac{1}{2}. \quad (A14)$$

A third approach⁶ is to use the cross section given by Eq. (A3) for $\theta_{mq} \leq \theta \leq \pi$, and that given by Eq. (A12) for $0 \leq \theta \leq \theta_{mq}$. The integral corresponding to Eq. (A1) then becomes

$$2L^2 \left[\int_0^{\theta_{mq}} \frac{\sin^3(\theta/2) d\sin(\theta/2)}{[\sin^2(\theta/2) + a^2]^2} + \int_{\theta_{mq}}^{\pi} \frac{d\sin(\theta/2)}{\sin(\theta/2)} \right] = 2L^2 \Lambda''_q.$$

where

$$\Lambda''_q = \ln(2/\theta_{mq}) + \frac{1}{2}(\ln 2 - \frac{1}{2}). \quad (A15)$$

Comparing the expressions for Λ_q given by Eqs. (A11), (A14), (A15):

$$\Lambda'_q = \Lambda_q - 0.5; \quad \Lambda''_q = \Lambda_q + 0.097.$$

Since the numerical value of Λ is of the order of 10, it is seen that these different approaches lead to results that differ by only a few percent.

Substituting Eqs. (A4) and (A8) into the expressions in Eq. (A11), one can write

$$\kappa/2L = (\bar{E}/\bar{E}^*)^{1/2},$$

$$\Lambda_q = \ln[(2/\pi)^{1/2} (\hbar e)^{-1} (\mu \bar{E} k T_e / N_e)^{1/2}],$$

$$\Lambda_c = \Lambda_q + \frac{1}{2} \ln(\bar{E}/\bar{E}^*),$$

where

$$\bar{E} = \frac{1}{2} \mu v^2,$$

$$\bar{E}^* = 2\alpha^2 (ZZ_j)^2 \mu c^2,$$

$\alpha = e^2/\hbar c$ is the fine-structure constant, and c is light velocity. Now let v_j represent a mean speed of the thermal particles of species j in the

plasma and consider the two cases: (I) $v \gg v_j$; and (II) $v \ll v_j$. Then: in case (I), $v = v$ and $\bar{E} = (\mu/m)E$; and in case (II), $v = v_j$ and $\bar{E} = (\mu/m_j)E_j$, where $E_j = \frac{1}{2} m_j v_j^2$. Define

$$E^* = (\mu/m) \bar{E}^* = 2\alpha^2 (ZZ_j)^2 \mu c^2, \quad (A16)$$

and

$$E_j^* = (m_j/\mu) \bar{E}^* = 2\alpha^2 (ZZ_j)^2 m_j c^2. \quad (A17)$$

Then, in the two cases, Eq. (A11) can be written

Case (I), $v \gg v_j$:

$$\Lambda_j^I = \begin{cases} \Lambda_q^I = K^I + \frac{1}{2} \ln(E k T_e / N_e), & E > E^* \\ \Lambda_c^I = \Lambda_q^I + \frac{1}{2} \ln(E/E^*), & E < E^* \end{cases} \quad (A18)$$

$$K^I = \ln \left[(2/\pi)^{1/2} (\hbar e)^{-1} (\mu/m)^{1/2} \right].$$

Case (II), $v \ll v_j$:

$$\Lambda_j^{II} = \begin{cases} \Lambda_q^{II} = K^{II} + \frac{1}{2} \ln(E_j k T_e / N_e), & E_j > E_j^* \\ \Lambda_c^{II} = \Lambda_q^{II} + \frac{1}{2} \ln(E_j/E_j^*), & E_j < E_j^* \end{cases} \quad (A19)$$

$$K^{II} = \ln \left[(2/\pi)^{1/2} (\hbar e)^{-1} (\mu/m_j)^{1/2} \right].$$

In a fully ionized plasma, $N_e = \sum_j Z_j N_j$, and for a plasma made up of isotopes of hydrogen, $Z_j = 1$,

$$N_e = N_0 \rho/\rho_0, \quad N_0 = 5.1 \times 10^{22} \text{ cm}^{-3},$$

where N_0 is normal liquid density and ρ/ρ_0 is the compression relative to normal density. For loss of a fast ion to thermal ions in the plasma, case (I) applies for most of the range. Letting $j = D, T$ designate the species of ion, and substituting numerical values for the constants in Eqs. (A16) and (A18), one obtains

$$\Lambda_j = \begin{cases} \Lambda_{qj} = K_j - \frac{1}{2} \ln(\rho/\rho_0) + \frac{1}{2} \ln(E k T_e), & E > E^* \\ \Lambda_{cj} = \Lambda_{qj} + \frac{1}{2} \ln(E/E^*), & E < E^* \end{cases} \quad (A20)$$

$$K_j = 16.48639 + \ln(A^{1/2} A_j / (A + A_j)) , \quad (j = D, J)$$

$$E^* = 0.099927 Z^2 \text{ A MeV}, \quad (E \text{ and } kT_e \text{ in MeV})$$

where A and A_j are defined in Eqs. (5).

For loss to electrons, case (II) applies, and $E_j = E_s = \frac{3}{2} kT_e$, $m_j = \mu = m_e$, $Z_j = -1$. So substituting $E^* = \frac{3}{2} kT_e^*$ and numerical values for the constants in Eqs. (A17) and (A19), one obtains

$$\Lambda_e = \begin{cases} \Lambda_{qe} = 12.93142 - \frac{1}{2} \ln(\rho/\rho_0) \\ \quad + \ln(kT_e), \quad kT_e > kT_e^* \\ \Lambda_{ce} = \Lambda_{qe} + \frac{1}{2} \ln(kT_e/kT_e^*), \quad kT_e < kT_e^* \end{cases} \quad (A21)$$

$$kT_e^* = 3.62822 \times 10^{-5} Z^2 \text{ MeV}. \quad (kT_e \text{ in MeV})$$

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