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Macroscopic Analysis of
Magnetoplasmadynamic Arc Jets

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Neville A. Black
David B. Fradkin

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MACROSCOPIC ANALYSIS OF MAGNETOPLASMA DYNAMIC ARC JETS

by

Neville A. Black and David B. Fradkin

ABSTRACT

Scaling laws for coaxial applied-field and self-field magnetoplasma dynamic arc jets are obtained by dimensional analysis. The electromagnetic stress tensor is used to obtain formulae for the aggregate effect of $\vec{j} \times \vec{B}$ forces, both thrust and torque, throughout the discharge volume, in terms of the magnetic field over a bounding surface. A simple rotating plasma model for applied-field accelerators is developed. The present state of self-field accelerator research is reviewed. Efficiencies of idealized linear accelerators are compared with the characteristics of the corresponding coaxial devices.

Introduction

In the last few years, the magnetoplasma-dynamic (MPD) arc jet has emerged as a promising thruster for propulsion in space. In common with the ion engine, its basic advantage over chemical or nuclear rockets is high exhaust velocity. However, the MPD arc jet operates at lower voltage and much higher thrust density than does the ion engine.

The phenomena occurring in MPD arcs are complicated. There is as yet no comprehensive theory for their operation, and definitive experiments are few. It is therefore valuable to extract as much information as possible from general, macroscopic considerations. This we attempt to do in this paper.

In the first section, scaling laws are derived by dimensional analysis, using as the pertinent physical quantities those variables which are common to most experimental work. Next, we use the electromagnetic stress tensor to show how the aggregate $\vec{j} \times \vec{B}$ forces occurring throughout the volume can be completely determined from a knowledge

of the magnetic field over a bounding surface. The third section contains the formulation, and explores the consequences, of a simple rotating-plasma model for the operation of applied-field MPD arcs. While most work to date has been done on applied-field devices, the self-field accelerator concept looks promising at considerably higher power levels; the present state of self-field accelerator research is discussed in the fourth section. In the final section, we consider the question of efficiency, comparing the computed performance of idealized linear self-field and applied accelerators to the corresponding coaxial MPD arc jet devices.

1. DIMENSIONAL ANALYSIS

We assume that the behavior of an arc jet is determined by a restricted set of physical parameters. For the self-field arc jet, these parameters are:

- I: arc current,
- R: scale size of the device; e.g., anode radius,
- $\dot{m}$: mass flow,

$\alpha = q/m$: ion charge/mass ratio,
 μ_0 : the dimensional constant ($4\pi \times 10^{-7}$
 henries/meter) relating the current
 and self-magnetic field.

For the applied-field (Hall-current) devices, these parameters are:

I: arc current,
 B: characteristic magnetic field; e.g.,
 at the cathode tip,
 R: scale size,
 $\dot{m}$: mass flow,
 $\alpha = q/m$: ion charge/mass ratio.

For dimensional analysis to be valid and useful, one must correctly select the smallest set of physical parameters (corresponding to the "experimental variables") adequate to determine the physical problem. This choice is based on intuition, aided by visualization of the physical quantities that would occur if one were able to write down a complete set of governing equations. However, in visualizing such a complete set of equations to obtain a usefully small set of parameters, one must, on grounds of intuition and experience or experimental results, reject many physical quantities that could be important.

Thus, we assume that the scale size, R, is adequate to specify a geometrically similar set of devices, while we can certainly imagine that other characteristic lengths, such as collision and ionization mean free paths, are important and do not necessarily scale with R. Nevertheless, we proceed on the basis that the variables selected above are the obvious ones under the control of the experimenter, with the exception of μ_0 (see Ref. 1 for a discussion of such dimensional constants), and that these variables can be expected to be of primary importance. In defense of our choice of such a severely limited number of physical parameters, we note that a wide range of gas-dynamic situations are largely determined by specifying geometry, density, velocity, specific heats, and sound speed, while recognizing that such parameters as viscosity, mean free paths, and heat-transfer coefficients may not be ignored under some circumstances.

We want to determine how arc jet characteristics such as thrust (T), voltage (V), exhaust velocity (U), and power (P) vary with the selected parameters. With the dimensionless variable β

defined by

$$\dot{m} \alpha = \beta I, \quad (1-1)$$

we obtain from dimensional analysis:

SELF FIELD:

$$\begin{aligned} T &= \mu_0 I^2 f_1(\beta) \\ V &= \mu_0^2 I^2 \alpha f_2(\beta) \\ U &= \mu_0 I \alpha f_3(\beta) \\ P &= \mu_0^2 I^3 \alpha f_4(\beta) \end{aligned} \quad (1-2)$$

APPLIED FIELD:

$$\begin{aligned} T &= I B R g_1(\beta) \\ V &= R^2 B^2 \alpha g_2(\beta) \\ U &= R B \alpha g_3(\beta) \\ P &= I R^2 B^2 \alpha g_4(\beta), \end{aligned} \quad (1-3)$$

where the f's and g's are unknown functions of β .

Within the limits of our assumptions, the self-field arc jet exhibits no dependence on the scale size of the device. Similarly, for constant β , i.e., mass flow proportional to current, the voltage and exhaust velocity are independent of current for the applied-field device. The predicted linear dependence of thrust on the arc current and magnetic field for the applied-field arc particularly demands experimental verification.

2. ELECTRODYNAMICS

A complete description of the flow field in a magnetoplasmadynamic arc would necessitate knowledge of the distribution functions of ions, electrons, and neutrals in the 6-dimensional position-velocity space; from this knowledge we could calculate the electric charge-, momentum-, and energy-flux tensors at the electrodes and in the exhaust beam. Such a complete description will doubtless elude us for some time; however, some important general results can be obtained from the laws of electrodynamics by invoking the electromagnetic stress tensor.

The geometry dictates that we use cylindrical coordinates. Thus, the simplifications of Cartesian tensor analysis are not available. Accordingly, we first display the notation and some useful tensor relationships in cylindrical coordinates.

It is necessary to consider the covariant and contravariant components of tensors and to relate these to the usual "physical components" of vector and Cartesian-tensor analysis. For an arbitrary

vector $\bar{B}$, we denote its physical (ordinary vector) components by $\bar{B}_r$, $\bar{B}_\theta$, and $\bar{B}_z$. When considering $\bar{B}$ as a first-order tensor, we drop the bar and follow the usual convention that a superscript denotes the contravariant form and components. (A subscript is used for the covariant case.) In cylindrical coordinates these three descriptions of the same entity are related in the following way:

Physical Component	=	Contravariant Component	=	Covariant Component	
$\bar{B}_r$	=	B^r	=	B_r	
$\bar{B}_\theta$	=	rB^θ	=	B_θ/r	(2-1)
$\bar{B}_z$	=	B^z	=	B_z	

Unless otherwise noted, the Einstein summation convention applies, i.e., we sum over all values of a repeated index, so that

$$a_i b^i = a_r b^r + a_\theta b^\theta + a_z b^z. \quad (2-2)$$

(The R.H.S. repeated indices do not imply summations here.)

We shall need the contravariant Kronecker delta. The Kronecker delta is defined as a mixed tensor

$$\delta_j^i = 0, i \neq j \\ = 1, i = j \quad (2-3)$$

in all coordinate systems. For any mixed tensor we can raise an index by use of the conjugate metric tensor g^{rs} , so that the doubly-contravariant Kronecker tensor is given by

$$\delta^{rs} = g^{st} \delta_t^r \text{ (sum on } t\text{)}. \quad (2-4)$$

Therefore, in cylindrical coordinates

$$\delta^{\theta\theta} = g^{\theta\theta} \delta_\theta^\theta \text{ (unsummed)} = 1/r^2. \quad (2-5)$$

The covariant component of a vector cross product is found from

$$(\bar{A} \times \bar{B})_i = \epsilon_{ijk} A^j B^k, \quad (2-6)$$

where ϵ_{ijk} is the absolute thrice-covariant permutation tensor:

$$\epsilon_{r\theta z} = r \\ \epsilon_{rz\theta} = -r, \text{ etc., in cylindrical coordinates.} \quad (2-7)$$

In cylindrical coordinates the contravariant position vector ℓ^k has components

$$\ell^r = r, \ell^\theta = z, \ell^0 = 0 \quad (2-8)$$

and, in general,

$$\ell_{,j}^i = \delta_j^i \text{ (Ref. 2),} \quad (2-9)$$

where the comma denotes the operation of covariant differentiation.

The divergence theorem of vector analysis takes the form

$$\iiint_{\text{volume}} B_{,\alpha}^\alpha dV = \iint_{\text{surface}} B^\alpha \nu_\alpha dS, \quad (2-10)$$

where B^α is an arbitrary contravariant vector and ν_α is the covariant unit surface normal vector. (In order to have an equivalent theorem for a tensor of order 2 or greater, a more elaborate formulation is called for in which the differential elements of volume and surface are assigned a tensorial character. The above form, however, will suffice for our purposes.)

The magnetic stress tensor we take as a doubly-contravariant tensor, $T^{i\alpha}$, defined so that the $\bar{j} \times \bar{B}$ body force, $\bar{f}$, per unit volume is given by

$$(\bar{j} \times \bar{B})^i = f^i = T^{i\alpha}_{,\alpha}. \quad (2-11)$$

In Cartesian coordinates the stress tensor is given by

$$\mu_0 T^{rs} = B^{rs} - \delta^{rs} \frac{B^2}{2}, \quad (2-12)$$

where B^2 is the square of the magnitude of the physical vector $\bar{B}$. This is a proper tensor expression, and it has the same form in all coordinate systems. In cylindrical coordinates it is expressly given by

$$\mu_0 T^{rs} = \begin{pmatrix} B^r B^r - B^2/2 & B^r B^\theta & B^r B^z \\ B^\theta B^r & B^\theta B^\theta - B^2/2r^2 & B^\theta B^z \\ B^z B^r & B^z B^\theta & B^z B^z - B^2/2 \end{pmatrix} \quad (2-13)$$

We now utilize the above results to obtain a general result for components resulting from $\bar{j} \times \bar{B}$ interactions throughout a volume. Imagine a parallel unit vector field; i.e., to every point in space we attach a unit magnitude vector, λ_i , so that all such vectors are parallel in the geometric sense. The fundamental property of such a field is that its covariant derivative vanishes:

$$\lambda_{i,\alpha} = 0. \quad (2-14)$$

The body force per unit volume, by definition of the magnetic stress tensor, is given by

$$f^k = (T^{k\alpha})_{,\alpha}. \quad (2-15)$$

The force in the direction of the unit vector $\bar{\lambda}$ is then

$$\bar{f} \cdot \bar{\lambda} = (T^{k\alpha})_{,\alpha} \lambda_k. \quad (2-16)$$

Because $\lambda_{k,\alpha} = 0$,

$$(T^{k\alpha})_{,\alpha} \lambda_k = (\lambda_k T^{k\alpha})_{,\alpha}. \quad (2-17)$$

Therefore the total force throughout the volume in the $\bar{\lambda}$ -direction is

$$\iiint_V (\lambda_k T^{k\alpha})_{,\alpha} dV = \iint_S \lambda_k T^{k\alpha} \nu_\alpha dS, \quad (2-18)$$

where the right-hand side follows from the

divergence theorem. We are primarily interested in the axial (thrust) force component, so $\lambda_z = 1$, $\lambda_r = \lambda_\theta = 0$, and

$$\bar{F}_z = \iint_S T^{z\alpha} v_\alpha ds. \quad (2-19)$$

In a similar manner we consider the total torque on a volume. The torque about the origin due to the local body force $\bar{f}$ is given by

$$\bar{t} = \bar{r} \times \bar{f} = \epsilon_{ijk} \ell^j (T^{k\alpha})_{,\alpha}. \quad (2-20)$$

Therefore, the component of torque along a unit parallel vector field λ^i is

$$\bar{t}_\lambda = \epsilon_{ijk} \lambda^i \ell^j (T^{k\alpha})_{,\alpha}. \quad (2-21)$$

Thus, the total λ -component of torque is given by

$$\bar{N}_\lambda = \iiint_V \epsilon_{ijk} \lambda^i \ell^j (T^{k\alpha})_{,\alpha} dV. \quad (2-22)$$

Now, consider the quantity

$$[\epsilon_{ijk} \lambda^i \ell^j T^{k\alpha}]_{,\alpha}. \quad (2-23)$$

Expanding (2-23) we have

$$\begin{aligned} & \epsilon_{ijk,\alpha} \lambda^i \ell^j T^{k\alpha} + \epsilon_{ijk} \lambda^i_{,\alpha} \ell^j T^{k\alpha} \\ & + \epsilon_{ijk} \lambda^i \ell^j_{,\alpha} T^{k\alpha} + \epsilon_{ijk} \lambda^i \ell^j T^{k\alpha}_{,\alpha}. \end{aligned} \quad (2-24)$$

Because $\epsilon_{ijk,\alpha} = 0$ (a property of the absolute permutation tensor), the first term vanishes. The second term vanishes because $\lambda^i_{,\alpha} = 0$ (by definition of a parallel vector field). We know that $\ell^j_{,\alpha} = \delta_\alpha^j$; therefore, the third term is

$$\epsilon_{ijk} \lambda^i \delta_\alpha^j T^{k\alpha} = \epsilon_{ijk} \lambda^i T^{kj}. \quad (2-25)$$

Now, write the summations explicitly:

$$\sum_i \left(\sum_{j,k} \epsilon_{ijk} \lambda^i T^{kj} \right). \quad (2-26)$$

Consider the sum over j and k :

$$\sum_{j,k} \epsilon_{ijk} T^{kj} = \frac{1}{2} \epsilon_{ijk} T^{kj} + \frac{1}{2} \epsilon_{ikj} T^{jk}, \quad (2-27)$$

since j and k are dummy indices and the summation is invariant under interchange of dummy indices. Because $\epsilon_{ijk} = -\epsilon_{ikj}$, expression (2-26) becomes

$$\frac{1}{2} \epsilon_{ijk} (T^{kj} - T^{jk}), \quad (2-28)$$

which is zero because the magnetic stress tensor is symmetric: $T^{jk} = T^{kj}$. Thus the third term above vanishes, and we have

$$\epsilon_{ijk} \lambda^i \ell^j T^{k\alpha}_{,\alpha} = [\epsilon_{ijk} \lambda^i \ell^j T^{k\alpha}]_{,\alpha}. \quad (2-29)$$

The volume integral is now

$$\iiint_V [\epsilon_{ijk} \lambda^i \ell^j T^{k\alpha}]_{,\alpha} dV = \iint_S [\epsilon_{ijk} \lambda^i \ell^j T^{k\alpha}] v_\alpha ds, \quad (2-30)$$

where the R.H.S. follows from the divergence theorem.

We are interested in the torque about the axis, so $\lambda^z = 1$, $\lambda^r = \lambda^\theta = 0$. The integral is then

$$\iint_S \epsilon_{zjk} \ell^j T^{k\alpha} v_\alpha ds. \quad (2-31)$$

Summing the square-bracket term in the integrand over all values of j and k , we find that

$$\sum_{j,k} [\] = \epsilon_{zr\theta} \ell^r T^{\theta\alpha} + \epsilon_{z\theta r} \ell^\theta T^{r\alpha}. \quad (2-32)$$

But $\ell^r = r$, $\ell^\theta = 0$, and $\epsilon_{zr\theta} = \epsilon_{r\theta z} = r$; therefore, the final result for the total axial component of torque is

$$\bar{N}_z = \iint_S r^2 T^{\theta\alpha} v_\alpha ds. \quad (2-33)$$

Note that from Eqs. (2-19) and (2-33) a complete knowledge of the magnetic fields over the bounding surface of the discharge volume yields the axial force and torque in the discharge volume due to $\bar{j} \times \bar{B}$ interactions. For the situation depicted in Fig. 1, a schematic coaxial arc jet carrying current I from an anode at radius R to a cathode of radius r_c , the broken line indicates an appropriate bounding surface. If we assume complete azimuthal symmetry, then the only information required is the magnetic field components along radii of the cylinder ends, (1) and (3) in Fig. 1, and along a line extending the length of the surface (2). In the absence of magnetic-field measurements, we are able to calculate the $(\bar{j} \times \bar{B})_z$ thrust for a self-field accelerator and an approximation to the angular momentum input in an applied-field device.

Consider the situations in Fig. 1, with no external field and azimuthal symmetry. The magnetic field is only $\bar{B}_\theta(r, z)$. The only nonzero $T^{z\alpha}$ is T^{zz} , so there is no contribution to the integral (2-19) from surface (2). The magnetic field vanishes on surface (3), and we have

$$\bar{F}_z = \iint_1 T^{z\alpha} v_\alpha ds. \quad (2-34)$$

On surface (1), $v_\alpha = v_z = -1$, and $T^{zz} = -B_\theta^2/2\mu_0$ from Eq. (2-13); thus,

$$\begin{aligned} \bar{F}_z &= \int_0^R \left(\frac{B_\theta^2}{2\mu_0} \right) 2\pi r dr \\ &= \frac{\pi}{\mu_0} \left[\int_0^{r_c} r B_\theta^2 dr + \int_{r_c}^R r B_\theta^2 dr \right]. \end{aligned}$$

Fig. 1. Schematic of coaxial MPD accelerator.

In the interval r_c to R , $\bar{B}_\theta = -(\mu_0 I / 2\pi r)$; therefore,

$$\frac{\pi}{\mu_0} \int_{r_c}^R r \bar{B}_\theta^2 dr = \frac{\mu_0 I^2}{4\pi} \ln \frac{R}{r_c}. \quad (2-35)$$

If the current is imagined to flow into the cathode in a thin annulus, then the axial force is correctly given by Eq. 2-35. If the current flows into the cathode with uniform density, then

$$\bar{B}_\theta = \frac{\mu_0 I}{2\pi r} \left(\frac{r}{r_c} \right)^2; \quad 0 \leq r \leq r_c \quad (2-36)$$

and

$$\frac{\pi}{\mu_0} \int_0^{r_c} r \bar{B}_\theta^2 dr = \frac{\mu_0 I^2}{4\pi} \left(\frac{1}{4} \right). \quad (2-37)$$

The total force is therefore

$$\bar{F}_z = \frac{\mu_0 I^2}{4\pi} \left[\ln \frac{R}{r_c} + \frac{1}{4} \right]. \quad (2-38)$$

It should be emphasized that the above result assumes only azimuthal symmetry and uniform cathode current density, and the result is otherwise completely independent of the detailed current distribution within the discharge volume.

For the evaluation of torque in the applied-field case, we make the rather severe assumption

that the magnetic field has only a z-component and is uniform throughout the volume; i.e., $\bar{B} = B_0 \hat{e}_z$. Thus, we neglect the radial component of the applied field, the variation of the axial component, and the radial and axial field components caused by possible $\bar{j}_\theta$ currents. We wish to evaluate the integral in Eq. (2-33). On surface (2) $v_\alpha = v_r$, but $T^{\theta r} = 0$ since $B^r = 0$ and there is no contribution to the integral. Similarly, on surface (3) $v_\alpha = v_z$, but $T^{\theta z} = 0$ since $B^\theta = 0$ on this surface. Thus

$$\bar{N}_z = \iint_1 r^2 T^{\theta z} v_z dS. \quad (2-39)$$

Now $v_z = -1$, and axial symmetry reduces the problem to an integration over r :

$$\bar{N}_z = \frac{2\pi}{\mu_0} \int_0^R -r^3 B^\theta B^z dr. \quad (2-40)$$

Recall that $B^z = \bar{B}_z = B_0$ and $B^\theta = \bar{B}_\theta / r$. For $r_c \leq r \leq R$, $\bar{B}_\theta = -\mu_0 I / 2\pi r$, so there is a contribution

$$\bar{N}_z = \frac{2\pi}{\mu_0} B_0 \left(\frac{\mu_0 I}{2\pi} \right) \int_{r_c}^R r dr = \frac{B_0 I R^2}{2} \left[1 - \frac{r_c^2}{R^2} \right]. \quad (2-41)$$

If we again assume uniform current density over the

cathode, there is an additional contribution $IB_0 r_c^2/4$; the net torque then becomes

$$\bar{N}_z = \frac{B_0 I R^2}{2} \left[1 - 3/4 \frac{r_c^2}{R^2} \right]. \quad (2-42)$$

If we now admit that there is a positive $\bar{B}_r$ component (diverging field), there will be a contribution to the integral in Eq. (2-33) opposite in sign to (2-41). It is worth noting that the sign of $\bar{N}_z$ depends on the sign of the product $(B_0 I)$, and that the torque varies as the square of the characteristic dimension R .

3. ROTATING PLASMA MODEL FOR APPLIED FIELD MAGNETO-PLASMA DYNAMIC ARCS

We have seen from electrodynamic considerations that there must be an angular momentum input, and hence a rotation of the mass in the discharge volume, in an applied-field accelerator. A possible thrust mechanism for applied-field accelerators is the conversion of the kinetic energy of rotation to kinetic energy of axial motion by expansion in a magnetic nozzle. This mechanism has been mentioned by Ellis³ and Hess *et al.*;⁴ here we develop the consequences under some simplifying assumptions.

Consider a rotating cylindrical plasma element of length l and uniform density ρ . The fluid angular momentum is

$$P_\theta = \rho l \int_0^R r v(r) 2\pi r dr. \quad (3-1)$$

If we further assume that $v(r) = v_\theta$, a constant, then

$$P_\theta = \frac{2}{3} l \rho v_\theta R^2 = \frac{2}{3} M v_\theta R, \quad (3-2)$$

where M is the total mass of the plasma element: $M = \rho \pi R^2 l$. It follows that the kinetic energy of rotation is

$$W_\theta = \frac{1}{2} M v_\theta^2. \quad (3-3)$$

As this rotating mass expands through a magnetic nozzle, to radius R_e , the angular momentum is conserved so that

$$v_{\theta_e} = v_\theta \left(\frac{R}{R_e} \right), \quad (3-4)$$

where the subscript e denotes the exit condition.

The energy converted to translation is

$$W_e = \frac{M}{2} (v_\theta^2 - v_{\theta_e}^2) = \frac{M}{2} v_\theta^2 \left[1 - \left(\frac{R}{R_e} \right)^2 \right], \quad (3-5)$$

and the exhaust velocity in the limit of large R_e/R is given by

$$v_e = v_\theta. \quad (3-6)$$

The torque due to $\bar{j} \times \bar{B}$ interactions is, approximately, given by

$$N = \frac{1}{2} B_0 I R^2, \quad (3-7)$$

and by equating the torque to time rate of change of angular momentum we have

$$N = \frac{d}{dt} \left(\frac{2}{3} M v_\theta R \right) = \frac{2}{3} R \dot{M} v_\theta \quad (3-8)$$

in steady state, where $\dot{M}$ is the mass flow. From the two equations for torque we obtain

$$v_e = \frac{3}{4} B_0 I R / \dot{M}, \quad (3-9)$$

and, therefore, we have the thrust

$$F = \dot{M} v_e = \frac{3}{4} B_0 I R. \quad (3-10)$$

The numerical factor $3/4$ is clearly a consequence of our specific assumptions, but it is not particularly sensitive to the form of $v(r)$; for example, if we assume $v(r) = \omega r$, i.e., "rigid body" rotation instead of constant velocity, the factor $3/4$ is replaced by $\sqrt{2}/2$.

The rotation of the plasma produces a contribution to the arc voltage, by virtue of the back emf:

$$V_r = \int_0^R E_r dr \quad (3-11)$$

$$E_r = v_\theta B_0 = v_e B_0. \quad (3-12)$$

With an additive constant for the electrode sheath drops and ohmic losses, the arc voltage becomes

$$V = V_0 + v_e B_0 R. \quad (3-13)$$

The thrust efficiency is defined as

$$\eta_T = \frac{T^2}{2\dot{M}IV} = \frac{T v_e}{2IV}. \quad (3-14)$$

Clearly, the efficiency is greater if V_0 can be neglected in comparison with $v_e B_0 R$; in that case we obtain

$$\eta_T = \frac{\left(\frac{3}{4} B_0 I R \right) v_e}{2 I v_e B_0 R} = \frac{3}{8}. \quad (3-15)$$

Despite the crudity of the assumptions in the above analysis, the formulae for thrust and voltage are in good agreement with data from other laboratories; 40% is the typical reported value for thrust efficiency. It should be noted, however, that thermal energy of the plasma, as well as the energy of ordered rotation, should be recoverable by expansion through the magnetic nozzle, and increased understanding of the detailed processes should lead to knowledge of how this may be achieved.

4. SELF-FIELD ACCELERATORS

Most research on the MPD arc has been done on applied-field devices. As far as we know, only at the Los Alamos Scientific Laboratory and in West Germany is there active interest in the coaxial self-field accelerator for space propulsion. Our remarks here are accordingly somewhat more parochial in nature than in the preceding sections.

The German work is associated with H. Hügel, G. Kruehle, and Th. Peters at the Deutsche Versuchsanstalt für Luft-und Raumfahrt Institut für Plasmadynamik (DVL), Stuttgart-Vaihingen.⁵ Their approach establishes an expanding supersonic arc-heated flow between a rod cathode tip and a downstream ring anode. Until recently, this work has been limited to 1000 amp at relatively high mass density, so electrodynamic interaction did not dominate the flow process. However, as an electro-thermal arc jet, the concept offers improvement over the usual process of heating followed by adiabatic expansion since, in theory, unlimited exhaust velocity can be achieved by heat addition to an expanding supersonic stream.

Stratton's free-particle model⁶ is at the opposite extreme from the DVL continuum formulation. Unfortunately, it is most likely that an operational device will operate in the collision-significant, yet noncontinuum, regime least susceptible to analysis.

Recall now Eq. (2-38), the thrust expression for the self-field accelerator:

$$T = \frac{\mu_0 I^2}{4\pi} \left(\frac{1}{4} + \ln \frac{R}{r_c} \right) \quad (2-38)$$

The thrust depends only on total arc current and the ratio of the electrode dimensions; neither mass flow nor absolute size is involved. However, we expect scale effects to manifest themselves in such details of the arc as anode and cathode sheaths and ionization kinetics, which presumably depend on particle density, the distribution of particle velocities in the beam, and the portion of mass flow effectively utilized. Such effects would then affect performance through anode and cathode power losses and the effective specific impulse achieved. Nevertheless, the problem of electrode erosion and energy transfer must be met in any MPD accelerator, and it would be most helpful if overall accelerator performance proved to be relatively independent of size. With

this independence the size could be increased to reduce the energy flux per unit area to the electrodes.

Arc-cathode current densities vary from 10^3 to 10^7 amp/cm²; such a wide range would, for a given current, have a strong effect on thrust through the variation of the effective cathode radius, r_c , in Eq. (2-38). The very high densities are generally associated with unstable cathode spots. We can expect perhaps 10^4 amp/cm² for a steady arc, though every effort should be made to operate in a stable, high current-density mode.

Because of the relatively unambiguous thrust mechanism of the self-field arc, in comparison with applied-field accelerators, the self-field device should be susceptible to empirical development. That is, if we can produce a stable, symmetric arc at several thousand amperes, we are guaranteed a useful amount of thrust. The effective specific impulse then is determined by how much the mass flow can be reduced while still maintaining the arc and avoiding electrode erosion and recirculation of gas in the test tank. The DVL group uses a tank pressure of several hundred microns, but they believe that recirculation is avoided by the general principle of no upstream propagation in a supersonic flow. We use condensable (lithium) propellant, with a noncondensable background pressure $< 10^{-6}$ mm; pumping is aided by the gettering action of the lithium deposited on the tank walls. Since we find it desirable to start arcs with gas, prior to introducing lithium, we should be able to explore the gas-arc situation as well as metal-vapor operation.

The questions of performance must be resolved by testing in a very high capacity facility; NASA Lewis Laboratory has a 25- x 70-ft vacuum chamber, originally built for cesium ion-engine testing and, therefore, lithium-compatible. This chamber has a 1 MW beam stop, instrumentation access ports, probe traverses, and a digital data recording system, together with adequate high-current dc power. Ultimately, orbital testing should be recognized as being simpler, more definitive, and probably cheaper.

A direct measurement of beam exhaust velocity would be most valuable. We hope to accomplish this with a Fabry-Perot Doppler-shift measurement.

Direct measurement of thrust, in principle straightforward, poses severe problems: a thrust stand of a few grams sensitivity is required, while several thousand amperes of current and many kilowatts of cooling capacity must be provided." But even without a direct measurement of thrust or exhaust velocity, valuable information can be gleaned from gross electrical and calorimetric measurements.

If R/r_c is independent of total current I , then the thrust equation becomes

$$T = KI^2 = \dot{m} v \quad (4-1)$$

Defining the effective exhaust velocity U so that

$$U = \frac{KI^2}{\dot{m}} \quad (4-2)$$

we see that the lower limit of $\dot{m}$ for stable non-eroding arc operation must be investigated. In a low-density regime, it seems a reasonable hypothesis that the current be equated to the mass flow rate expressed in atoms per second, assuming singly-charged ions. Thus $\dot{m} = \gamma I$. Then

$$U = \frac{K}{\gamma} I, \quad (4-3)$$

and neglecting all losses we equate beam power to arc power:

$$IV = \frac{T^2}{2\dot{m}} = \frac{K^2 I^3}{2\gamma}; \quad V = \left(\frac{K^2}{2\gamma}\right) I^2. \quad (4-4)$$

If we now lump all losses into a dissipation term, D , it follows that

$$V = \left(\frac{K^2}{2\gamma}\right) I^2 + \frac{D}{I}, \quad (4-5)$$

where we must recognize that D is certainly a function of I . A portion of D , D_E , can be measured directly by electrode calorimetry, the rest, D_B , is accounted for by nondirected beam energy.

$$\frac{dV}{dI} = \frac{K^2 I}{\gamma} + \frac{d}{dI} \left(\frac{D_E}{I} \right) + \frac{d}{dI} \left(\frac{D_B}{I} \right). \quad (4-6)$$

For efficient operation the first term should dominate, and this point bears investigating by mapping the V - I characteristic of the device while holding $\dot{m} = \gamma I$.

Let us take a concrete example. We assume $I = 10,000$ amp, mass flow equal to the equivalent current, cathode current density $= 10^4/\pi$ amp/cm² so that $r_c = 1$ cm, and we assume $R = 25$ cm, then

$$K = \frac{\mu_0}{4\pi} \left(\frac{3}{4} + \ln 25 \right) \quad (4-7a)$$

$$\approx 4 \times 10^{-7} \text{ newtons/amp}^2$$

$$T = KI^2 = 40 \text{ newtons} \approx 4000 \text{ grams}.$$

For lithium, 1 mg/sec corresponds to 13.9 amp,

For lithium, 1 mg/sec corresponds to 13.9 amp,

$$\lambda = \frac{10^{-6}}{13.9} = 7.2 \times 10^{-8} \text{ kg/amp-sec} \quad (4-7b)$$

$$\dot{m} = 0.72 \text{ g/sec} = 2.6 \text{ kg/hr}.$$

Without losses:

$$U = \frac{4 \times 10^{-7}}{7.2 \times 10^{-8}} \times 10^4 \left(\frac{\text{newtons}}{\text{amp}^2} \right) \left(\frac{\text{amp-sec}}{\text{kg}} \right) (\text{amp}) \quad (4-7c)$$

$$= 5.56 \times 10^4 \text{ meters/sec}$$

$$\approx 5700 \text{ seconds } I_{sp}.$$

$$V = \frac{16 \times 10^{-14} \times 10^8}{2 \times 7.2 \times 10^{-8}} = 111 \text{ volts} \quad (4-7d)$$

$$\text{Power} = 1.11 \text{ megawatts}. \quad (4-7e)$$

We can expect an anode dissipation of about 10 W/amp, or 100 kW total which is equal to 640 W/cm of anode perimeter. Thus, if the anode attachment is diffused over 5 mm, the anode power density is of the order 1.5 kW/cm². So far, we have been reluctant to exceed a few hundred W/cm² using a NaK-cooled, thin stainless steel anode. However, in high-power microwave practice, dissipation of several kW/cm² is standard, using demineralized water under high pressure (~600 psi) flowing through multiple fine coolant passages in OFHC anodes.

One could hope to reduce the anode cooling requirements, and the overall losses, by accomplishing the vaporization and ionization of the lithium at the anode. And it would be most desirable to develop a radiation-cooled heat pipe anode. Such an advance in heat pipes, involving a novel geometry and high thermal flux, would enormously simplify a thrust stand by removing the anode cooling loop, and it would be an important step towards a flight-ratable thruster.

5. EFFICIENCY

We are indebted to Professor R. G. Jahn of Princeton University for the use of his unpublished analysis of the efficiencies of idealized one-dimensional linear accelerators.⁷ While the cases considered are at first glance far removed from the presumably noncontinuum, coaxial devices discussed previously, the points of similarity are suggestive enough to warrant their inclusion here.

Consider the situation shown in Fig. 2: a one-dimensional channel of length L and unit area, across which are applied a uniform electric field, E , and a uniform magnetic field B perpendicular to

Fig. 2. Ideal linear $\vec{j} \times \vec{B}$ accelerator.

E. A constant mass flow, $\dot{m}$, enters at velocity u_0 , and is accelerated to exit velocity u_e by the interaction of B and the current density j .

We assume scalar conductivity, so that the current is parallel to the electric field. The efficiency can then be written

$$\eta = \frac{\int_0^L u j B \, dx}{\int_0^L j E \, dx}, \quad (5-1)$$

where the numerator represents the integral of the work done throughout the volume, and the denominator is the total power input. Equivalently,

$$\eta = \frac{\dot{m} (u_e^2 - u_0^2)/2}{\int_0^L j E \, dx}. \quad (5-2)$$

Assuming negligible pressure gradient along the duct, the equation of motion is simply

$$\dot{m} \frac{du}{dx} = j B,$$

which enables us to evaluate the total power input

$$\int_0^L j E \, dx = \frac{\dot{m} E}{B} \int_0^L \frac{du}{dx} \, dx = \frac{\dot{m} E}{B} (u_e - u_0). \quad (5-3)$$

The efficiency becomes

$$\eta = \frac{\dot{m} (u_e^2 - u_0^2)/2}{\dot{m} E (u_e - u_0)/B} = \frac{B}{2E} (u_e + u_0). \quad (5-4)$$

It should be noted that this result is independent of the current density $j(x)$. However, if positive acceleration of the flow is to be maintained throughout, then we must have $uB \leq E$, in particular at the exit where u is maximum. Therefore,

$$\eta \leq \frac{1}{2} \left(1 + \frac{u_0}{u_e} \right).$$

Thus the efficiency is 50% for this applied-field accelerator for negligible inlet velocity.

Now consider a similar flow configuration where the magnetic field is not applied from an outside source but arises solely from the current $j(x)$, according to Maxwell's equation

$$\frac{dB}{dx} = -\mu_0 j. \quad (5-5)$$

For the energy input we then have

$$\int_0^L j E \, dx = \frac{-E}{\mu_0} \int_0^L \frac{dB}{dx} \, dx = \frac{EB_0}{\mu_0} = EI. \quad (5-6)$$

The momentum equation is now

$$\dot{m} \frac{du}{dx} = j B = \frac{-B}{\mu_0} \frac{dB}{dx} = \frac{-1}{2\mu_0} \frac{d}{dx} (B^2), \quad (5-7)$$

which integrates directly to

$$u = u_0 + \frac{1}{2\mu_0 \dot{m}} (B_0^2 - B^2) = u_e - \frac{B^2}{2\mu_0 \dot{m}}. \quad (5-8)$$

The efficiency is expressed by

$$\eta^2 = \frac{\mu_0 \dot{m} u_e^3}{8E^2} \left(1 + \frac{u_0}{u_e} \right)^2 \left(1 - \frac{u_0}{u_e} \right), \quad (5-9)$$

where we must still invoke the constraint $uB \leq E$.

However, since $dB/dx < 0$ and $du/dx > 0$, the location of the maximum value of (uB) is not immediately apparent, but it can be found from Eq. (5-8):

$$\frac{d}{dx} (uB) = u_e \frac{dB}{dx} - \frac{3B^2}{2\mu_0 \dot{m}} \frac{dB}{dx} = 0 \text{ at } (uB) \text{ maximum}. \quad (5-10)$$

A star is used to denote quantities at (uB) maximum:

$$(B^*)^2 = \frac{2\mu_0 \dot{m}}{3} u_e \quad (5-11)$$

$$(u^* B^*)^2 = \left[u_e B^* - \frac{(B^*)^3}{2\mu_0 \dot{m}} \right]^2 = \frac{8}{27} \mu_0 \dot{m} u_e^3 \leq E^2. \quad (5-12)$$

Substituting Eq. (5-12) into Eq. (5-9) we obtain

$$\eta^2 \leq \frac{27}{64} \left(1 + \frac{u_0}{u_e} \right)^2 \left(1 - \frac{u_0}{u_e} \right)^{1/2}, \quad (5-13)$$

and, therefore, for negligible inlet velocity

$$\eta \leq \left(\frac{27}{64} \right)^{1/2} \approx 65\%. \quad (5-14)$$

Our knowledge of thrust and efficiency is summarized below.

Applied Field		
Ideal Linear		Coaxial
$T \propto I$	Thrust	$T \propto I$
$\eta = 50\%$	Efficiency	$\eta \approx 50\%$

<u>Self-Field</u>		
<u>Ideal Linear</u>		<u>Coaxial</u>
$T \propto I^2$	Thrust	$T \propto I^2$
$\eta = .65\%$	Efficiency	$\eta = ?$

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