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OAK RIDGE SCHOOL OF REACTOR TECHNOLOGY

"ESTIMATION OF RESONANCE INTEGRALS"

Edward Kuhn

August 1957

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PREFACE

In September, 1956, a group of men experienced in various scientific and engineering fields embarked on the twelve months of study which culminated in this report. For nine of those months, formal classroom and student laboratory work occupied their time. At the end of that period, this student was presented with a problem in reactor design. He studied it for ten weeks, the final period of the school term.

This is a summary report of his efforts. It must be realized that, in so short a time, a study of this scope can not be guaranteed complete or free of error. This "thesis" is not offered as a polished engineering report, but rather as a record of the work done by him under the leadership of his advisor. It is issued for use by those persons competent to assess the uncertainties inherent in the results obtained in terms of the preciseness of the technical data and analytical methods employed in the study. In the opinion of the students and faculty of ORSORT, the problem has served the pedagogical purpose for which it was intended.

The faculty joins the author in an expression of appreciation for the generous assistance which various members of the Oak Ridge National Laboratory gave. In particular, the guidance of his consultant, L. R. Dresner, is gratefully acknowledged.

Lewis Nelson
for
The Faculty of ORSORT

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I would like to express sincere appreciation to L. Dresner of the Applied Nuclear Physics Division, Oak Ridge National Laboratory, for his kind assistance in this work.

E. Kuhn

ABSTRACT

A statistical model which takes into account reduced neutron width fluctuations is used to calculate resonance integrals for non fissile isotopes of mass number ≥ 100 . Calculated values are then compared with the latest experimental values.

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I. INTRODUCTION

In a recent paper (Dresner, 1955), a statistical estimate of the resonance integral for non fissile elements in infinite dilution was given in terms of certain average resonance parameters. In this estimate, the effect of the statistical distribution of the reduced neutron widths (Porter-Thomas, 1956) was not included, and neighboring resonances had essentially the same reduced neutron widths. It is the purpose of this paper to correct this deficiency and revise the previous estimate.

II. FORMULATION OF RESONANCE INTEGRAL

In the work of Dresner (1955) the resonance integral I_s is

$$I_s = \frac{1.3 \times 10^6 \text{ barns}}{E_1} \left(\frac{2\pi \bar{\Gamma}_\gamma}{D^*} \right) \left(\frac{B - \ln(1+B)}{B^2} \right) \quad (1)$$

where

$$B = \left(\frac{\bar{\Gamma}_\gamma}{\bar{\Gamma}_n^0} \right) E_1^{-\frac{1}{2}}, \quad (2)$$

E_1 , expressed in eV, is the energy of the first resonance, $\bar{\Gamma}_\gamma$ is the average[†] radiation width, D^* is the average spacing between resonances of the same spin and parity, $\bar{\Gamma}_n^0$ is the average reduced neutron width.

Equation (1) is derived by integration from E_1 to infinity of

$$\left(\frac{1}{\Delta E} \right) \left(\frac{\Delta E}{D} \right) \frac{2\pi^2 \lambda^2 g \bar{\Gamma}_n \bar{\Gamma}_\gamma}{\bar{\Gamma} E} \quad (3)$$

which is the average contribution to the resonance integral per unit energy of the resonances of zero orbital angular momentum in a small logarithmic interval $\frac{\Delta E}{E}$ of energy. The restriction to zero orbital angular momentum is justified since these resonances supply almost all of the value of the resonance integral. In Equation (3) D is the average spacing between resonances of zero orbital angular momentum, $\bar{\Gamma}$ is the average total width equal to $\bar{\Gamma}_\gamma + \bar{\Gamma}_n$, $\bar{\Gamma}_n$ is the average neutron width equal to $\bar{\Gamma}_n^0 E^{\frac{1}{2}}$, λ is the neutron wavelength divided by 2π , and g is a statistical factor equal to $\frac{2J+1}{2(2I+1)}$, where I is the spin of the target nucleus in units of $\hbar$, and J is the total spin of the compound nucleus equal to $I \pm \frac{1}{2}$.

[†] A bar signifies a local average over many resonances.

In the derivation of (1) it was assumed that $D/g = D^*$ and that $\bar{\Gamma}_y/D^*$ is independent of energy. The resonances are assumed sufficiently close to each other so that summation over individual resonances can be replaced by integration over E.

Recently (Porter and Thomas, 1956) it has been shown that a chi-squared distribution with one degree of freedom in the random variable $\frac{\Gamma_n^0}{\bar{\Gamma}_n^0}$ is the best measure of the fluctuations of the reduced neutron width consistent with the data and some theoretical evidence. The chi-squared distribution with one degree of freedom is

$$P(x) dx = (2\pi x)^{-\frac{1}{2}} e^{-x/2} dx \quad (4)$$

The statistical estimate of the resonance integral can now be formulated as

$$I_S^C = \int \int \frac{2\pi^2 \lambda^2}{D} g \frac{\Gamma_n \bar{\Gamma}_y}{\Gamma_n + \bar{\Gamma}_y} \frac{dE}{E} P\left(\frac{\Gamma_n^0}{\bar{\Gamma}_n^0}\right) d\left(\frac{\Gamma_n^0}{\bar{\Gamma}_n^0}\right) \quad (5)$$

Setting $D/g \approx D^*$, $\Gamma_n = \Gamma_n^0 E^{\frac{1}{2}}$ and $\pi\lambda^2 = \frac{C}{E}$ where C is a constant equal to $.65 \times 10^6$ barns-ev, Equation (5) becomes

$$(6) \quad I_S^C = \left(\frac{2\pi \bar{\Gamma}_y}{D^*}\right) C \int_{E_1}^{\infty} \frac{dE}{E^2} \int_0^{\infty} \frac{\Gamma_n^0 \sqrt{E}}{\Gamma_n^0 \sqrt{E} + \bar{\Gamma}_y} P\left(\frac{\Gamma_n^0}{\bar{\Gamma}_n^0}\right) d\left(\frac{\Gamma_n^0}{\bar{\Gamma}_n^0}\right)$$

Let $\Gamma_n^0 / \bar{\Gamma}_n^0 = x$; $y = E/E_1$; then (6) becomes

$$(7) \quad I_s^c = \left(\frac{2\pi \sqrt{\bar{r}_x}}{D^*} \right) \frac{C}{E_1} \int_0^\infty P(x) dx \int_1^\infty \frac{1}{y^2} \frac{\sqrt{y} dy}{\sqrt{y} + \frac{B}{x}}$$

Integrating (7) over y we get

$$I_s^c = \frac{(1.3) \times 10^6 \text{ barns}}{E_1} \left(\frac{2\pi \sqrt{\bar{r}_x}}{D^*} \right) \int_0^\infty \frac{\frac{B}{x} - \ln \left(1 + \frac{B}{x} \right)}{(B/x)^2} P(x) dx \quad (8)$$

or

$$I_s^c = \int_0^\infty I_s \left(\frac{B}{x} \right) P(x) dx \quad (9)$$

Equation (9) can be written as

$$I_s^c = \frac{A}{\sqrt{2\pi}} B^{-2} \int_0^\infty (x)^{3/2} \left\{ \frac{B}{x} - \ln \left(1 + \frac{B}{x} \right) \right\} e^{-x/2} dx \quad (10)$$

where $A = \left(\frac{1.3 \times 10^6}{E_1} \right) \left(\frac{2\pi \sqrt{\bar{r}_x}}{D^*} \right)$

In a formal way, Equation (10) gives the statistical resonance integral I_s^c in terms of an average contribution from each resonance, allowing for fluctuations in $\sqrt{r_n}$. It is to be expected that the statistical model holds best for the case when the resonance integral is primarily due to the cumulative effect of many resonances closely spaced and approximately of the same order of magnitude.

The techniques used in calculating I_s^c are discussed in the next section.

III. TECHNIQUES OF CALCULATION

From Equation (10) it is clear the I_s^c is equal to $\frac{A}{\sqrt{2\pi}} B^{-2} f(B)$ where $f(B)$ is given by

$$f(B) = \int_0^{\infty} (x)^{3/2} \left\{ \frac{B}{x} - \ln \left(1 + \frac{B}{x} \right) \right\} e^{-x/2} dx \quad (11)$$

Three methods of calculation for $f(B)$ were used according to the value of the parameter B , namely

- (a) for $B \neq 1$, numerical integration
- (b) for $B \geq 10$, asymptotic series in $\frac{1}{B}$
- (c) for $B \leq 0.1$, power series in $B^{\frac{1}{2}}$

The methods are fully described in Appendices I, II, III.

The results are:

Case (a):

$$f(B) = \sqrt{2\pi} \left[2 + 3 F\left(\frac{1}{2}\right) + 3 \ln 2 \right] + \sqrt{2\pi} [B - 3 \ln B] - 2B^{5/2} J(B) \quad (12)$$

where $F\left(\frac{1}{2}\right)$ is $\left[\frac{d}{dz} (\ln Z!) \right]_{Z=\frac{1}{2}}$ and equals 0.03649 and $J(B)$ is given by

$$J(B) = \int_0^{\infty} y^4 \ln(1+y^2) e^{-B/2 y^2} dy \quad (13)$$

For $B = 1, 2, 3$, $J(B)$ was evaluated numerically. For $B = 1/2$, $f(B)$ was evaluated numerically in the form

$$f(B) = \int_0^{\infty} \left\{ y^2 - y^4 \ln \left(\frac{1+y^2}{y^2} \right) \right\} e^{-B/2 y^2} dy \quad (14)$$

Case (b):

$$f(B) = \sqrt{2\pi} \left[2 + 3 F\left(\frac{1}{2}\right) + 3 \ln 2 \right] + \sqrt{2\pi} B (2)^{5/2} \Gamma(5/2) \ln B + (2)^{5/2} \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \frac{2^k \Gamma(5/2 + k)}{B^k} \quad \dots (15)$$

Case (c):

$$f(B) = \sqrt{2\pi} B^2 \left[\frac{1}{2} - \frac{\sqrt{2\pi}}{5} B^{\frac{1}{2}} + \frac{1}{3} B - \frac{\sqrt{2\pi}}{14} B^{3/2} + \dots \right] \quad (16)$$

Equations (12), (13), (14), (15), (16) were used to compute the points of Figure 1. From the above equations it can be seen that

$$\text{or } \lim_{\substack{B \rightarrow \infty \\ B \rightarrow 0}} \left[\frac{I_B^c}{I_B} \right] = 1$$

IV. COMPARISON WITH EXPERIMENTAL RESONANCE INTEGRALS

In Table 1 a comparison is made between the calculated value of the resonance integral I_g^c and the latest experimental values. The calculated value is obtained by using Equations (1), (2) of Section II and Figure 1, and the indicated parameters of Table 1.

With these equations it is also possible to calculate the contribution to the resonance integral only from the unresolved resonances. Adding this contribution to the contribution of a finite number of resolved resonances gives another estimate of the total resonance integral.

TABLE 1

Nucleus	$\bar{\gamma}_x (10^{-3} \text{ ev})$	$\bar{\gamma}_n^o (10^{-3} \text{ ev})$	$E_1 (\text{ev})$	$\frac{\bar{\gamma}_n^o}{D^2} \times 10^4$	Experimental Resonance Integral (barns)	I_g^c Estimated Resonance Integral (barns)
$^{238}\text{U}_{92}$	24(a)	2.2(a)	6.68(b)	1.3(a)	$281 \pm 20^{(d)}$	178
$^{232}\text{Th}_{90}$	30(a)	1.5(a)	22(b)	1.0(a)	$67 \pm 5^{(d)}$	78
$^{236}\text{U}_{92}$	29(a)	1.9(a)	5.49(b)	1.3(a)	$350 \pm 90^{(a)}$	239
$^{234}\text{U}_{92}$	27(a)	1.1(a)	5.20(b)	1.2(a)	None	270
$^{107}\text{Ag}_{47}$	140(b)	1.57(b)	16.6(b)	0.5(c)	74(e)	75
$^{181}\text{Ta}_{73}$	50(b)	1.67(b)	4.28(b)	1.8(c)	590(e)	490
$^{127}\text{I}_{53}$	100(b)	2.96(b)	20.5(b)	1.2(c)	140(e)	125
$^{121}\text{Sb}_{51}$	75(b)	1.38(b)	6.24(b)	0.45(c)	162(e)	109

- (a) Harvey, J. A. and Schwartz, R. B., Thermal and Resonance Neutron Cross Sections of Heavy Nuclei, Phys. Rev. (to be published)
 (b) BNL-325, Supp. I.
 (c) Schwartz, R. B., Pilcher, V. E., Hughes, D. J., Zimmerman, R. L., (1956) Bull. A.P.S. Series II, 7, 347.
 (d) Macklin, R., and Pomerance H., (1956) Nuc. Energy, 2, 243.
 (e) Macklin, R. and Pomerance H., 1956, Prog. in Nuc. Energy, Vol. I., McGraw-Hill, New York, Pergamon Press, London.

Table 2 lists the results of such a calculation for ${}_{92}\text{U}^{238}$, ${}_{90}\text{Th}^{232}$, ${}_{92}\text{U}^{236}$.

TABLE 2

Nucleus	Resonance Integral (barns) Resolved Resonances	I_g^c (barns) Unresolved resonances	Experimental Resonance Integral (barns)	Estimated Resonance Integral (barns)
${}_{92}\text{U}^{238}$	$\begin{matrix} 257 \\ (6.68 \rightarrow 191)_{\text{ev}} \end{matrix}$	15	281 ± 20	272
${}_{90}\text{Th}^{232}$	$\begin{matrix} 71 \\ (22 \rightarrow 130)_{\text{ev}} \end{matrix}$	19	67 ± 5	90
${}_{92}\text{U}^{236}$	$\begin{matrix} 292 \\ (5.49 \rightarrow 133)_{\text{ev}} \end{matrix}$	25	350 ± 90	317

The contribution of the unresolved resonances is calculated from Equations (1), (2) of Section II and Figure 1. The parameters used in this calculation were $\bar{f}_v$, D^* , $\bar{f}_n^o$, of the Harvey-Schwartz paper. The value of the resonance integral from resolved resonances is also taken from this paper, and from calculations of J. A. Harvey of Oak Ridge National Laboratory.

In Figure 2 is plotted the value of I_g^c as a function of E_1 for elements of mass number ≥ 100 , with several choices of the nuclear parameters. The fairly close agreement between curves A and B, C and D indicates that I_g^c is relatively insensitive to $\bar{f}_v/D^*$. Curves C and D appear to give the best calculated value for the resonance integral for the type of resonance structure and elements considered. The parameters selected for the curves A, B, C, D, are average quantities for elements of mass number ≥ 100 .

V. CONCLUSIONS

1) Recalling that the statistical model is only valid for the elements whose resonance integral is due to the cumulative effect of many resonances all approximately of the same order of magnitude, the order of magnitude agreement of Tables 1 and 2 can be understood.

2) For mass numbers ≥ 100 and non-magic nuclei, these nuclei have resonance integrals virtually independent of $\sqrt{A}/D^*$.

3) Calculated I_s^c are within 37 per cent agreement of measured values.

4) From Table 2 it appears that the better calculated value for I_s^c , in the case of elements whose resonance integral mainly comes from a few large resonances, is obtained by using the statistical model for the unresolved resonances (small and closely spaced), and the area method for the large resolved resonances.

B

0.01 0.02 0.03 0.04 0.05 0.06 0.07 0.08 0.09 0.10 0.11 0.12 0.13 0.14 0.15 0.16 0.17 0.18 0.19 0.20 0.21 0.22 0.23 0.24 0.25 0.26 0.27 0.28 0.29 0.30 0.31 0.32 0.33 0.34 0.35 0.36 0.37 0.38 0.39 0.40 0.41 0.42 0.43 0.44 0.45 0.46 0.47 0.48 0.49 0.50 0.51 0.52 0.53 0.54 0.55 0.56 0.57 0.58 0.59 0.60 0.61 0.62 0.63 0.64 0.65 0.66 0.67 0.68 0.69 0.70 0.71 0.72 0.73 0.74 0.75 0.76 0.77 0.78 0.79 0.80 0.81 0.82 0.83 0.84 0.85 0.86 0.87 0.88 0.89 0.90 0.91 0.92 0.93 0.94 0.95 0.96 0.97 0.98 0.99 1.00

90

1.0

$\frac{1}{2}$

80

60

40

FIG. 1

-16-

406 016

FIG. II

Resonance Integral (barns)

FIG II -17-

APPENDIX I

NUMERICAL INTEGRATION

The first term in the braces in Equation (11) can be integrated to give $\sqrt{2\pi} B$; by breaking up the logarithm in the second term Equation (11) can be written as

$$f(B) = \sqrt{2\pi} B + \int_0^{\infty} (x)^{3/2} \ln x e^{-x/2} dx - \int_0^{\infty} (x)^{3/2} \ln (x+B) e^{-x/2} dx \quad (1)$$

The $\ln x$ integral can be written as

$$2\sqrt{2\pi} + 3 \int_0^{\infty} (x)^{1/2} \ln x e^{-x/2} dx \quad (2)$$

by partial integration with $u = x^{3/2} \ln x$, $dv = e^{-x/2} dx$; the integral in Equation (2) can be expressed as the derivative of the factorial function and is equal to

$$2\sqrt{2} \left[\frac{d}{dz} (Z!) \right]_{Z=\frac{1}{2}} + \sqrt{2\pi} \ln 2$$

After a variable change, $x = By^2$ the $\ln(x+B)$ integral in Equation (1) takes the form

$$3\sqrt{2\pi} \ln B + 2B^{5/2} J(B), \text{ where} \\ J(B) = \int_0^{\infty} y^4 \ln (y^2 + 1) e^{-B/2 y^2} dy \quad (3)$$

$J(B)$ was evaluated by Simpson's Rule with an interval of 0.2 for $B = 1, 2, 3$. The derivative of the factorial was evaluated by using the properties of the digamma function, $F(Z) = \frac{d}{dz} [\ln Z!]$. Finally by combining terms,

$$f(B) = \sqrt{2\pi} \left[2 + 3 F\left(\frac{1}{2}\right) + 3 \ln 2 \right] + \sqrt{2\pi} \left[B + 3 \ln B \right] - 2B^{5/2} J(B)$$

For $B = 0.5$, $f(B)$ was evaluated by Simpson's Rule in the form

$$\int_0^{\infty} \left\{ y^2 - y^4 \ln \left(\frac{y^2 + 1}{y^2} \right) \right\} e^{-By^2/2} dy, \text{ obtained from Equation (11) with } x = By^2.$$

APPENDIX II

ASYMPTOTIC SERIES IN $1/B$

From Equation (1), Appendix I, $f(B)$ can be written as

$$f(B) = \sqrt{2\pi} B - \ln B \int_0^{\infty} x^{3/2} e^{-x/2} dx + \int_0^{\infty} x^{3/2} \ln x e^{-x/2} dx - \int_0^{\infty} x^{3/2} \ln\left(\frac{x}{B} + 1\right) e^{-x/2} dx \quad \dots (1)$$

The first two integrals on the right of Equation (1) give $(2)^{5/2} \Gamma(5/2) \ln B$ and $\sqrt{2\pi} \left[2 + 3 F(\frac{1}{2}) + 3 \ln 2 \right]$ respectively (where $F(\frac{1}{2}) = 0.03649$). The third integral on the right of Equation (1) is changed by a variable transformation, $x/B = y$ to

$$(B)^{5/2} \int_0^{\infty} y^{3/2} e^{-By/2} \ln(y+1) dy \quad (2)$$

$$\text{for } y^2 \ll 1, \quad \ln(y+1) = - \sum_{k=1}^{\infty} (-1)^k \frac{y^k}{k} \quad (3)$$

Substituting Equation (3) into (2) and performing the integration, (2) can be written as

$$-(2)^{5/2} \sum_{k=1}^{\infty} \left\{ \frac{(-1)^k}{k} \frac{2^k}{B^k} \right\} \frac{\Gamma(5/2+k)}{B^k} \quad (4)$$

which is an asymptotic series in $1/B$, and is divergent. However, the series can be made to approximate the integral to an order of accuracy (Morse and Feshbach), depending on the number of terms taken and the value of B . The optimal number of terms to select is the integer closest to k for the value

of k when $\left| \frac{u_{k+1}}{u_k} \right| = 1$ where $u_k = \left\{ \frac{(-1)^k 2^k}{k} \right\} \frac{\Gamma(5/2+k)}{B^k}$

It can be shown that the error is less than the last term neglected when the third integral on the right in Equation (1) is approximated by (4).

First we expand $\ln(y+1) = f(y)$ in a Maclaurin Series with remainder:

$$f(y) = \sum_{k=0}^n f^{(k)}(0) \frac{y^k}{k!} + \frac{y^{n+1}}{(n+1)!} f^{(n+1)}(\theta y)$$

where $0 < \theta < 1$. The (k) th derivative of $f(y)$ can be written as

$(-1)^{k+1} (k-1)! (y+1)^{-k}$, ($k > 0$); therefore we note that $|f^{(k)}(\theta y)| < |f^{(k)}(0)|$ ($k > 0$). Hence in the power series $\ln(y+1) = - \sum_{k=1}^{\infty} (-1)^k \frac{y^k}{k}$ the

error is less than the first term neglected.

Set

$$L = B^{5/2} \int_0^{\infty} y^{3/2} e^{-By/2} \left(\sum_{k=0}^n f^{(k)}(0) \frac{y^k}{k!} + \frac{y^{n+1}}{(n+1)!} f^{(n+1)}(\theta y) \right) dy$$

Then the error in approximating L with n terms of the series is

$$(B)^{5/2} \int_0^{\infty} \frac{y^{n+1}}{(n+1)!} f^{(n+1)}(\theta y) y^{3/2} e^{-By/2} dy$$

The first term neglected in approximating L , namely the $(n+1)$ term of the series is

$$(B)^{5/2} \int_0^{\infty} y^{3/2} e^{-By/2} \frac{y^{n+1}}{(n+1)!} f^{(n+1)}(0) dy.$$

Since $\left| r^{(n+1)}(0y) \right| < \left| r^{n+1}(0) \right|$

the error in approximating the integral is less than the last term neglected in (4).

Finally,

$$f(B) = \sqrt{2\pi} \left[2 + 3 F\left(\frac{1}{2}\right) + 3 \ln 2 \right] + \sqrt{2\pi} B - 2^{5/2} \Gamma(5/2) \ln B + (2)^{5/2} \sum_{k=1}^{\infty} \frac{(-1)^k 2^k}{k B^k} \Gamma(5/2 + k)$$

For $B \geq 10$ the series is evaluated taking one half the last term

(Morse Feshbach); the maximum error in the series is 10%, and the maximum

error in I_s^c is 2%.

APPENDIX III
POWER SERIES IN $\sqrt{B}$

For convenience we will deal with I_s^c directly in the form

$$I_s^c = A \left(\frac{2B}{\pi} \right)^{\frac{1}{2}} \int_0^{\infty} \left\{ y^2 - y^4 \ln \left(\frac{y^2+1}{y^2} \right) \right\} e^{-By^2/2} dy \quad (1)$$

obtained from Equation (10) Section II by the variable transformation $x = By^2$. The quantity $\left\{ y^2 - y^4 \ln \left(\frac{y^2+1}{y^2} \right) \right\}$ approaches $\frac{1}{2} + O\left(\frac{1}{y^2}\right)$ as y approaches ∞ . Hence if we subtract $\frac{1}{2}$ from the curly bracket the result will be integrable between 0 and ∞ . By adding and subtracting $\frac{1}{2}$ to the curly bracket we can write Equation (1) as:

$$I_s^c = A \left[\frac{1}{2} - \left(\frac{2B}{\pi} \right)^{\frac{1}{2}} \int_0^{\infty} \left\{ \frac{1}{2} - y^2 + y^4 \ln \left(\frac{1+y^2}{y^2} \right) \right\} e^{-By^2/2} dy \right] \quad (2)$$

Now we can employ partial integration in the form

$$\int_0^{\infty} u dv = uv \Big|_0^{\infty} - \int_0^{\infty} v du \quad \text{with } v(y) = \int_y^{\infty} \left\{ \frac{1}{2} - y^2 + y^4 \ln \left(\frac{1+y^2}{y^2} \right) \right\} dy$$

and $u(y) = e^{-By^2/2}$. Noting that $v(0) = \pi/5$, Equation (2) can be written as

$$I_s^c = A \left(\frac{1}{2} - \left(\frac{2B}{\pi} \right)^{\frac{1}{2}} \left[\pi/5 - B \int_0^{\infty} \left\{ \pi/5 - y^5/5 \ln \left(\frac{1+y^2}{y^2} \right) + \frac{y^3}{5} - y/10 - 2/5 \tan^{-1} y \right\} e^{-By^2/2} dy \right] \right) \dots \quad (3)$$

Since the quantity

$$y \left\{ \frac{\pi}{5} - \frac{y^5}{5} \ln \left(\frac{1+y^2}{y^2} \right) + y^{3/5} - y/10 - 2/5 \tan^{-1} y \right\} \text{ approaches}$$

$1/3 + O\left(\frac{1}{y^2}\right)$ as y approaches ∞ , Equation (3) can be written as

$$I_s^c = A \left[\frac{1}{2} - \frac{\sqrt{2\pi}}{5} B^{\frac{1}{2}} + 1/3 B - \left(\frac{2}{\pi}\right)^{\frac{1}{2}} B^{3/2} \int_0^{\infty} \left\{ 1/3 - y \left(\frac{\pi}{5} - \frac{y^5}{5} \ln \left(\frac{1+y^2}{y^2} \right) + y^{3/5} - \frac{y}{10} - \frac{2}{5} \tan^{-1} y \right) \right\} e^{-\frac{By^2}{2}} dy \right] \quad (4)$$

by adding and subtracting $1/3$ from y times the curly bracket. Since the curly bracket in the integral in Equation (4) is integrable one can again

integrate by parts using $v = \int_0^{\infty} \left\{ 1/3 - y \left(\frac{\pi}{5} - \frac{y^5}{5} \ln \left(\frac{1+y^2}{y^2} \right) + y^{3/5} - \frac{y}{10} - \frac{2}{5} \tan^{-1} y \right) \right\} dy$
 $- \frac{B}{2} y^2$

and $u = e^{-\frac{By^2}{2}}$. Each repetition of this procedure will generate two terms in the power series, one from the addition and subtraction required to make the bracket integrable, and one which is the integrated term, $u v \Big|_0^{\infty}$. If after the addition and subtraction we set $B = 0$ in the exponential we also obtain the same two terms, e.g., if we set $B = 0$ in the integral in Equation (4) we would obtain

$$I_s^c = A \left[\frac{1}{2} - \frac{\sqrt{2\pi}}{5} B^{\frac{1}{2}} + 1/3 B - \frac{\sqrt{2\pi}}{14} B^{3/2} \right] \quad (5)$$

which are the first four terms we would obtain by two repetitions of the outlined partial integration procedure. Since $e^{-B/2 y^2} \leq 1$ the last term is an overestimate in absolute magnitude. It follows then that the error in the sum of the first three terms is less than the absolute value of the fourth.

APPENDIX IV

SPIN DEPENDENCE

In Dresner's (1955) paper it was assumed that $\bar{\Gamma}_{\gamma J}$, $\bar{\Gamma}_{nJ}^0/D_J$, and D_J were independent of J , the spin of the compound nucleus. Here $\bar{\Gamma}_{\gamma J}$ is the average radiative width of levels of spin J , $\bar{\Gamma}_{nJ}^0$ is the average reduced neutron width of these levels, and D_J is their average spacing. Recently, it has been suggested (Lane, 1956) that while $\bar{\Gamma}_{\gamma J}$ and $\bar{\Gamma}_{nJ}^0/D_J$ are, in fact, independent of J , D_J is proportional to $(2J+1)^{-1}$. It is of interest to see what effect this change will have on the work presented in this paper. Before we proceed it is convenient to define quite generally two symbols, viz. (i) $D^* = (2 - \delta_{I,0}) D_{\text{obs}}$ where D_{obs} is the observed average spacing of resonances excited by low energy neutrons, I is the target spin, and $\delta_{I,0}$ is Kronecker's delta (ii) $\bar{\Gamma}_n^0 = \sum_{J=I-\frac{1}{2}}^{I+\frac{1}{2}} \frac{D_{\text{obs}}}{D_J} \bar{\Gamma}_{nJ}^0$.

The contribution to the resonance integral of the resonances in a small logarithmic range of energy $\frac{\Delta E}{E}$ is given now by Equation (5) of Section II, generalized to two spin states, viz.:

$$\frac{dI_s^c}{dE} = \frac{2\pi^2 \chi^2}{E} \sum_{J=|I-\frac{1}{2}|}^{I+\frac{1}{2}} \frac{\delta_J}{D_J} \left\langle \frac{\bar{\Gamma}_{nJ}}{\bar{\Gamma}_{nJ} + \bar{\Gamma}_{\gamma J}} \right\rangle \quad (1)$$

where the bracket signifies an average over the distribution of neutron widths. In case we take D_J independent of J it is easily shown from (i) that $D_J = D^*$. Moreover, since $\bar{\Gamma}_{nJ}^0$ is then independent of J it also follows from (ii) that $\bar{\Gamma}_{nJ} = \bar{\Gamma}_n$ since $\sum \frac{D_{\text{obs}}}{D_J} = \text{always}$. Hence one gets for $\frac{dI_s^c}{dE}$ the result

$$\frac{dI_s^c}{dE} = \frac{\pi \lambda^2}{E} \left(\frac{2\pi \bar{\Gamma}_g}{D^*} \right) \left\langle \frac{\bar{\Gamma}_n/D^*}{\bar{\Gamma}_n/D^* + \bar{\Gamma}_g/D^*} \right\rangle \quad (2)$$

which is Dresner's result. Finally, this justifies his crude procedure of setting $D/g \approx D^*$.

In case we take $D_J \propto (2J+1)^{-1}$ we proceed as follows: First, we can easily show that $g_J D_J = D_{\text{obs.}} = (2 - S_{I,0})^{-1} D^*$. Secondly, if we divide (11) by this last equation we obtain $\frac{\bar{\Gamma}_n^0}{D^*} = (2 - S_{I,0})^{-1} \sum_{J=|I-\frac{1}{2}|}^{I+\frac{1}{2}} \frac{\bar{\Gamma}_{nJ}^0}{D_J}$. Since $\bar{\Gamma}_{nJ}/D_J$ is assumed independent of J it follows that $\bar{\Gamma}_{nJ}^0/D_J = \bar{\Gamma}_n^0/D^*$. With these results one can write Equation (1) as

$$\frac{dI_s^c}{dE} = \frac{\pi \lambda^2}{E} \left(\frac{2\pi \bar{\Gamma}_g}{D^*} \right) (2 - S_{I,0}) \sum_{J=|I-\frac{1}{2}|}^{I+\frac{1}{2}} g_J^2 \left\langle \frac{\bar{\Gamma}_n/D^*}{\bar{\Gamma}_n/D^* + \bar{\Gamma}_g/D^* (2 - S_{I,0}) g_J} \right\rangle \quad (3)$$

Four things are noteworthy about Equation (3).

- (a) It is formally the same as the sum of two terms each given by Equation (2). Hence, the analysis and results of this paper are valid.
- (b) For $I \approx 0$, Equation (3) and Equation (2) are identical, since then J can only equal $\frac{1}{2}$, and $g_J = 1$.

- (c) For $I \gg 1$, $g_J \sim \frac{1}{2}$ and $\sum_{J=|I-\frac{1}{2}|}^{I+\frac{1}{2}} g_J^2 \sim \frac{1}{2}$. With these values we find

Equation (3) and Equation (2) are again identical.

- (d) Because of the insensitivity of the resonance integral to $\bar{\Gamma}_g/D^*$ already noted we might expect Equation (2) to be a good approximation to Equation (3) even when $I \sim \frac{1}{2}$.

REFERENCES

1. Dresner, L., (1955), J. Nuclear Energy, 2, 118.
2. Macklin, R., and Pomerance, H., (1956), J. Nuclear Energy, 2, 243.
3. Macklin, R., and Pomerance, H., (1956), Progress in Nuclear Energy, Vol. 1, McGraw-Hill, New York, Pergamon Press, London.
4. BNL-325 and BNL-325, Supplement I.
5. Harvey, J. A., and Schwartz, R. B., Thermal And Resonance Neutron Cross Sections of Heavy Nuclei, Phys. Rev. (To be published).
6. Lane, A. M., (1956), Conference on Neutron Physics by Time-of-Flight held at Gatlinburg, Tennessee, ORNL-2309, page 113.
7. Schwartz, R. B., Pilcher, V. E., Hughes, D. J., Zimmerman, R. L., (1956), Bull. A.P.S., Series II, 7, 347.
8. Porter, C. E., and Thomas, R. G., (1956), Phys. Rev. 104, 483.
9. Morse, P. M., and Feshbach, H., (1953), Methods of Theoretical Physics, McGraw-Hill, New York.