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GENERALIZATION OF INTERNAL CENTRIFUGAL ZONE GROWTH
OF METAL-CERAMIC COMPOSITES:
ONE-DIMENSIONAL MODELS

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ABSTRACT

The electric and temperature fields within inductively heated materials is calculated in order to better understand the essential features of the crystal growth process called Internal Centrifugal Zone Growth (ICZG). A complicated two-dimensional problem is studied via a simple one-dimensional model which applies to infinitely long samples heated by infinitely long induction coils. Two such one-dimensional models are presented in this report. In the first model, all material properties of the sample are assumed to be independent of temperature. The coupled differential equations determining the electric field and temperature distributions within the sample are solved analytically. Resulting profiles of temperature and electromagnetic fields within the sample provide a basis for more sophisticated models. It is found that a sufficiently large surface-to-center temperature difference, needed for ICZG, is favored by small values of k/Lh whereas k is the thermal conductivity of the sample, h is the surface heat transfer coefficient, and L is the smaller of the sample radius or the RF skin depth. Our second model deals with materials (e. g., oxides) whose electrical conductivities are very low at room temperature but increase significantly with temperature. The differential equations for electromagnetic and temperature fields are consequently strongly coupled. These coupled nonlinear differential equations are

solved numerically on a computer by both a power series and a finite difference method. Solutions to the steady state equations display an interesting and perhaps technologically important heating instability phenomenon. For low values of an impressed RF field, there is very little heating of such a sample; whereas, after a critical value of the RF field has been exceeded, the slightly heated sample interacts with the RF field in an unstable manner, resulting in much higher steady state temperatures. Numerical solutions to linearized transient equations confirms this instability phenomenon.

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INTRODUCTION

In the early 1960's, a rather important crystal growth process was developed at Oak Ridge National Laboratories by A. T. Chapman and G. W. Clark. The process, named Internal Centrifugal Zone Growth (ICZG), was originally designed to grow single crystals of UO_2 for reactor fuel cells, but since then has been successfully extended to other oxide, mixed oxide, and oxide-metal systems. The process utilizes radio frequency induction heating to create, sustain, and translate a molten zone within a pressed oxide powder compact. Surface heat losses, predominantly by radiation, keep solid the sample surface which serves as a crucible to the liquid interior, isolating it from any contaminating substances. As the sample passes through the influence of the localized rf field, the internally contained liquid zone moves slowly through the interior of the sample followed by the unidirectional growth of a single crystal (see Figure 1).

Although many different oxide and oxide-metal systems have been successfully processed by ICZG, there have been many failures. The major problem in attempting to centrally melt oxides, MgO for example, is in achieving sufficient internal heating to establish a molten zone; this is largely attributable to the low electrical conductivities of oxides. On the other hand, the conceivable application of the ICZG process to metal powders presents the problem of obtaining surface and center temperatures that are sufficiently

FIGURE 1

Typical ICZG experimental arrangement employed by Chapman and Clark for UO_2 . Pressed and sintered UO_2 powder is caused to melt internally by direct coupling of an RF field through a gap in the molybdenum susceptor. The sample is then rotated and translated, causing crystal growth.

different to maintain a mechanically solid shell needed for containment purposes. By mathematically modeling such internal heating phenomena, we can determine the influence that material properties (e. g., electrical and thermal conductivity) and processing variables (e. g., rf generator frequency, sample size, magnetic field strength, sample pre-heat temperature) have upon internal power production and the variation of the temperature within the sample.

There is a class of problems associated with the actual growth process itself. There is presently little understanding of how the different variables of the process such as the induction coil configuration, rf generator frequency, and sample shape affect the size and shape of the liquid zone and, more important, the degree of perfection of single crystals grown from the melt. The internal temperature distribution, in several different ways, determines the quality of crystals grown. Some specific examples are:

1. In growing eutectic crystals, a flat solid/liquid interface, normal to the growth direction, is necessary to cause rods or lamellar plates to grow parallel to each other; otherwise, dislocations and segmented rod or plate growth (colony structure) results.
2. Unless the solid is cooled slowly, the strains resulting from extremely large thermal gradients can produce dislocations, deformation, or even cracking.

3. The thermal gradient in the melt at the solid/liquid interface determines, in part, if there is morphological instability and hence possibly dendritic solidification.
4. The size and shape of the liquid zone partially determines the degree of thermal mixing in the melt.

By modeling ICZG, we can predict the size, shape, and stability of the molten zone and the thermal gradients throughout the sample. This predictive capacity will be of considerable use in improving the degree of perfection and size of crystals grown from the melt.

An unavoidable consequence of the complexity of this process is that certain phenomena occur in the heated sample that are very difficult to model.

Examples are:

1. Voids in a heated oxide frequently migrate toward regions of higher temperature.
2. When a sintered powder compact which is less than fully dense is internally melted, a central void forms and increases in volume with further melting.
3. There often is extensive plastic deformation due to stresses from thermal gradients, density changes, and internal gas formation.
4. Internal voids in the solid may coalesce, forming cracks and weakening the wall containing the melt.

5. There are probably turbulent convection currents and magnetic stirring in the melt.
6. If a solute is present, the process of zone refining, and more generally, segregation may cause significant changes in electrical and thermal properties.

Incorporating some or all of these effects into our modeling would possibly enable us to understand why ICZG is successful in certain specific materials but not others; however, it would usually make the problem intractable and would not contribute very much to our understanding of the essentials of the process, in general. It is therefore our intention to model somewhat simplified situations that will shed light on the important and essential features of the ICZG process.

The effect of heat losses through sample ends and the localized nature of the external magnetic field are obviously necessary to have the type of internal hot (or molten) zones associated with ICZG; nevertheless, a great deal has been learned about the influence of system and material parameters upon internal heat generation, within the sample, from simple models in which samples, with lengths much greater than their radii, are heated via a very long and uniform induction coil.

This reduces a complex two-dimensional problem (with symmetry about the cylindrical axis) into a much simpler one-dimensional problem in which

the electromagnetic fields and heat transport within the material are fully described in terms of only the radial cylindrical coordinate.

Two such one-dimensional models are presented in this report. The first is a very simple model because the electrical conductivity of the sample material is assumed to be independent of temperature. The second model is far more interesting, because the electrical conductivity of the sample is allowed to vary with temperature in an Arrhenius fashion, i. e., $\exp(-\theta/T)$ where θ is constant, characteristic of an ionic conductivity like that of an oxide. This nonlinear behavior of the electrical conductivity is believed to be a very important feature of this internal heating process for oxides. Work is presently being done to model the more interesting two-dimensional problem in order to study actual zone formation.

A VERY SIMPLE ONE-DIMENSIONAL MODEL

A very simple model of the internal heating phenomena associated with ICZG assumes that material properties are independent of temperature and that the electromagnetic and temperature fields vary only in the radial direction. From this model, we have been able to study the basic skin depth phenomena associated with eddy current heating and the effect that different parameters of the system have upon internal power generation, center to surface temperature difference, and surface temperature. We can also anticipate the type of electromagnetic field and temperature distributions of more sophisticated models.

Whatever the degree of sophistication of the model being considered, our basic objective is to define, in terms of material properties and processing variables, the necessary conditions to establish a stable internal molten zone of suitable size within a material whose surface is cool enough to contain the liquid zone and support the sample as well.

In an actual heating situation there are many competing and coupled processes, but the two that most completely characterize the internal heating phenomena of ICZG are the heat transport and electromagnetic field penetration within the heated sample. Both are describable by differential equations. In the following section the equations are developed and, for the simple model being considered, analytic solutions are presented.

Electromagnetic Fields

The electric and magnetic fields within the material being induction heated are described by partial differential equations which are constructed from the fundamental relationships* of electrodynamics:

$$\text{curl } E = -\frac{\partial B}{\partial t} \quad \text{Faraday's law} \quad (1)$$

$$\text{curl } H = J + \frac{\partial D}{\partial t} \quad \text{Maxwell's form of Ampere's Law} \quad (2)$$

$$J = \sigma E \quad \text{Ohm's Law}$$

$$H = \frac{B}{\mu} \quad \left. \vphantom{\begin{matrix} H = \frac{B}{\mu} \\ D = \epsilon E \end{matrix}} \right\} \text{Constitutive relations} \quad (4)$$

$$D = \epsilon E \quad (5)$$

The wave equation for the electric field, Eq. (6), is derived for constant μ by taking the curl of Eq. (1) and eliminating the magnetic field from that equation with the use of Eq. (2). The wave equation for the magnetic field, Eq. (7),

* All electrodynamic quantities are expressed in MKS units.

- E = electric intensity field.
- B = magnetic induction field
- H = magnetic intensity field
- J = current density
- D = electric displacement field
- σ = electrical conductivity
- μ = magnetic permeability
- ϵ = dielectric constant

is developed in a similar manner by taking the curl of Eq. (2). Both wave equations contain damping terms proportional to the electrical conductivity of the material being induction heated:

$$\text{curl} (\text{curl } \mathbf{E}) + \mu\sigma \frac{\partial \mathbf{E}}{\partial t} + \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \quad (6)$$

$$\text{curl} (\text{curl } \mathbf{B}) + \mu\sigma \frac{\partial \mathbf{B}}{\partial t} + \mu\epsilon \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0 \quad (7)$$

Under steady state conditions, the electric and magnetic fields are harmonically varying functions of time ($\sim e^{i\omega t}$).^{*} With the time derivatives in Eqs. (6) and (7) evaluated, the steady state wave equations are

$$\text{curl} (\text{curl } \mathbf{E}) + (i\mu\sigma\omega - \mu\epsilon\omega^2) \mathbf{E} = 0 \quad (8)$$

$$\text{curl} (\text{curl } \mathbf{B}) + (i\mu\sigma\omega - \mu\epsilon\omega^2) \mathbf{B} = 0 \quad (9)$$

The terms in Eqs. (8) and (9), $\mu\epsilon\omega^2 \left\{ \frac{\mathbf{E}}{\mathbf{B}} \right\}$, arising from displacement currents,[†] are generally negligible compared to $\mu\sigma\omega \left\{ \frac{\mathbf{E}}{\mathbf{B}} \right\}$ and henceforth will be neglected.

* The electric field is represented as a complex quantity, but the actual electric field is

Real $\{ \mathbf{E}(r, t) \}$ which is equivalent to
Real $\{ \mathbf{E}(r) \} \cos \omega t$ - Imaginary $\{ \mathbf{E}(r) \} \sin \omega t$.

† The term $\partial \mathbf{D} / \partial t$ added to Ampere's Law by Maxwell, and called the displacement current, corresponds to the "current" flowing in the space between two charged condenser plates. A displacement current would give rise to electromagnetic radiation.

In general, the electrical conductivity in Eqs. (8) and (9) is a function of temperature, so solutions for the electric and magnetic fields are dependent upon how the temperature varies within the material. Because of this interdependence between the electromagnetic fields and the heat transport within the material, there are no apparent analytic solutions for the fields unless the electrical conductivity is independent of temperature.

By considering only samples that are in a uniform magnetic field and whose lengths are much greater than their radii, the three-dimensional differential equations are effectively reduced to one-dimensional equations, in the radial variable. Later a method is developed to solve for the electric field in the one-dimensional case when the electrical conductivity is a strong function of temperature.

Analytic solutions for the fields, for the simple one-dimensional case where the electrical conductivity is independent of temperature, are given in Eqs. (10) and (11);

$$E(r, t) = (1-i) \frac{\omega \delta B_s J_1 [(1+i)r/\delta] e^{i\omega t}}{2 J_0 [(1+i)R/\delta]} \quad (10)$$

$$B(r, t) = B_s \frac{J_0 [(1+i)r/\delta] e^{i\omega t}}{J_0 [(1+i)R/\delta]} \quad (11)$$

where J_0 and J_1 are Bessel functions.

The analytic solutions for the electric and magnetic fields are expressed in terms of the skin depth δ ($= \sqrt{2/\sigma\mu\omega}$), a parameter that roughly defines the depth to which eddy currents have penetrated into the sample material. The magnetic field at the surface of the material is equal to the externally imposed magnetic field, B_s . The electric field within the material satisfies one of Maxwell's equations, Eq. (1), in the form $\text{curl } E = -i\omega B$, and this equation evaluated at the surface determines the value of the electric field at the surface in terms of B_s .

The total magnetic field within the material, given by Eq. (11), can be thought of as a sum of the externally imposed field and the field due to the flow of eddy currents. In materials with high electrical conductivity like metals, the induced eddy currents flow in a region near the surface of the order of the skin depth and act like a solenoid, creating an induced magnetic field that is uniform over most of the interior of the material and falls to zero at the surface. The sum of the imposed and induced magnetic fields is a field that at the surface is equal to the field outside the material and exponentially decreases to zero at the center. In materials with low electrical conductivity like oxides at low temperature, there is virtually no interaction between the imposed magnetic field and the sample material, and therefore there are only very small induced electric or magnetic fields. Figures 2a and 2b illustrate the variation of the imposed and induced magnetic and electric fields within

FIGURE 2a

Variation of the magnitude of the magnetic fields (imposed, induced, and total) as a function of sample radius, r , for a material with a very large skin depth and one with a very small skin depth. In both cases, the electrical conductivity is constant and the sample is assumed to be a very long circular cylinder heated inductively by a very long solenoid.

Total Field

FIGURE 2b

Variation of the magnitude of the electric fields (imposed, induced, and total) as a function of sample radius, r , for a material with a very large skin depth and one with a very small skin depth. In both cases, the electrical conductivity is constant and the sample is assumed to be a very long circular cylinder heated inductively by a very long solenoid.

each of two materials, one with a large and the other with a small skin depth, but both with the same surface magnetic field.

One would expect that elements of the behavior for both large and small skin depths would be present when the electrical conductivity varies greatly within a sample. Consider, for example, an oxide in which the center is near the melting point and the surface is 1000°C cooler. In the surface region, the electrical conductivity is so small that the electric field is expected to vary approximately in a linear fashion (i. e., large skin depth limit). In the center region, the electrical conductivity is large, possibly as much as 100 times greater than at the surface; the electric field is expected to attenuate rapidly in this region. It will be shown later that actual solutions to the coupled differential equations describing the electric field and heat transport verify that this anticipated behavior is indeed correct.

Internal Temperature Distribution

Knowledge of the electromagnetic fields within the material being induction heated is only of indirect importance in modeling a crystal growth process like ICZG. It is actually the temperature distribution that is of primary interest. The steady state temperature distribution is the solution to the differential equation

$$\nabla \cdot (k \nabla T) + |E|^2 \frac{\sigma}{2} = 0^* \quad (12)$$

which in one-dimensional form is

$$\frac{1}{r} \frac{d}{dr} \left(k r \frac{dT}{dt} \right) + |E|^2 \frac{\sigma}{2} = 0 \quad (13)$$

where $|E|^2 \frac{\sigma}{2}$ is the effective power per unit volume induced within the sample.

The instantaneous power per unit volume is equal to $(\text{Real } \{E\})^2$; yet, because the material has considerable thermal inertia, the effective power per unit volume is $|E|^2 \frac{\sigma}{2}$ which is a time average of the instantaneous power.

If the induction heating supplies power preferentially to the inside of the sample, an internal zone is all the more likely to be achieved. In order to contain a molten zone within a solid, it is clear that external heat losses must be depended on to keep the outer portions of the sample solid. The most important loss mechanisms are surface radiation (possibly also convection at lower temperatures) and conduction through cooler parts of the sample (perhaps eventually to sample supports).

Analytic solutions to Eq. (13) (for σ and k independent of temperature) for two materials, one with large and the other small skin depth, are

* k is the thermal conductivity of the material.

$$T(r) = T|_{r=R} + \frac{B_s^2 R^4 \omega^2 \sigma}{128k} \left(1 - \frac{r^4}{R^4}\right) \quad \delta \gg R$$

$$T(r) = T|_{r=R} + \frac{B_s^2 \delta^4 \omega^2 \sigma}{16k} \left(1 - e^{2(r-R)/\delta}\right) \quad \delta \ll R$$

These solutions are displayed in Figure 3.

The surface temperature, $T|_{r=R}$, is determined by surface heat losses. A linearized cooling law is used to account for both surface radiation and convection losses: $-k \frac{dT}{dr} = h(T - T_{\text{ambient}})$ evaluated at $r=R$.

With T at $r=R$ calculated, the equations for the two limiting behaviors become

$$T(r) = T_{\text{ambient}} + \frac{B_s^2 R^4 \omega^2 \sigma}{128k} \left(\frac{4k}{Rh} + 1 - \frac{r^4}{R^4}\right) \quad \delta \gg R$$

$$T(r) = T_{\text{ambient}} + \frac{B_s^2 \delta^4 \omega^2 \sigma}{16k} \left(\frac{2k}{\delta h} + 1 - e^{2(r-R)/\delta}\right) \quad \delta \ll R$$

In either limiting case, if the term $\frac{4k}{Rh}$ or $\frac{2k}{\delta h}$ is large compared to 1, the center to surface temperature difference is small. This suggests that it would be difficult to centrally melt a sample that has a large thermal conductivity and/or a small radius or skin depth.

FIGURE 3

Power per unit volume, P_v , and steady state temperature, T , produced by induction heating of a material with a very large skin depth and one with a very small skin depth. All material properties are assumed to be constant and, in both cases, equal amounts of power are generated internally.

Dimensionless Coupled Differential Equations

The differential equations for the electric field and temperature distribution (in one-dimension with constant thermal conductivity) in dimensionless form are

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{dT}{dr} \right) + \frac{\Pi}{2} |E|^2 \sigma = 0 \quad (14)$$

$$\frac{d}{dr} \frac{1}{r} \left(\frac{d}{dr} rE \right) + 2i \Lambda \sigma E = 0 \quad (15)$$

Dimensionless variables in terms of dimensional variables are

$$r = \frac{r}{R}$$

$$T = \frac{T}{T_M}, \quad T_M \text{ is the melting temperature measured in degrees Kelvin}$$

$$E = \frac{E}{B_s \omega R}$$

$$\sigma = \frac{\sigma}{\sigma_{T_M}}, \quad \sigma_{T_M} \text{ is } \sigma \text{ of the solid at } T_M$$

$$\Pi = \frac{B_s^2 R^4 \omega^2 \sigma_{T_M}}{k T_M}$$

$$\Lambda = \mu \omega R^2 \sigma_{T_M}$$

For materials whose electrical conductivity is ionic in nature, for example oxides, the electrical conductivity is a very strong function of

temperature and can vary considerably within the solid. The differential equations determining the temperature and the electric field, Eqs. (14) and (15), are strongly coupled through the temperature dependent electrical conductivity, so they must be solved simultaneously.

A MODEL WITH TEMPERATURE DEPENDENT ELECTRICAL CONDUCTIVITY

There are several numerical methods that have been successful in solving the coupled equations, but a power series method produced the fastest computer solution (about 1/2 second execution time on the Univac 1108 for each simultaneous solution). The theoretical justification for using a power series method is based upon Frobenius' theorem* which says that Eqs. (14) and (15) are of the proper form to have power series solutions.

In solving Eqs. (14) and (15) using a power series method, all quantities which vary with temperature and position are expressed as power series in position. The undetermined coefficients in the power series are evaluated from recursion formulas developed from the differential equations. The recursion formulas express all the coefficients in terms of the first term in each of the power series for the temperature and electric field, and these two terms are evaluated by applying the boundary conditions. Because the algebraic equations determining the coefficients are so complicated, solution by computer is necessary. A detailed development of this method of solution follows.

* Frobenius' theorem says that any differential equation of the form

$$y'' + \frac{a(x)}{x} y' + \frac{b(x)}{x^2} y = 0, \text{ where the functions } a(x) \text{ and } b(x) \text{ are analytic}$$

at $x = 0$, has at least one solution which can be presented in the form

$$y(x) = x^a \sum_{m=0}^{\infty} b_m x^m \text{ where the exponent } a \text{ may be any real or complex}$$

number (and is chosen so b_0 is not equal to zero).

The electric field is an odd function of r and its power series is represented as $\sum_{j=1,3,5,\dots} A_j r^j$. The temperature distribution is an even function of r and its power series is represented as $\sum_{j=0,2,4,\dots} B_j r^j$. An electrical conductivity with some general functional dependence upon temperature can be expressed as a Taylor series about the center temperature of the rod.

$$\sigma = \sum_{n=0,1,2,3,\dots} \left. \frac{d^n \sigma}{dT^n} \right|_{T \text{ at } r=0} \frac{(T - T \text{ at } r=0)^n}{n!} \quad (16)$$

Because the temperature is represented as a power series in r , the electrical conductivity can be represented as a power series in r in terms of the B_j 's:

$$\sigma = \sum_{n=0,1,2,\dots} \left. \frac{d^n \sigma}{dT^n} \right|_{B_0} \frac{\left(\sum_{i=2,4,6,\dots} B_i r^i \right)^n}{n!} = \sum_{n=0,1,2,\dots} C_j r^j \quad (17)$$

Expansion of this complicated expression for a few terms and collection of like powers of r yields

$$\begin{aligned} \sigma = & \sigma|_{B_0} + \left. \frac{d\sigma}{dT} \right|_{B_0} B_2 r^2 + \left(\left. \frac{d\sigma}{dT} \right|_{B_0} B_4 + \frac{1}{2} \left. \frac{d^2\sigma}{dT^2} \right|_{B_0} B_2^2 \right) r^4 \\ & + \left(\left. \frac{d\sigma}{dT} \right|_{B_0} B_6 + \left. \frac{d^2\sigma}{dT^2} \right|_{B_0} B_2 B_4 + \frac{1}{6} \left. \frac{d^3\sigma}{dT^3} \right|_{B_0} B_2^3 \right) r^6 \end{aligned} \quad (18)$$

+ . . .

A computer program has been developed that performs the double expansions in Eq. (17) and groups the terms of like powers, up to large powers of r . Using the power series for the temperature, electric field, and electrical conductivity, Eqs. (14) and (15) transform into

$$\sum_j j^2 B_j r^{j-2} + \frac{\pi}{2} (\sum_j C_j r^j) (\sum_j A_j^* r^j) (\sum_j A_j r^j) = 0 \quad (19)$$

$$\sum_j (j^2 - 1) A_j r^{j-2} + 2i \Lambda (\sum_j A_j r^j) (\sum_j C_j r^j) = 0 \quad (20)$$

Expansion of Eqs. (19) and (20) in terms of like powers of r is for Eq. (19)

$$\begin{aligned} & r^0 \left[4 B_2 + \frac{\pi}{2} \cdot 0 \right] + r^2 \left[16 B_4 + \frac{\pi}{2} C_0 A_1^* A_1 \right] + \\ & r^4 \left[36 B_6 + \frac{\pi}{2} C_0 (A_1 A_3^* + A_3 A_1^*) + \frac{\pi}{2} C_2 A_1^* A_1 \right] \\ & + \dots = 0 \end{aligned} \quad (21)$$

and for Eq. (20)

$$\begin{aligned} & r^1 \left[8 A_3 + 2i \Lambda C_0 A_1 \right] + r^3 \left[24 A_5 + 2i \Lambda (C_0 A_3 + C_2 A_1) \right] + \\ & r^5 \left[48 A_7 + 2i \Lambda (C_0 A_5 + C_2 A_3 + C_4 A_1) \right] + \dots = 0 \end{aligned} \quad (22)$$

The A's and B's are independent of r ; therefore, the sum of the coefficients of each power of r in Eqs. (21) and (22) must be zero in order that the two power series in r are identically zero.

Thus, from Eq. (21)

$$B_2 = 0$$

$$B_4 = \frac{-\Pi}{32} C_0 A_1^* A_1$$

$$B_6 = \frac{-\Pi}{72} C_0 (A_1 A_3^* + A_1^* A_3) - \frac{\Pi}{72} C_2 A_1^* A_1$$

etc.

and, from Eq. (22)

$$A_3 = \frac{-i\Lambda}{4} C_0 A_1$$

$$A_5 = \frac{-i\Lambda}{12} (C_0 A_3 + C_2 A_1)$$

$$A_7 = \frac{-i\Lambda}{24} (C_0 A_5 + C_2 A_3 + C_4 A_1)$$

etc.

From Eqs. (8) and (9), it can be seen that the general recursion formulas for A and B are

$$A_j = \frac{-2i\Lambda}{j^2 - 1} \sum_{k=0, 2, \dots}^{j-3} C_k A_{j-2-k} \quad (23)$$

$$B_j = \frac{-\Pi}{2j^2} \sum_{k=0,2,\dots}^{j-4} C_k \left(\sum_{m=1,3,\dots}^{j-3} \text{Real} \{A_m\} \text{Real} \{A_{j-2-k-m}\} + \right. \\ \left. \text{Imaginary} \{A_m\} \text{Imaginary} \{A_{j-2-k-m}\} \right) \quad (24)$$

C_k is a non-linear function of all the B's, orders k and below (see Eq. (18)).

Because the recursion formulas relate all the A's and B's to preceding A's and B's, all the A's and B's are in terms of non-linear combinations of A_1 and B_0 . A_1 and B_0 are not determined by the differential equations but rather by the boundary conditions. If radiation is the principal loss mechanism, the boundary condition for the temperature differential equation is, in dimensional form, $-k \frac{dT}{dr} = \alpha (T^4 - T_{\text{ambient}}^4)$ which in dimensionless form is $-\frac{dT}{dr} = \Gamma (T_A^4 - T_A^4)$ at $r=1$ where

$$\alpha = (\text{emissivity}) (\text{Stefan-Boltzmann constant})$$

$$\Gamma = \alpha T_M^3 R/k$$

$$T_A = T_{\text{ambient}}/T_M$$

In terms of power series coefficients, the boundary condition is

$$\sum_{j=2,4,\dots} j B_j + \Gamma \left[\left(\sum_{j=0,2,4,\dots} B_j \right)^4 - T_A^4 \right] = 0 \quad (25)$$

This is a non-linear equation in B_0 and A_1 (all the B_j 's are reducible to expressions in B_0 and A_1).

The boundary condition for the electric field differential equation is, in dimensional form, $\frac{dE}{dr} + \frac{E}{r} = \omega B_s$ evaluated at $r=R$ which, in dimensionless form, is $\frac{dE}{dr} + E = 1$ at $r = 1$. In terms of A_j 's, it is

$$\sum_{j=1, 3, 5, \dots} A_j (1+j) = 1 \quad (26)$$

which is also a nonlinear equation in B_0 and A_1 .

These two coupled nonlinear algebraic equations determine B_0 and A_1 . Thus, the power series method reduces the problem of solving two coupled differential equations to a much simpler problem of solving two coupled algebraic equations.

A successful iteration scheme has been developed whereby successive guesses for B_0 and A_1 are made until the boundary conditions, Eqs. (25) and (26), are satisfied. With an initial guess for B_0 and A_1 , the recursion formulas, Eqs. (23) and (24), are used to solve for other A_j 's and B_j 's. Generally, the boundary conditions are not satisfied for the initial guess for B_0 and A_1 , so a second guess for B_0 and A_1 is made and the process repeated. Again, it is unlikely that the boundary conditions are satisfied. The third guess for B_0 and A_1 , however, can be based upon the knowledge of how satisfactory the previous two guesses were in attempting to satisfy the boundary conditions and usually proves to be better than the previous two. For example, if with the first B_0 , the temperature boundary condition is poorly satisfied,

but with the second B_0 , the boundary condition is nearly satisfied, linear interpolation chooses the third B_0 to be closer to the second B_0 than the first. The effect of A_1 upon the temperature boundary condition or the effect of B_0 upon the electric field boundary condition is not considered. More guesses are made in the same manner until the boundary conditions are satisfied. Usually five to ten guesses are all that are needed. Figure 5 shows a flow chart of a computer program that very efficiently obtains steady state solutions for the temperature and electric field via the power series method using the iteration scheme mentioned here.

Results

Solving the coupled differential equations by means of the power series method has proved to be one of the most efficient methods of solution when the surface to center temperature difference is small. If more than about twenty-five terms are required in the series expansion for any of the position varying quantities, the power series method becomes very awkward and it's best to use another method of solution (described in Appendix A). Because we now have two very different methods to solve the same differential equations, it has been possible to make a critical comparison of the solutions obtained by the two different methods and, to within the accuracy of each computer program (3 decimal places), both methods give identical solutions.

FIGURE 5

Flow chart of a computer program that solves the coupled differential equations determining the electric field and temperature distributions within a sample that has a temperature dependent electrical conductivity. A power series method is used to solve the differential equations-- Eqs. (14) and (15).

The computer programs have been written for any form of temperature dependent electrical conductivity, but, so far, we have considered only one— $\exp [\theta (1 - 1/T)]$,* the general form of an ionic electrical conductivity like that of oxides, nitrides, and carbides. We have studied θ 's in the range from 7 to 11 corresponding to oxides from UO_2 to MgO with values for other dimensionless parameters, except Π , kept fixed.

Five dimensionless parameters (Π , θ , T_A , Γ , Λ) are needed to specify the problem. However, if the center temperature is less than the melting temperature, temperatures can be scaled in terms of the activation temperature or the ambient temperature and only four dimensionless parameters are needed. In the work we have done so far, θ is varied from 7 to 11, Π is varied over a wide range, $\Gamma = 4.5$, $\Lambda = 100$, $T_A = .6$. Although we have collected very little data, we have discovered something of tremendous potential importance--there are multiple steady states for identical system parameters.

Plotting internal power generation against Π , for different θ 's results in "S" shaped curves (see Figure 6) that show, within certain ranges of Π , that there are as many as three separate steady states with very different

* All dimensionless temperatures are scaled in terms of the melting temperature, T_M ; so, $\theta = \text{activation temperature}/T_M$. The electrical conductivity is in dimensionless form and is equal to one at the melting point.

FIGURE 6

Variation of the internal power generated within an inductively heated material (whose electrical conductivity varies as $\exp [\theta (1 - 1/T)]$) as a function of $B_s^2 \omega^2 R^4 \sigma_{T_M} / k T_M$ for different values of θ . The dotted lines represent the anticipated behavior of data that has not been collected yet. Figures 7a, b, c, and d show the temperature, electric field, electrical conductivity, and power per unit volume distributions for points 1, 2, 3, and 4 indicated on this graph.

temperature and electric field distributions. This is substantiated by both methods for solving the differential equations. Figures 7a, 7b, 7c, and 7d show the temperature, electric field, electrical conductivity, and power per unit volume variations within the sample at different points of the "S" shaped curve for θ equal to 11. These curves are similar to those for θ 's of other values.

There are several important features of these curves. Solutions from the lower branch of the "S" have power per unit volume curves roughly parabolic in shape that do not peak within the sample. Also, d^2T/dr^2 is always negative. At the lower turning point of the "S", near point 2 on Figure 6, power per unit volume curves are horizontal near the sample surface. This marks a transition region between the lower branch and the middle and upper branches of the "S". In the middle and upper branches, the power per unit volume curves peak inside the sample. Corresponding to this phenomenon, d^2T/dr^2 is positive near the sample surface.

It's unlikely that all the multiple steady states are stable solutions. Multiple steady states are often obtained in solving nonlinear differential equations and a stability analysis usually proves that only one is stable and the others are unstable. In the next section, we will investigate the stability of solutions to the coupled differential equations and attempt to define a stability criterion.

FIGURE 7a

Temperature distributions for points 1, 2, 3, and 4 on Figure 6.

FIGURE 7b

Electric field distributions for points 1, 2, 3, and 4 on Figure 6.

FIGURE 7c

Electrical conductivity distributions for points 1, 2, 3, and 4 on Figure 6.

FIGURE 7d

Power per unit volume distributions for points 1, 2, 3, and 4 on Figure 6.

STABILITY ANALYSIS

In the previous section, it was shown that there are multiple steady states to the coupled differential equations which determine the temperature and electric field within the heated sample. The existence of multiple solutions suggests that some of the solutions are unstable with respect to small perturbations in the internal temperature and electric field from their steady state distributions. It cannot be determined a priori which solutions are stable. It's necessary to analyze the coupled differential equations to determine the effect that perturbations have upon the system. If all perturbations decay with time, the steady state is stable; if any perturbation increases with time, the steady state is unstable.

The stability of the solutions is determined by solving the time dependent differential equations which in dimensional form are

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r k \frac{\partial T}{\partial r} \right) + |E|^2 \frac{\sigma}{2} = \rho C_p \frac{\partial T}{\partial t} \quad (27)$$

$$\frac{\partial}{\partial r} \frac{1}{r} \left(\frac{\partial}{\partial r} r E \right) = - \mu \sigma \frac{\partial E}{\partial t} \quad (28)$$

where the time dependent temperature and electric field solutions are equal to their respective steady states plus time dependent perturbations:

$$T(r, t) = T_{ss}(r) + \delta T(r, t)$$

$$E(r, t) = \left[E_{ss}(r) + \delta E(r, t) \right] e^{i\omega t}$$

If $\left(\frac{\partial}{\partial t} \delta T\right) / \delta T$ is greater than zero, the steady state is unstable.

The steady state differential equations are successfully solved by a power series method; in a similar manner Eqs. (27) and (28) are solved. Eqs. (27) and (28) are much more complicated, however, because they involve time differentials. By linearizing the equations and their boundary conditions about the steady state and choosing an appropriate product solution, it's possible to reduce Eqs. (27) and (28) into a system of ordinary differential equations. A similar approach would be to Laplace transform the partial differential equations which would also eliminate the time derivatives. Both methods give solutions for δE and δT of the form $\sum_m \Phi_m(r) \exp(\lambda_m^2 t)$ where λ_m^2 can be either positive or negative. We are interested in determining if any of the λ_m^2 's are positive. If so, the perturbations increase with time, making the steady state unstable. In order to evaluate the λ_m^2 's, the $\Phi_m(r)$'s must be calculated. The power series method previously used to calculate steady state solutions is used to calculate the radial dependence of δE and δT .

Simplification and Linearization

The time derivative of the electric field in Eq. (28) is the derivative of actually two different time dependencies--the $e^{i\omega t}$ harmonic variation and the transient behavior of the electric field perturbation. Eq. (28), with the time derivative of the harmonic time variation already performed, is

$$\frac{\partial}{\partial r} \frac{1}{r} \left[\frac{\partial}{\partial r} r (E_{ss} + \delta E) \right] + i\omega\mu\sigma E_{ss} = -\mu\sigma \left(\frac{\partial \delta E}{\partial t} + i\omega\delta E \right) \quad (29)$$

The material being induction heated has considerable thermal inertia so it is expected that δT changes rather slowly with time. If it were not for the restraining influence of δT in Eq. (29), δE would damp out very rapidly.

The time evolution of δE is closely linked to that of δT through σ ; therefore, $\left| \frac{\partial \delta E}{\partial t} \right|$ is much less than $\omega\delta E$ and can be deleted from Eq. (29). With this simplification, Eqs. (27) and (29), in dimensionless form with constant thermal conductivity, are

$$\frac{1}{r} \frac{\partial}{\partial r} \left[r \frac{\partial}{\partial r} (T_{ss} + \delta T) \right] + \frac{\pi}{2} |E|^2 \sigma = \frac{\partial \delta T}{\partial t}^* \quad (30)$$

$$\frac{\partial}{\partial r} \frac{1}{r} \left[\frac{\partial}{\partial r} r (E_{ss} + \delta E) \right] + 2i\Lambda \sigma (E_{ss} + \delta E) = 0 \quad (31)$$

The electrical conductivity is a function of the steady state temperature distribution as well as the perturbing temperature distribution. In order to obtain an explicit expression in powers of δT , the electrical conductivity is expressed as a Taylor series about its steady state value, i. e.,

$$\begin{aligned} \sigma(r, t) = & \sigma_{ss}(r) + \left. \frac{d\sigma}{dT} \right|_{T_{ss}}(r) \delta T(r, t) \\ & + \left. \frac{d^2\sigma}{dT^2} \right|_{T_{ss}}(r) \frac{\delta T^2}{2}(r, t) + \dots \end{aligned}$$

* $t = \frac{tk}{\rho C_p R^2}$

Equations (30) and (31) then take the form

$$\begin{aligned}
 & \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \delta T}{\partial r} \right) + \frac{\pi}{2} \left(2 \sigma_{ss} \text{Real} (E_{ss} \delta E^*) + \right. \\
 & \left. \frac{d\sigma}{dT} \Big|_{T_{ss}} \delta T |E_{ss}|^2 + \text{terms in higher powers of } \delta E \text{ and } \delta T \right) \\
 & + \frac{1}{r} \frac{d}{dr} \left(r \frac{dT_{ss}}{dr} \right) + \frac{\pi}{2} |E_{ss}|^2 \sigma_{ss} = \frac{\partial \delta T}{\partial t} \quad (32)
 \end{aligned}$$

and

$$\begin{aligned}
 & \frac{\partial}{\partial r} \frac{1}{r} \left(\frac{\partial}{\partial r} r \delta E \right) = 2i\Lambda \left(\sigma_{ss} \delta E + \frac{d\sigma}{dT} \Big|_{T_{ss}} \delta T E_{ss} \right. \\
 & \left. + \text{terms in higher powers of } \delta E \text{ and } \delta T \right) + \frac{d}{dr} \frac{1}{r} \left(\frac{d}{dr} r E_{ss} \right) \quad (33) \\
 & + 2i\Lambda \sigma_{ss} E_{ss} = 0
 \end{aligned}$$

Since the perturbations are assumed to be very small, the terms in higher powers of δE and δT in Eqs. (32) and (33) are insignificantly small compared to linear terms in δE and δT and, therefore, can be ignored. Some terms of Eqs. (32) and (33) comprise, in sum, the homogeneous differential equations for the steady state temperature and electric field and, therefore, together they equal zero. Equations (32) and (33) are reduced to the form

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \delta T}{\partial r} \right) + \frac{\Pi}{2} \left(2 \sigma_{ss} \text{Real} (E_{ss} \delta E^*) + \frac{d\sigma}{dT} \bigg|_{T_{ss}} \delta T |E_{ss}|^2 \right) = \frac{\partial \delta T}{\partial t} \quad (34)$$

and

$$\frac{\partial}{\partial r} \frac{1}{r} \left(\frac{\partial}{\partial r} r \delta E \right) + 2i\Lambda \left(\sigma_{ss} \delta E + \frac{d\sigma}{dT} \bigg|_{T_{ss}} \delta T E_{ss} \right) = 0 \quad (35)$$

Product Solution

Equation (34) and (35) can be solved by seeking solutions for δT and δE of the form

$$\delta T = \sum_m \delta T_{\lambda_m} \exp(\lambda_m^2 t)$$

$$\delta E = \sum_m \delta E_{\lambda_m} \exp(\lambda_m^2 t)$$

If any of the λ_m^2 are positive, δT increases exponentially with time. If any perturbation in temperature increases with time, the steady state temperature distribution is proved to be unstable with respect to small perturbations. If the largest (i. e., most positive) λ_m^2 is negative, other λ_m^2 are even more negative and the steady state distribution is stable.

Substitution of the product solutions for δT and δE into Eqs. (34) and (35), with the summation over m moved outside the differential operators, yields

$$\left\{ \sum_m \left[\frac{1}{r} \frac{d}{dr} \left(r \frac{d}{dr} \delta T_{\lambda_m} \right) + \frac{\pi}{2} \left(\frac{d\sigma}{dT} \right)_{T_{ss}} |E_{ss}|^2 \delta T_{\lambda_m} \right. \right. \\ \left. \left. + 2 \sigma_{ss} \operatorname{Real} \left(E_{ss} \delta E_{\lambda_m}^* \right) - \lambda_m^2 \delta T_{\lambda_m} \right] \exp (\lambda_m^2 t) \right\} = 0 \quad \text{and}$$

$$\left\{ \sum_m \left[\frac{d}{dr} \frac{1}{r} \left(\frac{d}{dr} r \delta E_{\lambda_m} \right) + 2i\Lambda \left(\sigma_{ss} \delta E_{\lambda_m} + \frac{d\sigma}{dT} \right)_{T_{ss}} E_{ss} \delta T_{\lambda_m} \right] \right. \\ \left. \exp (\lambda_m^2 t) \right\} = 0$$

These two equations are both of the form

$$\left\{ \sum_m L_{\lambda_m} \left(\delta E_{\lambda_m}, \delta T_{\lambda_m} \right) \exp (\lambda_m^2 t) \right\} = 0$$

where L_{λ_m} is a linear differential operator.

Because the coefficient of each time mode, L_{λ_m} , is independent of time, each must be equal to zero in order that the series in time exponentials be identically zero. Therefore for each m ,

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{d}{dr} \delta T_{\lambda_m} \right) + \frac{\Pi}{2} \left[\frac{d\sigma}{dT} \Big|_{T_{ss}} |E_{ss}|^2 \delta T_{\lambda_m} + 2 \sigma_{ss} \operatorname{Real} \left(E_{ss} \delta E_{\lambda_m}^* \right) \right] - \lambda_m^2 \delta T_{\lambda_m}^2 = 0 \quad (36)$$

and

$$\frac{d}{dr} \frac{1}{r} \left(\frac{d}{dr} r \delta E_{\lambda_m} \right) + 2i\Lambda \left(\sigma_{ss} \delta E_{\lambda_m} + \frac{d\sigma}{dT} \Big|_{T_{ss}} E_{ss} \delta T_{\lambda_m} \right) = 0 \quad (37)$$

Power Series Method

The eigenvalues, λ_m , of Eq. (36) are determined by the linearized surface boundary condition which is of Sturm-Liouville form,

$$-\frac{d}{dr} \delta T_{\lambda_m} = h_R \delta T_{\lambda_m} \text{ evaluated at } r = 1 \text{ (} h_R \text{ is } 4 \Gamma T_{ss}^3 \text{ at } r = 1 \text{ if}$$

radiation is the principal loss mechanism). The surface boundary condition for Eq. (37) is $-\frac{d}{dr} \delta E_{\lambda_m} = \delta E_{\lambda_m}$.

All we are interested in determining from the two differential equations and their respective boundary conditions is the value of the largest λ_m^2 .

Calculating the largest λ_m^2 is a very similar problem to calculating the steady state temperature and electric field. The two surface boundary

conditions, $-\frac{d}{dr} \delta T_{\lambda_m} = h_R \delta T_{\lambda_m}$ and $-\frac{d}{dr} \delta E_{\lambda_m} = \delta E_{\lambda_m}$, are used to determine

the largest λ_m^2 and the first term in the series expansion for δE_{λ_m} , respectively. The first term in the series expansion for δT_{λ_m} can be set equal to 1. This can be done because δT_{λ_m} is actually proportional to a basic solution to Eq. (36), i. e., $\delta T_{\lambda_m} = \delta T_{\lambda_m}^0 \Psi(r, \lambda_m)$, where $\Psi(r, \lambda_m)$ is a solution to Eq. (36) whose leading term in its power series expansion is 1. $\delta T_{\lambda_m}^0$ is a constant that is a measure of the magnitude of the initial temperature perturbation of harmonic λ_m . Because the differential equations and boundary conditions are completely linearized, the eigenvalues are independent of δT_{λ}^0 . Therefore, the eigenvalues are independent of the size of the initial perturbations.

All position varying quantities in Eqs. (36) and (37) are represented as power series expansions in order to derive recursion formulas for terms in the series expansions of δT_{λ_m} and δE_{λ_m} . Power series expressions for the different quantities are

$$\begin{aligned}\delta T_{\lambda_m} &= \sum_{j=0, 2, \dots} F_j r^j \\ \delta E_{\lambda_m} &= \sum_{j=1, 3, \dots} G_j r^j\end{aligned}$$

$$\left. \frac{d\sigma}{dT} \right|_{T_{ss}} = \sum_{j=0,2,\dots} S_j r^j$$

$$\sigma_{ss} = \sum_{j=0,2,\dots} C_j r^j$$

$$|E_{ss}|^2 = \sum_{j=2,4,\dots} H_j r^j$$

$$E_{ss} = \sum_{j=1,3,\dots} A_j r^j$$

The recursion formulas for F_j and G_j are

$$F_j = \frac{\lambda_m^2 F_{j-2}}{j^2} - \frac{\pi}{2j^2} \sum_{p=0,2,\dots}^{j-2} \left[S_p \left(\sum_{k=2,4,\dots}^{j-p-2} H_k F_{j-p-k-2} \right) + 2 C_p \operatorname{Real} \left\{ \sum_{k=1,3,\dots}^{j-p-3} A_k G_{j-p-k-2}^* \right\} \right]$$

$$G_j = \frac{-2i\Lambda}{(j^2-1)} \sum_{p=0,2,\dots}^{j-2} \left[G_{j-p-2} C_p + S_p \sum_{k=1,3,\dots}^{j-p-2} A_k F_{j-k-p-2} \right]$$

Figure 8 shows a flow chart of a computer program that solves for the largest λ_m^2 using the iteration scheme mentioned earlier.

FIGURE 8

Flow chart of a computer program that solves the coupled partial differential equations determining the stability of steady state electric field and temperature distributions within a sample that has a temperature dependent electrical conductivity. A power series method is used to solve the differential equations--Eqs. (36) and (37).

Results

The stability of the steady state solutions to the coupled differential equations, Eqs. (14) and (15), has been determined using the preceding method of analysis. Figure 9 shows the largest eigenvalues (the determining factor of stability or instability) for θ equal to 11 over a continuous range of Π . The results prove that steady states on the lower branch of the "S" in the case of θ equal to 7, 9, and 11 are stable. Steady states on the middle branch of the "S" for θ equal to 11 are unstable (middle branches for θ equal to 7 and 9 have not been obtained yet). It is anticipated, but yet unproved, that steady states on the upper branch of the "S" are stable. This type of stability behavior is characteristic of solutions to nonlinear differential equations; solutions on the middle branch of the "S" are generally found to be unstable.

Approximate Stability Criterion

On the lower branch of the "S" curve before instability is encountered, the electrical conductivity is very small. Therefore, the temperature distribution (through the electrical conductivity) only weakly influences the electric field distribution. Thus, the time dependent differential equations are effectively decoupled. From the differential equation for the temperature, it is possible to state approximately under what conditions perturbations increase with time causing the steady state to be unstable.

FIGURE 9

Variation of the internal power generated within an inductively heated material whose electrical conductivity varies as $\exp [11 (1 - 1/T)]$ as a function of $B_s^2 \omega^2 R^4 \sigma_{T_M} / k T_M$. The dotted line represents the variation of the largest eigenvalue squared, λ^2 , as a function of $B_s^2 \omega^2 R^4 \sigma_{T_M} / k T_M$. When λ^2 is negative, the steady state solution is stable; when λ^2 is positive, the steady state solution is unstable.

Because the $2i\Lambda_0 \mathbf{E}$ term in Eq. (31) is very small, the differential equation for the electric field is approximately $\frac{d}{dr} \frac{1}{r} \frac{d}{dr} (r\mathbf{E}) = 0$. A general solution which is zero at $r = 0$ is $A r$ where A is determined by the surface boundary condition $\frac{d}{dr} \mathbf{E} = 1 - \mathbf{E}$. A is determined to be $1/2$ and $\mathbf{E}$ therefore equals $r/2$. The time dependent temperature equation, linearized about the steady state temperature solution is

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \mathbf{T}}{\partial r} \right) + \frac{\pi}{8} r^2 \left[\sigma_{ss} + \left. \frac{d\sigma}{dT} \right|_{T_{ss}} (\mathbf{T} - T_{ss}) \right] = \frac{\partial \mathbf{T}}{\partial t}.$$

With the steady state part of this partial differential equation subtracted out,

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \delta \mathbf{T}}{\partial r} \right) + \frac{\pi}{8} r^2 \left. \frac{d\sigma}{dT} \right|_{T_{ss}} \delta \mathbf{T} = \frac{\partial \delta \mathbf{T}}{\partial t} \quad \text{where } \delta \mathbf{T} \text{ is the perturbation in}$$

temperature. The linearized surface boundary condition is $-\frac{\partial \delta \mathbf{T}}{\partial r} = h_R \delta \mathbf{T}$ at $r=1$

where $h_R = 4 \pi T_{ss}^3$ at $r=1$.

To determine the condition for which $\delta \mathbf{T}$ increases with time, the differential equation is integrated over the cross sectional area of the sample,

$$2\pi \int_0^1 r dr \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \delta \mathbf{T}}{\partial r} \right) + \frac{r^2}{8} \pi \left. \frac{d\sigma}{dT} \right|_{T_{ss}} \delta \mathbf{T} = \frac{\partial \delta \mathbf{T}}{\partial t} \right].$$

The first integral simplifies, by integration by parts, to $2\pi \left. \frac{\partial \delta \mathbf{T}}{\partial r} \right|_{r=1}$ which

is equal to $2\pi h_R \delta \mathbf{T}|_{r=1}$ from the surface boundary condition. When

$2\pi \int_0^1 \frac{\partial \delta T}{\partial t} r dr / \delta T$ is greater than zero, more power is internally generated than is radiated and the steady state solution becomes unstable. Therefore, the limit of stability is given by the inequality

$$h_R \delta T|_{r=1} > \frac{\pi}{8} \int_0^1 \left. \frac{d\sigma}{dT} \right|_{T_{ss}} \delta T r^3 dr$$

This inequality contains several terms that are difficult to evaluate without further extensive analysis. The inequality can be enormously simplified by making two approximations.

1. $\left. \frac{d\sigma}{dT} \right|_{T_{ss}}$ is nearly constant and can be moved outside the integral.
2. δT is nearly constant and can also be moved outside the integral.

With the help of these two assumptions, the inequality simplifies to

$$\frac{\pi}{32 h_R} \left. \frac{d\sigma}{dT} \right|_{T_R} < 1. \quad \text{Both } \left. \frac{d\sigma}{dT} \right|_{T_R} \text{ and } h_R \text{ are expressed in terms of}$$

the steady state surface temperature. Since it's very close to the ambient temperature, in the spirit of the approximation, we assume they are at the same temperature—the ambient temperature. In terms of the dimensionless parameters previously defined, the criterion for stability, in a very approximate form is

$$\frac{\Pi \theta \exp [\theta (1 - 1/T_A)]}{T_A^5 \cdot 128 \Gamma} < 1.$$

and, in terms of dimensional parameters,

$$\frac{B_s^2 \omega^2 R^3 T_{\text{activation}} \sigma_{T_{\text{ambient}}}}{128 \text{ emissivity Stefan-Boltzman constant } T_{\text{ambient}}^5} < 1.$$

Figure 6 plotted against $\frac{\Pi \theta \exp [\theta (1 - 1/T_A)]}{128 T_A^5 \Gamma}$ instead of Π is given in

Figure 10. Plotting against this new parameter tends to map the lower portion of the "S" curves into each other.

The approximations made throughout the derivation of this criterion for stability do introduce a sizable error, but our computer-aided modeling has shown that the error is consistently on the order of 25 to 40 percent. An exact stability criterion can be written if a factor f is introduced which contains the error due to approximations in the derivation:

$$\frac{B_s^2 \omega^2 R^3 T_{\text{activation}} \sigma_{T_{\text{ambient}}}}{128 \text{ emissivity Stefan-Boltzman constant } T_{\text{ambient}}^5} < f.$$

f ranges from .25 to .40.

FIGURE 10

Variation of the internal power generated within an inductively heated material (whose electrical conductivity varies as $\exp [\theta (1 - 1/T)]$) as a function of $B_s^2 \omega^2 R^3 \sigma_{T_A} T_{\text{activation}} / 128 \alpha T_{\text{ambient}}^5$ for different values of θ .

CONCLUSIONS

Two of the essential features of ICZG are that the interior of the sample material is liquid and that the surface is cool enough to contain the liquid. From the two one-dimensional models contained in this report, we have been able to define these two necessary requirements in terms of material and processing variables. In particular, we have established what parameters are most important in determining the center-to-surface temperature difference for heating in metals and the amount of eddy current heating in ionic materials.

The first model represents the induction heating of materials whose electrical conductivities are independent of temperature. This model is a fairly realistic representation of one-dimensional heating in metals. Analytic solutions for the electric field and temperature within the sample have been obtained from their respective differential equations. Contained in the expression for the temperature distribution (page 16) is the parameter $\frac{k}{Lh}$ where L is the smaller of either the skin depth or sample radius. When $\frac{k}{Lh}$ is large compared to 1, there is very little radial dependence of the temperature and the sample is nearly isothermal. For there to be internal melting, $\frac{k}{Lh}$ must be less than about 5. It's possible to infer from the magnitude of $\frac{k}{Lh}$ if internal melting is plausible for a particular heating situation.

It has been proven experimentally that materials with low thermal conductivities can be centrally melted, but it is unknown if materials with large thermal conductivities like metals can be centrally melted. To succeed, it would probably be necessary to depress the surface temperature with forced convection. A strong flow of hydrogen or helium along the surface is an easy way to increase surface convection. Cooling the surface with a stream of liquid metal will result in even greater connective losses.

Metals can be induction heated very easily, but, because of their large thermal conductivities, it is difficult to have a sizable thermal gradient. The problem in centrally melting low-conductivity oxides is just the opposite. The difficulty is in getting the sample hot at all. The electrical conductivities of many oxides, pre-heated to even one-half of their melting point, are so low that there is very little eddy current heating within the sample.

The second model represents the induction heating of ionic materials such as oxides and semiconductors. The electric conductivities of these materials increase significantly when heated and this property results in a heating instability. As the temperature of the sample increases due to greater impressed external field, the electrical conductivity of the sample increases and therefore so does the eddy current heating; increased heating tends further to raise the temperature of the sample. Continued heating

becomes self-generating but radiation losses are a stabilizing influence. Up to some critical value of the impressed field, as the temperature of the sample increases, the sample radiates more power and the heating is stable. At this critical value, the sample cannot radiate all the power that is being generated. There would be catastrophic heating (Figure 11a) if it were not for the fact that as the electrical conductivity increases, the electric field decreases.

Figure 11b illustrates the actual effect of the heating instability upon eddy current heating within the sample. The importance of this instability phenomenon is in its relevance to the problem of achieving significant internal heating in low-conductivity oxides. At the point of instability, there is a tremendous increase in the sample temperature. Our modeling has provided us with a method to calculate under what conditions instability will occur. The analytic expression for the onset of instability, developed from simplified differential equations, clearly states how the different material parameters and processing variables affect the heating stability and consequently the temperature of the material being induction heated.

The approximate stability criterion (page 49) shows that the chances of being able to create significant internal heating within a sample are greatly improved if a high pre-heat temperature is used (where the

(a) $|B_{\text{surface}}|^2$ (b) $|B_{\text{surface}}|^2$

Figure 11

The induction heating of a material whose electrical conductivity increases with temperature can show a heating instability when the sample cannot radiate all the power being internally generated. This instability takes the form of a sharp increase in the sample temperature. If it were not for the fact that the electric field within the sample decreases as the sample heats, the sample would draw all the power its source could supply (Figure 11a). An actual heating curve is shown in Figure 11b.

pre-heat temperature is approximately equivalent to the ambient temperature). Although a high pre-heat temperature assures greater surface losses, which are undesirable if one is trying to raise the sample's temperature, its effect upon the electrical conductivity of the material is enormous. A 20°C increase in the pre-heat temperature can increase the electrical conductivity of the sample by a factor of 2. Another parameter that has an immense effect upon the heating ability of any material is the value of the rf frequency, but a change to a higher frequency is much more costly to engineer than use of a high pre-heat temperature. So far there has been little work done at Oak Ridge in attempting to use higher pre-heat temperatures to extend ICZG to oxides (such as MgO) heretofore impossible to centrally melt by rf heating.

Our one-dimensional modeling has provided us with some ideas as to how we can improve ICZG and extend its use to other materials. The merit of these ideas and others we will obtain from further modeling will be tested experimentally at Oak Ridge and also at Carnegie-Mellon University.

APPENDIX A

Another way to solve the coupled differential equations determining the temperature and electric field distributions is to use a finite difference method of solution in which differentials are represented as discrete quantities, e. g., $\Delta r = r_{\text{at point } j+1} - r_{\text{at point } j}$, $\Delta T = T_{\text{at point } j+1} - T_{\text{at point } j}$. To solve Eqs. (14) and (15) we use Euler's modified self-starting method in combination with an iteration scheme similar to the one used with the power series method of solution. With Euler's method, one can calculate the slope and value of the temperature or electric field at a point within the sample from the slope and value at a neighboring point. If Euler's method is applied, point by point, from one boundary to the opposite boundary, the slope and value at each point in the sample can be evaluated.

Euler's method is particularly well suited to problems which have slope and value specified at one boundary. The boundary conditions for Eqs. (14) and (15) are slope or value at each boundary; so to start, it's necessary to guess either the slope or value depending on which is already known. The results of two guesses can be used to make a good third guess and so forth.

Euler's method approximates slopes and values with truncated Taylor series expansions in order to relate them to neighboring slopes and values. For example, first approximations for T'_{j+1} and T_{j+1} are

$$T'_{j+1} = T'_j + T''_j \Delta r \text{ and } T_{j+1} = T_j + T'_j \Delta r + T''_j \frac{(\Delta r)^2}{2} \text{ where } T''_j \text{ is}$$

computed from the differential equation using values of T_j and T'_j ; in the case of Eq. (14), $T''_j = -\frac{\pi}{2} \sigma(T_j) |E_j|^2 - \frac{T_j}{r_j}$. Euler's method

compensates for over- and undershoot by using the newly calculated values for T'_{j+1} and T_{j+1} to improve the first approximations. A better approximation for T'_{j+1} and T_{j+1} is obtained by averaging T''_{j+1} and T''_j (where T''_{j+1} is calculated with values of T'_{j+1} and T_{j+1})

$$T'_{j+1} = T'_j + \frac{(T''_{j+1} + T''_j)}{2} \Delta r$$

$$T_{j+1} = T_j + T'_j \Delta r + \frac{(T''_{j+1} + T''_j)}{2} \frac{(\Delta r)^2}{2}$$

The interaction scheme for this method is somewhat more involved than the one for the power series method. Initial values for the center temperature, surface electric field, and power per unit volume distribution are guessed. The temperature slope at the center is known to be zero. With starting values for the center temperature and its slope, Euler's method is used to calculate the temperature slope and value at each point from the center to the surface. Next, Euler's method is used to calculate the slope and value of the electric field at each point. The value of the electric field at the center is known to be

zero and the surface value of the electric field was originally assured to be some value, so to start Euler's method, we need either a center or surface slope. We choose to start Euler's method at the center and work towards the surface. A center slope is guessed; Euler's method is applied point by point from the center to the surface. The surface value of the electric field obtained in this manner is generally not equal to the value originally chosen, so it is necessary to try different center slopes to find the one that is complementary to the surface value originally guessed. An iteration is performed to determine which center temperature and surface electric field cause the surface boundary conditions to be satisfied.