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DYNAMICS OF AEROSOLS AND HYDROSOLS
CLASS NOTES

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INTRODUCTION

1.1 SCOPE

The behavior of suspended particulate matter has become a subject of considerable interest to the engineer and scientist. The single most important source of current interest is, undoubtedly, environmental concern. Suspended particles in the atmosphere have obvious and subtle deleterious effects on man and the environment. The magnitude of these effects depend upon many factors including the concentration, size and chemical composition of the pollutants.

An understanding of the dynamic behavior of these particles is necessary to the engineer and scientist in this field. This knowledge is essential to the description of particle transport, collection and deposition. The meaning of many measurements, as well as the validity of conclusions drawn from them, depend on this understanding of the motion of suspended particles. The proper design of equipment to remove suspended particles from the suspending fluid requires that the designer be familiar with the dynamics of particulate matter.

In addition to the well known atmospheric applications of suspended particle technology, there are numerous other sources of significant problems. These include such disparate fields as xerography, spraying, pneumatic conveying, sedimentation, combustion, printing, agriculture, cloud physics, and medical diagnosis.

The term, aerosol, as used in this text simply means a small particle suspended in air. Associated with the small size is a small sedimentation velocity in a gravitational field. The notion of smallness here can be made less vague and imprecise by expressing it in terms of the Reynolds number for the particle motion.

At the risk of oversimplification, we can say that the Reynolds number describing a moving particle is a dimensionless number expressing the relative importance of inertial and viscous forces on the particle. If the Reynolds number is large, much greater than one, the effects of fluid inertia dominate the viscous effects. If the Reynolds number is much less than one, the converse is true. The motions that interest us here are normally characterized by Reynolds numbers less than one. The Reynolds number is discussed further in the next chapter.

A hydrosol is an analogous particle in water, defined in the same way. The dynamic behavior of aerosols and hydrosols are alike in many ways. Many analyses and methods of analysis apply equally well to either case. Conclusions drawn from these analyses depend, of course, on the magnitudes of physical properties in gases and in liquids.

Differences do arise, largely due to the presence of a mean free path in gases. The molecules of a gas are widely separated compared to their dimensions. Consequently, a gas molecule may travel many multiples of its diameter before having a strong interaction or collision with another molecule. The average distance traveled by a gas molecule between collisions is its mean free path. If a typical aerosol dimension is much larger than this mean free path, the gas may be considered a continuum and an explicit acknowledgement of the molecular nature of the medium is unnecessary. If the aerosol dimension is smaller than the mean free path, the continuum model breaks down and the distribution of molecular velocities must be considered. The ratio of mean free path to the aerosol dimension is called the Knudsen number.

In liquids, the molecules are much more closely packed and are in continuous interaction. The mean free path is zero. Continuum models have been applied with reasonable success to particles having molecular dimensions. Consequently, small Knudsen number or continuum models are applicable to particles in liquids and sufficiently large particles in gases. Large Knudsen number analyses apply only to the motion of small aerosols. The extensive applicability of small Knudsen number (continuum) results warrant the simultaneous treatment of the dynamics of aerosols and hydrosols throughout most of this volume.

The term particle is used to mean solid or fluid particle. The methods of analysis are valid in either case. Differences in the behavior of the two kinds of particles are readily accounted for. We shall consider solid particles in gas or in liquid, liquid particles or drops in gas or in liquid, and gaseous particles or bubbles in liquids.

1.2 .CLASSIFICATION OF AEROSOLS

The reader should become familiar with some of the terminology utilized in fields associated with aerosols or hydrosols. Some of the distinctions expressed in the definitions have no effect on the dynamics of the particle. Consequently it will not be necessary for us to retain many of these distinctions in the formulation of the particle dynamics.

A dispersion aerosol is formed without a change in phase by dispersing solids or liquids as by spraying or by suspending a finely divided solid. In contrast, a condensation aerosol is formed by the condensation of a gas or by chemical reaction. Condensation aerosols tend to be smaller than dispersion aerosols. Liquid aerosols, whether condensation or dispersion aerosols, tend to assume a spherical shape as a result of surface tension. Solid aerosols are far more irregularly shaped.

Liquid aerosols are characterized as mists or sprays. Sprays are dispersion aerosols and are the largest drops. Mists ordinarily are taken to be smaller drops whether formed by condensation or by atomization.

Dusts are large solid dispersion aerosols. A smoke is a solid condensation aerosol, frequently formed by combustion. From a dynamic standpoint, only the distinction that dusts are larger is important.

Each of these aerosol classifications is vague. Beyond this, the definitions are, by no means, uniformly agreed upon. Different authors may give completely contradictory definitions of some of these terms. Our classification, in terms of size, may be found in Figure 1.1

DIAMETER (microns)

Figure 1.1
1-4

2.1 INTRODUCTION

The dynamic behavior of suspended particles is determined by the forces acting on the particle. In response to an externally applied force, the particle will move through the fluid. This motion is resisted by the drag force exerted on the particle by the fluid. Obtaining expressions for this drag force is a major objective in the study of aerosols and hydrosols. In order to accomplish this task, it is necessary to examine the fluid motion in the neighborhood of the particle. This chapter provides a review of some of the more basic concepts of fluid dynamics.

Any fluid motion of interest involves the motion of great numbers of atoms. In dealing with these motions, we have no desire or need to follow the motion of individual atoms. If the knowledge of individual molecular positions and velocities was necessary in order to predict the macroscopic behavior of a system, the exercise would not be useful because such detailed knowledge is not available. Furthermore, if predictions of macroscopic behavior resulting from such calculations depended in any significant way on the position and velocity of each molecule, such predictions would be severely restricted in their applicability.

Fortunately, we can be content to obtain statistical knowledge of the behavior of an assembly of molecules. Information of this nature is adequate for our purpose and expresses our incomplete knowledge of the microscopic state of the system. The type of model used depends on the relative magnitudes of the particle's dimensions and the mean free path.

2.2 MEAN FREE PATH AND KNUDSEN NUMBER

The average distance traveled by a gas molecule between collisions with other molecules is an important parameter in the description of gas flows. The distance traveled by a molecule between successive collisions is called the free path. The average length of this path is the mean free path. In a liquid, molecules are continuously in strong interaction so that the concept of mean free path is not useful there. The mean free path in a liquid is zero.

Momentum transfer in a gas is accomplished by molecular collisions. If the distance that a molecule travels between collisions is ordinarily very large compared to the dimensions of an aerosol particle, then disturbances in the gas

due to aerosol motion will occur in a diffuse way over relatively large distances from the particle. Consequently, the molecules approaching the particle will have a distribution of velocities that is affected in a negligible way by the presence of the moving particle.

If the mean free path is much smaller than a typical particle dimension, molecules rebounding from the particle may be expected to collide with other molecules near the particle. These and succeeding collisions influence the molecular velocity distribution near the particle. In particular, the velocity distribution of molecules that are about to strike the particle will be different from the distribution observed when the mean free path is comparatively large. Consequently, the calculations of the force exerted by the gas on the particle and the resultant expressions are different in the two cases.

The relative magnitudes of the mean free path, ℓ , and a characteristic particle dimension, a , are significant in classifying the flow encountered. The ratio of these two quantities is the Knudsen number, Kn .

$$Kn \equiv \frac{\ell}{a} \quad (1)$$

Flows where the Knudsen number is large are termed free molecule flows. Such flows are treated using methods of statistical mechanics. An extensive background in statistics is unnecessary in the solution of the problems considered here. An understanding of some of the more simple concepts will suffice and may be developed as we proceed.

Flows where the Knudsen number is much less than unity have been quite successfully analyzed using continuum models. Because this approach is so generally successful in liquids and valid for the larger aerosols, we shall first use the methods of continuum mechanics to predict the behavior of particles.

The region of Knudsen number near one is termed the transition regime. As may be expected, analysis is most difficult in this regime and empirical drag relations are ordinarily used. For small Knudsen numbers, a continuum slip flow approximation, where effects of the non-zero mean free path are confined to a region close to the surface, has been employed to yield first order corrections in the Knudsen number.

We should now obtain an estimate of the mean free path in a gas. A rough estimate may be obtained using a very simple model. Consider a gas consisting of hard sphere molecules which collide like billiard balls. The path of such a molecule will be composed of a series of straight lines of varying lengths, connected end to end and oriented in a random fashion.

Further consider all of the molecules to be stationary except for a single test molecule and determine the mean free path of that molecule. It should be clear that this simplification is most accurate for very fast molecules which see the other molecules as essentially stationary. For simplicity, the segments of the path may be mentally unfolded and aligned as in Figure 2.1 where the solid line is the test molecule's trajectory.

All of the molecules in this gas have the same molecular radius, r_0 . The moving molecule will touch or collide with another molecule when their centers are one molecular diameter apart. Accordingly, the moving molecule will collide with every molecule whose center lies within a circular cylinder whose axis is the path of the moving molecule and whose radius is twice the molecular radius. This cylinder is shown by the dotted lines in Figure 2.1.

When the test molecule travels a distance, D , it will strike every molecule having its center in a cylinder of volume $4\pi r_0^2 D$. If the average number density of the molecules is n , then there are, on the average, $4\pi n r_0^2 D$ molecules in the cylinder. The average distance between collisions, or mean free path, for very fast molecules is then

$$\lambda = \frac{D}{4\pi n r_0^2 D} = \frac{1}{4\pi n r_0^2} \quad (2)$$

Figure 2.1 Molecular collisions

<u>Molecule</u>	<u>$2r_o^o, \text{\AA}$</u>
H ₂	2.40
O ₂	2.98
N ₂	3.15
CO	3.19
CO ₂	3.34

Table 2.1 Molecular diameters from viscosity measurements
(From Handbook of Chemistry and Physics)

Slower molecules will travel shorter distances between collisions because the motion of the other molecules is not negligible. A stationary atom will be struck, sooner or later, without travelling any distance. As a result, the mean free path in the gas is somewhat smaller than the quantity just calculated. An analysis that accounts for the distribution of molecular velocities yields

$$\ell = \frac{1}{4\sqrt{2} \pi n r_0^2} \quad (3)$$

This is the mean free path of a hard-sphere Maxwellian gas. In terms of the collision cross-section, σ , the cross sectional area of the cylinder of Figure 2.1, we have

$$\ell = \frac{1}{\sqrt{2} n \sigma} \quad (4)$$

Calculation of the mean free path requires a knowledge of the molecular size and number density. Estimates of molecular dimensions are not precise but depend upon the technique used. Typical values, determined from viscosity measurements, are shown in Table 2.1.

The number density of any sufficiently rarefied gas may be calculated from the ideal gas equation of state. Statistical mechanics predicts, and experiment confirms, the validity of the ideal gas equation of state in this regime.

$$p = n k T \quad (5)$$

p is the pressure; k , Boltzmann's constant; and T is the absolute or thermodynamic temperature. Boltzmann's constant has the value

$$k = 1.38 \times 10^{-16} \text{ erg / K} \quad (6)$$

As an example, let us choose a gas at a pressure of 10^6 dyne/cm² (~ 1 atmosphere) and a temperature of 300 K. The ideal gas equation of state yields

$$n = 2.42 \times 10^{19} / \text{cm}^3 \quad (7)$$

If the gas is composed of N_2 molecules, (3) predicts a mean free path

$$\ell = 9.5 \times 10^{-2} \text{ microns} \quad (8)$$

There are other models for the mean free path, some more accurate and some less. Since in any experimental work, the mean free path is not measured but inferred from relations like (4) or other indirect measurements, it is important to know what relation the experimenter used to get the mean free path. Unfortunately, authors do not always report this important piece of information.

2.3 VISCOSITY, STRESS AND STRAIN RATE

The effects of fluid friction are important in the dynamics of aerosols and hydrosols. Normally, the dominant forces on a fluid element, and consequently on the particle, are the pressure and viscous forces. For this reason, we should review the action of fluid viscosity.

Consider a simple experiment as illustrated in Figure 2.2. The lower plate is held stationary. A tangential force F is applied to the upper plate causing it to move relative to the lower plate with velocity U . The plates are separated by a distance L which is much larger than the mean free path, so that the fluid between the plates may be considered to adhere to the plates.

When this experiment is performed, many fluids are observed to exhibit the following characteristic. The force per unit area applied to the plates is proportional to the resulting velocity U and inversely proportional to the separation distance.

$$\frac{F}{A} \propto \frac{U}{L} \quad (1)$$

The proportionality constant is different for different fluids. For any single fluid, the proportionality constant varies significantly only when the temperature changes. The proportionality constant is a fluid transport property, is called the viscosity and is denoted by μ

$$\frac{F}{A} = \mu \frac{U}{L} \quad (2)$$

The viscosity obviously has dimensions of $F \cdot t / L^2$. The force per unit area is a stress, denoted by σ having units of F / L^2 .

$$\sigma = \mu \frac{U}{L} \quad (3)$$

Since the flow is steady, the stress must be the same on the upper surface and the lower surface and any other parallel imaginary surface drawn through the fluid. Consider such a surface a distance L' above the lower plate. The velocity at this imaginary surface is U' . Since the stress is the same at all such surfaces, (3) yields

$$\frac{U'}{L'} = \frac{U}{L} = \text{constant} = \frac{du}{dy} \quad (4)$$

where u is the velocity of the fluid a distance y above the stationary plate. We conclude that the velocity distribution between the plates is linear

$$u = \frac{y}{L} U \quad (5)$$

Figure 2.2 Parallel viscous flow between two plates

Figure 2.3 Stresses on a fluid element

and that

$$\sigma = \mu \frac{du}{dy} \quad (6)$$

While this simple expression for the stress in terms of the velocity gradient describes the result of this parallel flow, the simple expression is inadequate to describe more general flows. In general, the tangential stress on a surface is not simply the product of the viscosity and the normal derivative of the tangential velocity.

Details of the demonstration are left until the section on Cartesian tensors. The argument is as follows. An expression for the stress, in order to be general, must be correct regardless of the orientation of the Cartesian axes. By considering a force balance on an infinitesimal element, one can determine how the stresses in one set of Cartesian coordinates are related to the stresses in another frame. The result shows that (6) does not transform in the right way. The net force on an element would depend on the frame selected which is, of course, impossible. Consequently, (6) does not have the necessary generality.

Expressions for the tangential or shearing stress that transform in the proper manner are

$$\begin{aligned} \sigma_{xy} = \sigma_{yx} &= \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \\ \sigma_{xz} = \sigma_{zx} &= \mu \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \\ \sigma_{yz} = \sigma_{zy} &= \mu \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \end{aligned} \quad (7)$$

u , v , and w are velocity components in the x , y , and z directions, respectively. The first of these reduces to (6) in the parallel flow. σ_{xy} is the stress acting in the y direction on a surface whose normal is the x direction.

The transformation rules obeyed by the stress are the transformation rules of a second order tensor. The stress is a tensor. It has nine components. The six components just enumerated are the tangential or shearing stress components. The other three components correspond to the normal stresses as the subscript convention indicates. For an incompressible Newtonian fluid, these

normal stresses are

$$\begin{aligned}\sigma_{xx} &= -p + 2\mu \frac{\partial u}{\partial x} \\ \sigma_{yy} &= -p + 2\mu \frac{\partial v}{\partial y} \\ \sigma_{zz} &= -p + 2\mu \frac{\partial w}{\partial z}\end{aligned}\quad (8)$$

p is the fluid pressure. Figure 2.3 shows these stresses acting on a fluid element. Tensors and tensor transformations are treated in greater detail in the next chapter.

2.4 VORTICITY AND FLUID ROTATION

In addition to the translation and straining associated with fluid motion, a fluid element will normally rotate during its motion. The rotation or angular velocity of a fluid element can be related to the fluid velocity distribution by consideration of the variation of velocity along the boundary of the element. Consider the plane element of Figure 2.4. The fluid velocity along the sides of the square is not uniform. Consequently, as time proceeds, each of the sides will be observed to rotate. The edge that coincides with the y axis has an angular velocity (in the z direction) equal to $-\frac{\partial u}{\partial y}$. The edge coinciding with the x axis has an instantaneous angular velocity, also in the z direction, equal to $\frac{\partial v}{\partial x}$. The average angular velocity of the element is simply the average of these two angular velocities

$$\omega = \frac{1}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \hat{k}$$

the unit vector noting that the rotation vector is normal to the plane of rotation. Extending this concept to three dimensions

$$\vec{\omega} = \frac{1}{2} \left[\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) \hat{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) \hat{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \hat{k} \right]$$

or in vector notation

$$\vec{\omega} = \frac{1}{2} \nabla \times \vec{V} \quad (1)$$

Following Lamb, who first used the term, vorticity, to describe this measure of rotation rate, we shall define the vorticity to be twice this average angular velocity

$$\vec{\zeta} \equiv \nabla \times \vec{V} \quad (2)$$

Figure 2.4 Rotation of a fluid element

A flow for which the vorticity is everywhere zero is said to be irrotational. A fluid element in such a flow, although it may become distorted as a result of the shearing strain, will retain the same orientation with respect to the coordinate axes. It does not rotate. Irrotationality is a necessary and sufficient condition that the flow should have a velocity potential

$$\vec{V} = - \nabla \phi$$

That any flow described by a velocity potential must be irrotational follows immediately from the vector identity

$$\nabla \times \nabla \phi = 0$$

Viscous flows are rarely irrotational because the no-slip condition normally imposed at a particle surface is usually incompatible with irrotational flow.

2.5 CONTINUITY

The conservation of fluid mass is expressed by a relation known as the continuity equation. Both the global and local form of this equation are widely used and will be presented. Their equivalence follows immediately upon application of the divergence theorem.

Let us develop the continuity equation by examining the flow entering and leaving an infinitesimal element of space. For generality, an unsteady compressible flow is considered although the simpler form that results for incompressible flow normally suffices for the study of particle motions through fluids.

In Figure 2.5, an elemental region of space is shown along with the fluid velocity components normal to each of its faces. Only the normal components of velocity correspond to flow across the bounding surface so only these need be considered. The continuity equation is generated simply by accounting for the flows across the surface and equating the net influx of mass to the net increase of mass contained in the region.

The net mass flow rate into the element is the sum of the mass flow rates across each face. On each face, the mass flow rate is the product

Figure 2.5 Mass flow through an elemental cube

of the density, normal velocity component and the area. Accordingly, the net mass flow in is

$$(\rho_f u|_x - \rho_f u|_{x+\Delta x}) \Delta y \Delta z + (\rho_f v|_y - \rho_f v|_{y+\Delta y}) \Delta x \Delta z \\ + (\rho_f w|_z - \rho_f w|_{z+\Delta z}) \Delta x \Delta y = \frac{\partial \rho_f}{\partial t} \Delta x \Delta y \Delta z$$

The right hand side is simply the rate of accumulation of mass in the region. Now, dividing by $\Delta x \Delta y \Delta z$, letting each of the increments go to zero.

$$- \lim_{\Delta x \rightarrow 0} \frac{\rho_f u|_{x+\Delta x} - \rho_f u|_x}{\Delta x} - \lim_{\Delta y \rightarrow 0} \frac{\rho_f v|_{y+\Delta y} - \rho_f v|_y}{\Delta y} \\ - \lim_{\Delta z \rightarrow 0} \frac{\rho_f w|_{z+\Delta z} - \rho_f w|_z}{\Delta z} = \frac{\partial \rho_f}{\partial t}$$

Rearranging and using the definition of the derivative, the local form of the continuity equation results.

$$\frac{\partial \rho_f}{\partial t} + \frac{\partial}{\partial x} (\rho_f u) + \frac{\partial}{\partial y} (\rho_f v) + \frac{\partial}{\partial z} (\rho_f w) = 0$$

Alternatively, we write this in vector form

$$\frac{\partial \rho_f}{\partial t} + \nabla \cdot (\rho_f \vec{V}) = 0 \quad (1)$$

Reduction to the special case of an incompressible fluid results from holding the density constant. The result

$$\nabla \cdot \vec{V} = 0 \quad (2)$$

states that the net volume flow into the region is zero for any incompressible flow, steady or not.

By means of Gauss' Theorem

$$\int_V \nabla \cdot \vec{B} \, dv = \int_A \vec{B} \cdot \hat{n} \, dA \quad (3)$$

and using $\rho_f \vec{V}$ as $\vec{B}$, the control volume or global form of the continuity equation results

$$\frac{d}{dt} \int_V \rho_f \, dv + \int_A \rho_f \vec{V} \cdot \hat{n} \, dA = 0 \quad (4)$$

An alternative derivation of the continuity equation can be based on Reynolds' transport theorem which relates the rate of change of any property b in a material volume v' that moves with the fluid to changes within the region of space v , instantaneously coincident with v' , and flow across the surface of v .

$$\frac{D}{Dt} \int_{v'} b \, dv' = \frac{d}{dt} \int_v b \, dv + \int_A b \vec{V} \cdot \hat{n} \, dA \quad (5)$$

The volume v' always contains the same fluid. Choosing b to be the fluid density and recognizing that the mass of the material volume v' does not change

$$\frac{D}{Dt} \int_{v'} \rho_f \, dv' = 0 \quad (6)$$

the control volume form of the continuity equation results

$$\frac{d}{dt} \int_V \rho_f \, dv + \int_A \rho_f \vec{V} \cdot \hat{n} \, dA = 0$$

Applying Gauss' theorem

$$\int_V \left[\frac{\partial \rho_f}{\partial t} + \nabla \cdot (\rho_f \vec{V}) \right] dv = 0$$

Since the choice of control volume is arbitrary, the integrand must itself be zero and the local form of the continuity equation is recovered.

The continuity equation has been generated two different ways proceeding first from the local form to the global form and then proceeding in the opposite manner. The approaches are equivalent. One's preference is a matter of choice.

2.6 ACCELERATION

The acceleration of a fluid may be described using either of two commonly employed methods. The methods are widely known as the Lagrangian and Eulerian methods, although Euler in fact introduced each of them. The Lagrangian method uses material coordinates, coordinates fixed to fluid particles. Each fluid particle is described by a particular set of coordinates which move forever with that particle. Eulerian coordinates are fixed in space rather than fixed to matter. The Eulerian coordinates of a point in space remain unchanged regardless of the fluid motion at that point.

The Lagrangian acceleration is the acceleration experienced by an observer riding on a fluid particle passing a point. The Eulerian acceleration is the rate of change of velocity at that point as seen by an observer who remains at the spatial point of interest. The difference may be realized by considering the steady flow through a nozzle. A fluid particle undergoes a continuous acceleration as it passes through the nozzle. The Lagrangian acceleration is everywhere nonzero. The Eulerian acceleration, however, is everywhere zero because the flow is steady. At any fixed point in space the velocity remains constant. The Eulerian and Lagrangian accelerations differ in this case because the flow is non-uniform. There is a gradient of velocity in the direction of motion.

In the usual notation, the temporal derivative of a quantity in Lagrangian coordinates is written as D/Dt and is known as a material derivative or substantial derivative. The temporal derivative in Eulerian coordinates is $\partial/\partial t$. It is desirable to relate these two derivatives to facilitate understanding of derived results and to permit the use of either method.

A property b may be considered to change with position and time. At time t , a selected fluid element has a value $b(x,y,z,t)$ of this property. A small time Δt later, the element has moved to Eulerian coordinates $x+\Delta x$, $y+\Delta y$, $z+\Delta z$ and the property has the value $b(x+\Delta x, y+\Delta y, z+\Delta z, t+\Delta t)$, so that the Lagrangian change of this property is

$$\Delta b = b(x+\Delta x, y+\Delta y, z+\Delta z, t+\Delta t) - b(x,y,z,t)$$

This expression may be altered to read

$$\begin{aligned} \Delta b = & b(x+\Delta x, y+\Delta y, z+\Delta z, t+\Delta t) - b(x,y,z,t+\Delta t) \\ & + b(x,y,z,t+\Delta t) - b(x,y,z,t) \end{aligned}$$

Dividing by Δt , letting Δt approach zero so as to obtain derivatives, and expressing the first derivative on the right hand side by means of the chain rule

$$\frac{Db}{Dt} = \frac{\partial b}{\partial x} \frac{Dx}{Dt} + \frac{\partial b}{\partial y} \frac{Dy}{Dt} + \frac{\partial b}{\partial z} \frac{Dz}{Dt} + \frac{\partial b}{\partial t}$$

The perspicacious student will note that the Lagrangian derivatives of position are the fluid velocities

$$u = \frac{Dx}{Dt}, \quad v = \frac{Dy}{Dt}, \quad w = \frac{Dz}{Dt} \quad (1)$$

so that the relation between Lagrangian and Eulerian derivatives reduces to

$$\frac{Db}{Dt} = \frac{\partial b}{\partial t} + u \frac{\partial b}{\partial x} + v \frac{\partial b}{\partial y} + w \frac{\partial b}{\partial z}$$

or

$$\frac{Db}{Dt} = \frac{\partial b}{\partial t} + (\vec{V} \cdot \nabla) b \quad (2)$$

The substantial derivative or material derivative is equal to the local derivative plus the convective derivative.

The same form of relation applies when the property is a vector quantity such as velocity. The immediate result is

$$\frac{D\vec{V}}{Dt} = \frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla) \vec{V} \quad (3)$$

and the terms on the right hand side are the local acceleration and convective acceleration. In the previously cited example of steady flow through a nozzle, the local acceleration is zero while the convective acceleration is not.

2.7 THE NAVIER-STOKES EQUATIONS

The Navier-Stokes equations are basically momentum equations applied to the motion of a fluid. They differ from a purely momentum equation in that a phenomenological relation between the stresses and strain rates is used in the Navier-Stokes equations. The stress-strain rate relation is not fundamental as is conservation of momentum. Different fluids possess different stress-strain rate relations. The most common relation, one that adequately describes all fluids of interest here, is that of a Newtonian fluid. We shall consider only Newtonian fluids. Furthermore, the fluid flow in the neighborhood of a particle, resulting from the particle motion, is to an excellent degree of approximation, incompressible. The flow is a low Mach number unconfined flow and consequently, density changes are slight. Accordingly, we will use the stress-strain rate relations for an incompressible Newtonian fluid, equations 2.3-(7) and 2.3-(8). We will further take the viscosity to be constant since viscosity depends most strongly on temperature and differences of temperature in the immediate neighborhood of a particle are ordinarily small.

Consider again the fluid element shown in Figure 2.3. Let the dimensions of this element be Δx , Δy and Δz . Then applying Newton's second law to the fluid (not to the region of space occupied by the fluid) we have

$$\vec{F} = m \frac{D\vec{V}}{Dt}$$

or

$$\vec{F} = \rho_f \Delta x \Delta y \Delta z \frac{D\vec{V}}{Dt} \quad (1)$$

But the force results from the stresses on the surface and from any body forces, $\bar{f}$, acting on the fluid. Summing forces in the x direction

$$\begin{aligned} F_x = & [(\sigma_{xx}|_{x+\Delta x} - \sigma_{xx}|_x) \Delta y \Delta z \\ & + (\sigma_{yx}|_{y+\Delta y} - \sigma_{yx}|_y) \Delta x \Delta z \\ & + (\sigma_{zx}|_{z+\Delta z} - \sigma_{zx}|_z) \Delta x \Delta y] \hat{i} \\ & + f_x \Delta x \Delta y \Delta z \hat{i} \end{aligned}$$

and equating this expression to the x component of (1), dividing by $\Delta x \Delta y \Delta z$, and letting each dimension approach zero so as to obtain derivatives, we find

$$\rho_f \frac{Du}{Dt} = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + f_x \quad (2)$$

Substituting the stress-strain rate relations 2.3-(7) and 2.3-(8), and rearranging

$$\begin{aligned} \rho_f \frac{Du}{Dt} = & - \frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + f_x \\ & + \mu \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \end{aligned}$$

The last term in parentheses may be recognized as the divergence of velocity which vanishes for incompressible flow. The x component of the Navier-Stokes equations for an incompressible Newtonian fluid with constant viscosity is

$$\begin{aligned} \rho_f \frac{Du}{Dt} = & \rho_f \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) \\ = & - \frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + f_x \end{aligned} \quad (3)$$

The y and z components are

$$\rho_f \frac{Dv}{Dt} = \rho_f \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) \quad (4)$$

$$= - \frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) + f_y$$

and

$$\rho_f \frac{Dw}{Dt} = \rho_f \left(\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) \quad (5)$$

$$= - \frac{\partial p}{\partial z} + \mu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) + f_z$$

respectively. In vector notation, these are

$$\rho_f \frac{D\vec{V}}{Dt} = \rho_f \left(\frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla) \vec{V} \right) \quad (6)$$

$$= - \nabla p + \mu \nabla^2 \vec{V} + \vec{f}$$

The student is cautioned to carefully account for derivatives of unit vectors when evaluating these vector expressions in non-Cartesian coordinate systems. The vector identities

$$(\vec{V} \cdot \nabla) \vec{V} = \frac{1}{2} \nabla (\vec{V} \cdot \vec{V}) - \vec{\nabla}_x (\nabla_x \vec{V}) \quad (7)$$

and

$$\nabla^2 \vec{V} = \nabla (\nabla \cdot \vec{V}) - \vec{\nabla}_x (\nabla_x \vec{V}) \quad (8)$$

are frequently useful.

2.8 REYNOLDS NUMBER

The Reynolds number is a dimensionless group whose magnitude expresses the relative importance of inertial and viscous effects in a fluid flow. Depending on the value of the Reynolds number, inertial effects or viscous effects may be neglected in the analysis of the fluid flow in certain regions. The ability to safely discard terms of the Navier-Stokes equations is important because these equations are nonlinear and few exact solutions of the complete Navier-Stokes equations have been found.

Consider a particle, having a characteristic dimension d , travelling with a velocity U through a fluid. The flow is steady so the fluid acceleration is the convective acceleration. We wish to estimate the relative magnitude of the effects of this acceleration and the viscous drag in the Navier-Stokes equation, 2.7-(6). Accordingly, an order of magnitude estimate of $\rho_f(\vec{V} \cdot \nabla)\vec{V}$ yields $\rho_f U^2 / d$ if we estimate that velocity changes of the order of U occur over distances of order d . Similarly, we estimate $\mu \nabla^2 \vec{V}$ to have a magnitude of the order of $\mu U / d^2$. Neither of these estimates is precise, since the actual magnitudes vary from point to point. The purpose of these estimates is only to give an average indication of relative importance.

The ratio of the two expressions is dimensionless and is defined as the Reynolds number

$$Re = \frac{\rho_f U^2 / d}{\mu U / d^2}$$

or

$$Re = \frac{\rho_f U d}{\mu} = \frac{U d}{\nu} \tag{1}$$

ν is the kinematic viscosity of the fluid, μ/ρ_f . When the Reynolds number is much less than one, inertial effects are negligible in the neighborhood of the particle and need not be considered in determining the drag. For low Reynolds number flow, 2.7-(6) reduces to

$$\rho_f \frac{\partial \vec{V}}{\partial t} = - \nabla p + \mu \nabla^2 \vec{V} + \vec{f}$$

It should be noticed that this equation, unlike 2.7-(6), is linear. Its solution is very much easier than the solution of 2.7-(6), warranting use of the simplification when it is valid. For steady motion without body forces, the equation of motion reduces simply to

$$0 = -\nabla p + \mu \nabla^2 \vec{V}$$

When the Reynolds number is large and viscous effects are neglected in 2.7-(6), we obtain, in the absence of body forces,

$$\rho_f \frac{D\vec{V}}{Dt} = -\nabla p$$

the Euler equation. Even at high Reynolds numbers, there is a region of fluid adjacent to the surface of any moving body where viscous forces are of the same magnitude as inertia forces. This layer is termed the boundary layer.

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PROBLEMS

1. Find an expression for the mean free path in a gas composed of two species having different number densities and radii.
2. Before Kelvin, certain flows were described as motions where $u dx + v dy + w dz$ is an exact differential. How would we describe such flows today?
3. Find the equation governing the velocity potential in an irrotational incompressible flow.
4. Derive the continuity equation for an incompressible fluid in spherical coordinates by accounting for the flow entering and leaving an infinitesimal element whose surfaces are coordinate surfaces in spherical coordinates.
5. Consider a fluid rotating as a rigid body. Determine the convective acceleration of the fluid. Does your answer make sense? (Depending upon the approach used, the student may easily miss an important point.)
6. Obtain the radial component of the Navier-Stokes equations in cylindrical coordinates. Assume the fluid is Newtonian, incompressible, and has constant viscosity. Obtain this result by considering the forces acting on an element of fluid whose surfaces coincide with the coordinate surfaces. You should also demonstrate that the fluid acceleration expression is

$$\rho_f \left(\frac{Dv_r}{Dt} - \frac{v_\theta^2}{r} \right)$$

7. An alternative derivation of the Navier-Stokes equations may be constructed by using Reynolds transport theorem to obtain a control volume form of the momentum equation, then applying this relation to an elemental region of space. Derive the Navier-Stokes equations for an incompressible fluid in this manner.

8. A spherical particle of radius a sits on a smooth plane surface. An amount of liquid whose volume is very much less than the particle volume symmetrically fills the space between the particle and plane near the contact point. If the liquid angle of contact is zero at both the particle surface and the planar surface, what is the net force, due to surface tension, holding the particle to the surface.

3.1 ORTHOGONAL CURVILINEAR COORDINATES

The governing equations for field problems of mathematical physics are, in general, most simply expressed in a Cartesian coordinate system. The solutions of problems of interest are, however, frequently most readily obtained by using other coordinate systems. The problem is solved by finding the solution of the field equation that satisfies the imposed boundary conditions. As a result, the shape of the boundary should be considered in choosing a coordinate system consistent with the geometry of the problem.

Use of a well chosen coordinate system can simplify the problem in several ways. The boundary location is very easily expressed if the physical surface corresponds to the surface defined by holding one of the coordinates constant. The boundary condition itself may be more readily expressed in the non-Cartesian coordinates. The coordinate system may be chosen to take advantage of symmetry.

As an illustrative example, let us momentarily consider a problem that we will treat in detail shortly. A spherical particle is at rest with respect to an observer. An unbounded fluid stream, uniform far from the particle, flows past the particle. The boundary shape obviously suggests the use of a spherical coordinate system. The surface of the particle is more simply represented by

$$r = a$$

than by the Cartesian equivalent,

$$x^2 + y^2 + z^2 = a^2$$

The boundary condition of no relative fluid velocity normal to the particle surface is simply

$$v_r = 0$$

in spherical coordinates while the Cartesian equivalent involves three velocity components as well as the coordinates of the position on the boundary. As a rotational coordinate system, the spherical coordinates take advantage of the rotational symmetry of this problem. Only two components of velocity are non-zero and they are functions of only two coordinates.

There are disadvantages as well in the use of non-Cartesian coordinates. In addition to the generally more complex form of the equations to be solved, methods of solution that have proven valuable in Cartesian coordinates may fail in another coordinate system. As an example, a differential equation that can be solved using the method of separation of variables in Cartesian coordinates is not necessarily separable in a different coordinate system.

Rather than individually consider each coordinate system that may be of use to us, it is preferable to consider a general class of coordinate systems. Without specifying a coordinate system, it is possible to obtain results useful in the employment of several different coordinate systems. We will limit the generality of this discussion only by restricting consideration to those curvilinear coordinate systems that are termed orthogonal. The meaning of this term is explained below. The restriction to orthogonal curvilinear coordinates is made because of resulting simplification in the analysis and because orthogonal coordinate systems are, by far, the most useful. All coordinate systems in common use are orthogonal.

The location of any point in a Cartesian coordinate system is specified by the three Cartesian coordinates (x, y, z) of that point. The location of this point may also be expressed in terms of its coordinates (u_1, u_2, u_3) in any other coordinate system. The sets of coordinates are related by

$$\begin{aligned} u_1 &= u_1(x, y, z) \\ u_2 &= u_2(x, y, z) \\ u_3 &= u_3(x, y, z) \end{aligned} \tag{1}$$

or, the inverse,

$$\begin{aligned} x &= x(u_1, u_2, u_3) \\ y &= y(u_1, u_2, u_3) \\ z &= z(u_1, u_2, u_3) \end{aligned} \tag{2}$$

The surface defined by holding one such coordinate constant is called a coordinate surface. Figure 3.1 shows sets of coordinate surfaces in three familiar coordinate systems and also shows a general representation of such surfaces. Note that in each of the familiar cases, the coordinate surfaces at any point, P, are perpendicular to one another. When the coordinate surfaces of a curvilinear coordinate system intersect at right angles, the coordinate system is said to be orthogonal.

The position of the point P can be specified by a position vector $\vec{R}$.

$$\vec{R} = \vec{R}(u_1, u_2, u_3) \quad (3)$$

An infinitesimal change in position $d\vec{R}$ resulting from an infinitesimal change in one of the coordinates, e.g. du_1 , while the other coordinates are held constant, is given by

$$d\vec{R} = \frac{\partial \vec{R}}{\partial u_1} du_1 \quad (4)$$

Since u_2 and u_3 are held constant, the vector $d\vec{R}$ is tangent to the u_1 curve, the curve lying along the intersection of the u_2 and u_3 coordinate surfaces. Since the coordinate surfaces are orthogonal, the vector $d\vec{R}$ is also normal to the u_1 coordinate surface. The unit vector that is tangent to the u_1 coordinate curve and normal to the u_1 coordinate surface is denoted by $\hat{e}_1$. The change in position may accordingly be written

$$d\vec{R} = \left| \frac{\partial \vec{R}}{\partial u_1} \right| du_1 \hat{e}_1 \quad (5)$$

The scalar quantity, $\left| \frac{\partial \vec{R}}{\partial u_1} \right|$, is the ratio of the distance travelled, $|d\vec{R}|$, to the change, du_1 , in the coordinate u_1 . This quantity is called a scale factor or metric coefficient and is designated h_1 . We have then

$$d\vec{R} = h_1 du_1 \hat{e}_1 \quad (6)$$

In general, the differential vector $d\vec{R}$ is

$$d\vec{R} = \frac{\partial \vec{R}}{\partial u_1} du_1 + \frac{\partial \vec{R}}{\partial u_2} du_2 + \frac{\partial \vec{R}}{\partial u_3} du_3 \quad (7)$$

or

a. Cartesian Coordinates

b. Cylindrical Coordinates

Figure 3.1 Coordinate surfaces

c. Spherical Coordinates

d. Unspecified Orthogonal Curvilinear Coordinates

Figure 3.1 Coordinate Surfaces

$$d\vec{R} = h_1 du_1 \hat{e}_1 + h_2 du_2 \hat{e}_2 + h_3 du_3 \hat{e}_3 \quad (8)$$

Note that the three unit vectors of an orthogonal coordinate system are mutually perpendicular.

The scale factors can be easily calculated in Cartesian, cylindrical, and spherical coordinates using elementary geometry. In Cartesian coordinates, when x is changed by an infinitesimal amount, dx , the infinitesimal arc length or distance travelled is also dx .

$$h_x = \left| \frac{dx}{dx} \right| = 1$$

h_y and h_z are also unity by the same argument. In the cylindrical coordinate system,

$$h_\rho = h_z = 1$$

but when the angle ϕ is changed by an amount $d\phi$, the differential arc length is $\rho d\phi$. Consequently,

$$h_\phi = \left| \frac{\rho d\phi}{d\phi} \right| = \rho$$

It is easily verified that the metric coefficients of the spherical coordinate system are

$$h_r = 1, \quad h_\theta = r, \quad h_\phi = r \sin\theta$$

The demonstration is left as an exercise to the reader. In more complex coordinate systems, the magnitude of the scale factors is not so obvious. The scale factors may be calculated, however, using a straight forward procedure.

The differential arc length, ds , of an arbitrary infinitesimal displacement is given by

$$(ds)^2 = d\vec{R} \cdot d\vec{R} = h_1^2 (du_1)^2 + h_2^2 (du_2)^2 + h_3^2 (du_3)^2 \quad (9)$$

where (8) has been used. We may also write

$$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2, \quad (10)$$

the equivalent expression in Cartesian coordinates. Comparing these expressions, we note that their equivalence requires

$$\begin{aligned} h_1^2 &= \left(\frac{\partial x}{\partial u_1} \right)^2 + \left(\frac{\partial y}{\partial u_1} \right)^2 + \left(\frac{\partial z}{\partial u_1} \right)^2 \\ h_2^2 &= \left(\frac{\partial x}{\partial u_2} \right)^2 + \left(\frac{\partial y}{\partial u_2} \right)^2 + \left(\frac{\partial z}{\partial u_2} \right)^2 \\ h_3^2 &= \left(\frac{\partial x}{\partial u_3} \right)^2 + \left(\frac{\partial y}{\partial u_3} \right)^2 + \left(\frac{\partial z}{\partial u_3} \right)^2 \end{aligned} \quad (11)$$

The scale factors can be found by differentiating equations of the form of (2).

In most coordinate systems of interest, the scale factors are already determined and the results tabulated. The availability of tabulated properties for a coordinate system are an important factor in assessing the possible advantages of using that coordinate system. See Appendix B for some properties of several coordinate systems. Moon and Spencer [1] have developed very extensive tables.

3.2 VECTOR OPERATIONS IN ORTHOGONAL CURVILINEAR COORDINATES

In order to easily use orthogonal curvilinear coordinates, expressions for vector operations and commonly encountered differential equations should also be tabulated or otherwise readily found. We shall derive a general expression for the components of the gradient of a scalar and then simply list expressions for other vector operations.

Ω is a scalar function of position

$$\Omega = \Omega(u_1, u_2, u_3) \quad (1)$$

Its gradient, $\nabla\Omega$, is a vector whose direction and magnitude are those corresponding to the maximum rate of change of Ω with distance. In the (u_1, u_2, u_3) coordinate system, $\nabla\Omega$ can be written in terms of its components.

$$\nabla\Omega = a_1 \hat{e}_1 + a_2 \hat{e}_2 + a_3 \hat{e}_3 \quad (2)$$

Expressions for the a 's in terms of operations on Ω are our goal. To find these expressions, consider the change in Ω , $d\Omega$, resulting from some change of

position, $d\vec{R}$.

$$d\Omega = \nabla\Omega \cdot d\vec{R} = a_1 h_1 du_1 + a_2 h_2 du_2 + a_3 h_3 du_3 \quad (3)$$

Equations (2) and 3.1 - (8) were used. Alternatively, a chain rule expansion yields

$$d\Omega = \frac{\partial\Omega}{\partial u_1} du_1 + \frac{\partial\Omega}{\partial u_2} du_2 + \frac{\partial\Omega}{\partial u_3} du_3 \quad (4)$$

Equating these results, we obtain

$$a_1 = \frac{1}{h_1} \frac{\partial\Omega}{\partial u_1} ; a_2 = \frac{1}{h_2} \frac{\partial\Omega}{\partial u_2} ; a_3 = \frac{1}{h_3} \frac{\partial\Omega}{\partial u_3} \quad (5)$$

or

$$\nabla\Omega = \frac{1}{h_1} \frac{\partial\Omega}{\partial u_1} \hat{e}_1 + \frac{1}{h_2} \frac{\partial\Omega}{\partial u_2} \hat{e}_2 + \frac{1}{h_3} \frac{\partial\Omega}{\partial u_3} \hat{e}_3 \quad (6)$$

The following relations may also be derived and are listed here for convenience. For a vector, $\vec{A}$, written in terms of its components in the (u_1, u_2, u_3) coordinate system

$$\vec{A} = A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3 \quad (7)$$

we have for the divergence,

$$\nabla \cdot \vec{A} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} (h_2 h_3 A_1) + \frac{\partial}{\partial u_2} (h_1 h_3 A_2) + \frac{\partial}{\partial u_3} (h_1 h_2 A_3) \right] \quad (8)$$

and for the curl,

$$\nabla \times \vec{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix} \quad (9)$$

in the form of a determinant.

These basic operations combine to yield other useful results, e.g.

$$\begin{aligned} \nabla^2 \Omega &= \nabla \cdot (\nabla \Omega) \\ &= \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial \Omega}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left(\frac{h_3 h_1}{h_2} \frac{\partial \Omega}{\partial u_2} \right) + \frac{\partial}{\partial u_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial \Omega}{\partial u_3} \right) \right] \end{aligned} \quad (10)$$

3.3 ROTATIONAL COORDINATE SYSTEMS

Because of the common occurrence of axisymmetric flows in the analysis of particle motion, rotational coordinate systems are of particular interest to us. A rotational coordinate system has coordinates (u_1, u_2, ϕ) . Figure 3.2 shows such a coordinate system. The angle ϕ is an azimuthal angle, similar to longitude. The ϕ coordinate surface, defined by ϕ equal to a constant, is a plane called a meridian plane. The axis of revolution lies in all meridian planes. The unit vector, $\hat{e}_\phi$, is normal to the meridian plane. ϕ here is the same ϕ as we have already used in spherical and cylindrical coordinates.

Because of their orthogonality to $\hat{e}_\phi$, the unit vectors, $\hat{e}_1$ and $\hat{e}_2$, lie in the meridian plane. The u_1 and u_2 curves also lie in the meridian plane. The u_2 coordinate surface and the u_1 coordinate surface are formed by rotating the u_1 and u_2 curves about the axis of revolution. These surfaces are surfaces of revolution.

The scale factor, h_ϕ , has been found in the discussion of the cylindrical and spherical coordinate systems. For all rotational coordinates,

$$h_\phi = \rho$$

h_ϕ is equal to the perpendicular distance from the axis of revolution. It is frequently desirable to express this distance in terms of u_1 and u_2

$$h_\phi = \rho(u_1, u_2)$$

as when we write

$$h_\phi = r \sin \theta$$

in spherical coordinates.

Figure 3.2 Rotational Orthogonal Coordinate System

The other scale factors, h_1 and h_2 , can be found using equations 3.1-(11) which utilize the equations 3.1-(2) relating Cartesian coordinates to (u_1, u_2, u_3) coordinates. It is left as an exercise to show that the scale factors may also be calculated using a transformation

$$\begin{aligned} u_1 &= u_1(z, \rho) \\ u_2 &= u_2(z, \rho) \\ \phi &= \phi \end{aligned} \quad (1)$$

from cylindrical coordinates. The result is

$$\begin{aligned} h_1^2 &= \left(\frac{\partial z}{\partial u_1} \right)^2 + \left(\frac{\partial \rho}{\partial u_1} \right)^2 \\ h_2^2 &= \left(\frac{\partial z}{\partial u_2} \right)^2 + \left(\frac{\partial \rho}{\partial u_2} \right)^2 \\ h_\phi^2 &= \rho^2 \end{aligned} \quad (2)$$

The coordinate system (u_1, u_2, ϕ) is said to be right handed if

$$\hat{e}_1 \times \hat{e}_2 = \hat{e}_\phi$$

Similar expressions generated by a cyclic permutation of coordinates also correspond to right handedness.

$$\hat{e}_2 \times \hat{e}_\phi = \hat{e}_1$$

$$\hat{e}_\phi \times \hat{e}_1 = \hat{e}_2$$

Unless otherwise stated, we will always take our coordinate systems to be right handed.

3.4 CONFORMAL TRANSFORMATIONS

Many rotational coordinate systems may be generated by using a conformal transformation in a meridian plane. Any position in the meridian plane is determined by values of z and ρ in cylindrical coordinates or by values of u_1 and u_2 in (u_1, u_2, ϕ) coordinates. These coordinates are related by relations of the form 3.3-(1). When the transformation between these coordinates can be represented by an analytic function

$$z + i\rho = f(u_1 + iu_2) \quad (1)$$

the transformation is said to be conformal. The most important characteristic of conformal transformations is that they are angle preserving. The angle formed by the intersection of two lines in a plane will have the same value in the transformed plane. Of considerable importance to us is the simple form resulting for the metric coefficients.

The metric coefficients of a rotational coordinate system may be found using 3.3-(2). When the relation between the coordinates is of the form (1), then the terms appearing on the right hand side of 3.3-(2) are readily related. Defining

$$w \equiv u_1 + iu_2$$

we have

$$\frac{\partial z}{\partial u_1} + i \frac{\partial \rho}{\partial u_1} = \frac{\partial f}{\partial u_1} = \frac{df}{dw} \frac{\partial w}{\partial u_1} = \frac{df}{dw} \quad (2)$$

$$\frac{\partial z}{\partial u_2} + i \frac{\partial \rho}{\partial u_2} = \frac{\partial f}{\partial u_2} = \frac{df}{dw} \frac{\partial w}{\partial u_2} = i \frac{df}{dw} \quad (3)$$

Multiplying (3) by $-i$, equating the result to (2), and separating real and imaginary parts, we obtain

$$\frac{\partial z}{\partial u_1} = \frac{\partial \rho}{\partial u_2} \quad (4)$$

and

$$\frac{\partial z}{\partial u_2} = - \frac{\partial \rho}{\partial u_1} \quad (5)$$

These are the Cauchy-Riemann equations. When substituted into 3.3-(2), they show that any rotational coordinate system formed by a conformal transformation in the meridian plane has the property that

$$h_1^2 = h_2^2$$

3.5 CARTESIAN TENSORS

The conservation equations of fluid flow can be written in a very compact and easily used form if tensor notation is employed. In particular, if we restrict ourselves to Cartesian axes when using tensor notation, the notation is exceptionally simple. The advantages of using tensor notation instead of vector notation arise from the ability to describe operations and quantities that may not be expressed in vector notation. The fluid stress and strain rate are examples of such quantities in fluid flow. Each of these quantities is what is termed a second order tensor.

Let us first consider tensors and tensor notation from a geometrical point of view. We will then see how these definitions and operations relate to physical quantities. Consider two sets of Cartesian coordinates, (x, y, z) and (x', y', z') having the same origin. A point described by the coordinates x, y, z can also be described in terms of the primed coordinate system. The primed coordinates of a point are related to the unprimed coordinates by the relations

$$\begin{aligned} x' &= l_1 x + m_1 y + n_1 z \\ y' &= l_2 x + m_2 y + n_2 z \\ z' &= l_3 x + m_3 y + n_3 z \end{aligned} \tag{1}$$

l_1 is the cosine of the angle between the x and x' axes. Each of the other coefficients, $m_1, n_1, l_2, \dots, n_3$, is also the cosine of the angle between the two axes that it relates in (1). Accordingly, n_2 is the cosine of the angle between the y' and z axes. Figure 3.3 shows these axes and angles in two dimensions.

In agreement with (1) we also have

$$\begin{aligned} x &= l_1 x' + l_2 y' + l_3 z' \\ y &= m_1 x' + m_2 y' + m_3 z' \\ z &= n_1 x' + n_2 y' + n_3 z' \end{aligned} \tag{2}$$

Rather than continue to use the common x, y, z notation, it proves convenient to develop a more compact notation. This notation uses subscripts to denote the various axes. For (x, y, z) , write (x_1, x_2, x_3) . For (x', y', z') , write (x'_1, x'_2, x'_3) . Then the unprimed coordinates are x_i where i takes on the value 1, 2, or 3. The primed coordinates are x'_j where $j = 1, 2, 3$. The direction cosines may be replaced by twice subscripted terms, a_{ij} , which then permit each of the equations (1) to be expressed by the single relation

Figure 3.3 Cartesian coordinates in two dimensions

$$x_j = a_{1j}x_1 + a_{2j}x_2 + a_{3j}x_3 = \sum_{i=1}^3 a_{ij}x_i \quad (3)$$

where j is 1, 2, or 3. The transformations are expressed in a more compact notation than (1), a notation that recognizes that the three equations of (1) are all of the same form. A further reduction in the amount that need be written is accomplished by now introducing the Einstein summation convention. The convention is that whenever a subscript appears twice in a term, then a summation over all values of the subscript is understood to occur. Similarly, when a subscript appears only once, no summation occurs. The utility of this convention can be seen in (3) where the subscript i is repeated, the subscript j appears only once, and the summation is over all values of i . Using this notation, (1) or (3) is written

$$x'_j = a_{ij}x_i \quad (4)$$

and (2) becomes

$$x_i = a_{ij}x'_j \quad (5)$$

Any quantity that transforms in the same manner as position is termed a vector. Its components in primed and unprimed coordinates are related by

$$v'_j = a_{ij}v_i$$

and

$$v_i = a_{ij}v'_j$$

Because each component is identified by a single subscript, such a quantity is also called a first order tensor. The tensor v_i is a first order tensor with components v_1 , v_2 , and v_3 .

A scalar has a single value whose magnitude is independent of the choice or orientation of the axes. Since no subscript is required, a scalar is also termed a tensor of zero order.

The product of a vector and a scalar is a vector because the product transforms like a vector

$$bv'_j = a_{ij}(bv_i)$$

The sum of two vectors is a vector because the sum transforms like a vector

$$\hat{v}_j + \hat{w}_j = a_{ij} (v_i + w_i)$$

Other operations are simply written using the summation convention. The inner product or dot product or scalar product of two vectors is $v_i w_i$. The divergence of a vector v_i is $\partial v_i / \partial x_i$. Accordingly, the continuity equation for an incompressible fluid is in tensor, vector, and scalar notations

$$\frac{\partial v_i}{\partial x_i} = \nabla \cdot \vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

If we take two vectors v_i and w_k and form the set of products of their components we obtain nine quantities that may be denoted by $v_i w_k$. The subscripts may each take on any of the three permissible values. In a different frame (primed axes). this set becomes

$$\hat{v}_j \hat{w}_\ell = a_{ij} v_i * a_{k\ell} w_k$$

since v_i and w_k are each vectors. The order of terms on the right hand side is immaterial to the operation, so we can write

$$\hat{v}_j \hat{w}_\ell = a_{ij} a_{k\ell} v_i w_k \quad (6)$$

The subscript convention is used twice on the right hand side since both i and k are repeated, indicating summation over each of these indices.

The transformation rule expressed by (6) is the transformation rule of a second order tensor. Any quantity that transforms in this manner is a second order tensor. As the name implies, a second order tensor has two indices. If

$$\gamma_{j\ell} = a_{ij} a_{k\ell} \gamma_{ik} \quad (7)$$

then γ_{ik} is a second order tensor. By induction, one can produce the transformation rules for higher order tensors.

In section 2.3, it was stated that the stress was a second order tensor. In order to verify this statement, it is necessary to show that the stress, a set of nine quantities denoted by σ_{ij} , transforms according to (7)

$$\sigma'_{j\ell} = a_{ij}a_{k\ell}\sigma_{ik} \quad (8)$$

and retains the same physical significance. The quantities $\sigma'_{j\ell}$ are the physical stresses measured with respect to the primed coordinates. The mere fact that $\sigma'_{j\ell}$ is a doubly subscripted expression representing nine quantities is insufficient to qualify the stress as a second order tensor.

We can demonstrate that the stress is indeed a second order tensor by means of the following argument. The net force on any infinitesimal element must be independent of the choice of coordinates used to express that force. An infinitesimal surface on this element may have both normal and tangential forces exerted on it. The product of the component of stress σ_{ij} and the area ds is $\sigma_{ij}ds$ and, by definition of the stress, is the force acting in the j direction on an element of surface whose normal lies in the i direction. The transformation rule for the stress can now be obtained by balancing the forces acting on the faces of an infinitesimal element shaped like a tetrahedron.

Figure 3.4 shows an element with three orthogonal surfaces, one normal to each of the unprimed axes, and one inclined surface whose normal lies in the x'_1 direction. By resolving the forces along the primed axes, the stress components in the primed coordinates are related to the stress components in the unprimed coordinates.

As an example, the normal stress on the inclined face may be calculated. First, all of the forces on the other faces are found, then these are resolved into components parallel or perpendicular to the x'_1 axis.

Figure 3.4 Tetrahedral element

If the area of the face normal to the x_1 axis is s_1 then the forces on this face in the x_1 , x_2 , and x_3 directions are $\sigma_{11}s_1$, $\sigma_{12}s_1$ and $\sigma_{13}s_1$ respectively. Since the force is a vector, the components of these forces in the x'_1 direction are $a_{11}\sigma_{11}s_1$, $a_{21}\sigma_{12}s_1$ and $a_{31}\sigma_{13}s_1$. The area s_1 of this surface can also be expressed in terms of the area s'_1 of the inclined surface. s_1 is the projected area of the inclined surface on a plane normal to x_1 . Accordingly

$$s_1 = a_{11} s'_1$$

and the net component of force in the x'_1 direction exerted on the face normal to x_1 is

$$(a_{11}\sigma_{11} + a_{21}\sigma_{12} + a_{31}\sigma_{13}) s'_1$$

The surface areas are not the components of a first order tensor, the subscript notation implying nothing in that regard.

The force on each of the other two orthogonal faces is resolved in the x'_1 direction. The sum of these forces is

$$\begin{aligned} & [a_{11}\sigma_{11} + a_{21}\sigma_{22} + a_{31}\sigma_{33} \\ & + a_{11}\sigma_{12} + a_{21}\sigma_{21} + a_{11}\sigma_{13} + a_{31}\sigma_{31} \\ & + a_{21}\sigma_{23} + a_{31}\sigma_{32}] s'_1 \end{aligned}$$

Equating this force to the normal force on the x'_1 face, there results

$$\begin{aligned} \sigma'_{11} = & a_{11}^2 \sigma_{11} + a_{21}^2 \sigma_{22} + a_{31}^2 \sigma_{33} \\ & + a_{11}a_{21}(\sigma_{12} + \sigma_{21}) + a_{11}a_{31}(\sigma_{13} + \sigma_{31}) \\ & + a_{21}a_{31}(\sigma_{23} + \sigma_{32}) \end{aligned} \quad (9)$$

which is written in tensor notation as

$$\sigma'_{ll} = a_{il} a_{kl} \sigma_{ik}$$

in agreement with (8). Each of the other stresses in the primed coordinates may be found in a similar manner. The result is that the stress transforms according to (8). The stress is a second order tensor.

If there are no body moments in the fluid, then consideration of the moments on an elementary block shows that

$$\sigma_{ij} = \sigma_{ji} \quad (10)$$

a result that can be used to simplify expressions such as (9). (10) states that the stress tensor is symmetric.

The discussion just completed demonstrates why an expression such as 2.3-(6) does not have the necessary generality to always be valid. Since the stress is a tensor, any relation between it and the fluid strain rate must also be tensorial.

A strain rate tensor, e_{ij} , is defined by

$$e_{ij} = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)$$

which is clearly a symmetric tensor. The corresponding antisymmetric tensor is

$$\omega_{ij} = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right)$$

the rotation tensor. The sum of these two tensors is the deformation rate tensor.

Another tensor of interest to us is the Kronecker delta or substitution tensor which has the property

$$\delta_{ik} = 1, \quad k = i$$

$$\delta_{ik} = 0, \quad k \neq i$$

and may be shown to obey the transformation rules for a second order tensor using geometrical relations among the direction cosines.

The last tensor we need define here is the alternating tensor, ϵ_{ijk} , a third order tensor which is a set of twenty seven quantities whose values are determined as follows. In the event of any repeated subscript, the component has the value zero. If the subscripts are not repeated and appear in cyclic order, i.e., 123, 231 or 312 the value of the term is unity. If the subscripts are not repeated but do not appear in cyclic order, then the component is equal to minus one. The alternating tensor is useful, for example, in writing tensor expressions for moments.

Finally, we write the stress-strain rate relation for an incompressible Newtonian fluid. The phenomenological relation is

$$\sigma_{ij} = -p \delta_{ij} + 2\mu e_{ij}$$

which may be recognized as equations 2.3-(7) and 2.3-(8).

SUGGESTED SUPPLEMENTARY SOURCES

Aris, R., Vectors, Tensors, and the Basic Equations of Fluid Mechanics, Prentice-Hall, Englewood Cliffs, 1962

Jeffreys, H., Cartesian Tensors, Cambridge University Press, Cambridge, 1969

Moon, P. and Spencer, D.E., Field Theory Handbook, Springer-Verlag, Berlin, 1961

Morse, P.M. and Feshbach, H., Methods of Theoretical Physics, V. 1 and 2, McGraw-Hill, New York, 1953

PROBLEMS

- 1) Using the expressions for $\vec{\nabla} \times \vec{A}$ and $\vec{\nabla} \Omega$ in orthogonal curvilinear coordinates, verify that for a scalar function Ω , $\vec{\nabla} \times \vec{\nabla} \Omega = 0$.
- 2) Find the scale factors for inverse prolate spheroidal coordinates (η, θ, ϕ)

$$x = \frac{a \sinh \eta \sin \theta \cos \phi}{\cosh^2 \eta - \sin^2 \theta}$$

$$y = \frac{a \sinh \eta \sin \theta \sin \phi}{\cosh^2 \eta - \sin^2 \theta}$$

$$z = \frac{a \cosh \eta \cos \theta}{\cosh^2 \eta - \sin^2 \theta}$$

- 3) Identify the tensor below in terms of vector operations

$$\frac{\partial v_j}{\partial x_i} - \frac{\partial v_i}{\partial x_j}$$

- 4) Resolve forces on a tetrahedral element to relate a tangential stress in one set of Cartesian coordinates to the stresses in another Cartesian coordinate system.
- 5) Verify that the Kronecker delta is a second order tensor.
- 6) Verify that the alternating tensor is a tensor.
- 7) Contraction is the process of setting the subscripts of a tensor equal and summing all over all indices. What does contraction of the strain rate tensor yield if the fluid is incompressible.

AXISYMMETRIC FLOW

4.1 INTRODUCTION

Many flows of interest in the study of particle dynamics possess a rotational symmetry. Such flows are commonly encountered because the particle shapes of greatest interest are bodies of revolution. Clearly, knowledge of the behavior of a spherical particle is of greater utility than similar knowledge concerning a unique irregularly shaped jagged particle. Moreover, the characteristics of irregularly shaped particles are often described in terms of equivalent spherical particles. The sedimentation radius of a particle for example, refers to the radius of a sphere having the same density and sedimentation velocity.

The reason for noting the rotational symmetry of many important flows is that a great computational simplification can be obtained. In the absence of an apparent symmetry, the fluid velocity is specified by three scalar quantities, the velocity components along the coordinate axes, that are each functions of three position coordinates. Using a rotational coordinate system (u_1 , u_2 , ϕ) in the axisymmetric flow, we may simplify the description by recognizing that, by definition, no scalar quantity depends upon the azimuthal angle, ϕ .

$$\frac{\partial}{\partial \phi} = 0 \quad (1)$$

so that all scalar quantities depend only on the other two coordinates.

Furthermore, if the flow is incompressible, an excellent assumption in most cases of particle motion, then the fluid velocity components lying in a meridian plane, i.e. v_1 and v_2 , can be expressed in terms of a single scalar quantity, the stream function. If this stream function, ψ , is known

$$\psi = \psi(u_1, u_2) \quad (2)$$

then the velocities in the meridian plane can be calculated from it. This simplification does not depend upon the azimuthal velocity, v_ϕ , being zero.

The azimuthal velocity is zero in most cases of interest. Its existence is actually a matter of indifference when low Reynolds number motion in the meridian plane is examined because any swirling motion about the axis of

revolution is dynamically uncoupled from the streamline motion.

We find that the axisymmetric incompressible fluid flow can ordinarily be described by a single variable, the stream function, that depends only on two coordinates. The promise of considerable simplification in analysis is fulfilled. Most of the known solutions to flows resulting from particulate motion are solutions to problems of axisymmetric flow.

4.2 THE STREAM FUNCTION

Conservation of mass is expressed in control volume form by the relation

$$\frac{d}{dt} \int_V \rho_f dv + \int_A \rho_f \vec{V} \cdot \hat{n} dA = 0 \quad (1)$$

which simply states that the rate at which mass accumulates within a control volume is equal to the rate at which mass flows in across the bounding surface. This relation takes a particularly simple form for an incompressible fluid. In an incompressible flow, the mass within any fixed volume is constant, whether the flow is steady or not.

$$\frac{d}{dt} \int_V \rho_f dv = 0 \quad (2)$$

Since the density is constant for the incompressible fluid, it may be taken out of the flux integral, leaving

$$\int_A \vec{V} \cdot \vec{n} dA = 0 \quad (3)$$

In an incompressible flow, the net volume flow rate across any closed surface is zero.

Let us form a surface that will be helpful in the study of axisymmetric incompressible flow. Consider any point, P, as in Figure 4.1. A line, starting and terminating on the axis of revolution and passing through P, is drawn in the meridian plane. This line is shown as the solid line of Figure 4.1. A closed surface of revolution is generated by rotating this line about the axis. The dotted line of Figure 4.1 is an intersection of this surface and the meridian plane. P' lies on the hoop formed by rotating the point P about the axis. With the choice of a (u_1, u_2, ϕ) coordinate system of revolution, every point

Fig. 4.1 Projection of a closed surface of revolution

on that hoop has the same values of u_1 and u_2 . In an incompressible flow, the net volume flow rate across the closed surface of revolution vanishes.

The closed surface is composed of two open surfaces which join each other at the hoop formed by rotating the point P about the axis. The meridian curve formed by the intersection of the "top" surface and the meridian plane is the curve, P-A-P'. The meridian curve of the "bottom" surface is P-B-P'. The net volume flow rate across one of these surfaces is, for an incompressible fluid, equal to the net volume flow rate for the other surface. This follows, of course, from (3).

Since the two surfaces were chosen independently, it follows that the volume flow rate of an incompressible fluid must be the same across any such surface extending from the axis and bounded by the hoop formed by revolving P about the axis. This flow rate, consequently does not depend at all on the shape or position of the surface, but only at the location, P (u_1 , u_2), of its boundary.

$$Q = Q(u_1, u_2) \quad (4)$$

Rather than use Q as a measure of the volume flow rate, it will prove convenient to express the volume flow rate in terms of a stream function, ψ , defined by

$$\psi \equiv \frac{Q(u_1, u_2)}{2\pi} \quad (5)$$

The factor of 2π is placed in the definition to prevent 2π from appearing repeatedly in later calculations. The advantage is identical to that obtained when a factor of 4π is introduced into Coulomb's law and a rationalized system of units results.

The reason that the stream function is introduced is that it proves to be very easy to express the fluid velocity at any point in terms of the rate at which stream function changes with position.

The stream function defined here is the Stokes stream function. An analogous quantity used to analyze flow in a plane is the Lagrange stream function.

4.3 VELOCITY AND THE STREAM FUNCTION

The volume flow across the surfaces described in the previous section is due to the fluid having a velocity component perpendicular to the surfaces. For simplicity of argument, we have considered only surfaces of revolution. For any such surface, any azimuthal component of velocity, v_ϕ , is tangential to the surface and makes no contribution to the volume flow rate across the surface. Only the components of velocity in the meridian plane, v_1 and v_2 , can contribute to flow across the surface.

Since neither of these components may have any ϕ dependence in an axisymmetric flow,

$$v_1 = v_1(u_1, u_2), \quad v_2 = v_2(u_1, u_2) \quad (1)$$

the flow in any meridian plane is identical to that in any other meridian plane.

To relate the velocity components to volume flow rate and thus to the stream function, let us examine the flow across an infinitesimal surface element. Referring to Figure 4.2, let us examine the flow across any elemental surface of revolution whose intersection with the meridian plane is a line joining L and M

Fig. 4.2 Axisymmetric flow in an elemental region

The points L and M, are located at (u_1, u_2) and $(u_1 + du_1, u_2 + du_2)$ respectively. The separation is infinitesimal. The value of the stream function at L is ψ , while the stream function at M may have a slightly different value, $\psi + d\psi$, due to flow across the line joining them.

$$(\psi + d\psi) - \psi = d\psi = \frac{dQ}{2\pi} \quad (2)$$

where the differential form of 4.2-(5) is used.

The flow rate dQ across the surface generated by the line L-M is, for the incompressible flow, equal to the sum of the flow rates across the surfaces generated by coordinate curves L-N and N-M

$$d\psi = \frac{dQ}{2\pi} = \frac{1}{2\pi} (dQ_1 + dQ_2) \quad (3)$$

since the net flow into the closed surface formed by these three surfaces is zero.

The flow rate across the coordinate surfaces is expressed simply in terms of velocity components. The flow rate is the integral of the normal component of velocity over the surface area. In an axisymmetric flow, the normal component is not a function of ϕ , so

$$\begin{aligned} dQ_1 &= \vec{V} \cdot \hat{e}_2 dA_1 = v_2 dA_1 \\ dQ_2 &= -\vec{V} \cdot \hat{e}_1 dA_2 = -v_1 dA_2 \end{aligned} \quad (4)$$

dA_1 is the area of the surface generated by rotating the line L-N about the axis. Since the length of the line L-N is $h_1 du_1$,

$$dA_1 = (2\pi\rho) (h_1 du_1) \quad (5)$$

and, similarly

$$dA_2 = 2\pi\rho h_2 du_2 \quad (6)$$

substituting into (3)

$$d\psi = v_2 \rho h_1 du_1 - v_1 \rho h_2 du_2 \quad (7)$$

The reason for including 2π in the definition, 4.2-(5), of the stream function should now be apparent.

Now, if we simply expand $d\psi$ using the chain rule,

$$d\psi = \frac{\partial\psi}{\partial u_1} du_1 + \frac{\partial\psi}{\partial u_2} du_2 \quad (8)$$

and compare (7) and (8), recognizing that the magnitude of the infinitesimal increments, du_1 and du_2 , are arbitrary, we see that

$$v_1 = \frac{-1}{\rho h_2} \frac{\partial\psi}{\partial u_2} ; \quad v_2 = \frac{1}{\rho h_1} \frac{\partial\psi}{\partial u_1} \quad (9)$$

If v_ϕ is zero, the velocity becomes

$$\vec{V} = \frac{-1}{\rho h_2} \frac{\partial\psi}{\partial u_2} \hat{e}_1 + \frac{1}{\rho h_1} \frac{\partial\psi}{\partial u_1} \hat{e}_2 \quad (10)$$

The components of velocity are expressed in terms of operations on the stream function similar to the operations yielding the gradient, 3.1-(17). The components are, however, associated with different unit vectors in the two expressions. Recognizing that

$$\begin{aligned} -\hat{e}_1 &= \hat{e}_\phi \times \hat{e}_2 \\ \hat{e}_2 &= \hat{e}_\phi \times \hat{e}_1 \end{aligned} \quad (11)$$

it is possible to rearrange (10) in the form

$$\vec{V} = \frac{\hat{e}_\phi}{\rho} \times \left(\frac{1}{h_1} \frac{\partial\psi}{\partial u_1} \hat{e}_1 + \frac{1}{h_2} \frac{\partial\psi}{\partial u_2} \hat{e}_2 \right) \quad (12)$$

or

$$\vec{V} = \frac{\hat{e}_\phi}{\rho} \times \nabla \psi \quad (13)$$

Writing the velocity in this form is instructive if we reflect upon what this relation says. Because the velocity is written as the cross product of two vectors, it is perpendicular to each of them. Because $\hat{e}_\phi$ is normal to the

meridian plane, $\vec{V}$ lies, of course, in that plane, $\nabla\psi$ lies in the meridian plane as well. Because the velocity is also normal to the gradient of the stream function, the velocity is everywhere tangential to the lines of constant ψ . These lines are called streamlines. There is no flow across a streamline. The speed is proportional to the magnitude of $\nabla\psi$ and inversely proportional to ρ because the speed is the rate of change of the volume flow rate with respect to area.

$$|\vec{V}| = \frac{dQ}{dA} = \frac{2\pi d\psi}{dA} = \frac{2\pi d\psi}{2\pi\rho ds} = \frac{1}{\rho} \frac{d\psi}{ds} \quad (14)$$

ds is an infinitesimal change in position normal to a streamline.

In the commonly used cylindrical and spherical coordinate systems, the equations (9) for velocity components in terms of the stream function become

$$v_z = -\frac{1}{\rho} \frac{\partial\psi}{\partial\rho} ; \quad v_\rho = \frac{1}{\rho} \frac{\partial\psi}{\partial z} \quad (15)$$

and

$$v_r = -\frac{1}{r^2 \sin\theta} \frac{\partial\psi}{\partial\theta} ; \quad v_\theta = \frac{1}{r \sin\theta} \frac{\partial\psi}{\partial r} \quad (16)$$

respectively.

4.4 THE AZIMUTHAL VELOCITY AND THE SWIRL

If the azimuthal velocity, v_ϕ , is not zero in an axisymmetric flow, it does not affect the stream function. The velocity components in the meridian plane are given by 4.3-(9) and the velocity can be written

$$\vec{V} = \frac{\hat{e}_\phi}{\rho} \times \nabla\psi + v_\phi \hat{e}_\phi \quad (1)$$

Alternatively, a quantity, Ω , called the swirl, may be defined

$$\Omega \equiv \rho v_\phi \quad (2)$$

and

$$\vec{V} = \frac{\hat{e}_\phi}{\rho} \times \nabla\psi + \frac{\Omega}{\rho} \hat{e}_\phi \quad (3)$$

Because the condition of rotational symmetry requires that v_ϕ be independent of ϕ , the swirl depends only on the position in the meridian plane

$$\Omega = \Omega(u_1, u_2) \quad (4)$$

4.5 THE STREAM FUNCTION AND CONTINUITY

The definition of the stream function and the relation of velocity to the stream function were based on two kinematic restrictions. These are the condition of rotational symmetry and the condition, 4.2-(3), that there be no net volume flow rate into any region. It is reasonable, therefore, that any velocity distribution that can be written in the form

$$v_1 = -\frac{1}{\rho h_2} \frac{\partial \psi}{\partial u_2} ; \quad v_2 = \frac{1}{\rho h_1} \frac{\partial \psi}{\partial u_1} ; \quad v_\phi = v_\phi(u_1, u_2) \quad (1)$$

will automatically satisfy continuity, 4.2-(3)

$$\int_A \vec{V} \cdot \hat{n} dA = 0 \quad (2)$$

This is readily shown. Using Gauss' theorem, (2) becomes

$$\int_V \nabla \cdot \vec{V} dv = \int_A \vec{V} \cdot \hat{n} dA = 0 \quad (3)$$

Since this must be true for any volume, the integrand must itself be zero

$$\nabla \cdot \vec{V} = 0 \quad (4)$$

which is the continuity equation for an incompressible fluid as applied to an infinitesimal element. This relation could have been taken as a starting point instead of (2). Regardless, if we substitute the expressions (1) for the velocity components into the expression for the divergence in (u_1, u_2, ϕ) coordinates

$$\nabla \cdot \vec{V} = \frac{1}{h_1 h_2 \rho} \left[\frac{\partial}{\partial u_1} (h_2 \rho v_1) + \frac{\partial}{\partial u_2} (h_1 \rho v_2) + \frac{\partial}{\partial \phi} (h_1 h_2 v_\phi) \right]$$

there results

$$\nabla \cdot \vec{V} = \frac{1}{h_1 h_2 \rho} \left[-\frac{\partial^2 \psi}{\partial u_1 \partial u_2} + \frac{\partial^2 \psi}{\partial u_2 \partial u_1} + 0 \right]$$

which is identically zero since the order of differentiation is immaterial. Velocity components written in terms of ψ automatically satisfy the continuity equation for an incompressible flow. By expressing the velocity components in terms of ψ , the number of equations to be solved is reduced by one.

An alternate demonstration is obtained by using the vector expression for velocity, 4.4-(3) and recognizing that

$$\frac{\hat{e}_\phi}{\rho} = \frac{1}{\rho} \frac{\partial \phi}{\partial \phi} \hat{e}_\phi = \frac{1}{h_\phi} \frac{\partial \phi}{\partial \phi} \hat{e}_\phi = \nabla \phi$$

so that the velocity becomes

$$\vec{V} = \nabla \phi \times \nabla \psi + \Omega \nabla \phi$$

and

$$\nabla \cdot \vec{V} = \nabla \cdot (\nabla \phi \times \nabla \psi) + \nabla(\Omega \nabla \phi)$$

Using vector identities

$$\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot \nabla \times \vec{A} - \vec{A} \cdot \nabla \times \vec{B}$$

and

$$\nabla \times \nabla \phi = 0$$

the first term vanishes. The second term is expanded to yield

$$\nabla \cdot \vec{V} = \nabla \Omega \cdot \nabla \phi + \Omega \nabla^2 \phi$$

which is zero since the two gradients are normal to one another and since $\nabla^2 \phi$ is zero.

4.6 VORTICITY AND THE E^2 OPERATOR

The vorticity $\vec{\zeta}$ is defined by 2.4-(2)

$$\vec{\zeta} = \nabla \times \vec{V}$$

and, in an axisymmetric flow is readily expressed in terms of operations on the stream function and the swirl. The curl is given in orthogonal curvilinear coordinates by 3.2-(9) so we have

$$\vec{\zeta} = \frac{1}{h_1 h_2 \rho} \begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & \rho \hat{e}_\phi \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial \phi} \\ h_1 v_1 & h_2 v_2 & \Omega \end{vmatrix}$$

or

$$\vec{\zeta} = \frac{\hat{e}_1}{h_2 \rho} \frac{\partial \Omega}{\partial u_2} - \frac{\hat{e}_2}{h_1 \rho} \frac{\partial \Omega}{\partial u_1} + \frac{\hat{e}_\phi}{h_1 h_2} \left[\frac{\partial}{\partial u_1} (h_2 v_2) - \frac{\partial}{\partial u_2} (h_1 v_1) \right]$$

The terms involving the swirl are similar in form to the stream function expression 4.3-(10) and are simplified in the same manner

$$\vec{\zeta} = \nabla \Omega \times \frac{\hat{e}_\phi}{\rho} + \frac{\hat{e}_\phi}{h_1 h_2} \left[\frac{\partial}{\partial u_1} (h_2 v_2) - \frac{\partial}{\partial u_2} (h_1 v_1) \right]$$

The velocity components in the meridian plane are expressed in terms of the stream function in 4.3-(9) so the vorticity becomes

$$\vec{\zeta} = \nabla \Omega \times \frac{\hat{e}_\phi}{\rho} + \frac{\hat{e}_\phi}{h_1 h_2} \left[\frac{\partial}{\partial u_1} \left(\frac{h_2}{\rho h_1} \frac{\partial \psi}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left(\frac{h_1}{\rho h_2} \frac{\partial \psi}{\partial u_2} \right) \right]$$

The operation on the stream function in this expression is found to be a very common operation appearing repeatedly in the analysis of axisymmetric viscous flows. Consequently, it is given a symbol, E^2 ,

$$E^2 \equiv \frac{\rho}{h_1 h_2} \left[\frac{\partial}{\partial u_1} \left(\frac{h_2}{\rho h_1} \frac{\partial}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left(\frac{h_1}{\rho h_2} \frac{\partial}{\partial u_2} \right) \right] \quad (1)$$

and called simply the Stokes stream function operator. In terms of this operator

$$\vec{\xi} = \nabla \Omega \times \frac{\hat{e}_\phi}{\rho} + \frac{\hat{e}_\phi}{\rho} E^2 \psi \quad (2)$$

In a flow without swirl, the vorticity becomes

$$\vec{\xi} = \frac{\hat{e}_\phi}{\rho} E^2 \psi \quad (3)$$

Since we know that any flow described by ψ is an incompressible flow, it follows from this last result that the equation of motion for an axisymmetric incompressible, irrotational flow is

$$E^2 \psi = 0 \quad (4)$$

Again, we point out that viscous flows are rarely irrotational.

4.7 THE EQUATION OF MOTION

The flow of an incompressible Newtonian fluid is governed by the continuity and Navier-Stokes equations. Since use of the stream function automatically ensures that the continuity equation is satisfied, only the Navier-Stokes equation remains to be satisfied. When the viscosity is constant, the Navier-Stokes equations for an incompressible Newtonian fluid are 2.7-(6)

$$\begin{aligned}\rho_f \frac{D\vec{V}}{Dt} &= \rho_f \left(\frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla) \vec{V} \right) \\ &= -\nabla p + \mu \nabla^2 \vec{V} + \vec{f}\end{aligned}$$

which we will write using vector identities 2.7-(7) and 2.7-(8) and simplify using the continuity equation 2.5-(2)

$$\begin{aligned}\frac{\partial \vec{V}}{\partial t} + \frac{1}{2} \nabla (\vec{V} \cdot \vec{V}) - \vec{V} \times (\nabla \times \vec{V}) \\ = -\nabla p - \nu \nabla \times (\nabla \times \vec{V}) + \frac{\vec{f}}{\rho_f}\end{aligned}\tag{1}$$

By taking the curl of this equation, we can eliminate the pressure term. If the body force $\vec{f}$ is conservative, i.e. the gradient of a scalar field, then it is also eliminated by this operation. The force due to gravity is an example of a conservative body force. This leaves an equation with velocity as the sole dependent variable, an equation that may be used to yield an equation for the stream function and swirl. In terms of velocity and vorticity, we obtain

$$\frac{\partial \vec{\zeta}}{\partial t} - \nabla \times (\vec{V} \times \vec{\zeta}) + \nu \nabla \times (\nabla \times \vec{\zeta}) = \frac{\nabla \times \vec{f}}{\rho_f}\tag{2}$$

governing the transport, diffusion and generation of vorticity.

We shall now proceed to write each of the terms on the left hand side of this equation in terms of the stream function and swirl. Using 4.6-(2).

$$\frac{\partial \vec{\zeta}}{\partial t} = \frac{\partial}{\partial t} (\nabla \Omega) \times \frac{\hat{e}_\phi}{\rho} + \frac{\hat{e}_\phi}{\rho} \frac{\partial}{\partial t} (E^2 \psi) \quad (3)$$

The second term is constructed starting with

$$\vec{V} \times \vec{\zeta} = \left(\frac{\hat{e}_\phi}{\rho} \times \nabla \psi + \frac{\Omega}{\rho} \hat{e}_\phi \right) \times \left(\nabla \Omega \times \frac{\hat{e}_\phi}{\rho} + \frac{\hat{e}_\phi}{\rho} E^2 \psi \right)$$

where 4.4-(3) and 4.6-(2) are used.

Expanding

$$\begin{aligned} \vec{V} \times \vec{\zeta} = & \left(\frac{\hat{e}_\phi}{\rho} \times \nabla \psi \right) \times \left(\nabla \Omega \times \frac{\hat{e}_\phi}{\rho} \right) + \left(\frac{\hat{e}_\phi}{\rho} \times \nabla \psi \right) \times \frac{\hat{e}_\phi}{\rho} E^2 \psi \\ & + \frac{\Omega}{\rho} \hat{e}_\phi \times \left(\nabla \Omega \times \frac{\hat{e}_\phi}{\rho} \right) + \frac{\Omega}{\rho} \hat{e}_\phi \times \frac{\hat{e}_\phi}{\rho} E^2 \psi \end{aligned} \quad (4)$$

The last term vanishes. The second and third terms on the right hand side simplify

$$\left(\frac{\hat{e}_\phi}{\rho} \times \nabla \psi \right) \times \frac{\hat{e}_\phi}{\rho} E^2 \psi = \frac{E^2 \psi}{\rho^2} \nabla \psi$$

$$\frac{\Omega}{\rho} \hat{e}_\phi \times \left(\nabla \Omega \times \frac{\hat{e}_\phi}{\rho} \right) = \frac{\Omega}{\rho^2} \nabla \Omega$$

Using the vector identity

$$(\vec{A} \times \vec{B}) \times (\vec{C} \times \vec{D}) = \vec{B} (\vec{A} \cdot \vec{C} \times \vec{D}) - \vec{A} (\vec{B} \cdot \vec{C} \times \vec{D})$$

to simplify the first term on the right hand side of (4), we find

$$\left(\frac{\hat{e}_\phi}{\rho} \times \nabla \psi \right) \times \left(\nabla \Omega \times \frac{\hat{e}_\phi}{\rho} \right) = - \frac{\hat{e}_\phi}{\rho} (\nabla \psi \cdot \nabla \Omega \times \frac{\hat{e}_\phi}{\rho})$$

But we have the identity

$$\vec{A} \cdot \vec{B} \times \vec{C} = - \vec{C} \cdot \vec{B} \times \vec{A}$$

so this becomes

$$\left(\frac{\hat{e}_\phi}{\rho} \times \nabla \psi \right) \times \left(\nabla \Omega \times \frac{\hat{e}_\phi}{\rho} \right) = \frac{\hat{e}_\phi}{\rho} \left(\frac{\hat{e}_\phi}{\rho} \cdot \nabla \Omega \times \nabla \psi \right) \quad (5)$$

Noting that the product, $\nabla \psi \times \nabla \Omega$, is a vector in ϕ direction, this last result can be simplified to read

$$\left(\frac{\hat{e}_\phi}{\rho} \times \nabla \psi \right) \times \left(\nabla \Omega \times \frac{\hat{e}_\phi}{\rho} \right) = \frac{\nabla \Omega \times \nabla \psi}{\rho^2}$$

Gathering terms

$$\vec{V}_{x\zeta} = \frac{\nabla \Omega \times \nabla \psi}{\rho^2} + \frac{E^2 \psi}{\rho^2} \nabla \psi + \frac{\Omega}{\rho^2} \nabla \Omega \quad (6)$$

The curl of this expression appears in (2) so we form

$$\nabla \times (\vec{V}_{x\zeta}) = \nabla \times \left(\frac{\nabla \Omega \times \nabla \psi}{\rho^2} \right) + \nabla \times \left(\frac{E^2 \psi}{\rho^2} \nabla \psi \right) + \nabla \times \left(\frac{\Omega}{\rho^2} \nabla \Omega \right)$$

The identity

$$\nabla \times (\gamma \vec{A}) = \gamma (\nabla \times \vec{A}) + \nabla \gamma \times \vec{A} \quad (7)$$

simplifies the last two terms since the curl of the gradient of a scalar is zero

$$\nabla \times (\vec{V}_{x\zeta}) = \nabla \times \left(\frac{\nabla \Omega \times \nabla \psi}{\rho^2} \right) + \nabla \left(\frac{E^2 \psi}{\rho^2} \right) \times \nabla \psi + \nabla \left(\frac{\Omega}{\rho^2} \right) \times \nabla \Omega$$

Simplifying the last term further

$$\nabla \left(\frac{\Omega}{\rho^2} \right) = \frac{\nabla \Omega}{\rho^2} - \frac{2\Omega}{\rho^3} \nabla \rho$$

so that

$$\nabla \times (\vec{\nabla} \times \vec{\zeta}) = \nabla \times (\frac{\nabla \Omega \times \nabla \psi}{\rho^2}) + \nabla (\frac{E^2 \psi}{\rho^2}) \times \nabla \psi - \frac{2\Omega}{\rho^3} \nabla \rho \times \nabla \Omega \quad (8)$$

We must now proceed to find a suitable expression for the third term of (2). We start with

$$\vec{\nabla} \times \vec{\zeta} = \nabla \times (\nabla \Omega \times \frac{\hat{e}_\phi}{\rho} + \frac{\hat{e}_\phi}{\rho} E^2 \psi)$$

which uses 4.6-(2). Using (7), there results

$$\vec{\nabla} \times \vec{\zeta} = \nabla \times (\nabla \Omega \times \frac{\hat{e}_\phi}{\rho}) + \nabla (E^2 \psi) \times \frac{\hat{e}_\phi}{\rho}$$

since

$$\nabla \times \frac{\hat{e}_\phi}{\rho} = 0 \quad (9)$$

The first term of the expression for the curl of vorticity is also simplified by noting that for any scalar γ which is independent of the azimuthal angle

$$\nabla \times (\frac{\hat{e}_\phi}{\rho} \times \nabla \gamma) = \frac{\hat{e}_\phi}{\rho} E^2 \gamma \quad (10)$$

This relation follows from the development 4.3-(13), 2.4-(2) and 4.6-(3). Accordingly

$$\vec{\nabla} \times \vec{\zeta} = - \frac{\hat{e}_\phi}{\rho} E^2 \Omega + \nabla (E^2 \psi) \times \frac{\hat{e}_\phi}{\rho} \quad (11)$$

The final operation, which again makes use of (7) and (10), yields

$$\nabla \times \vec{\nabla} \times \vec{\zeta} = - \nabla (E^2 \Omega) \times \frac{\hat{e}_\phi}{\rho} - \frac{\hat{e}_\phi}{\rho} E^4 \psi$$

The operator E^4 is just E^2 operating on E^2 .

Gathering the expressions developed for the terms of (2), (2) becomes

$$\begin{aligned} & \frac{\partial}{\partial t} (\nabla \Omega) \times \frac{\hat{e}_\phi}{\rho} + \frac{\hat{e}_\phi}{\rho} \frac{\partial}{\partial t} (E^2 \psi) \\ & - \nabla \times (\frac{\nabla \Omega \times \nabla \psi}{\rho^2}) - \nabla (\frac{E^2 \psi}{\rho^2}) \times \nabla \psi + \frac{2\Omega}{\rho^3} \nabla \rho \times \nabla \Omega \\ & + v [- \nabla (E^2 \Omega) \times \frac{\hat{e}_\phi}{\rho} - \frac{\hat{e}_\phi}{\rho} E^4 \psi] = \frac{\nabla \times \vec{f}}{\rho_f} \end{aligned}$$

Since each of the the terms on the left hand side represents a vector either in the meridian plane or perpendicular to it, the equation of motion may be separated into two equations. Collecting terms representing vectors normal to the meridian plane

$$\begin{aligned} & \frac{\hat{e}_\phi}{\rho} \frac{\partial}{\partial t} (E^2 \psi) - \nabla (\frac{E^2 \psi}{\rho^2}) \times \nabla \psi + \frac{2\Omega}{\rho^3} \nabla \rho \times \nabla \Omega \\ & - v \frac{\hat{e}_\phi}{\rho} E^4 \psi = \frac{\nabla \times \vec{f}}{\rho_f} \end{aligned} \quad (12)$$

For simplicity of notation and because the more general case is not obviously of interest, we have taken the force $\vec{f}$ to lie in the meridian plane.

In the meridian plane, we have

$$\frac{\partial}{\partial t} (\nabla \Omega) \times \frac{\hat{e}_\phi}{\rho} - \nabla \times (\frac{\nabla \Omega \times \nabla \psi}{\rho^2}) - v \nabla (E^2 \Omega) \times \frac{\hat{e}_\phi}{\rho} = 0 \quad (13)$$

It should be noted that the terms that couple the streamline motion and the swirl are terms involving products of velocity. These terms originate in the convective acceleration. If the Reynolds number is sufficiently small, they may be neglected and the streamline motion and swirling motion uncouple.

The coupled motion at high Reynolds numbers may be familiar from past study of the flow near a rotating disc. Schlichting [1] describes this flow. When the Reynolds number is small, (12) governs the streamline motion and (13) the swirling motion.

Equation (13) may be simplified further and interpreted if the coupling term is written in a form similar to the other terms. Thus, from (5)

$$\nabla \times \left(\frac{\nabla \Omega \times \nabla \psi}{\rho^2} \right) = \nabla \times \left[\left(\frac{\hat{e}_\phi}{\rho} \cdot \nabla \Omega \times \nabla \psi \right) \frac{\hat{e}_\phi}{\rho} \right]$$

and using (7) and (9)

$$\nabla \times \left(\frac{\nabla \Omega \times \nabla \psi}{\rho^2} \right) = \nabla \left(\frac{\hat{e}_\phi}{\rho} \cdot \nabla \Omega \times \nabla \psi \right) \times \frac{\hat{e}_\phi}{\rho}$$

Substituting into (13), cancelling like terms, and integrating

$$\frac{\partial \Omega}{\partial t} + \frac{\hat{e}_\phi}{\rho} \cdot \nabla \psi \times \nabla \Omega = \nu E^2 \Omega + C$$

The constant of integration vanishes and the meaning of this equation is clear when it is recognized that this equation is the ϕ component of the Navier-Stokes equation (1) using substitutions from (6) and (11)

$$\frac{\partial \Omega}{\partial t} + \frac{\hat{e}_\phi}{\rho} \cdot \nabla \psi \times \nabla \Omega = \nu E^2 \Omega \quad (14)$$

We could have obtained (14) more quickly by extracting it directly from (1). The cross products appearing in (12) and (14) have magnitudes that are well expressed in terms of Jacobians. Since, for example

$$\nabla \psi = \frac{1}{h_1} \frac{\partial \psi}{\partial u_1} \hat{e}_1 + \frac{1}{h_2} \frac{\partial \psi}{\partial u_2} \hat{e}_2$$

and

$$\nabla \left(\frac{E^2 \psi}{\rho^2} \right) = \frac{1}{h_1} \frac{\partial}{\partial u_1} \left(\frac{E^2 \psi}{\rho^2} \right) \hat{e}_1 + \frac{1}{h_2} \frac{\partial}{\partial u_2} \left(\frac{E^2 \psi}{\rho^2} \right) \hat{e}_2$$

their cross product is

$$\nabla \psi \times \nabla \left(\frac{E^2 \psi}{\rho^2} \right) = \frac{1}{h_1 h_2} \left[\frac{\partial \psi}{\partial u_1} \frac{\partial}{\partial u_2} \left(\frac{E^2 \psi}{\rho^2} \right) - \frac{\partial \psi}{\partial u_2} \frac{\partial}{\partial u_1} \left(\frac{E^2 \psi}{\rho^2} \right) \right] \hat{e}_\phi$$

or

$$\nabla \psi \times \nabla \left(\frac{E^2 \psi}{\rho^2} \right) = \frac{1}{h_1 h_2} \frac{\partial \left(\psi, \frac{E^2 \psi}{\rho^2} \right)}{\partial (u_1, u_2)} \hat{e}_\phi$$

Consequently, (12) becomes

$$\begin{aligned} \frac{\partial}{\partial t} \left(E^2 \psi \right) + \frac{\rho}{h_1 h_2} \frac{\partial \left(\psi, \frac{E^2 \psi}{\rho^2} \right)}{\partial (u_1, u_2)} + \frac{2\Omega}{h_1 h_2 \rho^2} \frac{\partial \left(\rho, \Omega \right)}{\partial (u_1, u_2)} \\ = \nu E^4 \psi + \rho \frac{|\nabla \times \bar{f}|}{\rho_f} \end{aligned} \quad (15)$$

and (14) becomes

$$\frac{\partial \Omega}{\partial t} + \frac{1}{h_1 h_2 \rho} \frac{\partial \left(\psi, \Omega \right)}{\partial (u_1, u_2)} = \nu E^2 \Omega \quad (16)$$

We will not have occasion to solve (15) and (16) in their full generality. Rather we will ordinarily be examining flows with Reynolds numbers small enough to permit neglect of convective acceleration. For such flows, and in the absence of non-conservative body forces, there results

$$\frac{\partial}{\partial t} \left(E^2 \psi \right) = \nu E^4 \psi \quad (17)$$

which governs the streamline flow and

$$\frac{\partial \Omega}{\partial t} = \nu E^2 \Omega \quad (18)$$

which governs the swirling flow. Unlike their higher Reynolds number counterparts, these two equations are linear and consequently, amenable to exact solution by classical techniques.

When the flow is steady,

$$E^4 \psi = 0 \quad (19)$$

and

$$E^2 \Omega = 0 \quad (20)$$

govern the motion.

The task of finding the steady drag on a particle moving in axisymmetric flow is largely a matter of finding the solution of (19). Determining the torque on a steadily rotating particle requires the solution of (20).

4.8 SEPARATION OF THE EQUATION OF MOTION

The problem of finding the solution of a partial differential equation such as 4.7-(19) is considerably easier if the problem can be reduced to one of finding solutions of ordinary differential equations. One classical technique of accomplishing this reduction is the method of separation of variables. This method does not work in all instances but on those occasions when it does work it proves to be quite powerful.

Consider, for example, the stream function $\psi = \psi(u_1, u_2)$ which is the solution of 4.7-(19). If the governing equation is separated into two ordinary differential equations following substitution of the trial form $\psi = \psi_1(u_1) \psi_2(u_2)$, then the equation is simply separable. Note that the product form $\psi_1(u_1) \psi_2(u_2)$ is not as general an expression as $\psi(u_1, u_2)$ so that equations need not always separate.

If an equation is not simply separable, it may be separable in another way called R separability. If the partial differential equation can be separated into two ordinary differential equations by using the form $\psi = \psi_1(u_1) \psi_2(u_2) / R(u_1, u_2)$, then the equation is R separable. R is here a known, non-constant function of the independent variables, u_1 and u_2 .

The type of separability, if any, depends on the equation and the coordinate system employed. Laplace's equation is simply separable in Cartesian coordinates, R separable in bispherical coordinates, and inseparable in hyperbolic coordinates. The Helmholtz equation is simply separable in Cartesian coordinates, but inseparable in both bispherical and hyperbolic coordinates.

As an example, let us separate the Laplace equation in Cartesian coordinates.

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$$

is the equation in two dimensions. We seek a solution of the form

$$\Phi = f(x) g(y)$$

Substituting this trial form into the Laplace equation, we obtain

$$g \frac{d^2 f}{dx^2} + f \frac{d^2 g}{dy^2} = 0$$

which may be rearranged into a form convenient for separation

$$\frac{1}{f} \frac{d^2 f}{dx^2} + \frac{1}{g} \frac{d^2 g}{dy^2} = 0 \quad (1)$$

Now, examine the two terms of this equation. The first term does not depend on y , but depends only on x . The second term does not depend on x , but only on y . The sum of these two terms must, however, remain constant and equal to zero. Therefore, the expression $\frac{d^2 f}{dx^2} / f$ cannot change its value when x changes. Otherwise, the equality expressed by (1) could not be maintained. Similarly, although g may vary with y , $\frac{d^2 g}{dy^2} / g$ cannot. If by changing the value of y , while holding x and consequently $\frac{d^2 f}{dx^2} / f$ constant, one could change

the value of $\frac{d^2g}{dy^2} / g$, then (1) could not remain valid. The conclusion is that each of the expressions in (1) is a constant and that

$$\frac{1}{f} \frac{d^2f}{dx^2} = k^2 \quad ; \quad \frac{1}{g} \frac{d^2g}{dy^2} = -k^2 \quad (2)$$

satisfies that condition as well as (1). The choice of k^2 is largely arbitrary. Choice of the separation constant is usually based on the form of the resulting ordinary differential equation and also on the imposed boundary conditions.

The solutions to these two equations are well known.

$$f = e^{kx} \quad \text{or} \quad f = e^{-kx}$$

satisfies the first of (2). Because the equation is linear, any linear combination of these two solutions is also a solution.

$$f = ae^{kx} + be^{-kx}$$

or

$$f = a \sinh kx + b \cosh kx$$

are equally acceptable ways of writing the general solution. The choice between these two forms is made based on the boundary conditions. a and b are constants.

Similarly, the general solution to the second of (2) is

$$g = ce^{iky} + de^{-iky}$$

or

$$g = c \sin ky + d \cos ky$$

where c and d are constants.

Having obtained solutions to each of (2), the solution to (1) is

$$\Phi = (a \sinh kx + b \cosh kx) (c \sin ky + d \cos ky)$$

or if solutions exist for several discrete values of k , the sum of these solutions is also a solution since (1) is linear

$$\Phi = \sum_n (a_n \sinh k_n x + b_n \cosh k_n x) (c_n \sin k_n y + d_n \cos k_n y)$$

Now, the equation of primary interest to us is 4.7-(19)

$$E^4 \psi = 0 \quad (3)$$

whose separation in spherical coordinates is due to Sampson [2]. Happel and Brenner [3] describe the separation analysis of Haberman and Sayre [4]. The separation of this equation is of particular importance in spherical coordinates because of the interest in the behavior of spherical particles.

We shall base our separation of (3) in spherical coordinates on a method used by Stimson and Jeffery [5] to separate (3) in bispherical coordinates. First, we demonstrate that

$$E^2 \psi = 0 \quad (4)$$

is simply separable in spherical coordinates and we find the solution. The Stokes stream function operator, defined by 4.6-(1), becomes

$$E^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin \theta}{r^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \right) \quad (5)$$

in spherical coordinates. If we define here

$$\mu \equiv \cos \theta$$

then we may also write this operator as

$$E^2 = \frac{\partial^2}{\partial r^2} + \frac{(1-\mu^2)}{r^2} \frac{\partial^2}{\partial \mu^2} \quad (6)$$

A trial separation of the form

$$\psi = F(r) G(\mu)$$

when substituted into (4) yields

$$\frac{d^2F}{dr^2} G + \frac{F}{r^2} (1 - \mu^2) \frac{d^2G}{d\mu^2} = 0$$

Dividing by FG/r^2 , the equation is put into the form of a function of r only plus a function of μ only.

$$\frac{r^2}{F} \frac{d^2F}{dr^2} + \frac{(1 - \mu^2)}{G} \frac{d^2G}{d\mu^2} = 0$$

In order that the separated ordinary differential equations appear in a recognizable form, the separation constant is chosen to be $n(n-1)$. We have

$$r^2 \frac{d^2F}{dr^2} - n(n-1) F = 0 \quad (7)$$

and

$$(1-\mu^2) \frac{d^2G}{d\mu^2} + n(n-1) G = 0 \quad (8)$$

The equation governing the radial dependence is recognized as an example of Euler's equation or Cauchy's equation. The equation governing the angular dependence may be recognized as an equation having Gegenbauer or Ultraspherical polynomials as a solution [6].

The general form of Euler's equation is

$$r^n \frac{d^n F}{dr^n} + a_{n-1} r^{n-1} \frac{d^{n-1} F}{dr^{n-1}} + \dots + a_1 r \frac{dF}{dr} + a_0 F = f(r) \quad (9)$$

The equation is linear, but has variable coefficients. The exponent of the coefficient of each term is equal to the order of the derivative. Euler's equation can be transformed into a linear equation with constant coefficients through use of the transformation

$$r = e^x$$

Using this transformation

$$\frac{dr}{dx} = e^x$$

so that

$$\frac{dx}{dr} = e^{-x}$$

and

$$\frac{d^2x}{dr^2} = e^{-x} \frac{dx}{dr} = e^{-2x}$$

In order to transform (7) , first observe

$$\frac{dF}{dr} = \frac{dF}{dx} \frac{dx}{dr} = e^{-x} \frac{dF}{dx}$$

$$\frac{d^2F}{dr^2} = \frac{d}{dx} \left(e^{-x} \frac{dF}{dx} \right) \frac{dx}{dr} = e^{-2x} \left(\frac{d^2F}{dx^2} - \frac{dF}{dx} \right)$$

so that (7) becomes the second order ordinary linear differential equation with constant coefficients

$$\frac{d^2F}{dx^2} - \frac{dF}{dx} - n(n-1) F = 0 \quad (10)$$

Introducing the differential operator D ,

$$D \equiv \frac{d}{dx}$$

(10) may be factored

$$(D-n) (D+n-1) F = 0$$

so that the solutions of

$$(D-n) F = 0$$

and

$$(D+n-1) F = 0$$

are solutions of (10). These solutions are obviously

$$F = e^{nx}$$

and

$$F = e^{(1-n)x}$$

so that a general solution of (10) is

$$F = ae^{nx} + be^{(1-n)x}$$

In terms of the spherical radius, we now have

$$F = a_n r^n + b_n r^{1-n} \quad (11)$$

the general solution of (7).

We now have to find the solution to (8), the relation governing the angular dependence. From [6], we find a differential equation

$$(1-x^2)y'' - (2\alpha + 1)xy' + n(n + 2\alpha)y = 0$$

having as a solution, the Gegenbauer or Ultraspherical polynomial of order α and degree n , $C_n^{(\alpha)}(x)$. n is an integer. If we set α equal to $-1/2$, then this equation reduces to (8) and we find

$$G = C_n^{-\frac{1}{2}}(\mu) \quad (12)$$

Because (8) is second order, it possesses two solutions. The second solution [2] however is unbounded on the axis of revolution and consequently of marginal interest to the dynamics of hydrosols and aerosols. (12) is sufficiently general for all of our purposes. From (11) and (12) we find that

$$\psi = \sum_{n=0}^{\infty} (a_n r^n + b_n r^{1-n}) C_n^{-\frac{1}{2}}(\mu) \quad (13)$$

is a solution of (4). The sum need not extend over negative values of n since this would only repeat terms already in the sum. To demonstrate this, set $n' = 1-n$. Then positive values of n correspond to negative values of n' . Since $1-n = n'$, $n = 1-n'$ and $n(n-1) = n'(n'-1)$, terms in the sum (13) correspond exactly to terms that would be added by letting the index range over negative values. Since the coefficients a_n and b_n are as yet undetermined, nothing new would be added.

Equation (4) has been separated and a solution found. It is however (3), the equation for low Reynolds number streamline motion, whose solution we seek. Let us consider the argument used by Stimson and Jeffery [5] to find solutions of (3) from solutions of (4). The theorem may be stated as follows. If ψ_1 is a solution of $E^2\psi = 0$, then $\psi = z\psi_1$ is a solution of $E^4\psi = 0$. z is the cylindrical axial position.

The proof is easily given using cylindrical coordinates. In cylindrical coordinates, the operator E^2 becomes

$$E^2 = \rho \frac{\partial}{\partial \rho} \left(\frac{1}{\rho} \frac{\partial}{\partial \rho} \right) + \frac{\partial^2}{\partial z^2} \quad (14)$$

It is given that ψ_1 satisfies $E^2\psi_1 = 0$ and is to be shown that $E^4(z\psi_1) = 0$. First,

$$\begin{aligned} E^2(z\psi_1) &= z\rho \frac{\partial}{\partial \rho} \left(\frac{1}{\rho} \frac{\partial \psi_1}{\partial \rho} \right) + \frac{\partial}{\partial z} \left(\psi_1 + z \frac{\partial \psi_1}{\partial z} \right) \\ &= zE^2\psi_1 + 2 \frac{\partial \psi_1}{\partial z} \\ &= 2 \frac{\partial \psi_1}{\partial z} \end{aligned}$$

so

$$\begin{aligned} E^4(z\psi_1) &= E^2E^2(z\psi_1) \\ &= E^2 \left(2 \frac{\partial \psi_1}{\partial z} \right) \\ &= 2 \frac{\partial}{\partial z} (E^2\psi_1) \\ &= 0 \end{aligned}$$

The proof is complete.

We now make use of this result. (13) is a solution of (4). Multiplying (13) by z , which is ru in spherical coordinates, we find a solution to (3)

$$\psi = \sum_{n=0}^{\infty} (a_n r^{n+1} + b_n r^{2-n}) \mu C_n^{-\frac{1}{2}}(\mu) \quad (15)$$

This is not the most general solution since (13) itself is also a solution of (3) that remains bounded on the axis. Any linear combination of (13) and (15) is also a solution and represents the most general solution when the axis is in the flow field. Before forming this linear combination, (15) can be simplified using the recurrence relation [6]

$$2(n+\alpha) \mu C_n^{(\alpha)}(x) = (n+1) C_{n+1}^{(\alpha)}(x) + (n+2\alpha-1) C_{n-1}^{(\alpha)}(x) \quad (16)$$

so that

$$\mu C_n^{-\frac{1}{2}}(\mu) = \frac{(n+1) C_{n+1}^{-\frac{1}{2}}(\mu) + (n-2) C_{n-1}^{-\frac{1}{2}}(\mu)}{2n-1} \quad (17)$$

With this substitution, (15) becomes

$$\begin{aligned} \psi = \sum_{n=0}^{\infty} [& (a_n r^{n+1} + b_n r^{2-n}) \frac{n+1}{2n-1} C_{n+1}^{-\frac{1}{2}}(\mu) \\ & + (a_n r^{n+1} + b_n r^{2-n}) \frac{n-2}{2n-1} C_{n-1}^{-\frac{1}{2}}(\mu)] \end{aligned}$$

Now, change the accounting by substituting n for $n+1$ in the first term and n for $n-1$ in the second term. The result is that the coefficients of $C_n^{-\frac{1}{2}}(\mu)$ are r^n and r^{1-n} as in (13) and also r^{3-n} and r^{n+2} . The general solution of (3) in spherical coordinates is consequently

$$\psi = \sum_{n=0}^{\infty} (a_n r^n + b_n r^{1-n} + c_n r^{n+2} + d_n r^{3-n}) c_n^{-\frac{1}{2}}(\mu) \quad (18)$$

for any flow with the axis of revolution in the flow field.

With the special values [6]

$$c_0^{-\frac{1}{2}}(\mu) = 1$$

$$c_1^{-\frac{1}{2}}(\mu) = -\mu$$

and the recurrence relation (17), the Gegenbauer polynomial expressions are readily generated. Setting $n=1$ in (17), we obtain

$$c_2^{-\frac{1}{2}}(\mu) = \frac{1-\mu^2}{2}$$

This latter polynomial proves to be of considerable importance to us. In many technically important particle motions, the stream function of the fluid flow about the particle is described by this single term of the series (18)

$$\psi = (a_2 r^2 + b_2 r^{-1} + c_2 r^4 + d_2 r) \frac{1-\mu^2}{2}$$

or simply

$$\psi = \frac{1}{2} (a_2 r^2 + b_2 r^{-1} + c_2 r^4 + d_2 r) \sin^2 \theta \quad (19)$$

This solution will prove sufficient to solve a wide variety of problems dealing with the motion of a spherical particle in an unbounded fluid.

4.9 SOME BOUNDARY CONDITIONS

The solution for the stream function ψ that describes the fluid flow about a particle is made unique by requiring that the solution of the governing differential equation satisfy conditions at the boundaries of the fluid domain. Because of the many flow regimes encountered and types of fields that may be applied to the particle, different boundary conditions may be applicable. The range of boundary conditions is much larger than the range normally applied to larger bodies. One reason for this variety is the phenomena observed when the mean free path of a gas is of the order of the aerosol radius. We will describe most of these boundary conditions later, in conjunction with the analysis of the corresponding particle motion. Here we will examine only the common no normal flow boundary condition, the no-slip boundary condition, and the condition of uniform flow far from the particle.

The condition of no flow normal to a particle boundary is a kinematic boundary condition. Its validity depends only on the impermeability of the particle, and not on any model of fluid motion. This boundary condition does not appear in a purely swirling motion because the only velocity component is the azimuthal velocity so that the condition is automatically satisfied.

In streamline motion, this condition is not automatically satisfied but must be imposed. The form of the boundary condition depends on the frame chosen since it is not necessarily the absolute normal fluid velocity that vanishes at the surface. The relative velocity of the fluid normal to the surface vanishes at this boundary. If the particle surface is moving normal to itself, then the normal component of the fluid velocity must be the same as the normal component of the particle surface velocity.

Consider first the no normal flow boundary condition in the frame of the particle at rest. In this frame, the observer moves with the particle. The particle is stationary and the fluid flows past.

To such an observer, the particle surface is a stream surface. There is no fluid velocity component normal to a stream surface. The stream function is constant on such a surface, so the boundary condition becomes

$$\psi = \text{constant}$$

Normally the particle surface intersects the axis of revolution and so the value of the stream function on this surface is the same as that assigned to the stream function on the axis of revolution. By convention, this value is zero so the no normal flow condition becomes

$$\psi = 0 \quad (1)$$

Alternatively, we may obtain this boundary condition by explicitly examining the velocity component expression. Assume, for definiteness, that the particle surface corresponds to the u_1 coordinate surface. For example, the surface of a spherical particle corresponds to a constant value of the radius. In the frame of the particle, the no normal flow condition is

$$v_1 = 0 \quad (2)$$

4.3-(9) gives this in terms of the stream function

$$\frac{-1}{\rho h_2} \frac{\partial \psi}{\partial u_2} = 0$$

or simply

$$\frac{\partial \psi}{\partial u_2} = 0 \quad (3)$$

Since this boundary condition is valid everywhere on the surface, and since integration with respect to u_2 is integration over the surface, we may integrate from the axis to obtain (1)

$$\psi = \int \frac{\partial \psi}{\partial u_2} du_2 = 0$$

The no-slip boundary condition, applied at the surface of a rigid particle, is not so obvious as the no normal flow condition. The early literature of fluid dynamics contains considerable discussion of this matter. It is now accepted that, except when the mean free path in a gas is large enough compared

to body dimensions to permit significant changes over a mean free path, the fluid may be taken to adhere to the surface.

For the particle whose surface is a u_1 coordinate surface, this condition requires that the tangential component of fluid velocity, v_2 , be equal to the tangential component of surface velocity. For a stationary particle, this is

$$v_2 = 0 \quad (4)$$

or, using 4.3-(9) ,

$$\frac{\partial \psi}{\partial u_1} = 0 \quad (5)$$

This expression may not be integrated, as was (3) since integration with respect to u_1 is integration along a path normal to the surface and the boundary condition ceases to be valid away from the boundary.

For a spherical particle, the boundary conditions at the surface, corresponding to a particular value of the radius, are found simply by writing in spherical coordinates the expressions just presented. The no normal flow condition, in the particle frame, is

$$\psi = 0 \quad \text{or} \quad v_r = 0 \quad (6)$$

The no-slip condition, also in the frame of the particle, is

$$\frac{\partial \psi}{\partial r} = 0 \quad \text{or} \quad v_\theta = 0 \quad (7)$$

In addition to conditions imposed at the particle surface, boundary conditions are imposed at all fluid surfaces. If the fluid is not bounded, but extends indefinitely away from the particle, the velocity distribution in the fluid as the distance from the particle increases without bound must be specified. Any disturbance to the fluid flow caused by the presence of the particle must vanish far from the particle.

Frequently, the motion of interest consists of a particle or assembly of particles moving through an otherwise uniform fluid flow. In axisymmetric flow, this uniform flow is parallel to the axis of revolution and consequently, most easily expressed in cylindrical coordinates. A uniform flow of magnitude U in the negative z direction

$$\vec{V} = -U \hat{e}_z \quad (8)$$

has velocity components related to the stream function by 4.3-(15)

$$v_z = -U = -\frac{1}{\rho} \frac{\partial \psi}{\partial \rho} ; \quad v_\rho = 0 = \frac{1}{\rho} \frac{\partial \psi}{\partial z}$$

so that ψ depends only on ρ , which is obvious since the streamlines in the uniform flow are parallel to the axis.

$$\frac{1}{\rho} \frac{d\psi}{d\rho} = U$$

Rearranging

$$d\psi = U \rho d\rho$$

and integrating from the axis, remembering the convention that ψ is assigned the value zero on the axis.

$$\psi = \frac{U \rho^2}{2} \quad (9)$$

In spherical coordinates, the stream function for a uniform flow is

$$\psi = \frac{U r^2 \sin^2 \theta}{2} \quad (10)$$

While the no-slip and no normal flow boundary conditions have been examined and expressed in the frame of the particle, it is necessary on

occasion to use a different frame. This change of frames becomes necessary when analyzing transient motions. The equation of motion, and consequently the boundary conditions, should be written in an inertial frame. The frame of an accelerating particle is not inertial. If the fluid far from the particle is stationary, then that frame is inertial and may be used. The frame of the stationary fluid far from the moving particle is often chosen. The form of the particle boundary conditions in this frame is needed.

If an observer stationary with respect to a particle sees a distant uniform flow of magnitude U in the negative z direction, then an observer at rest with respect to the distant fluid will see the particle moving with a velocity component U in the positive z direction. The particle surface is not a stream surface in this frame.

The unit vector normal to the particle surface is $\hat{e}_1$. The normal velocity component of the surface therefore is $U \hat{e}_z \cdot \hat{e}_1$. The condition of no relative fluid motion normal to the surface, the impenetrability condition, becomes in this frame

$$v_1 = U \hat{e}_z \cdot \hat{e}_1 \quad (11)$$

Expressing the fluid velocity in terms of the stream function

$$-\frac{1}{\rho h_2} \frac{\partial \psi}{\partial u_2} = U \hat{e}_z \cdot \hat{e}_1 \quad (12)$$

Noting from the orthogonality of the unit vectors that

$$\hat{e}_z \cdot \hat{e}_1 = \hat{e}_2 \cdot \hat{e}_\rho$$

so that

$$\begin{aligned} \hat{e}_z \cdot \hat{e}_1 &= \hat{e}_2 \cdot \nabla \rho \\ &= \frac{1}{h_2} \frac{\partial \rho}{\partial u_2} \end{aligned}$$

With this substitution, (12) is in a form

$$\frac{1}{\rho h_2} \frac{\partial \psi}{\partial u_2} + \frac{U}{h_2} \frac{\partial \rho}{\partial u_2} = 0$$

that may be rearranged and integrated from the axis along the particle surface to yield

$$\psi + \frac{U \rho^2}{2} = 0 \quad (13)$$

or

$$\psi + \frac{U r^2 \sin^2 \theta}{2} = 0 \quad (14)$$

as the impenetrability condition at the surface of a moving particle.

In a similar manner, the component of particle surface velocity parallel to the surface is $U \hat{e}_z \cdot \hat{e}_2$. The no-slip condition requires that the relative tangential velocity between the fluid and surface be zero so the tangential fluid velocity is

$$v_2 = U \hat{e}_z \cdot \hat{e}_2 \quad (15)$$

In terms of the stream function

$$\frac{1}{\rho h_1} \frac{\partial \psi}{\partial u_1} = U \hat{e}_z \cdot \hat{e}_2$$

Examining the stream function expression, we note the desirability of writing the scalar product of the unit vectors as a derivative with respect to u_1 ,

$$\begin{aligned} \hat{e}_z \cdot \hat{e}_2 &= - \hat{e}_1 \cdot \hat{e}_\rho \\ &= - \hat{e}_1 \cdot \nabla \rho \\ &= - \frac{1}{h_1} \frac{\partial \rho}{\partial u_1} \end{aligned}$$

Thus,

$$\frac{1}{\rho h_1} \frac{\partial \psi}{\partial u_1} + \frac{U}{h_1} \frac{\partial \rho}{\partial u_1} = 0$$

or

$$\frac{\partial}{\partial u_1} \left(\psi + \frac{U \rho^2}{2} \right) = 0 \quad (16)$$

is the no-slip condition in this frame. For a spherical particle, this is

$$\frac{\partial}{\partial r} \left(\psi + \frac{U r^2 \sin^2 \theta}{2} \right) = 0 \quad (17)$$

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PROBLEMS

1. Show that the E^2 operator is related to the Laplacian ∇^2 by

$$E^2 = \nabla^2 - \frac{2}{\rho} \nabla \rho \cdot \nabla$$

2. A point source is a point in space with a net volume flow rate, q , flowing from it. The fluid flows out radially and uniformly. Find the stream function corresponding to the flow from a point source.
3. The low Reynolds number flow through a circular pipe is called Poiseuille flow. q is the total volume flow rate through the pipe. A is the pipe radius. Analyze Poiseuille flow, using the stream function, by
 - a) Writing the differential equation for ψ in an appropriate coordinate system.
 - b) Explaining physically the boundary conditions you will impose.
 - c) Writing these boundary conditions in terms of ψ .
 - d) Finding ψ .
 - e) Finding each of the velocity components in your coordinate system.
4. Separate $E^2 \Omega = 0$ in cylindrical coordinates and find the solutions of the separated ordinary differential equations.
5. Consider the fluid flow from a point source at $r = 0$ on a solid plane wall, $\theta = \pi / 2$. The stream function is

$$\psi = - \frac{q}{2\pi} (1 - \cos^3 \theta)$$

- a) Verify that this expression satisfies $E^4\psi = 0$.
- b) Write each boundary condition in terms of ψ and explain what it means physically.
- c) Show that the above expression for ψ satisfied these boundary conditions.
- d) Sketch the streamlines in the region $\theta < \pi / 2$.

6. The metric coefficients in bispherical coordinates (ξ, η, ϕ) are

$$h_{\xi} = h_{\eta} = \frac{c}{\cosh \xi - \cos \eta}$$

$$h_{\phi} = \rho = \frac{c \sin \eta}{\cosh \xi - \cos \eta}$$

Consider a voltage $V = V(\xi, \eta)$ having an axisymmetric distribution. Show that Laplace's equation, governing V , is R separable by trying

$$V = (\cosh \xi - \cos \eta)^{1/2} F$$

Find the differential equation governing F . Show that this equation for F is simply separable. Separate and solve the resulting ordinary differential equations in ξ and η .

7. Verify that the following expression for Stokes stream function represents a viscous flow by showing that it satisfies $E^4\psi = 0$

$$\psi = Aa^2 \left[\left(\frac{a}{r} \right)^2 - 1 \right] \sin^2 \theta \cos \theta$$

A and a are constant. Sketch the streamlines for $r \geq a$. This fluid flow may be generated by applying an electric field to a liquid drop immersed in another liquid.

8. A point source of strength q is located at a position $z = a$ on the axis of revolution. A sink of equal strength is located at $z = -a$. Find the stream function for a doublet by finding the stream function for this pair and letting a approach zero while holding $m \equiv 2aq$ constant.
9. Show that Laplace's equation is simply separable in spherical coordinates. Find the solutions of the three resulting ordinary differential equations.

MOTION IN A CONTINUUM

5.1 INTRODUCTION

Having formulated the basic equations governing incompressible axisymmetric Newtonian fluid flow, we are able to apply these relations to the fluid flow generated by the motion of aerosols and hydrosols. By determining the fluid velocity distribution in the region surrounding the particle, we are then capable of calculating the stresses in the fluid and, consequently, the force or drag on the particle. The problem is essentially one of finding the stream function that satisfies the equation of motion and the boundary conditions imposed. Having found the stream function, the stresses may be determined and integrated by quite straight forward though tedious procedures. The necessity to determine the drag in this manner, however, has been eliminated by development of simple expressions for drag in terms of the stream function.

The types of particle motions to be considered are quite varied. They vary due to the different particle shapes. The particles move as a result of applied gravitational fields, electric fields, temperature gradients and concentration gradients. The particles move alone through bounded or unbounded fluid or their motion is affected by neighboring particles.

We start by developing the simplest, and yet the single most important, relation in the dynamics of aerosols and hydrosols, Stokes Law.

5.2 STEADY FLOW PAST A SPHERE

Stokes law [1] expresses the drag on a rigid sphere moving steadily with no slip through an unbounded fluid. The fluid is considered a continuum and inertial effects within the fluid are neglected. Stokes law is a result of the solution of the equation for creeping flow. This drag expression accurately describes the drag experienced by an aerosol or hydrosol particle provided that the particle is large enough that the continuum model is valid and yet moves slowly enough that inertial effects are negligible.

These two limits are more accurately described by the use of dimensionless groups. The continuum approach is valid if the Knudsen number based on particle

radius is much less than one. The Knudsen number and mean free path in a gas are described in Section 2.2. The mean free path in air under normal atmospheric conditions is of the order of a tenth of a micron. A particle moving through air under these conditions must be larger than this for Stokes law to be a good approximation. The accuracy of course depends on the actual form of the drag's dependence on Knudsen number, a matter we have yet to examine.

Similarly, validity of the creeping flow equation of axisymmetric flow, 4.7-(19) requires a small particle Reynolds number. Calculation of Reynolds number based on Stokes law will be done later. Experimental results showing the dependence of drag on Reynolds number will also be examined to reveal how the drag deviates from Stokes law.

The fact that Stokes law is derived using an unbounded fluid should not be interpreted as a severe restriction on its applicability. As long as a particle has no neighbors within several diameters, the distance necessary for disturbances caused by the particle to attenuate, the particle is for all practical purposes alone.

The governing equation for fluid in axisymmetric creeping flow, 4.7-(19)

$$E^4\psi = 0 \quad (1)$$

is to be solved subject to boundary conditions applied at the surface of the sphere and far from the sphere. The expression of these boundary conditions and the form of the resulting stream function depend on the frame chosen. We choose, somewhat arbitrarily, a frame at rest with respect to the particle. The choice of spherical coordinates is obvious.

Denoting the particle radius by a , the no normal flow condition is 4.9-(1)

$$\psi = 0 \quad \text{at} \quad r = a \quad (2)$$

in this frame. The no-slip condition, 4.9-(7) is

$$\frac{\partial \psi}{\partial r} = 0 \quad \text{at} \quad r = a \quad (3)$$

A uniform flow far from the particle having velocity U in the negative z direction gives

$$\psi \rightarrow \frac{Ur^2 \sin^2 \theta}{2} \text{ as } r \rightarrow \infty \quad (4)$$

While 4.8-(18) is a more general solution of (1) in spherical coordinates, only the terms having $\sin^2 \theta$ dependence do not vanish so that we may seek a solution of the form 4.8-(19). This form is also suggested by the angular dependence of (4). We rearrange 4.8-(19), changing notation, so that we try the solution

$$\psi = \frac{U}{2} (Ar^2 + Br + \frac{C}{r} + Dr^4) \sin^2 \theta \quad (5)$$

The coefficients are now determined from the boundary conditions. The boundary condition at infinity (4) offers the greatest immediate simplification. The coefficient of r^4 must vanish since otherwise the corresponding velocities are unbounded as r increases without bound.

$$D = 0$$

The leading term corresponds to the uniform flow and, by comparison,

$$A = 1$$

Although the Br term of (5) will increase as the radius increases, the velocities calculated from this term approach zero as the distance from the sphere increases. As a result, (4) says nothing about the value of B . Similarly, C is unaffected.

Applying condition (2), it is clear that ψ can be zero everywhere on the surface only if the expression in parentheses is zero there

$$a^2 + Ba + \frac{C}{a} = 0$$

A second equation in the unknown coefficients, B and C, results from applying condition (3)

$$2a + B - \frac{C}{a^2} = 0$$

Solving these equations, we find

$$B = - \frac{3a}{2}$$

and

$$C = \frac{a^3}{2}$$

so that Stokes stream function in this frame is

$$\psi = \frac{Ua^2}{4} \left[2 \left(\frac{r}{a} \right)^2 - 3 \frac{r}{a} + \frac{a}{r} \right] \sin^2 \theta \quad (6)$$

Plotting this stream function distribution is left as an exercise. It should be noted however, that the stream function has symmetry fore and aft. Unlike higher Reynolds number flows, there is no wake.

The corresponding velocities are calculated using 4.3-(16)

$$v_r = - \frac{U}{2} \left[2 - 3 \frac{a}{r} + \left(\frac{a}{r} \right)^3 \right] \cos \theta \quad (7)$$

and

$$v_\theta = \frac{U}{4} \left[4 - 3 \frac{a}{r} - \left(\frac{a}{r} \right)^3 \right] \sin \theta \quad (8)$$

It is interesting to note that the disturbance velocity of the particle dies off as r^{-1} . In an irrotational or potential flow about a body, the disturbance velocity would asymptotically vanish as r^{-3} . The asymptotic behavior of the disturbance velocity will be of further interest to us later for several reasons. It is important in the Oseen criticism of the creeping flow equations, in developing a simple drag expression, and in the interaction

of particles in dilute suspension.

(6) describes the fluid flow past the sphere and is to be used to find the drag on the particle. Before finding the drag, however, it is instructive to repeat the calculation of the stream function using a different frame. The other frame of interest is the frame of the fluid far from the particle. If we denote this frame by II and the frame of the particle by I, then velocities in the two frames are related by

$$\vec{V}_{II} = \vec{V}_I + U\hat{e}_z \quad (9)$$

so that $\vec{V}_{II}$ is zero far from the particle and $\vec{V}_{II} = U\hat{e}_z$ at the particle surface. It follows that the stream functions in these frames differ by the stream function of a uniform flow with velocity component U in the z direction

$$\psi_{II} = \psi_I - \frac{U r^2 \sin^2 \theta}{2} \quad (10)$$

Using (6), this yields

$$\psi_{II} = \frac{Ua^2}{4} \left[-3 \frac{r}{a} + \frac{a}{r} \right] \sin^2 \theta \quad (11)$$

and the velocity components in spherical coordinates differ from those previously calculated by the uniform flow component

$$v_{r_{II}} = \frac{U}{2} \left[3 \frac{a}{r} - \left(\frac{a}{r} \right)^3 \right] \cos \theta \quad (12)$$

and

$$v_{\theta_{II}} = -\frac{U}{4} \left[3 \frac{a}{r} + \left(\frac{a}{r} \right)^3 \right] \sin \theta \quad (13)$$

These velocity components and the stream function (11) were found using the results of calculation in the other frame. If one was to proceed initially in frame II, then the boundary conditions applied to (5) would be no normal flow, 4.9-(14)

$$\psi + \frac{Ua^2 \sin^2 \theta}{2} = 0 \quad \text{at} \quad r = a$$

no-slip, 4.9-(17)

$$\frac{\partial \psi}{\partial r} + Ua \sin^2 \theta = 0 \quad \text{at} \quad r = a$$

and zero velocity far from the particle

$$v_r = \frac{-1}{r^2 \sin \theta} \frac{\partial \psi}{\partial \theta} \rightarrow 0 \quad \text{as} \quad r \rightarrow \infty$$

$$v_\theta = \frac{1}{r \sin \theta} \frac{\partial \psi}{\partial r} \rightarrow 0 \quad \text{as} \quad r \rightarrow \infty$$

Using these conditions to evaluate the coefficients of (5) cause only one change

$$A = 0$$

in the results, the only difference being due to the uniform velocity U .

Now, let us find the drag, the force acting on the sphere. To do this, we may integrate the stress component in the axial direction over the surface of the particle. Clearly, the only net force is axial. Since σ_{rz} is the stress acting in the z direction on a surface whose normal points in the r direction, the net force is

$$F = \int_A \sigma_{rz} dA \quad (14)$$

The integration is over the particle surface.

Actually, since the flow is steady and the effects of fluid acceleration have been assumed negligible, the net force on any closed surface surrounding the particle is the same. It is not necessary to integrate at the particle surface. We will pursue this point further. For the moment, we will be content to calculate the net force on a spherical envelope of arbitrary radius

centered on the particle. The resultant force will be found to be independent of radius and consequently equal to the value that would be obtained by setting r equal to a . The independence of the answer from the radius is an example of the more general surface independence.

The stress σ_{rz} is conveniently written in spherical coordinates by resolving the radial and tangential forces on the surface in the axial direction.

$$\sigma_{rz} = \sigma_{rr} \hat{e}_r \cdot \hat{e}_z + \sigma_{r\theta} \hat{e}_\theta \cdot \hat{e}_z$$

or

$$\sigma_{rz} = \sigma_{rr} \cos\theta - \sigma_{r\theta} \sin\theta \quad (15)$$

The conversion is necessary because we have no stress-strain rate relation for nonorthogonal (r, z) coordinates. In any orthogonal curvilinear coordinate system, the stresses and strain rate components for an incompressible Newtonian fluid are related by

$$\sigma_{ij} = -p \delta_{ij} + 2\mu e_{ij} \quad (16)$$

so that the stress components of (15) are

$$\sigma_{rr} = -p + 2\mu e_{rr}$$

and

$$\sigma_{r\theta} = 2\mu e_{r\theta}$$

Finding the strain rate expressions in Appendix B, we have

$$\begin{aligned} \sigma_{rz} = & -p \cos\theta + 2\mu \frac{\partial v_r}{\partial r} \cos\theta \\ & - \mu \left[r \frac{\partial}{\partial r} \left(\frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} \right] \sin\theta \end{aligned} \quad (17)$$

to be integrated to yield the drag. The integration over a spherical surface centered on the particle is of the form

$$F = 2\pi r^2 \int_0^\pi \sigma_{rz} \sin \theta \, d\theta \quad (18)$$

The velocities in either frame could be used to evaluate the strain rate expressions. The strain rate components, e_{rr} and $e_{r\theta}$, are independent of this choice. Velocities in the two frames differ only by the uniform velocity, $U \hat{e}_z$. There is no straining in a uniform flow. Consequently, we have in either frame

$$e_{rr} = \frac{\partial v_r}{\partial r} = \frac{3U}{2a} \left[\left(\frac{a}{r}\right)^4 - \left(\frac{a}{r}\right)^2 \right] \cos \theta \quad (19)$$

and

$$e_{r\theta} = \frac{1}{2} \left[r \frac{\partial}{\partial r} \left(\frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} \right] \quad (20)$$

which becomes

$$e_{r\theta} = \frac{3U}{4a} \left(\frac{a}{r}\right)^4 \sin \theta \quad (21)$$

Substituting into (17) and then into (18), the force becomes

$$\begin{aligned} F = & - 2\pi r^2 \int_0^\pi p \sin \theta \cos \theta \, d\theta \\ & + \frac{6\pi\mu r^2 U}{a} \left[\left(\frac{a}{r}\right)^4 - \left(\frac{a}{r}\right)^2 \right] \int_0^\pi \cos^2 \theta \sin \theta \, d\theta \\ & - \frac{3\pi\mu r^2 U}{a} \left(\frac{a}{r}\right)^4 \int_0^\pi \sin^3 \theta \, d\theta \end{aligned} \quad (22)$$

Evaluating the last two integrals

$$\int_0^{\pi} \cos^2 \theta \sin \theta \, d\theta = - \frac{\cos^3 \theta}{3} \Big|_0^{\pi} = \frac{2}{3} \quad (23)$$

$$\int_0^{\pi} \sin^3 \theta \, d\theta = \frac{\cos^3 \theta}{3} - \cos \theta \Big|_0^{\pi} = \frac{4}{3} \quad (24)$$

The force is greatly simplified

$$F = - 2\pi r^2 \int_0^{\pi} p \sin \theta \cos \theta \, d\theta = 4\pi \mu a U \quad (25)$$

The term just evaluated, $4\pi \mu a U$, is often called the viscous drag to distinguish this contribution from the 'pressure drag' which we have yet to find. Such a distinction, of course, has no operational significance.

Before calculation of the 'pressure drag', it is worthwhile to note that constant pressure, whatever its value, over the surface of a body exerts no net force. Only variations of pressure along the surface contribute to a drag. The fact, therefore, that we cannot calculate the absolute pressure, but only differences in pressure, need not concern us. See Problem 1.

The integral in (25) can be written in terms of the pressure variation along the spherical surface using the identity

$$p \sin \theta \cos \theta = \frac{1}{2} \frac{\partial}{\partial \theta} (p \sin^2 \theta) = \frac{1}{2} \frac{\partial p}{\partial \theta} \sin^2 \theta$$

Substituting into (25), noting that one of the resulting integrals is zero, we have

$$F = \pi r^2 \int_0^{\pi} \frac{\partial p}{\partial \theta} \sin^2 \theta \, d\theta = 4\pi \mu a U \quad (26)$$

relating the force to the pressure gradient. The pressure gradient is found from the equation of motion. Under the restrictions imposed on this flow, the Navier-Stokes equation is

$$\nabla p = \mu \nabla^2 \vec{V}$$

which is, using 2.7-(8) and continuity,

$$\nabla p = -\mu \nabla \times \vec{\xi}$$

4.7-(11) gives this expression in terms of the stream function

$$\nabla p = -\mu \nabla (E^2 \psi) \times \frac{\hat{e}_\phi}{\rho} \quad (27)$$

so that the term we need in spherical coordinates is

$$\frac{\partial p}{\partial \theta} = \frac{\mu}{\sin \theta} \frac{\partial}{\partial r} (E^2 \psi)$$

Operating on the stream function, (6) or (11), we find

$$E^2 \psi = \frac{3Ua \sin^2 \theta}{2r}$$

so that

$$\frac{\partial p}{\partial \theta} = -\frac{3\mu a U \sin \theta}{2r^2}$$

Substituting into (26) and integrating, we obtain Stokes Law

$$F = -6\pi\mu a U \quad (28)$$

This force is the drag, the force exerted on the particle by the surrounding fluid. For steady motion, this force would have to be balanced by an externally applied, equal and opposite, force f .

$$f = 6\pi\mu aU \quad (29)$$

The applied force and the resultant velocity of the particle relative to the fluid are colinear

$$\vec{f} = 6\pi\mu a\vec{U} \quad (30)$$

5.3 MOBILITY AND SEDIMENTATION

The mobility, b , of a particle is defined by the relation

$$\vec{U} = b \vec{f}$$

For a constant force, the greater the mobility of a particle, the greater is its velocity, the more mobile it is. For a spherical particle obeying Stokes law, the mobility is

$$b = \frac{1}{6\pi\mu a} \quad (1)$$

a function only of the fluid viscosity and the particle size.

If the particle is in a gravitational field, then a gravitational force acts on the particle

$$f = (M-m)g$$

M is the mass of the particle, m is the mass of the displaced fluid and g is the acceleration of gravity. Equating this force to the Stokes drag, and expressing the fluid and solid masses in terms of the radius and material

densities, the steady settling velocity of the particle results

$$U = \frac{2 (\rho_p - \rho_f) a^2 g}{9 \mu} \quad (2)$$

ρ_p and ρ_f are the particle and fluid densities. If the particle density greatly exceeds the fluid density, as is the case when a solid particle falls through the air, the settling velocity is more simply given by the approximate expression

$$U = \frac{2 \rho_p a^2 g}{9 \mu} \quad (3)$$

It may be noted that the settling velocity of particles increases with increasing particle size. The rate of increase of gravitational force exceeds the decrease in mobility. The result is that larger particles more rapidly fall out of suspension. The collection of the smaller particles from a fluid is the greater technical problem.

As a sample calculation, we may determine the settling velocity of a 10 micron spherical particle having a density of 2 g/cm³. We will consider first the particle settling in atmospheric air and then in water. The density and viscosity of both fluids are tabulated briefly in Appendix A. A temperature of 20C is chosen. Standard gravitational acceleration is about 980 cm/sec².

The settling velocity in air is adequately given by (3). Substituting the selected values, we obtain

$$U = 0.60 \text{ cm/sec}$$

Equation (2) must be used to find the settling velocity in water. Primarily because of the much larger liquid viscosity, the settling velocity is much lower

$$U = 5.5 \times 10^{-3} \text{ cm/sec}$$

In each of these cases, we can calculate a Reynolds number assuming the validity of Stokes law. If the Reynolds number is much smaller than one, then the Stokes drag expression is valid. The Reynolds number for Stokes law settling through a gas is, based on particle diameter,

$$Re = \frac{\rho_f U d}{\mu} = \frac{\rho_f \rho_p d^3 g}{18 \mu^2}$$

while the more accurate expression, necessary when the fluid and solid densities are comparable, is

$$Re = \frac{\rho_f (\rho_p - \rho_f) d^3 g}{18 \mu^2}$$

Note that this settling Reynolds number varies as the cube of the particle diameter. For the 10μ particle just used in the example, the settling Reynolds number in air is

$$Re = 4.0 \times 10^{-3}$$

while that in water is

$$Re = 5.4 \times 10^{-4}$$

Each of these values is small enough to warrant use of Stokes law. Since the Reynolds number varies as the cube of the diameter, it is also easy to see how rapidly the Reynolds number increases with particle size. For a 100μ particle in air, the error associated with use of Stokes law would be unacceptable.

5.4 INERTIAL EFFECTS - EXPERIMENTAL

The derivation of Stokes law is based on the neglect of inertial effects. As the Reynolds number increases and these effects become more significant, the actual drag experienced by a spherical particle must deviate from the values predicted by Stokes law. The experimental determination of the drag as a function of Reynolds number has been the subject of many experiments. Fuchs [2]

describes several of these studies. The recent work of Pruppacher and Steinberger [3] provides accurate measurements in the low Reynolds number range of interest here.

Pruppacher and Steinberger give the empirical relation

$$D_{\text{exp}} / D_{\text{Stokes}} = 1 + 0.102 \text{ Re}^{.95} \quad (1)$$

as a representation of the ratio of their experimentally determined drag to the Stokes drag when the Reynolds number is less than 2 and greater than 10^{-3} . $\text{Re} = 10^{-3}$ represents the lower bound of their data. Their results indicate that the drag lies between the theoretical Stokes and Oseen [4] drag expressions. The Oseen expression, which is discussed in more detail in the next section, is

$$D_{\text{Oseen}} / D_{\text{Stokes}} = 1 + \frac{3}{16} \text{ Re} \quad (2)$$

More exact analysis indicates that the drag should approach the Oseen drag for sufficiently small Reynolds numbers.

As the Reynolds number decreases, their results indicate that the drag does approach the Oseen drag, rather than the Stokes drag. In this regard, the results also agree with other recent experiments of Maxworthy [5]. At a Reynolds number as large as 1, their results indicate that both the Oseen and Stokes drags are in error by about 10%.

Particularly at higher Reynolds numbers, the drag on a body is often expressed in terms of the drag coefficient. The drag coefficient of a body is defined as

$$C_D = \frac{D}{\frac{1}{2} \rho_f U^2 A} \quad (3)$$

D is the drag and A is the projected area of the body. The drag coefficient is defined in this manner because at high Reynolds numbers it is relatively constant

and has a value of order unity. The denominator is an estimate of the drag based on the pressure increase at the front of the body due to fluid stagnation there.

For a particle obeying Stokes law, the drag coefficient is

$$C_D = \frac{6\pi\mu aU}{\frac{1}{2} \rho_f U^2 \pi a^2} = \frac{24}{\rho_f U d / \mu} = \frac{24}{Re} \quad (4)$$

According to Oseen's analysis, the drag coefficient for small Reynolds numbers becomes

$$C_D = \frac{24}{Re} + \frac{9}{2} \quad (5)$$

Figure 5.1 shows how the experimental results compare with these analyses. Because of the strong variation of drag coefficient with Reynolds number in the low Reynolds number range, drag is seldom expressed in terms of drag coefficient there.

Although Stokes law is strictly valid only for zero Reynolds number, its simplicity is appealing and errors associated with its use in the dynamics of aerosols and hydrosols are often of little importance.

5.5 INERTIAL EFFECTS - ANALYTICAL

The derivation of Stokes law was based on the neglect of inertial effects throughout the flow field. It is instructive to examine the resulting solution to see if it is in fact consistent with this simplification. Assuming the uniform validity of the solution expressed by 5.2-(6), 5.2-(7) and 5.2-(8), it is a simple matter to estimate the magnitude of the terms in the Navier-Stokes equation for steady motion of an incompressible Newtonian fluid having constant viscosity.

$$\rho_f (\vec{V} \cdot \nabla) \vec{V} = - \nabla p + \mu \nabla^2 \vec{V}$$

Figure 5.1 Drag Coefficient for a Sphere

For the assumption leading to the Stokes flow solution to be uniformly valid, the inertial terms must be everywhere negligible compared to the viscous terms given a sufficiently small Reynolds number. Examination of the velocity distribution, 5.2-(7) and 5.2-(8), shows that sufficiently far from the particle, the velocity is of the form

$$\vec{V} = -U \hat{e}_z + O\left[U \frac{a}{r}\right] \quad (1)$$

The disturbance velocity dies off as r^{-1} . Accordingly at a large distance from the particle, the magnitude of the inertial term, $\rho_f (\vec{V} \cdot \nabla) \vec{V}$, is $\rho_f U^2 a / r^2$. Using the asymptotic form of the velocity distribution, the magnitude of the viscous term, $\mu \nabla^2 \vec{V}$, is $\mu U a / r^3$. The ratio of magnitude of inertia to viscous terms is

$$\frac{\rho_f U^2 a / r^2}{\mu U a / r^3} = \frac{1}{2} \frac{\rho_f U (2a)}{\mu} \frac{r}{a} = \frac{1}{2} \text{Re} \left(\frac{r}{a} \right)$$

So, given any Reynolds number no matter how small, beyond a distance of the order a/Re from the particle, the inertia terms resulting from the Stokes solution dominate the viscous terms. Near the particle the viscous terms are always dominant provided the Reynolds number is much less than one. The Stokes approximation is not self consistent over the entire flow field. The solution resulting from the use of the creeping flow equations shows these equations to be invalid far from the particle. This inconsistency of the Stokes assumption was first noted by Oseen [4]. Because the solution near the sphere is consistent and because this region of validity can be made as large as desired by reducing the Reynolds number, the drag obtained from the Stokes solution is correct in the zero Reynolds number limit.

The first attempt to extend the Stokes solution to higher Reynolds numbers was made by Whitehead [6]. Whitehead's approach was the seemingly reasonable one of using the Stokes solution to evaluate the inertia terms in the Navier-Stokes equation and solving this linear equation for the second approximation

to the flow field. Because the Stokes solution is not valid far from the particle, the second approximation generated in this manner is invalid there as well and, in fact, would not satisfy the condition of uniform flow far from the particle.

When Oseen explained the origin of this problem, he also proposed a means of avoiding it. Rather than neglect the inertial term $\rho_f(\vec{V} \cdot \nabla)\vec{V}$ in the Navier-Stokes equation, Oseen replaced this expression by the linear expression $\rho_f(-U\hat{e}_z \cdot \nabla)\vec{V}$. Far from the particle, where inertia effects are important, the fluid velocity approaches the streaming velocity $-U\hat{e}_z$ so that the linearized expression accurately represents the convective acceleration there. Near the particle, both the actual convective acceleration and its linearized replacement are small compared to the viscous term when the Reynolds number is small. The error due to the linearization is of order Re near the particle so that the Oseen equation is no worse than the Stokes equation in this region.

The equations used by Oseen were the linearized Navier-Stokes equation

$$\rho_f (-U\hat{e}_z \cdot \nabla) \vec{V} = -\nabla p + \mu \nabla^2 \vec{V} \quad (2)$$

in place of the creeping flow equation of Stokes and the continuity equation as before. Oseen's solution, in terms of the stream function, is

$$\begin{aligned} \psi = & \frac{Ua^2}{4} \left[2 \left(\frac{r}{a}\right)^2 + \frac{a}{r} \right] \sin^2\theta \\ & - \frac{3Ua^2}{Re} (1 + \cos\theta) \left(1 - e^{-\frac{Re}{4} \frac{r}{a}} (1 - \cos\theta) \right) \end{aligned}$$

which satisfies the governing equation, the boundary condition at infinity, but fails by a term of order Re to satisfy the boundary condition at the surface. This is easily seen by expanding the exponential term

$$\psi = \frac{Ua^2}{4} \left[2 \left(\frac{r}{a}\right)^2 - 3 \frac{r}{a} + \frac{a}{r} \right] \sin^2\theta + O \left[Ua^2 Re \left(\frac{r}{a}\right)^2 \right]$$

The first expression is recognized as the Stokes solution, 5.2-(6). Since the governing equation is in error at the surface to the same order as the Oseen solution, no more accurate solution of the Oseen equation can be justified on the grounds of yielding a more accurate answer. Goldstein [7] did find the exact solution of flow past a sphere governed by (2).

Since the Oseen solution is correct only to zeroth order in the Reynolds number, it would seem that it could only be used to verify the Stokes drag and not to provide a first order correction in terms of the Reynolds number. In fact, more exact analysis [8] reveals that the drag calculated by Oseen from his solution

$$D = 6\pi\mu aU \left(1 + \frac{3}{16} Re \right)$$

is correct to first order in Reynolds number. The drag due to Goldstein

$$D = 6\pi\mu aU \left(1 + \frac{3}{16} Re - \frac{19}{1280} Re^2 + \dots \right)$$

is likewise only correct to the first order in the Reynolds number.

The problem of finding inertial corrections to the Stokes drag on a spherical particle was again undertaken by Proudman and Pearson [8]. Their elegant solution, obtained by matching asymptotic expansions valid near to and far from the particle, gave two terms of the Stokes drag correction.

$$D = 6\pi\mu aU \left(1 + \frac{3}{16} Re + \frac{9}{160} Re^2 \ln \frac{Re}{2} + \dots \right)$$

The first correction term is again that of Oseen. Their method is conceptually capable of generating continually higher order approximations to the drag. The task of finding these higher order terms, however, also becomes continually more arduous.

At least through the first order correction in Reynolds number, the more recent experimental results [3] agree with these analyses. The necessity of

including any correction at all to Stokes law depends, of course, on the range of Reynolds numbers encountered as well as the desired accuracy.

5.6 BROWNIAN DIFFUSION - THE EINSTEIN RELATION

Although diffusion is not a steady process, it is included here because the Brownian diffusivity of a particle is easily calculated from the mobility of the particle in steady flow. The relation follows from the conditions necessary for thermodynamic equilibrium to exist. The basic argument is due to Einstein [9] who published several papers on Brownian motion early in his career.

Consider a collection of particles suspended in a quiescent fluid. A conservative force field such as the force due to gravity acts on the particles causing a net motion in one direction. For definiteness, we will consider a gravitational field causing the particles to sink through the fluid. In addition to this directed motion, each particle will display a random motion resulting from statistical fluctuations present in any system. Although a thermodynamic system may be in a macroscopic state of equilibrium, its microscopic state is continually changing. The result is a fluctuating force acting on the particle producing the observed Brownian motion or thermal motion. The average kinetic energy of a particle undergoing Brownian motion is

$$\frac{1}{2} M \overline{v^2} = \frac{3}{2} kT \quad (1)$$

a result of the principle of equipartition of energy, which is itself a result of classical statistical thermodynamics. k is Boltzmann's constant, which appeared in Section 2.2 where the value 1.38×10^{-16} erg/K is given. T is the absolute temperature. The average kinetic energy of the particle is independent of any transport properties of the surrounding fluid. The root mean square velocity becomes

$$v_{\text{rms}} = \sqrt{\frac{3kT}{M}} \quad (2)$$

and is larger for smaller particles. A 10 micron diameter spherical particle with a density of 2 g/cm³ has, at 20c, a root mean square velocity of about

10^{-2} cm/sec. If the diameter is a tenth of a micron, the root mean square velocity is about 10 cm/sec, a considerable larger value.

The net effect of a random motion of this type is that the particles tend to diffuse from regions of high concentration to regions of low concentration. Diffusion acts to remove such disparities, tending toward a more uniform distribution. In a gravitational field acting to accumulate particles at the bottom of a container, diffusion acts to smooth out the distribution and thus produces an upward flux of particles. In thermal equilibrium, the downward flux due to gravity is balanced by the upward flux due to diffusion. The net flux across any imaginary surface such as surface A in Figure 5.2 is zero. The balancing is, of course, a time average. The instantaneous flux need not be zero. The walls of the container are taken to be reflecting surfaces rather than surfaces to which the particles adhere.

In this field, the flux in the z direction due to the applied force f is, using the definition of mobility b ,

$$j_f = nbf \quad (3)$$

The flux j_f is the number of particles that would cross the surface A per unit area and unit time if the gravitational force alone caused motion across the surface. n is the number density of particles in the neighborhood of the imaginary surface.

The flux due to diffusion is expressed by Fick's first law

$$j_d = -D \frac{dn}{dz} \quad (4)$$

The constant D is the diffusion coefficient or diffusivity. A primary objective of this section is to obtain an expression for the magnitude of the diffusivity.

Since the net flux vanishes in thermal equilibrium,

$$j_f + j_d = 0 \quad (5)$$

Figure 5.2 Thermal Equilibrium in a Gravitational Field

or

$$\rho \frac{dn}{dz} = nbf \quad (6)$$

Now, the force in any conservative field can be expressed in terms of the gradient of the energy

$$f = - \frac{dE}{dz} \quad (7)$$

In a gravitational field whose strength varies little over the dimensions of the container, the potential energy of a particle is simply $(M-m)gz$. Substituting this expression for energy into (6), then separating and integrating we find an expression for the number density distribution

$$n = n_0 e^{-bE/\rho} \quad (8)$$

in terms of the number density n_0 at the position where the energy is taken to be zero.

Statistical thermodynamics, however, provides us with another expression for the number density distribution. Applying the Boltzmann equation for the population of thermodynamic states to the distribution of particles, we have

$$n = n_0 e^{-E/kT} \quad (9)$$

where k again is the Boltzmann constant. The distribution (9) is termed the Boltzmann distribution.

Comparing these two expressions for number density, we note that they are identical once we set

$$D = b k T \quad (10)$$

This is the Einstein relation between a particle's diffusivity and its mobility. For a spherical particle obeying Stokes law, the diffusivity is, using 5.3-(1)

$$D = \frac{kT}{6\pi\mu a} \quad (11)$$

the Stokes-Einstein relation. For a 10 micron diameter particle in atmospheric air at 20C, the diffusivity is $2.38 \times 10^{-8} \text{ cm}^2/\text{sec}$.

The reader should note that although the development was phrased in terms of a gravitational field, the argument is equally applicable to any conservative field. As an example, an electric field exerts a force, analogous to (7)

$$f = - q \frac{dV}{dz}$$

on a particle carrying a charge q . V is here the voltage. The corresponding energy is qV . As a result, the distribution of ions in an electrolytic liquid

is governed by (9) near a charged surface. The surface charge repels ions of like sign and attracts ions of opposite sign. Diffusion opposes this separation and the Boltzmann distribution results. The mobility and diffusivity of each ionic specie are related by (10).

5.7 THE DIFFUSION EQUATION

The number density distribution $n(x, y, z, t)$ of a collection of suspended particles obeys a simple linear equation when the transport of particles is due solely to diffusion. The derivation of this equation is based on the conservation of species where the specie flux through a stagnant fluid is given by the general form of 5.6-(4)

$$\vec{j} = - D \nabla n \quad (1)$$

For simplicity, we consider only a monodisperse suspension, one having a single particle size, so that all particles have the same diffusion coefficient. The extension to the more general case of a polydisperse suspension is trivial because the relations are linear and the solution is obtained directly by superposition. We further neglect coagulation which results in loss of particles of one size and generation of particles of another size as well as any other interaction among the particles. This restriction implies a dilute suspension.

By an argument identical in form to the argument of Section 2.5 that led to 2.5-(1) expressing conservation of mass, we obtain an expression conserving numbers of particles. Consider an elemental region of space as shown in Figure 2.5. The net number of particles entering the volume per unit time is equal to the rate of change of the number of particles in the volume

$$\begin{aligned} (j_x|_x - j_x|_{x + \Delta x}) \Delta y \Delta z + (j_y|_y - j_y|_{y + \Delta y}) \Delta x \Delta z \\ + (j_z|_z - j_z|_{z + \Delta z}) \Delta x \Delta y = \frac{\partial n}{\partial t} \Delta x \Delta y \Delta z \end{aligned}$$

Dividing by the element's volume and taking limits in the usual manner, the conservation equation becomes

$$-\nabla \cdot \vec{j} = \frac{\partial n}{\partial t} \quad (2)$$

With the flux of particles given by Fick's first law, we finally have

$$\nabla \cdot (D \nabla n) = \frac{\partial n}{\partial t}$$

With uniform diffusivity, which implies from (10) a uniform temperature, this is

$$D \nabla^2 n = \frac{\partial n}{\partial t} \quad (3)$$

This relation is known as the diffusion equation or as Fick's second law. It's solution has been subject to extremely extensive investigation. Techniques for solving the diffusion equation are highly developed. The considerable interest in obtaining solutions of this equation is due to that fact that equations of the same form govern a wide variety of phenomena including the transfer of heat by conduction. Carslaw and Jaeger [10] and Crank [11] give several solutions. The Carslaw and Jaeger book is particularly comprehensive.

The number density represented in these equations is an average value or expected value rather than the value that would result from any single experiment. The average is an ensemble average, an average of the number densities obtained in an infinite number of similar experiments. The statistical nature of these equations results from the flux expression (1) which itself represents an average. In equilibrium, as described in the last section, an ensemble average is assumed to yield the same value as a time average. This assumption is known as the ergodic hypothesis. Physically, the random nature of Brownian motion requires that we be content with statistical information.

We may readily obtain several solutions of wide applicability by means of superposition. We start with the solution of the diffusion equation describing diffusion of particles from a finite point source.

Consider an infinite fluid, devoid of suspended particles, except in a small region where N particles may be found. Conceptually let the size of this region become ever smaller approaching a point in the limit. The number density of the N particles increases as the volume occupied by the particles is reduced.

The finite dimensions of real particles would, of course, prevent an arbitrarily close approach to the limit. The limit of N particles located at a single point in an otherwise empty stagnant fluid represents the initial condition. At a certain instant, say $t=0$, the particles are released and proceed to diffuse through the medium. It may be verified that the resulting (expected) number density of particles is given as a function of position and time by

$$n = \frac{N}{(4\pi Dt)^{3/2}} e^{-r^2/4Dt} \quad (4)$$

where r is the distance from the source. The solution satisfies the governing equation (3), as well as the initial condition that n vanish everywhere but at the origin where it is infinite. The number of particles in the fluid at any time is, of course, N .

$$\begin{aligned} \int_V n dV &= 4\pi \int_0^\infty r^2 n dr \\ &= \frac{N}{2\pi^{1/2}(Dt)^{3/2}} \int_0^\infty r^2 e^{-r^2/4Dt} dr \\ &= N \end{aligned}$$

The definite integral,

$$\int_0^\infty x^n e^{-a^2 x^2} dx = \Gamma\left(\frac{n+1}{2}\right) / 2a^{n+1} \quad (5)$$

valid for $n+1 > 0$ and $a > 0$, was used.

The solution (4) shows how the particles diffuse from the source in a spherically symmetric manner. The number density distribution becomes progressively more uniform. This solution is of considerable interest because more complicated transient diffusion distributions can be considered to result from a distribution of sources and sinks.

A excellent measure of the extent to which particle dispersion results from diffusion is the mean square displacement of the particles at some time after the diffusion commences. In an unbounded fluid, the distribution of particles at any time can be considered as the sum of an appropriate number of point sources. The resulting mean square displacement of the particles from their initial positions is then simply the same as the mean square displacement from any single finite point source. Accordingly, we calculate the mean square displacement from (4). From the definition of the average, the mean square displacement is

$$\overline{r^2} = \int_V r^2 n dV / N \quad (6)$$

so that

$$\overline{r^2} = \frac{1}{2\pi^{1/2}(\mathcal{D}t)^{3/2}} \int_0^\infty r^4 e^{-r^2/4\mathcal{D}t} dr$$

which, using (5) is

$$\overline{r^2} = 6\mathcal{D}t \quad (7)$$

The mean square displacement in terms of Cartesian components is

$$\overline{r^2} = \overline{x^2 + y^2 + z^2}$$

so that, using symmetry, we find the mean square displacement along a Cartesian coordinate axis

$$\overline{x^2} = 2\mathcal{D}t \quad (8)$$

The mean square displacement is proportional to time and the particle diffusivity. For a particle whose diffusivity is given by the Stokes-Einstein relation, 5.6-(11), the mean square displacement in any direction is

$$\overline{x^2} = \frac{kTt}{3\pi\eta a} \quad (9)$$

The root mean square displacement is

$$\sqrt{\overline{x^2}} = \sqrt{\frac{kTt}{3\pi\eta a}} \quad (10)$$

Shortly after Einstein's derivation of these relations, Perrin [12] provided experimental verification by observation of the displacements of mastic and gamboge globules in suspension. He also verified the Boltzmann distribution, 5.6-(9), for the particles by counting the number of particles at several depths in the liquid.

The mean square displacement after one second is calculated from (9) for a 10 micron particle in atmospheric air at 20C. The result is $4.76 \times 10^{-8} \text{ cm}^2$. The root mean square displacement is $2.18 \times 10^{-4} \text{ cm}$ or roughly one fifth of the particle diameter. The root mean square displacement in a given time, resulting from Brownian diffusion, increases as the particle size decreases. In the Stokes law range, (10) indicates that the root mean square displacement varies inversely as the square root of the particle radius.

Having determined an expression for the root mean square displacement of a suspended particle, it is informative to estimate the mean height of a suspension in a gravitational field above a reflecting surface. It seems reasonable to propose that the effective height, h , of a suspension is a distance whose magnitude is of the order of the root mean square linear displacement of a particle during the time that a particle moving solely due to gravity would require to fall through that height.

The time required for a particle having mobility b to fall through a distance h is, with a gravitational force of $(M-m)g$

$$t = \frac{h}{b(M-m)g}$$

In this time, the mean square displacement resulting from Brownian motion is, from (8) and 5.6-(10)

$$\overline{z^2} = \frac{2kTh}{(M-m)g}$$

Equating this mean square displacement to h^2 , our rough estimate of the average height of a particle from the lower surface becomes simply

$$h = \frac{2kT}{(M-m)g} \quad (11)$$

Fortunately, we may compare this estimate with an exact calculation using the Boltzmann distribution, 5.6-(9) with the potential energy given by

$$E = (M-m)gz$$

The mean height is

$$\bar{z} = \frac{\int_0^\infty z n dz}{\int_0^\infty n dz}.$$

The denominator normalizes the Boltzmann distribution for n . Substituting, we have

$$\bar{z} = \frac{\int_0^\infty z e^{-(M-m)gz/kT} dz}{\int_0^\infty e^{-(M-m)gz/kT} dz}$$

which is integrated using

$$\int_0^\infty x^n e^{-ax} dx = n! / a^{n+1}$$

which is valid for $a > 0$ and non-negative integer n . The mean height of the suspension is

$$\bar{z} = \frac{kT}{(M-m)g} \quad (12)$$

which is exactly half the order of magnitude estimate (11). This agreement is quite satisfactory for an estimate such as led to (11). The argument producing (11) gives a physical picture of how the directed motion due to gravity and the random Brownian motion produce a Boltzmann distribution in this case.

Equation (12) can be used to calculate the mean height of the Boltzmann distribution for a given particle size. For a spherical particle, (12) becomes

$$\bar{z} = \frac{3 k T}{4\pi a^3 (\rho_p - \rho_f) g}$$

showing that $\bar{z}$ depends strongly on the particle size. For a 10 micron particle with a density of 2g/cm^3 suspended in atmospheric air at 20°C , we find

$$\bar{z} = 3.94 \times 10^{-4} \mu$$

a small fraction of the particle radius. However, a 0.1 micron particle under the same conditions yields

$$\bar{z} = 394 \mu$$

which is several thousand radii.

The conclusion is that effects of Brownian diffusion are normally negligible for larger aerosols but become considerably more important for finer dispersions. Brownian diffusion becomes very important, for instance, in the collection of small aerosols by filtration.

5.8 FURTHER SOLUTIONS OF THE DIFFUSION EQUATION

The solution, 5.7-(4), describing the diffusion of particles from a finite point source can, using superposition, be used to construct solutions to other important cases.

The solution describing the diffusion of particles from an instantaneous line source is obtained by constructing the line source from an infinite chain of

infinitesimal point sources. Consider a point source located at a position described by $z = z'$ and $\rho = 0$ in cylindrical coordinates. In these coordinates, 5.7-(4), the solution for diffusion from a finite point source, is

$$n = \frac{N}{(4\pi Dt)^{3/2}} e^{-[\rho^2 + (z-z')^2]/4Dt}$$

Consequently, the number density distribution resulting from an infinitesimal line source of length dz' containing $N' dz'$ particles is

$$dn = \frac{N' dz'}{(4\pi Dt)^{3/2}} e^{-[\rho^2 + (z-z')^2]/4Dt}$$

where N' is the number of particles per unit length of source. If we now sum (integrate) over an infinite chain of such sources, we obtain the solution for the instantaneous line source.

$$n = \frac{N'}{(4\pi Dt)^{3/2}} e^{-\rho^2/4Dt} \int_{-\infty}^{\infty} e^{-(z-z')^2/4Dt} dz'$$

which, using 5.7-(5), is

$$n = \frac{N'}{4\pi Dt} e^{-\rho^2/4Dt} \quad (1)$$

The solution for the instantaneous plane source is obtained in the same manner. In Cartesian coordinates, the particle distribution about an instantaneous line source located at x' and y' is

$$n = \frac{N'}{4\pi Dt} e^{-[(x-x')^2 + (y-y')^2]/4Dt}$$

Accordingly, the distribution about an infinitesimal plane source bounded by y' and $y' + dy'$ is

$$dn = \frac{N'' dy'}{4\pi Dt} e^{-[(x-x')^2 + (y-y')^2]/4Dt}$$

The source is a ribbon, infinitely long in the z direction, with width dy' in the y direction and with no thickness in the x direction. N'' is the number of particles per unit area of the plane source. Integration over all values of y' yields the distribution of particles emanating from an instantaneous plane source of infinite extent.

$$n = \frac{N''}{(4\pi Dt)^{1/2}} e^{-(x-x')^2/4Dt} \quad (2)$$

Now, let us use these results to solve a problem dealing with diffusional deposition. Consider the semi-infinite region, $x>0$, initially containing a uniformly distributed suspension with particle number density n_0 . The region is bounded at x equal to zero by an adsorbing wall. Any particle reaching this wall will adhere to it so that the number of particles in suspension is continually being depleted by removal of those particles that diffuse to the wall. The wall is vertical so that motion due to gravity is parallel to the wall and need not be considered. (See Problem 4.)

The initial distribution of particles in the region, $x>0$, can be considered as a uniform distribution of infinitesimal instantaneous point sources. Alternatively, it can be considered as a uniform distribution of instantaneous plane sources.

At the wall to which diffusing particles adhere, the particle concentration drops to zero. This is the boundary condition that one would apply to the diffusion equation. Alternatively, the wall can be conceptually removed and replaced with a uniform distribution of sinks in the region $x<0$.

The sinks are equivalent to the sources in every sense except that they detract from the net number density rather than add to it. Whereas an instantaneous point source has a strength N , an instantaneous point sink of equal magnitude has strength $-N$. The net number density at any point surrounded by sources and sinks is the sum of the number densities at that point resulting from each of those sources and sinks. The number densities resulting from sinks are negative. Of course, in the region of physical interest, $x>0$ in this case,

only non-negative net number densities may be found.

Instead of a region of initially uniform number density n_0 bounded by an adsorbing wall, we will consider the infinite region having initially uniform number density n_0 for x greater than zero and initially uniform number density $-n_0$ for x less than zero. By symmetry, the number density at x equal to zero is then zero for all succeeding time. This is the same condition as results from the adsorbing wall. Consequently, in the right half space, $x > 0$, the two cases are identical. This equivalence permits the solution of the case of diffusion to an adsorbing surface by superposition.

The particles initially lying between x' and $x' + dx'$ diffuse as if from a plane source located at x' and having strength $n_i dx'$. Accordingly, from (2) we find

$$dn = \frac{n_i dx'}{(4\pi Dt)^{1/2}} e^{-(x-x')^2/4Dt}$$

n_i represents the initial number density. Summing over all space, the resulting expression for the number density distribution in terms of position and time is

$$n = \frac{1}{(4\pi Dt)^{1/2}} \int_{-\infty}^{\infty} n_i(x') e^{-(x-x')^2/4Dt} dx' \quad (3)$$

For our case

$$n_i = n_0 \text{ when } x > 0$$

and

$$n_i = -n_0 \text{ when } x < 0$$

so that the integral form of the solution is

$$n = \frac{n_0}{(4\pi Dt)^{1/2}} \left[\int_0^{\infty} e^{-(x-x')^2/4Dt} dx' - \int_{-\infty}^0 e^{-(x-x')^2/4Dt} dx' \right]$$

Making the substitution

$$\xi = (x-x')/(4Dt)^{1/2}$$

in each of these integrals and rearranging

$$n = \frac{n_0}{\pi^{1/2}} \left[\int_{-\infty}^{x/(4Dt)^{1/2}} e^{-\xi^2} d\xi - \int_{x/(4Dt)^{1/2}}^{\infty} e^{-\xi^2} d\xi \right]$$

Since the integrand is an even function of ξ , this becomes

$$n = \frac{2n_0}{\pi^{1/2}} \int_0^{x/(4Dt)^{1/2}} e^{-\xi^2} d\xi \quad (4)$$

This integral is well known with well tabulated properties [13]. By definition, the error function is

$$\text{erf } x = \frac{2}{\pi^{1/2}} \int_0^x e^{-\xi^2} d\xi \quad (5)$$

so that we may more succinctly write the solution as

$$n = n_0 \text{erf} \left[\frac{x}{(4Dt)^{1/2}} \right] \quad (6)$$

For small values of the argument, the error function can be evaluated by replacing the integrand of (5) by its power series representation and then integrating term by term. Accordingly, sufficiently near the wall and for sufficiently large time, the number density distribution can be written

$$n = \frac{2n_0}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1) n!} \left[\frac{x}{(4Dt)^{1/2}} \right]^{2n+1} \quad (7)$$

The rate at which particles are deposited on the wall is given by the number flux toward the wall at x equal to zero

$$j_{\text{wall}} = D \left. \frac{\partial n}{\partial x} \right|_{x=0}$$

The gradient is readily calculated from (4) or (7), giving

$$j_{\text{wall}} = n_0 \left(\frac{D}{\pi t} \right)^{1/2} \quad (8)$$

The flux to the wall decreases with time as the number density of particles in the fluid near the wall is depleted. The total number of particles deposited on a unit area of the wall is

$$N = 2n_0 \left(\frac{Dt}{\pi} \right)^{1/2} \quad (6)$$

The last case of diffusion to be presented in this section is the case of the continuous point source. A continuous point source, rather than containing a fixed number of particle which are simultaneously released, releases particles continuously at a specified rate. The distribution resulting from an continuous point source is also obtainable directly from the solution, 5.7-(4), for an instantaneous point source by using superposition.

At any time t , the distribution of particles released from an instantaneous point source at time τ ($\tau < t$) is

$$n = \frac{N}{[4\pi D(t-\tau)]^{3/2}} e^{-r^2/4D(t-\tau)}$$

If instead, the particles were released at a rate of $\dot{N}$ particles per unit time for an infinitesimal period $d\tau$, then the distribution is

$$dn = \frac{\dot{N} d\tau}{[4\pi D(t-\tau)]^{3/2}} e^{-r^2/4D(t-\tau)}$$

at the later time t . If particles are liberated in this manner since $\tau=0$, then at time t , the number density distribution is

$$n(r,t) = \frac{1}{(4\pi D)^{3/2}} \int_0^t \frac{\dot{N}(\tau) e^{-r^2/4D(t-\tau)}}{(t-\tau)^{3/2}} d\tau \quad (7)$$

For a continuous point source of constant strength, $\dot{N}$ constant, the substitution

$$\xi = r/[4D(t-\tau)]^{1/2}$$

produces

$$n = \frac{1}{2Dr\pi^{3/2}} \int_{r/(4Dt)^{1/2}}^{\infty} e^{-\xi^2} d\xi \quad (8)$$

Related to the error function (5) is the complementary error function

$$\text{erfc } x = 1 - \text{erf } x = \frac{2}{\pi^{1/2}} \int_x^{\infty} e^{-\xi^2} d\xi \quad (9)$$

so that the distribution surrounding the constant continuous point source can be written

$$n = \frac{1}{4\pi Dr} \text{erfc} \left[\frac{r}{(4Dt)^{1/2}} \right] \quad (10)$$

As the time increases, the argument of the complementary error function approaches zero, the value of the complementary error function approaches one,

and the particle distribution approaches the steady distribution

$$n = \frac{1}{4\pi Dr} \quad (11)$$

where the number density varies as the reciprocal of the distance from the source.

Further consideration of diffusion is relegated to the problems and to sections dealing with particular applications.

5.9 DRAG ON AN ISOLATED PARTICLE IN AXISYMMETRIC MOTION

In the calculation of the Stokes law drag, we demonstrated that the net force on any spherical envelope surrounding the particle was the same. It was not necessary to calculate the drag by integrating the stresses on the particle surface. Here, we will explain and extend this result, then use it to derive a very simple means of calculating the drag on an isolated particle in steady axisymmetric motion, given its stream function. This means of calculating the drag is due to Payne and Pell [14] who derived it in a different manner.

Consider the particle shown in Figure 5.3. The particle is surrounded by a control volume which excludes the particle. The surface of the control volume consists of an 'inner' surface coincident with the particle surface and an outer surface lying entirely in the fluid. Only fluid is contained in the control volume. The net force exerted on the fluid by the surroundings is equal to the rate of change of fluid momentum. The control volume form of the momentum equation results from applying Reynolds transport theorem, 2.5-(5), to Newton's second law.

Figure 5.2 Control Volume Surrounding a Particle

$$\sum \vec{F} = \frac{\partial}{\partial t} \int_V \rho_f \vec{V} dv + \int_A \rho_f \vec{V} \vec{V} \cdot \hat{n} dA$$

The net force on the control volume is equal to the rate of change of momentum stored within the control volume plus the net momentum flux across the control surface. In steady flow, the momentum within the control volume is constant and the first of the integrals vanishes. In the Stokes approximation, e.g. massless fluid, the second term vanishes as well. The net force on the fluid is zero.

$$\sum \vec{F} = 0$$

It follows immediately, in the absence of body forces, that the force exerted by the surroundings at the outer surface is exactly balanced by the force exerted on the fluid by the particle at the inner control surface. The force exerted on the fluid by the particle is, of course, equal and opposite to the force exerted by the fluid on the particle. The force exerted by the fluid on the particle may be found by calculating the force exerted by the surroundings on the outer control surface. This is exactly the procedure that was used to calculate the Stokes drag on the sphere. The spherical outer boundary that was used in that calculation was convenient, but not necessary. Any closed surface surrounding the particle would suffice.

Since the shape and location of the outer control surface are arbitrary, they may be chosen to simplify the calculation. As before, let us choose a spherical surface defined by constant r in a spherical coordinate system.

We restrict our attention now to an axisymmetric particle in slow steady motion along its axis of revolution. The force exerted by the fluid on this particle is also directed along this axis. This force is determined by integration of the axial stress component over the spherical outer control surface as in 5.2-(18)

$$F = 2\pi r^2 \int_0^\pi \sigma_{rz} \sin\theta d\theta$$

Since the axisymmetric fluid flow possesses a stream function, the stress can be expressed in terms of operations on the stream function. First, we recall 5.2-(17), the expression for the stress component σ_{rz} in terms of velocity components in a spherical coordinate system

$$\begin{aligned}\sigma_{rz} = & -p \cos\theta + 2\mu \frac{\partial v_r}{\partial r} \cos\theta \\ & - \mu \left[r \frac{\partial}{\partial r} \left(\frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} \right] \sin\theta\end{aligned}$$

The velocity components are expressed in terms of the stream function by 4.3-(16), so that the stress becomes

$$\begin{aligned}\sigma_{rz} = & -p \cos\theta - 2\mu \frac{\partial}{\partial r} \left(\frac{1}{r^2} \frac{\partial \psi}{\partial \theta} \right) \frac{\cos\theta}{\sin\theta} \\ & - \mu \left[r \frac{\partial}{\partial r} \left(\frac{1}{r^2} \frac{\partial \psi}{\partial r} \right) - \frac{1}{r^3} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin\theta} \frac{\partial \psi}{\partial \theta} \right) \sin\theta \right]\end{aligned}$$

and the force becomes

$$\begin{aligned}F = & -2\pi r^2 \int_0^\pi p \sin\theta \cos\theta \, d\theta - 2\pi r^2 \mu \int_0^\pi \left[2 \frac{\partial}{\partial r} \left(\frac{1}{r^2} \frac{\partial \psi}{\partial \theta} \right) \cos\theta + r \frac{\partial}{\partial r} \left(\frac{1}{r^2} \frac{\partial \psi}{\partial r} \right) \sin\theta - \frac{1}{r^3} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin\theta} \frac{\partial \psi}{\partial \theta} \right) \sin^2\theta \right] d\theta\end{aligned}$$

The pressure integral is expressed in terms of the stream function through the substitutions

$$p \sin \theta \cos \theta = \frac{1}{2} \frac{\partial}{\partial \theta} \left(p \sin^2 \theta \right) - \frac{1}{2} \frac{\partial p}{\partial \theta} \sin^2 \theta$$

and

$$\frac{\partial p}{\partial \theta} = \frac{\mu}{\sin \theta} \frac{\partial}{\partial r} (E^2 \psi),$$

the θ component of 5.2-(27). With these substitutions,

$$\begin{aligned} F = \pi r^2 \mu \int_0^\pi & \left[\frac{\partial}{\partial r} (E^2 \psi) \sin \theta - 4 \frac{\partial}{\partial r} \left(\frac{1}{r^2} \frac{\partial \psi}{\partial \theta} \right) \cos \theta \right. \\ & \left. - 2r \frac{\partial}{\partial r} \left(\frac{1}{r^2} \frac{\partial \psi}{\partial r} \right) \sin \theta + \frac{2}{r^3} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin \theta} \frac{\partial \psi}{\partial \theta} \right) \sin^2 \theta \right] d\theta \end{aligned} \quad (1)$$

expressing F entirely in terms of the stream function. The most general solution of the equation governing the fluid motion

$$E^4 \psi = 0 \quad (2)$$

that remains bounded on the axis of revolution is the solution, 4.8-(18)

$$\psi = \sum_{n=0}^{\infty} (a_n r^n + b_n r^{1-n} + c_n r^{n+2} + d_n r^{3-n}) c_n^{-\frac{1}{2}(\mu)} \quad (3)$$

in spherical coordinates. μ here is $\cos \theta$.

Of course, not every possible solution of the equation of motion can appear as a solution to the flow past an isolated particle in an otherwise uniform flow. First, the stream function must satisfy the streaming flow boundary condition far from the particle. In the frame of the particle

$$\psi \rightarrow \frac{U r^2 \sin^2 \theta}{2} \quad \text{as} \quad r \rightarrow \infty$$

so that no term of (3) involving a power of r greater than 2 can appear in the solution. Additionally, only that r^2 term that is multiplied by $\sin^2\theta$ may remain. The Gegenbauer polynomial $C_2^{-1/2}(\mu)$ is $\sin^2\theta/2$. All remaining terms of the solution must involve powers of r that are one or less. As we noted in Section 5.2, a uniform flow is a rigid body motion. There is no fluid strain rate associated with this motion. The $Ur^2\sin^2\theta/2$, or $Up^2/2$, term makes no contribution to the drag.

Notice now that each of the terms on the right hand side of (1) is, when the r^2 outside of the integral is included, of the order of ψ/r . Consider, then, the contribution to the integral (1) of any term of (3) involving a power of r , r^m , where m is less than one. The contribution to the integral of this term is of the order r^{m-1} . Since the value of r has not been specified, it may be chosen as large as desired. By letting r increase without bound, the contribution of each of these terms to the integral is seen to vanish and these terms need not be considered further in the calculation of the drag.

At this stage, we have eliminated from consideration all solutions of (2) that have other than a linear dependence on the radius. Only one solution involving a higher power can be present in flow past an isolated particle and it exerts no force. An infinite number of solutions with lower powers of r can be present but they do not contribute to the drag. We must now examine those terms having the linear dependence on the radius. From (3), we see that there are only three such terms. Writing only these terms,

$$\psi = a_1 r C_1^{-1/2}(\mu) + b_0 r C_0^{-1/2}(\mu) + d_2 r C_2^{-1/2}(\mu)$$

is the only part of ψ that need be considered further. This is also written

$$\psi = -a_1 r \cos\theta + b_0 r + \frac{d_2}{2} r \sin^2\theta$$

Now, let us consider each of these terms in order. The first term, $r \cos\theta$, corresponds to an infinite continuous line source of constant strength located along the axis of symmetry. In cylindrical coordinates, $r \cos\theta$ is simply z and the stream surfaces are planes perpendicular to the axis. The fluid flow is in the cylindrical radial direction. Since no source is permitted in the fluid

surrounding the particle, this term is not part of any solution to flow about a particle.

The second term, r , describes the flow due to a continuous uniform line source on the positive z axis and a continuous uniform line sink of equal strength on the negative z axis. The stream surfaces are concentric spheres. Again, since no source appears in the fluid surrounding the particle, the stream function describing the flow past the particle can not include this term.

We are left with a single term, $r \sin^2 \theta$, that determines the steady drag on an isolated axisymmetric particle in motion along its axis of revolution through an otherwise uniform fluid. This term appears in the stream function describing the Stokes flow about a rigid sphere that was obtained in Section 5.2. In that case, the contribution to the stream function due to this term was written as

$$\psi = \frac{U}{2} B r \sin^2 \theta \quad (4)$$

which form will be retained.

Substituting this expression into (1) and performing the differentiation, r vanishes from the force relation. Integrating, using 5.2-(23) and 5.2-(24), we obtain

$$F = 4\pi\mu UB \quad (5)$$

This is the expression sought. It permits immediate calculation of the drag given the stream function. Its use will greatly simplify many calculations yet to do.

As a check, recall that B was $-3a/2$ in the analysis leading to Stokes law. Inserting this value into (5) gives 5.2-(28), Stokes law.

An alternative way of expressing (5) is the expression used by Payne and Pell. If ψ is the stream function in the frame of the distant fluid, then (5) can be written

$$F = 8\pi\mu \lim_{r \rightarrow \infty} \frac{r\psi}{\rho}$$

In the frame of the particle, the stream function differs by $U_0^2/2$ from the stream function in the frame of the distant fluid. Using the stream function in the frame of the particle,

$$F = 8\pi\mu \lim_{r \rightarrow \infty} \frac{r (\psi - U_0^2/2)}{\rho^2}$$

The force on an isolated particle or the net force on an isolated collection of particles can be determined with any of these expressions provided these additional conditions are met. The creeping flow equations apply, inertial effects being negligible. The motion is steady and axisymmetric. The fluid is unbounded. The flow far from the particle is uniform streaming.

The derivation of Payne and Pell utilizes the additional restrictions of no-slip and no normal flow at the particle surface. These conditions are relaxed here because there are many cases that we wish to treat where these conditions are not satisfied. As an example, translation of a bubble or drop produces a continuous tangential deformation of the surface.

5.10 THE STOKESLET

In the last section, the far flow field of an isolated particle was examined and the dominant portion of the stream function was related to the drag. In terms of the axial force exerted by the fluid on the particle, this distant flow field is

$$\psi = \frac{F r \sin^2 \theta}{8\pi\mu} \quad (1)$$

Note that the particle shape and dimensions do not enter this expression. If we now consider the particle getting ever smaller while maintaining a constant force, we see that this stream function corresponds to the stream function of the flow produced by a point force. This point force is called a Stokeslet. In terms of the force, f , exerted on the fluid in the positive axial direction, the stream function of a Stokeslet is

$$\psi = - \frac{f r \sin^2 \theta}{8\pi\mu} \quad (2)$$

The axisymmetric motion of any finite particle can be considered to result from distributed Stokeslets, sources and sinks. The disturbance caused by the motion of a rigid sphere in the Stokes law regime is given by 5.2-(11). This stream function is composed of the stream functions of a doublet (see Problem 4.8) and a Stokeslet.

Because the Stokeslet contribution to the fluid flow dominates at large distances from particles, the interactions of widely spaced particles in steady creeping flow can be approximated by considering only the flow produced by the Stokeslets. Blake [15] determined the image system necessary to produce no slip on a plane boundary in the presence of a Stokeslet.

5.11 MOTION OF A BUBBLE OR A DROP

The low Reynolds number translation of a spherical bubble or drop can readily be analyzed by the techniques developed in this chapter. The resulting expression for the mobility of the fluid particle was first obtained by Hadamard [16] and independently by Rybczynski [17].

A gas bubble moving through a liquid or a liquid drop moving through a liquid or gas presents a fluid dynamic system consisting of two fluids separated by an interface. The shape of this boundary is an important factor in determining the tractability of the flow to analysis. Indeed, because the boundary shape is determined by the combined effects of fluid stress and surface tension, the shape of the boundary is, *a priori*, unknown. We shall first consider a limiting case where the boundary shape is known and later examine how this boundary may become distorted. The limiting case, for low Reynolds number flow is the spherical bubble. While surface tension clearly acts to restore a bubble to a spherical shape, we shall also find, surprisingly enough, that in the creeping flow regime, the fluid stresses at the interface of a spherical bubble do not act so as to distort the bubble shape.

Consider now the steady creeping motion of a fluid sphere whose surface is located at $r = a$. The fluid external to the sphere is different from the internal fluid. A prime will be used to denote quantities interior to the sphere. This axisymmetric motion of each of the fluids is governed by

$$E^4 \psi = 0 \quad (1)$$

Solution of the governing equations will yield two stream functions, one describing the interior motion and the other describing the flow of the exterior fluid about the sphere.

The stream functions are subject to boundary conditions imposed at the limits of their domains of validity. The boundary conditions imposed at the interface are similar in part to those imposed on the flow past a rigid sphere. First, we choose to write the boundary conditions in the frame of the fluid sphere. The surface of this fluid particle then has no component of velocity normal to itself. There is no normal component of fluid velocity at the surface for either fluid. Following the previous analysis, we express this no normal flow condition by

$$\psi = 0 \text{ at } r = a \quad (2)$$

for the external fluid and

$$\psi' = 0 \text{ at } r = a \quad (3)$$

for the interior fluid.

The no-slip condition at the interface takes a different form than it did at the surface of a rigid sphere. In that case, the tangential fluid velocity was equated to the tangential solid velocity. Because the solid is rigid, this tangential velocity was zero. Here the interior fluid is free to move and could not support any tangential stress at the interface without such motion. The interior and exterior fluid each have tangential velocities at the interface but do not slip relative to one another.

$$v_{\theta} = v'_{\theta} \text{ at } r = a$$

or in terms of the stream function

$$\frac{\partial \psi}{\partial r} = \frac{\partial \psi'}{\partial r} \text{ at } r = a \quad (4)$$

Unlike the case of the rigid sphere, a force balance at the interface provides an additional boundary condition. The stresses exerted by the fluid on

the solid sphere were balanced without continuous straining of the solid. Equating those stresses yields no information useful in determining the fluid velocity distribution. In balancing forces exerted on the interface of the fluid sphere, useful information is extracted. We proceed by balancing forces first in the tangential direction, then in the normal direction.

The tangential stress exerted by the external fluid on the boundary of a fluid sphere is balanced by the tangential stress exerted by the interior fluid and by any gradient in the interfacial surface tension. If the surface tension varies with θ , then it exerts a net tangential force on any element of surface that must be balanced by tangential fluid stresses. In the absence of concentration gradients and temperature gradients along the surface, the surface tension is constant and does not affect the tangential force balance. In that case

$$\sigma_{r\theta} = \sigma_{r\theta}' \quad \text{at } r = a$$

Writing these stresses in terms of the strain rates and writing the velocities in the strain rate expressions in terms of the stream functions, this boundary condition becomes

$$\mu \frac{\partial}{\partial r} \left(\frac{1}{r^2} \frac{\partial \psi}{\partial r} \right) = \mu' \frac{\partial}{\partial r} \left(\frac{1}{r^2} \frac{\partial \psi'}{\partial r} \right) \quad \text{at } r = a. \quad (5)$$

This expression was simplified by noting that $\partial v_r / \partial \theta$ is zero at the surface in this frame.

In practice, surface-active contaminants are frequently adsorbed onto the interface producing a non-uniform surface tension and retarding the motion of this surface. This point is examined further later in this section. The effect of a temperature gradient on the motion of a bubble is examined in the next section.

Balancing the forces normal to the interface yields no additional information. The interior and exterior normal stresses should differ by a constant amount due to surface tension acting over the spherical boundary. Any variations of the difference between the normal stresses would result in distortions of the boundary from a spherical shape and the surface tension would again balance the stresses.

By assumption, these distortions are presumed small.

Far from the fluid sphere, the flow is uniform streaming in the negative z direction. The stream function of the exterior fluid accordingly has the limit

$$\psi \rightarrow \frac{U r^2 \sin^2 \theta}{2} \quad \text{as } r \rightarrow \infty \quad (6)$$

The final condition to be imposed is that the fluid velocity is everywhere bounded.

Following the lead of the Stokes law analysis, we attempt to describe the fluid motions with stream functions of the form

$$\psi = \frac{U}{2} (A r^2 + B r + \frac{C}{r} + D r^4) \sin^2 \theta \quad \text{for } r \geq a \quad (7)$$

and

$$\psi' = \frac{U}{2} (A' r^2 + B' r + \frac{C'}{r} + D' r^4) \sin^2 \theta \quad \text{for } r \leq a \quad (8)$$

which satisfy the governing equation, (1).

The limiting condition (6) yields

$$A = 1 \quad ; \quad D = 0$$

The boundedness of the fluid velocities imposed at the origin, the center of the fluid sphere, requires

$$B' = C' = 0$$

With these simplifications, the stream functions are

$$\psi = \frac{U}{2} (r^2 + B r + \frac{C}{r}) \sin^2 \theta$$

and

$$\psi' = \frac{U}{2} (A' r^2 + D' r^4) \sin^2 \theta$$

The four unknown constants are determined from the remaining four boundary conditions, (2) through (5). The no normal flow conditions give

$$a^2 + Ba + \frac{C}{a} = 0$$

and

$$A'a^2 + D'a^4 = 0$$

The condition of no-slip between the phases (4) gives

$$2a + B - \frac{C}{a^2} = 2A'a + 4D'a^3$$

Equality of the tangential stresses requires

$$\mu \left(-\frac{2}{a^2} - \frac{2B}{a^3} + \frac{4C}{a^5} \right) = \mu' \left(-\frac{2A'}{a^2} + 4D' \right)$$

Solution of these four equations in the coefficients yields for the external fluid

$$B = -\frac{a}{2} \frac{2\mu + 3\mu'}{\mu + \mu'} \quad (9)$$

and

$$C = \frac{a^3}{2} \frac{\mu'}{\mu + \mu'}$$

and for the internal fluid

$$A' = -\frac{1}{2} \frac{\mu}{\mu + \mu'}$$

and

$$D' = \frac{1}{2a^2} \frac{\mu}{\mu + \mu'}$$

The nature of the motion is determined by the viscosity ratio of the two fluids. If the viscosity of the internal fluid greatly exceeds the viscosity of the surrounding medium, then B and C approach the rigid sphere values, $-3a/2$ and $a^3/2$. In the same limit, the coefficients of the internal flow vanish so that the fluid there remains stationary. A liquid drop falling through air behaves nearly like a rigid sphere because of the large liquid to gas viscosity ratio. For example, the viscosity of water is roughly two orders of magnitude greater than the viscosity of air at 20C.

The case of small internal fluid viscosity relative to the viscosity of the external fluid corresponds to a gas bubble rising through a liquid. In that case, both internal and external flows differ significantly from the motion associated with translation of a rigid sphere. The velocities resulting from the internal circulation of the bubble are of the order of the translation velocity of the bubble.

The force exerted by the surrounding fluid on the fluid sphere is found immediately using 5.9-(5)

$$F = 4 \pi \mu U B$$

for the creeping, steady, axisymmetric motion of an isolated particle. Substituting (9), there results

$$F = - 6\pi\mu a U \frac{2\mu/3 + \mu'}{\mu + \mu'} \quad (10)$$

giving the force for an arbitrary viscosity ratio. For a liquid drop moving through gas, the force is nearly the Stokes law force, $-6\pi\mu a U$. A gas bubble moving through a liquid has a strong internal circulation, thus producing less severe velocity gradients in the external fluid and a smaller force resisting the motion. For small μ'/μ , the force is approximately $-4\pi\mu a U$, two thirds of the Stokes law expression.

In a gravitational field, the force acting on the sphere is

$$f = (M-m)g = \frac{4}{3} \pi a^3 (\rho_f' - \rho_f) g$$

so that the settling velocity of the sphere is

$$U = \frac{2(\rho_f' - \rho_f) a^2 g}{9 \mu} \frac{\mu + \mu'}{2\mu/3 + \mu'} \quad (11)$$

A liquid drop falls through air with a velocity indistinguishable from the velocity of a solid particle of equal size. A gas bubble, $\rho_f' \ll \rho_f$ and $\mu' \ll \mu$, rises through a liquid with the velocity nearly equal to

$$U = \frac{\rho_f a^2 g}{3 \mu} \quad (12)$$

according to (11).

In practice, small gas bubbles are seldom observed to rise through a liquid with velocities as large as predicted by (12). Rather, their velocities are frequently much closer to the value, $2\rho_f a^2 g / 9\mu$, that a Stokes drag of $6\pi\mu aU$ would produce. Additionally, the internal circulation of these bubbles is not as strong as predicted by this model. The accepted explanation is that the surface of the bubble accumulates surface-active substances (surfactants) transported to it from the surrounding liquid. Because of the tangential surface velocity associated with internal circulation, these surfactants are carried to the rear of the bubble. The concentration of surfactants is consequently greater at the rear than at the bubble front where new surface is continuously being formed. Because of this higher concentration of surfactants at the rear, the surface tension is lower there. A net force on the surface due to the surface tension gradient results. This force opposes the surface motion causing the bubble to move with reduced internal circulation, more nearly like a rigid sphere.

The first attempt at an analytical explanation of the observed reduction of circulation was made by Frumkin and Levich [18]. This work, as well as a related paper by Levich [19] are described by Levich [20]. These models, however, do not describe the cap of rigid surface observed at the rear of the bubble by Savic [21]. These observations reveal that the fluid motion is not simply a slower motion with fore and aft symmetry. Instead, the circulation and surrounding flow are asymmetric. The rear of the bubble is a rigid spherical cap resulting from the surface convection of relatively insoluble surfactant. The remainder of the

bubble surface has negligibly small tangential stress because of the low concentration of surfactant and the negligible internal viscosity. Davis and Acrivos [22] and Harper [23], as well as Savic, have analyzed the motion with stagnant caps.

Now, let us turn to the matter of distortion of a bubble or drop from a spherical shape. For sufficiently large surface tension, the bubble or drop will retain a nearly spherical shape regardless of the stress distribution on the surface. Here we wish to demonstrate the result in creeping flow, anticipated at the beginning of this section, that the stress distribution on the surface of a fluid sphere moving as a result of gravity does not act to deform the sphere. This result is valid only in the creeping flow regime where inertial effects are negligible. As fluid inertia becomes more important, a rising bubble will assume an oblate spheroidal shape and at higher Reynolds numbers, the form of a spherical cap. Taylor and Acrivos [24] and Pan and Acrivos [25] have analytically investigated the deformation of a drop or bubble due to inertial effects in the low Reynolds number range.

Deformation of the surface of a fluid sphere would result from variations along the surface of the normal stress difference across the surface. The normal stresses exerted by the fluids on either side of the interface are not equal, even in equilibrium. The difference is due to the surface tension, σ . Consider a stationary fluid sphere of radius a in the absence of a gravitational field. The pressure p' of the internal fluid is uniform as is the pressure p outside the sphere. A hemispherical section of the sphere constitutes a convenient element for a force balance. The internal pressure produces a net force $p'\pi a^2$, on this section. This force is balanced by the force, $p\pi a^2$, exerted by the exterior fluid pressure acting over the same area and the force, $2\pi a\sigma$, in the same direction that the surface tension exerts on the edge of the hemispherical shell. The result is an internal fluid pressure that exceeds the external pressure by $2\sigma/a$.

For a fluid sphere moving through a gravitational field, the surface will not distort provided the difference in normal stresses remains constant. Since symmetry permits variations only with the polar angle, θ , we need only show that

$$\left. \frac{\partial \sigma_{rr}}{\partial \theta} \right|_{r=a} = \left. \frac{\partial \sigma'_{rr}}{\partial \theta} \right|_{r=a} \quad (13)$$

to verify that the creeping flow stresses do not cause deformation of the bubble or drop.

The normal stresses on either side of the interface are

$$\sigma_{rr} = -p + 2\mu \frac{\partial v_r}{\partial r} \quad (14)$$

and

$$\sigma'_{rr} = -p' + 2\mu' \frac{\partial v'_r}{\partial r} \quad (15)$$

While the derivatives of velocity are easily calculated from the known stream functions of the motion, the pressure distribution is worthy of some discussion.

In a gravitational field, a body force, $\rho_f g$, is exerted on the fluid in, say, the negative z direction. The Navier - Stokes equation for incompressible viscous flow in this field is

$$\rho_f \frac{D\vec{V}}{Dt} = -\nabla p - \rho_f g \hat{e}_z + \mu \nabla^2 \vec{V}$$

For the creeping flow treated here, this is

$$\nabla p + \rho_f g \hat{e}_z = \mu \nabla^2 \vec{V}$$

which may alternatively be written as

$$\nabla(p + \rho_f g z) = \mu \nabla^2 \vec{V}$$

Defining

$$P \equiv p + \rho_f g z, \quad (16)$$

the equation of motion becomes

$$\nabla P = \mu \nabla^2 \vec{V} \quad (17)$$

Following Batchelor [26], we call P the modified pressure. The equation of motion of an incompressible fluid in a gravitational field is of the same form as the equation of motion in the absence of such a field. In the absence of gravity, a pressure gradient is associated with fluid motion. With gravity, a pressure gradient also arises due to the gravitational force, $\rho_f g$, so that only part of the pressure variation, the variation of modified pressure, results from fluid motion.

Normally we ignore gravity in calculating the drag on a particle. Gravity is later accounted for by applying a force, $(M-m)g$, to the particle. Here, we need to determine the actual pressure distribution about the bubble or drop in order to determine any tendency toward deformation.

In the same manner that the governing equation

$$\nabla p = \mu \nabla^2 \vec{V}$$

let to 5.2-(27) in the absence or neglect of a gravitational field, (17) gives us

$$\nabla P = - \mu \nabla (E^2 \psi) \times \frac{\hat{e}_\phi}{\rho}$$

which yields the variation of modified pressure with polar angle

$$\frac{\partial P}{\partial \theta} = \frac{\mu}{\sin \theta} \frac{\partial}{\partial r} (E^2 \psi)$$

The actual pressure variation at the surface of the fluid sphere is

$$\frac{\partial p}{\partial \theta} = \frac{\partial P}{\partial \theta} - \rho_f g \frac{\partial z}{\partial \theta}$$

or

$$\frac{\partial p}{\partial \theta} = \frac{\mu}{\sin \theta} \frac{\partial}{\partial r} (E^2 \psi) \Big|_{r=a} + \rho_f g a \sin \theta$$

The normal stress variation on the outside of the interface is now

$$\frac{\partial \sigma_{rr}}{\partial \theta} = 2\mu \frac{\partial^2 v_r}{\partial \theta \partial r} - \frac{\mu}{\sin \theta} \frac{\partial}{\partial r} (E^2 \psi) - \rho_f g a \sin \theta$$

evaluated at r equal to a . The stream function for the external flow is (7). Operating on this stream function, we find that in terms of the non-vanishing coefficients

$$\frac{\partial \sigma_{rr}}{\partial \theta} = - \left(\frac{3\mu B U}{a^2} + \frac{6\mu C U}{a^4} + \rho_f g a \right) \sin \theta$$

U is the sphere velocity in the z (upward) direction. Calculating the normal stress variation on the inside of the interface in the same manner, we find

$$\frac{\partial \sigma'_{rr}}{\partial \theta} = - \left(6D' \mu' a U + \rho'_f g a \right) \sin \theta$$

The variation of normal stress difference across the interface is

$$\begin{aligned} \frac{\partial}{\partial \theta} (\sigma_{rr} - \sigma'_{rr}) &= (\rho'_f - \rho_f) g a \sin \theta \\ &+ \frac{\mu U}{a} \left(\frac{6\mu' D' a^2}{\mu} - \frac{3B}{a} - \frac{6C}{a^3} \right) \sin \theta \end{aligned}$$

Substituting the values of the coefficients, this expression reduces to

$$\begin{aligned} \frac{\partial}{\partial \theta} (\sigma_{rr} - \sigma'_{rr}) &= (\rho'_f - \rho_f) g a \sin \theta \\ &+ \frac{9\mu U}{2a} \frac{2\mu/3 + \mu'}{\mu + \mu'} \sin \theta \end{aligned}$$

Finally, we note that the settling velocity, that is, the velocity in the negative z direction, is given by (11). The velocity here is the negative of that in (11) so that

$$\frac{\partial}{\partial \theta} (\sigma_{rr} - \sigma'_{rr}) = 0$$

The difference between the normal stresses is constant and from previous argument equal to $2\sigma/a$.

The drag on spherical bubbles moving with large Reynolds numbers through viscous liquids has been analyzed elegantly by Chao [27]. Chao's analysis is applicable to flow with internal circulation and also with negligible separation. For

$$\rho_f \mu' \ll \rho_f \mu$$

as for a gas bubble in liquid, his predicted drag coefficient is

$$C_D = \frac{32}{Re} \left(1 + 2 \frac{\mu'}{\mu} - 0.314 \frac{1+4\mu'/\mu}{Re^{1/2}} \right)$$

Winnikow and Chao [28] experimentally investigated high Reynolds number droplet motion in very pure liquid. They found the effect of flow separation to be significant and proposed a correction to Chao's unseparated flow drag coefficient.

5.12 MOTION OF A BUBBLE IN A TEMPERATURE GRADIENT

The bubble motion described in the last section was analyzed under the assumption of constant surface tension. It was noted there that variations of temperature or concentration gradients would produce a non-uniform surface tension. Here, we will examine the motion of a spherical bubble resulting from the application of a temperature gradient to the fluid surrounding the bubble.

A temperature gradient in a liquid containing a small bubble will cause the bubble to move through the liquid. This motion may be attributed to the variation of surface tension with temperature. The surface tension of a liquid is a strong function of temperature, normally decreasing linearly with increasing

temperature over a broad range. The temperature distribution on the surface of the bubble consequently produces a variation of surface tension along the surface. Because the surface must be in mechanical equilibrium, the tangential stresses exerted by the fluids on the interface may not vanish but must exert a force balancing the force due to surface tension. Tangential stresses are generated only through fluid motion. The resulting fluid motion causes translation of the bubble at a speed that we wish to determine. Young, Goldstein, and Block [29] have investigated this phenomenon analytically and experimentally. The bubble motion is easily demonstrated using a test tube containing a 20 centistoke kinematic viscosity silicone oil. After shaking the test tube to introduce small bubbles, application of a hot soldering iron to the outside of the test tube causes bubble migration toward the hot wall.

The first step in describing the fluid motion is to determine the temperature distribution at the bubble surface. While the temperature and velocity distributions are, in fact, coupled, the effect of slow fluid motion on the temperature distribution is small and may be safely neglected.

The first law of thermodynamics for an infinitesimal fluid element is

$$\dot{Q} - \dot{W} = \rho_f \frac{De}{Dt}$$

where $\dot{Q}$ is the heat transfer rate to the element from the surroundings, $\dot{W}$ is the rate at which the element does work on its surroundings and $\rho_f \frac{De}{Dt}$ is the material derivative of the stored energy. See Section 2.6.

If the fluid is incompressible and if viscous dissipation is neglected, then $\dot{W}$ vanishes. The most important mode of heat transfer is normally conduction heat transfer and is governed by Fourier's law.

$$\vec{q} = -k \nabla T$$

$\vec{q}$ is the heat transfer per unit time per unit area normal to the direction of heat flow. k is the thermal conductivity of the medium, a material property whose value depends primarily on temperature. T is the local temperature. Expressing the net heat transfer rate, $\dot{Q}$, in terms of the divergence of the flux, $\vec{q}$,

$$-\nabla \cdot \vec{q} = \rho_f \frac{De}{Dt}$$

The expression for the net heat transfer rate is obtained in the same manner as net mass flow rate in Section 2.5 and net diffusion to an element in Section 5.7.

Substituting Fourier's law, we find

$$\nabla \cdot (k \nabla T) = \rho_f \frac{De}{Dt}$$

or for constant thermal conductivity

$$k \nabla^2 T = \rho_f \frac{De}{Dt}$$

Neglecting changes in kinetic and potential energy in comparison to changes in internal energy

$$\frac{De}{Dt} = \frac{Du}{Dt}$$

The internal energy of an incompressible fluid depends only on its temperature. In terms of the specific heat, c , we have

$$k \nabla^2 T = \rho_f c \frac{DT}{Dt}$$

which is written as

$$k \nabla^2 T = \rho_f c \left(\frac{\partial T}{\partial t} + \vec{V} \cdot \nabla T \right) \quad (1)$$

in Eulerian coordinates.

The relative importance of the convective terms and conduction in determining the temperature distribution can be estimated using order of magnitude arguments. The ratio of the estimated convective term to a conduction term is of the order $\rho_f c U a / k$. Alternately, the ratio is written

$$Pe = \frac{\mu c}{k} \cdot \frac{2\rho_f Ua}{\mu} = Pr \cdot Re \quad (2)$$

Pe is the Peclet number and is the product of the Prandtl number, Pr, and the Reynolds number. The Prandtl number is a constant whose value depends only on fluid properties. For most fluids, the Prandtl number is roughly of order unity. For water at 20C, the Prandtl number is about 7.1 and for air at the same temperature, the Prandtl number is about 0.72. There are fluids whose Prandtl number differs significantly. Glycerin, for example, has a Prandtl number of the order of ten thousand at temperatures near 20C. Normally the condition of low Reynolds number is sufficient to ensure a low Peclet number. For Peclet number much less than unity, we neglect convective heat transfer and find the temperature distribution is governed by Laplace's equation

$$\nabla^2 T = 0 \quad (3)$$

Consider now the temperature distribution in and around a spherical bubble in a liquid with an otherwise uniform temperature gradient. The temperature distribution outside the bubble obeys (3) while the internal temperature distribution, T' , also obeys Laplace's equation

$$\nabla^2 T' = 0 \quad (4)$$

Here, as before, the prime refers to the fluid in the bubble.

At the interface between the two phases, the temperatures are equal.

$$T = T' \quad \text{at} \quad r=a \quad (5)$$

Additionally, the heat flux into the interface from one phase is equal to the heat flux out of the interface into the other phase.

$$k' \frac{\partial T'}{\partial r} = k \frac{\partial T}{\partial r} \quad \text{at} \quad r=a \quad (6)$$

Far from the bubble, the disturbance due to the presence of the bubble vanishes and the temperature distribution corresponds to the uniform applied temperature gradient. Aligning the axis of the coordinate system with the temperature gradient, we have

$$T \rightarrow T_0 + \frac{dT}{dz} z \text{ as } r \rightarrow \infty$$

or

$$T \rightarrow T_0 + \frac{dT}{dz} r \cos\theta \text{ as } r \rightarrow \infty \quad (7)$$

in spherical coordinates. T_0 is the temperature far from the bubble on the plane of symmetry. dT/dz is the magnitude of the applied temperature gradient. The final condition imposed on the temperature is that it remain bounded at the origin.

The solution of these equations is obtained in a manner similar to the solution of the equation of motion for viscous flow. Laplace's equation is simply separable in spherical coordinates. (Problem 4.9.) The only terms of the resulting series that we need retain are a constant, which we set equal to T_0 , and the axisymmetric solution whose θ dependence agrees with (7).

We have then

$$T = T_0 + \left(Ar + \frac{B}{r^2} \right) \cos\theta$$

satisfying (3) and

$$T' = T_0 + \left(A'r + \frac{B'}{r^2} \right) \cos\theta$$

satisfying (4).

The boundedness condition requires that B' vanish. The asymptotic condition (7) shows that A is the applied temperature gradient, dT/dz . Substitution into the remaining boundary conditions, (5) and (6), and solution of the two resulting equations in the coefficients yields

$$B = a^3 \frac{k - k'}{2k + k'} \frac{dT}{dz}$$

and

$$C = \frac{3k}{2k + k'} \frac{dT}{dz}$$

The temperature distributions are

$$T = T_0 + \frac{dT}{dz} \left[r + \frac{a^3}{r^2} \left(\frac{k - k'}{2k + k'} \right) \right] \cos\theta \quad (8)$$

and

$$T' = T_0 + \frac{3k}{2k + k'} \frac{dT}{dz} r \cos\theta \quad (9)$$

It is interesting to note that the temperature gradient within the bubble is constant and equal to $3k/(2k + k')$ times the applied gradient. The thermal conductivity, k' , of the gas in a bubble is normally very much smaller than the thermal conductivity, k , of the liquid. As a result, the temperature gradient in the bubble is approximately 3/2 of the applied gradient.

Having determined the temperature distribution, the variation of surface tension follows immediately. From symmetry, the surface temperature and surface tension vary only with the polar angle θ .

$$\frac{d\sigma}{d\theta} = \frac{d\sigma}{dT} \frac{dT}{d\theta} \Big|_{r=a}$$

$d\sigma/dT$ is the rate of change of surface tension with temperature. It is a fluid property. $d\sigma/dT$ is negative and tends to be very constant over wide temperature ranges. For an air-water surface, $d\sigma/dT$ is approximately -0.15 dyne/cmC. Finding $dT/d\theta$ at the surface from (8) or (9), there results

$$\frac{d\sigma}{d\theta} = - \frac{3 k a}{2k + k'} \frac{d\sigma}{dT} \frac{dT}{dz} \sin\theta \quad (10)$$

For a bubble thermal conductivity much less than the liquid conductivity, this becomes approximately

$$\frac{d\sigma}{d\theta} = - \frac{3a}{2} \frac{d\sigma}{dT} \frac{dT}{dz} \sin\theta \quad (11)$$

We now must determine the tangential stress distribution at the surface that exerts the force necessary to produce mechanical equilibrium of the surface. Since the gas viscosity is much less than the liquid viscosity, we may neglect the tangential stress exerted by the gas and consider only the tangential stress exerted by the liquid. The more general analysis is relegated to the Problems.

Let us balance the forces in the θ direction on the infinitesimal element of surface shown in Figure 5.3. The element is a piece of a spherical surface. Its boundaries correspond to constant θ and constant ϕ . The radius of curvature of the surface is a . Forces in the θ direction are exerted by the tangential stress acting over the area, by the surface tension pulling on the constant θ boundaries and also by the the surface tension acting on the constant ϕ boundaries. While the unit vectors $\hat{e}_\theta$ and $\hat{e}_\phi$ are orthogonal at any point, the unit vector in the surface perpendicular to the $\phi + d\phi$ border is in the $\hat{e}_\phi + (\partial\hat{e}_\phi / \partial\phi) d\phi$ direction. This last vector has a component in the ρ direction as we shall now show.

The derivative of the unit vector is evaluated by considering the definition of the derivative

$$\frac{\partial\hat{e}_\phi}{\partial\phi} = \lim_{\Delta\phi \rightarrow 0} \frac{\hat{e}_\phi|_{\phi + \Delta\phi} - \hat{e}_\phi|_\phi}{\Delta\phi}$$

The difference between the two unit vectors on the right hand side is found easily by drawing a vector triangle. The resultant vector is in the $-\hat{e}_\rho$ direction and has magnitude $\Delta\phi$. Consequently,

$$\frac{\partial\hat{e}_\phi}{\partial\phi} = -\hat{e}_\rho$$

Figure 5.3 Equilibrium of Bubble Surface

The length of each of the constant ϕ boundaries is $a d\theta$ so that the net force exerted by surface tension on these boundaries is

$$-\sigma (a d\theta) \hat{e}_\phi + \sigma (a d\theta) (\hat{e}_\phi - \hat{e}_\rho d\phi) = -\sigma a d\theta d\phi \hat{e}_\rho$$

The net force exerted on these boundaries is in the negative cylindrical radial direction. This net force has a component in the θ direction since

$$\hat{e}_\rho \cdot \hat{e}_\theta = \cos\theta$$

The component of this force in the θ direction is $-\sigma a d\theta d\phi \cos\theta$.

Now, we may proceed to sum all the forces on the surface element acting in the θ direction. The length of the constant θ boundary is $(a \sin\theta d\phi)$. We consider only the liquid to exert a tangential stress.

$$\begin{aligned} & (\sigma a \sin\theta d\phi)|_{\theta+d\theta} - (\sigma a \sin\theta d\phi)|_\theta \\ & - \sigma a d\theta d\phi \cos\theta + \sigma_{r\theta} a^2 \sin\theta d\theta d\phi = 0 \end{aligned}$$

Dividing by the incremental angles and letting their magnitudes approach zero, we find

$$a \frac{d}{d\theta} (\sigma \sin\theta) - a \sigma \cos\theta + a^2 \sigma_{r\theta} \sin\theta = 0$$

which reduces to

$$\frac{d\sigma}{d\theta} + a \sigma_{r\theta} = 0 \quad (12)$$

giving an expression for the distribution of tangential stress exerted by the liquid on the interface. In terms of the previously calculated variation of surface tension (10),

$$\sigma_{r\theta} = \frac{3k}{2k + k'} \frac{d\sigma}{dT} \frac{dT}{dz} \sin\theta \quad \text{at } r=a \quad (13)$$

This is the boundary condition on tangential stress. It replaces the equal tangential stress boundary condition applied to bubble motion in the last section. That equal stress boundary condition reduces to $\sigma_{r\theta}$ equal to zero when the internal viscosity is neglected as it is here.

The problem has reduced to one of fluid mechanics. The governing equation for the axisymmetric creeping flow is

$$E^4 \psi = 0$$

for the external fluid motion. Because the internal viscosity is neglected, the external fluid flow is uncoupled from the internal flow. The external flow field may be found without finding the internal distribution. The boundary conditions imposed on the external flow at the interface are no normal flow, which gives

$$\psi = 0 \quad \text{at } r = a \quad (14)$$

in the frame of the bubble and also the prescribed stress boundary condition (13) which reduces to

$$\frac{\partial}{\partial r} \left(\frac{1}{r^2} \frac{\partial \psi}{\partial r} \right) = \frac{3k}{(2k + k')a\mu} \frac{d\sigma}{dT} \frac{dT}{dz} \sin^2\theta \quad \text{at } r = a \quad (15)$$

when the stress is expressed in terms of the strain rate and the velocities in terms of the stream function. The flow also satisfies the streaming flow condition far from the bubble.

$$\psi \rightarrow \frac{Ur^2 \sin^2 \theta}{2} \quad \text{as} \quad r \rightarrow \infty \quad (16)$$

From the polar angle dependence exhibited in (15) and (16), we again anticipate a solution of the form

$$\psi = \frac{U}{2} (Ar^2 + Br + \frac{C}{r} + Dr^4) \sin^2 \theta$$

The streaming flow condition requires that A be unity and D vanish. Solution of the two algebraic equations resulting from substitution into boundary conditions (14) and (15) yields the remaining coefficients.

$$B = -a - \frac{a^2}{\mu U} \frac{k}{2k + k'} \frac{d\sigma}{dT} \frac{dT}{dz} \quad (17)$$

$$C = \frac{a^4}{\mu U} \frac{k}{2k + k'} \frac{d\sigma}{dT} \frac{dT}{dz}$$

The stream function of the liquid surrounding the bubble is

$$\begin{aligned} \psi = & \frac{Ua^2}{2} \left[\left(\frac{r}{a} \right)^2 - \left(\frac{r}{a} \right) \right] \sin^2 \theta \\ & + \frac{a^3}{2\mu} \frac{k}{2k + k'} \frac{d\sigma}{dT} \frac{dT}{dz} \left(\frac{a}{r} - \frac{r}{a} \right) \sin^2 \theta \end{aligned} \quad (18)$$

The first part of this stream function expression is the stream function of a bubble having negligible internal viscosity and moving in the absence of a temperature gradient through the liquid. The stream function of the previous section reduces to this result when internal viscosity is negligible. The second part of the stream function describes the flow about a stationary bubble in a temperature gradient. The complete stream function describes the liquid motion about a moving bubble in a temperature gradient.

In order to calculate the velocity of a bubble resulting from the applied temperature gradient, consider a horizontal temperature gradient. There is no externally applied force on the bubble in the horizontal direction. The force exerted by the liquid must then also vanish. Using 5.9-(5),

$$F = 4\pi\mu UB = 0$$

so that, with (17), there results

$$U = - \frac{a}{\mu} \frac{k}{2k + k'} \frac{d\sigma}{dT} \frac{dT}{dz} \quad (19)$$

giving the velocity in terms of the applied temperature gradient. Because $d\sigma/dT$ is negative, a positive temperature gradient causes a positive bubble velocity. The bubble moves in the direction of increasing temperature. If the thermal conductivity of the gas is neglected, the velocity is

$$U = - \frac{a}{2\mu} \frac{d\sigma}{dT} \frac{dT}{dz}$$

The thermal conductivity of air at 20C is about 2.5×10^{-4} joule/cm sec K while that of water is about 5.9×10^{-3} joule/cm sec K at the same temperature. The error introduced by neglecting bubble conductivity is a few percent.

Notice that the bubble velocity is a linear function of the radius while the velocity of rise in a gravitational field, 5.11-(12), has quadratic dependence on the bubble radius. The conclusion is that the effects of temperature gradient on bubble motion become more significant for smaller bubbles.

The vertical temperature gradient that will hold a bubble stationary in a gravitational field is found by setting the sums of the velocities due to gravity and to temperature gradient equal to zero.

$$\frac{dT}{dz} = \frac{\rho_f ag}{3 d\sigma/dT} \frac{2k + k'}{k} \sim \frac{2\rho_f ag}{3 d\sigma/dT}$$

The temperature increases downward so that experiments performed to determine this temperature gradient must be done at sufficiently small Rayleigh number that free convection is inhibited. The temperature gradient required to hold the bubble stationary decreases as the bubble radius decreases. The liquid density and temperature coefficient of surface tension are the only fluid properties strongly affecting the result. For an air bubble in water at 20C, values of these properties indicate that a 44C/cm gradient will hold a 200 μ diameter bubble stationary.

5.13 ELECTROPHORESIS

A solid particle in a liquid electrolyte is observed to move through the fluid when an electric field is imposed on the particle. This motion results from the electric forces acting on those regions on or near the particle where a charge imbalance exists. This motion is termed electrophoresis and is distinct from a different phenomena known as dielectrophoresis. Motion due to dielectrophoresis [30] appears only in non-uniform applied fields and does not require that the particle be charged.

When a solid particle is immersed in an electrolytic liquid, the particle ordinarily selectively adsorbs ions of one sign onto its surface. Normally, anions (negative ions) are adsorbed preferentially because they are generally less hydrated. On occasion, ions originating in the particle are dissolved in the fluid leaving a charged surface. The mechanism whereby the surface is charged is not important to the calculations that concern us here. Only the presence of the adsorbed layer is significant.

Because of the electrical attraction of counter-ions and repulsion of co-ions, the surface attracts ions of opposite sign in the liquid. Opposing the tendency to segregate the ions is the diffusion or random thermal motion of the ions. As a result, the particle will be surrounded by a diffuse layer, characterized by a preponderance of counter-ions, where the ion distribution is determined by the combined effects of electrostatic forces and thermal motion. The fixed adsorbed layer and the mobile diffuse layer together constitute a region known as the double layer. A particle with a double layer is shown schematically in Figure 5.4.

Figure 5.4 Particle With Double Layer

The distance that the mobile portion of the double layer extends into the fluid is determined by the effects of electrical attraction and repulsion and the diffusion of the ions. The surface of the particle is a reflecting or electrically insulating surface. The physical description is very similar to the description of particle distribution in thermal equilibrium in a gravitational field. Since the double layer is also in thermal equilibrium, the ion distribution is also the Boltzmann distribution, 5.6-(9)

$$n = n_0 e^{-E/kT} \quad (1)$$

n is the number density of the ions in the fluid about the particle. E is the energy of the ion in the electric field produced by the charges. Choosing the fluid far from the surface as a datum for the energy, n_0 is the number density of ions in the neutral fluid. If for simplicity, we consider a binary electrolyte with each ion carrying the same number, z , of charges, then n_0 is the same for the positive and negative ions. The net charge density in neutral fluid is zero.

With the choice of the neutral fluid as the zero for energy, the energy of an ion at any position in the double layer is equal to the work that would be required to move the ion from neutral fluid to that position. For positive ions, cations, the energy is $ze\psi$ at a point where the voltage is an amount ψ greater than the voltage in the neutral fluid. For anions, the energy is $-ze\psi$. Consequently, within the mobile portion of the double layer, the number density distributions of the two ionic species are given by

$$n_+ = n_0 e^{-ze\psi/kT} \quad (2)$$

and

$$n_- = n_0 e^{ze\psi/kT} \quad (3)$$

The number density at any point depends only on the local voltage.

The voltage distribution in the double layer is governed by the Poisson equation.

$$\nabla^2 \psi = -\rho_e / \epsilon \quad (4)$$

ρ_e is the electrical charge density and ϵ is the fluid permittivity.

$$\rho_e = ze (n_+ - n_-) \quad (5)$$

The geometry must be specified before the voltage and number density distributions can be found. We consider a plane solid surface bounding the diffuse portion of the double layer. The voltage and number densities then depend only on y , the distance from the surface. Poisson's equation becomes

$$\frac{d^2 \psi}{dy^2} = -\rho_e / \epsilon$$

Using (2), (3) and (5) to express the charge density in terms of the voltage and then substituting into Poisson's equation, a differential equation with voltage as the sole dependent variable results.

$$\frac{d^2}{dy^2} \left(\frac{ze\psi}{kT} \right) = \frac{2n_0 z^2 e^2}{\epsilon kT} \sinh \left(\frac{ze\psi}{kT} \right) \quad (6)$$

This is the Poisson-Boltzmann equation.

Defining the Debye length

$$\delta \equiv \left(\frac{\epsilon kT}{2n_0 z^2 e^2} \right)^{1/2} \quad (7)$$

and a dimensionless length

$$Y = y / \delta$$

the Poisson - Boltzmann equation is

$$\frac{d^2}{dy^2} \left(\frac{ze\psi}{kT} \right) = \sinh \left(\frac{ze\psi}{kT} \right) \quad (8)$$

In order to solve this equation simply so as to estimate the distance over which the voltage is sensibly different from zero, we will assume that $ze\psi/kT$ ranges only over small values. We assume that the potential difference across the mobile portion of the double layer, the zeta potential, is small enough that the approximation

$$\sinh \left(\frac{ze\psi}{kT} \right) \sim \frac{ze\psi}{kT}$$

is valid throughout the diffuse layer. Actually, the zeta potential is seldom this small. If an accurate description of the voltage distribution in the mobile portion of the double layer were necessary, (8) would be solved rather than the approximate

$$\frac{d^2}{dy^2} \left(\frac{ze\psi}{kT} \right) = \frac{ze\psi}{kT} \quad (9)$$

The solution of (9) is

$$\psi = c_1 e^{-Y} + c_2 e^Y$$

c_2 is zero because the voltage ψ is zero far from the wall. c_1 is the zeta potential, ζ , because the voltage is ζ at $Y = 0$. Substituting for dimensionless length, the voltage distribution is written

$$\psi = \zeta e^{-Y/\delta} \quad (10)$$

Accordingly, the potential is $1/e$ of the zeta potential when the distance from the wall is the Debye length. Thus, the Debye length, δ , is a measure of the double layer thickness.

Using (7), the Debye length for a 10^{-3} M aqueous solution of 1-1 electrolyte at 25C is 10^{-2} μ while in a stronger 10^{-1} M solution, the Debye length is 10^{-3} μ . As a comparison, a human red blood cell, an erythrocyte, is about 2μ thick and 10 to 20 μ in diameter. A white blood cell, a leucocyte, is about 7 μ in diameter. In these examples, the Debye length is much smaller than the particle dimensions. Of course for smaller particles, or in insulating liquids where the Debye length is much larger, the converse is true.

We will restrict further discussion to particles having a Debye length much smaller than typical particle dimensions. This restriction will simplify analysis considerably because the mobile portion of the double layer, the region of charged liquid, is confined to a thin layer adjacent to the particle surface. If the local radii of curvature at any point on the particle surface are much larger than the Debye length, the mobile portion of the double layer may be treated as a plane layer.

Under these restrictions, let us consider the fluid and particle motion resulting from application of a uniform electric field to the medium. Any applied field of reasonable magnitude will have little effect on the ionic distribution in the double layer. The electric field resulting from the equilibrium charge distribution is much stronger within the mobile portion of the double layer than the applied field. From (10), the electric field naturally present in the mobile portion of the double layer is of the order ζ/δ . For a zeta potential of 100 mv and a Debye length of 10^{-2} μ , the electric field strength is of the order of 10^7 v/m, greater than any conceivable applied field.

An additional effect to be considered is the so-called time-of-relaxation effect. When a particle moves through a liquid, it tends to move relative to the surrounding ionic cloud as if to leave it behind. The applied electric field also causes ionic motion (conduction) so as to increase the asymmetry. The resulting asymmetric charge distribution produces an electric field opposing this charge imbalance. Diffusion also acts to remove the asymmetry. The electric field resulting from the charge acts as to oppose the applied

electric field, decreasing its apparent strength. Using an order of magnitude analysis, Overbeek [31] demonstrated that time-of-relaxation effects have a negligible effect on a particle's electrophoretic velocity provided that the double layer is small compared with particle dimensions. We need not account for this double layer deformation.

The particle is considered to be electrically insulating. There is no current normal to the surface. Even materials that are normally conducting frequently behave as insulating particles because of inhibited charge transfer at the surface.

Before treating the flow of the bulk neutral fluid surrounding the particle, we consider the effect of the applied electric field on the charged liquid within the mobile portion of the double layer. Since the local radii of curvature have been restricted to values much larger than the Debye length, the double layer thickness, the mobile portion of the double layer can be treated as a plane layer. An applied electric field produces an electric field tangential to this surface. The tangential field causes the ions to migrate along the surface dragging fluid with them. This fluid motion is shown in Figure 5.5. The fluid flow in the double layer is governed by the Navier-Stokes equation with an electric body force

$$\mu \frac{d^2 u}{dy^2} = -\rho_e \mathcal{E}_t$$

u is the velocity tangential to the surface and $\mathcal{E}_t$ is the local tangential electric field. The charge and voltage distributions are governed by the Poisson equation

$$\rho_e = -\epsilon \frac{d^2 \psi}{dy^2}$$

Substituting this expression for charge density into the Navier-Stokes equation, then integrating twice,

$$\mu u = \epsilon \psi \mathcal{E}_t + C_1 y + C_2$$

At the inner boundary of the mobile portion of the double layer where y is zero, the velocity is also zero because there is no slip and the voltage is the zeta potential. Accordingly,

$$C_2 = -\epsilon \zeta \mathcal{E}_t$$

Consistent with the thin double layer restriction, any velocity gradient produced at the outer boundary of the double layer by external fluid motion is negligible. The velocity gradient vanishes as y increases and

$$C_1 = 0$$

Combining, the velocity within the mobile portion of the double layer is

$$u = \frac{\epsilon (\psi - \zeta) \mathcal{E}_t}{\mu} \quad (11)$$

and, in particular, the relative tangential velocity at the outer boundary of the double layer is

$$u_t = -\frac{\epsilon \zeta \mathcal{E}_t}{\mu} \quad (12)$$

Figure 5.5 Velocity Distribution within the Double Layer

The particle now is seen to have a thin layer on its surface. The velocity at the outer boundary of this layer is proportional to the local electric field. The local electric field is not, in general, equal to the applied field but will vary along the surface. A system consisting of the particle and the double layer is electrically neutral, possessing no net charge. The uniform applied electric field produces no net electrical force on the system. The problem of a charged particle with an electrical force driving it through charged liquid can be replaced by the problem of a neutral particle with no applied force, but with a prescribed velocity distribution (12) at the surface. This simplification results from moving the effective particle boundary from the solid surface to the outer boundary of the double layer, a negligible distance according to the thin double layer restriction. This means of simplifying the formulation of thin double layer electrophoretic motion originated with Levich [20].

In order to find the velocity of a particle, the electric field distribution is first determined. The electric field distribution generates the apparent slip velocity (12) which is a boundary condition for the bulk neutral liquid motion. For sake of example, we choose a spherical particle of radius a . The voltage distribution in neutral fluid obeys Laplace's equation

$$\nabla^2 V = 0$$

Far from the particle, the electric field approaches the uniform applied field, $\mathcal{E}_0$,

$$V = -\mathcal{E}_0 z \quad \text{as} \quad r \rightarrow \infty$$

while at the insulating particle surface, neglect of surface conductivity arising from the increased ionic concentration in the double layer requires that the normal electric current density vanish. Consequently, the normal electric field vanishes.

$$\frac{\partial V}{\partial r} = 0 \quad \text{at} \quad r = a$$

The equation and boundary conditions on the voltage may be seen to be analogous to those obeyed by the external fluid temperature in the last section in the event that the thermal conductivity of the bubble is neglected. Using this analogy and referring to the temperature distribution, 5.12-(8), the voltage distribution results immediately.

$$V = - \mathcal{E}_0 \left[r + \frac{a^3}{2r} \right] \cos\theta$$

From this voltage distribution, only the tangential electric field at the particle surface is needed.

$$\mathcal{E}_\theta \Big|_{r=a} = - \frac{1}{a} \frac{\partial V}{\partial \theta} \Big|_{r=a} = - \frac{3}{2} \mathcal{E}_0 \sin\theta \quad (13)$$

This tangential electric field produces an apparent slip velocity at the surface according to (12)

$$v_\theta \Big|_{r=a} = \frac{3 \epsilon \zeta \mathcal{E}_0}{2\mu} \sin\theta \quad (14)$$

which in turn generates motion of the surrounding liquid.

Since the liquid outside of the double layer is neutral, it experiences no body force and obeys the creeping flow equation

$$E^4 \psi = 0 \quad (15)$$

The particle is impermeable so that at the surface there is no relative liquid motion normal to the surface. In the frame of the spherical particle, this yields

$$\psi = 0 \quad \text{at} \quad r = a$$

The prescribed velocity boundary condition (14), written in terms of the stream function, is

$$\frac{\partial \psi}{\partial r} = \frac{3 a \epsilon \zeta \mathcal{E}_0}{2\mu} \sin^2 \theta \quad \text{at} \quad r = a$$

The final condition is one of uniform streaming far from the particle.

$$\psi \rightarrow \frac{Ur^2 \sin^2 \theta}{2} \quad \text{as} \quad r \rightarrow \infty$$

The form of the boundary conditions again suggests that the flow may be described by the solution

$$\psi = \frac{U}{2} (Ar^2 + Br + \frac{C}{r} + Dr^4) \sin^2 \theta$$

of (15). The streaming flow boundary condition again yields

$$A = 1, \quad D = 0$$

while solution of the two equations in the coefficients resulting from the other two boundary conditions gives

$$B = \frac{3a}{2} \left(\frac{\epsilon \zeta \mathcal{E}_0}{\mu U} - 1 \right)$$

$$C = \frac{a^3}{2} \left(1 - \frac{3 \epsilon \zeta \mathcal{E}_0}{\mu U} \right)$$

The electrophoretic velocity results from setting the force exerted by the neutral liquid equal to zero. Using 5.9-(5), this is

$$F = 4 \pi \mu U B = 0$$

which means that B is zero so that

$$U = \frac{\epsilon \zeta \mathcal{E}_0}{\mu} \quad (16)$$

This equation is known as Smoluchowski's equation after its originator [32]. Smoluchowski asserted that the result was independent of the particle shape. Henry [33] made this assertion plausible by demonstrating that the equation described electrophoretic motion of a sphere and cylinder. Morrison [34] demonstrated that Smoluchowski's equation is valid for a particle of arbitrary shape. The conditions of a thin double layer, insulating particle, and negligible surface conductivity are also applied in analyzing the motion of the arbitrarily shaped particle.

The electrophoretic velocity (16) is normally of the order of one micron per second for each volt per centimeter of applied field.

5.14 CONTINUUM-SLIP FLOW: SMALL KNUDSEN NUMBER EFFECTS

The drag exerted on an aerosol particle as it moves relative to the surrounding gas is strongly dependent on the Knudsen number, the ratio, ℓ/a , of the mean free path of the gas molecules to a characteristic particle dimension. The mean free path and Knudsen number have been discussed in Section 2.2. In Section 5.2, Stokes law was derived with an assumed no-slip condition at the particle surface. As we shall see, this motion corresponds to aerosol motion in the zero Knudsen number limit. While an analytical solution of the fluid flow past the particle would ideally encompass the complete range of Knudsen numbers, the great difficulties encountered in the solution of the complete Boltzmann equation currently preclude this possibility. At best, available solutions are approximate and have limited ranges of validity.

Here, we wish to examine the effect of small non-zero Knudsen number, ranging up to about 0.25. The result will give the drag on smaller particles than can be described with Stokes law.

Because we seek to describe a transport process explicitly accounting for the molecular nature of the gas through its non-vanishing mean free path, it proves instructive to first examine how a transport property of the gas depends on the mean free path. The viscosity is the bulk transport property affecting particle drag and is chosen for examination on that basis. Let us estimate the gas viscosity using the simple arguments of elementary kinetic theory. We can then refine these results by reference to more exact analysis.

Consider a gas whose velocity, that is, the mean velocity of the molecules, is in the x direction and varies linearly in the y direction

$$\vec{V} = u(y) \hat{e}_x = (u_0 + \frac{du}{dy} y) \hat{e}_x$$

The gas is at a uniform absolute temperature T corresponding to a mean thermal speed, measured relative to the gas at rest, of $\bar{c}$. Equilibrium statistical thermodynamics gives

$$\bar{c} = \sqrt{\frac{8 k T}{\pi m}} \quad (1)$$

k is Boltzmann's constant and m is the molecular mass. The mean speed is of the same order of magnitude as the root mean square speed calculated using equipartition. For a more extensive exposure, see a textbook on statistical thermodynamics, statistical mechanics or kinetic theory. Some of these are listed as supplementary sources at the end of this chapter.

Consider an imaginary flat surface extending through the gas. This surface is defined by a constant value of y. Gas molecules are continuously crossing this surface from above and from below. The molecules crossing the surface from above have a mean tangential velocity component characteristic of the position of the last collision. This last collision occurred at a

distance above the surface of the order of the mean free path, ℓ . Let us call the average height of previous collisions above the surface by the expression $2 \phi \ell$. 2ϕ here is a dimensionless constant whose value is given by more exact analysis, but is of order one.

From equilibrium statistical mechanics, we also have the average number of molecules per unit time and area crossing the surface from above. This number flux is $n\bar{c}/4$. n is the number density of the molecules.

Combining these expressions, we can obtain the mean tangential momentum flux of the molecules as they cross the surface from above.

$$\frac{1}{4} n \bar{c} m u (y + 2 \phi \ell)$$

or

$$\frac{1}{4} n \bar{c} m [u(y) + 2 \phi \ell \frac{du}{dy}]$$

m is the molecular mass. Similarly, the momentum flux of molecules crossing the surface from below is

$$\frac{1}{4} n \bar{c} m [u(y) - 2 \phi \ell \frac{du}{dy}]$$

Consequently, the net rate of increase of momentum per unit time and per unit area beneath the surface is

$$\begin{aligned} & \frac{1}{4} n \bar{c} m [u(y) + 2 \phi \ell \frac{du}{dy}] - \frac{1}{4} n \bar{c} m [u(y) - 2 \phi \ell \frac{du}{dy}] \\ &= \phi n \bar{c} m \ell \frac{du}{dy} \end{aligned}$$

This net rate of change of momentum is equivalent to a force per unit area or stress acting tangential to the surface. This stress is σ_{yx} and, in macroscopic terms, is expressed in terms of the viscosity and strain rate.

Here

$$\sigma_{yx} = \mu \frac{du}{dy} = \phi n \bar{c} m \ell \frac{du}{dy} \quad (2)$$

Eliminating the velocity gradient from the last two expressions, we have the viscosity

$$\mu = \phi n \bar{c} m \ell \quad (3)$$

The constant ϕ has been determined by Enskog and by Chapman, see [35], to be 0.499 for a dilute hard sphere gas.

$$\mu = 0.499 n \bar{c} m \ell \quad (4)$$

These analyses are based on approximate solution of the Boltzmann equation describing the non-equilibrium, i.e., non-Maxwellian, distribution of velocities in a shearing gas. Viscosity, of course, appears only when a gas is perturbed from equilibrium. Measurements of viscosity are frequently used in conjunction with this expression to determine the mean free path in the gas.

An interesting consequence of this analysis is obtained by combining this expression with 2.2-(4) which gives the mean free path in terms of the number density and the molecular collision cross section

$$\mu = 0.499 \bar{c} m / \sqrt{2} \sigma \quad (5)$$

The result predicts the perhaps surprising fact that the viscosity of a dilute gas is independent of its pressure or density. The validity of this conclusion, first obtained by Maxwell, is well established experimentally.

Now, using the same sort of considerations, let us examine the molecular velocities in the vicinity of a solid surface bounding a shearing gas. The

molecules adjacent to this surface consist of those approaching the surface and about to collide with it and those receding from the surface following collision. We seek to estimate the mean tangential velocity component of these molecules, the slip velocity of the gas, u_0 . We assume that the velocity gradient in the gas remains constant up to the surface. The velocity gradient will in fact be affected by the presence of the wall.

The mean velocity of the gas molecules at the solid wall is the mean velocity of those approaching and receding. Considering first those approaching the wall, we note that they suffered their last collision on the average at a distance $2\phi\lambda$ from the wall. Their mean tangential velocity is $u_0 + 2\phi\lambda \frac{du}{dy}$ where y is the normal distance from the surface.

Now consider the molecules leaving the solid wall. These molecules did not originate from within the body at a depth $2\phi\lambda$. Rather, they are the reflected and scattered molecules that hit from above. In order to estimate how their velocities are distributed, let us consider how they interact with the surface. At one extreme, the surface could be considered smooth and reflecting so that any molecule striking the surface leaves with its tangential velocity component unaltered and its normal velocity component exactly reversed. In this case, the angle of incidence equals the angle of reflectance. The molecules are said to be specularly reflected.

At the other extreme, we could consider the angle of reflectance to be unrelated to the angle of incidence. The molecule could become adsorbed onto the surface, being emitted later in a random direction. By symmetry, the mean tangential velocity of such diffusely reflected molecules is zero.

In general, neither of these two extremes will apply but the particle surface interaction is more complicated. The usual procedure is to account for this more subtle behavior by assuming that some fraction f of the incoming molecules are diffusely reflected, the remainder $(1-f)$ being specularly reflected. f is called the scattering coefficient or the momentum accommodation coefficient.

With these simplifications, the mean tangential velocity of molecules leaving the surface is $(1-f) (u_0 + 2\phi\lambda \frac{du}{dy})$ resulting solely from the tangential velocity distribution of specularly reflected molecules.

Combining these results, the mean gas velocity at the surface is

$$u_0 = \frac{u_0 + 2\phi \ell \frac{du}{dy} + (1-f) (u_0 + 2\phi \ell \frac{du}{dy})}{2}$$

or

$$u_0 = 2\phi \frac{2-f}{f} \ell \frac{du}{dy} \quad (6)$$

While the estimate of the magnitude of this slip is subject to improvement, the existence of a slip is inescapable. A no-slip condition would require that the reflection at the surface on the average send the gas molecules back in the direction from which they came. No real surface would exhibit such remarkable behavior.

The momentum accommodation coefficient has been measured for some few gas-surface combinations and the results have been collected by Paul [36]. These coefficients are shown in Table 5.1. Theoretical values

Gas-Surface	f
Air or CO ₂ on mercury or old shellac	1.00
Air on oil	0.895*
CO ₂ on oil	0.92
H ₂ on oil	0.925
Air on glass	0.89
He on oil	0.874
Air on fresh shellac	0.79
Air on Ag ₂ O	0.98
He on Ag ₂ O	1.0
H ₂ on Ag ₂ O	1.0
O ₂ on Ag ₂ O	0.99

Table 5.1 Experimental values of momentum accommodation coefficient (From Paul [36])

*This value, from [38], is misprinted in [36].

of the momentum accommodation coefficient are not available. The momentum accommodation coefficient and the behavior of reflected molecules are important in determining the drag at any non-zero Knudsen number.

A different coefficient for the slip is frequently defined

$$C_m = 2\phi \frac{2-f}{f} \quad (7)$$

so that the slip velocity is written

$$u_o = C_m \ell \frac{du}{dy} \quad (8)$$

Values of C_m are of order unity. In terms of the tangential stress, σ_o , exerted by the fluid on the surface, the slip velocity is

$$u_o = C_m \ell \sigma_o / \mu \quad (9)$$

We conclude the general discussion by noting that an extrapolation of the velocity gradient beneath the solid surface gives a zero velocity, or an apparent no-slip surface, at a distance $C_m \ell$ beneath the actual surface. Consequently, we may anticipate that the effect of surface slip becomes important only when the object dimensions characterizing the flow are of the order of the mean free path. Those effects would appear for example, in the flow through a sufficiently small tube or in the motion of a sufficiently small aerosol particle

Now, let us examine how these slip flow considerations have been applied to the motion of aerosol particles. The effects of surface slip are non-continuum effects arising from the non-zero mean free path. The Navier-Stokes equations are equations for continuum motion, invalid for flow with large mean free paths where molecules rebounding from the surface travel large distances before collision. The use of slip flow boundary conditions with the Navier-Stokes equations is consequently only justifiable where the mean free path is small compared to a characteristic particle dimension. The description of small Knudsen number flows using the Navier-Stokes equations with slip flow

boundary conditions is known as continuum-slip analysis. While this technique is plausible, it certainly is not rigorous and so depends on agreement with experiment for its justification.

For isothermal motion then, continuum - slip analysis consists of replacing the no slip condition at the particle surface by a linear relation between the local tangential velocity and the local tangential stress

$$\beta v_t = \sigma_{nt} \quad (10)$$

β is the proportionality constant written in the form employed by Basset [37]. The subscripts t and n refer to tangential and normal. The coefficient β is related to the previous expression by comparison with (9)

$$\beta = \frac{\mu}{C_m \ell} \quad (11)$$

Equivalent to (10) is the alternative form

$$v_t = C_m \ell \sigma_{nt} / \mu$$

which is (9) with the more explicit notation.

Now, let us apply this approach to the creeping motion of a small spherical aerosol particle. The equation of motion is the Navier-Stokes equation so that

$$E^4 \psi = 0 \quad (12)$$

again describes this axisymmetric flow. Choosing a spherical coordinate system fixed to the particle, the no normal flow boundary is

$$\psi = 0 \quad \text{at} \quad r = a$$

Instead of a no-slip boundary condition, we have (10) which is here

$$\beta v_{\theta} = \sigma_{r\theta} \quad \text{at} \quad r = a \quad (13)$$

Using the constitutive equation for the stress in terms of the strain rate, noting that the radial surface velocity is zero in this frame, and then writing velocities in terms of the Stokes stream function, we have

$$\beta \frac{\partial \psi}{\partial r} = \mu a^2 \frac{\partial}{\partial r} \left(\frac{1}{r^2} \frac{\partial \psi}{\partial r} \right) \quad \text{at} \quad r = a$$

for the slip flow boundary condition.

The final boundary condition is one of uniform streaming far from the body

$$\psi \rightarrow \frac{U}{2} r^2 \sin^2 \theta \quad \text{as} \quad r \rightarrow \infty$$

The solution

$$\psi = \frac{U}{2} \left(Ar^2 + Br + \frac{C}{r} + Dr^4 \right) \sin^2 \theta$$

of (12) is again suggested. The uniform streaming boundary condition requires that A be one and D be zero. Simultaneous solution of the two algebraic equations resulting from the other two boundary conditions yields

$$B = - \frac{3a}{2} \frac{\beta a + 2\mu}{\beta a + 3\mu} \quad (14)$$

and

$$C = \frac{a^3}{2} \frac{\beta a}{\beta a + 3\mu} \quad (15)$$

The description of the flow is complete. Before proceeding, it is comforting to check these results by comparison to previous analyses. The limit as β approaches infinity corresponds to no slip at the surface and the coefficients reduce to their Stokes law values

$$B \rightarrow -\frac{3a}{2}, \quad C \rightarrow \frac{a^3}{2}$$

The limit as β approaches zero corresponds to no tangential stress at the surface. This is the boundary condition for liquid flow over an inviscid bubble and the coefficients agree in this limit as well.

$$B \rightarrow -a, \quad C \rightarrow 0$$

The force on this isolated particle in steady creeping flow is given by 5.9-(5)

$$F = 4\pi\mu UB$$

so that the force exerted by the gas on the particle is

$$F = -6\pi\mu aU \frac{\beta a + 2\mu}{\beta a + 3\mu} \quad (16)$$

in the form of Stokes law with a slip flow correction. This is Basset's expression for the drag on a spherical particle.

Using (11) to express the drag correction in terms of the Knudsen number, the force exerted by the gas becomes

$$F = -6\pi\mu aU \frac{1 + 2 C_m \ell/a}{1 + 3 C_m \ell/a} \quad (17)$$

The correction becomes steadily less important as the Knudsen number approaches zero, yielding Stokes law in the limit. For sufficiently small Knudsen number,

the denominator of the drag correction can be expanded using

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \dots$$

so that to first order in Knudsen number, (17) reduces to

$$F \approx - 6\pi\mu a U (1 - C_m \ell/a)$$

or

$$F \approx - 6\pi\mu (a - C_m \ell) U \quad (18)$$

This is exactly the result that would have been obtained had slip been accounted for by displacing the effective boundary a distance $C_m \ell$ beneath the actual solid surface. This distance beneath the surface is the location of the apparent no-slip surface found by extrapolating the gas velocity gradient.

We may also compare the predicted particle velocity resulting from an applied force to the Stokes law velocity, U_s , resulting from the same force.

$$F = - 6\pi\mu a U \frac{1 + 2 C_m \ell/a}{1 + 3 C_m \ell/a} = - 6\pi\mu a U_s$$

Thus

$$\frac{U}{U_s} = \frac{1 + 3 C_m \ell/a}{1 + 2 C_m \ell/a} \quad (19)$$

or to first order in the Knudsen number,

$$\frac{U}{U_{\text{Stokes}}} \approx 1 + C_m \ell/a \quad (20)$$

These expressions have been compared with experimental data [38, 39, 40].

For small ℓ/a , less than about 0.1, they predict nearly the same behavior and are well substantiated by experiment. Surprisingly, for larger ℓ/a , (20) continues to be quite accurate at least to a Knudsen number of 0.4. Equation (19) is seriously in error at these larger Knudsen numbers, underestimating the slip correction. This behavior is not anticipated from the analysis. The experimentally discovered wide range of validity for (20) is a pleasant but unexpected result.

When the force on a aerosol particle is written in the approximate form

$$F \approx - \frac{6\pi\mu aU}{1 + C_m \ell/a} \quad (21)$$

in agreement with (20), the correction is called the Cunningham correction after its proposer [41].

At yet larger Knudsen numbers, an empirical expression proposed by Millikan [42] and by Knudsen and Weber [43] is frequently used. The expression is

$$F = - \frac{6\pi\mu aU}{\{1 + \ell/a[\bar{A} + \bar{B} \exp(-\bar{C} a/\ell)]\}}$$

$\bar{A}$, $\bar{B}$, and $\bar{C}$ are empirical constants. For small Knudsen number, this expression reduces to (21) with $\bar{A}$ equal to C_m . In experiments on oil drops in air, Millikan [44] found agreement with this relation within 2 percent for Knudsen numbers up to about 100 in terms of a mean free path calculated from (4) and using viscosity measurements. Millikan actually used a viscosity expression employing ϕ equal to 0.3502 rather than the 0.499 used here. His mean free paths and coefficients have been recalculated in terms of the more recent value of ϕ . Calculating the mean free path from (4), Millikan's values for oil in air are changed to be

$$\bar{A} = 1.23 \quad , \quad \bar{B} = 0.413 \quad , \quad \bar{C} = 0.877$$

$\bar{A}$ is in excellent agreement with C_m calculated using (7) with ϕ equal to 0.895.

5.15 SMALL KNUDSEN NUMBER THERMOPHORESIS

A temperature gradient in a gas will cause suspended particles to move through the gas. An aerosol particle in an applied temperature gradient is observed to move in the direction of decreasing temperature. This motion, known as thermophoresis, finds widespread application as a sampling technique. In a normal thermophoretic precipitator, particle laden gas passes down a narrow channel past a filament electrically heated to a high temperature. In the gas between the hot filament and the cooler channel walls is a high temperature gradient. Particles carried into this region are driven toward the channel walls where they may be deposited and subsequently examined.

Here, we wish to examine the process whereby the particles move. An understanding of this phenomena and the resulting expression for thermophoretic velocity are necessary to determine how the deposited aerosol distribution differs from the distribution of suspended aerosol. We also obtain information useful in evaluating thermophoretic effects in other contexts.

Like the frictional slip examined in the last section, thermophoresis arises from gas-surface interaction. In a continuum-slip analysis, these interactions appear as boundary conditions. There are two additional boundary effects to be considered. They, as well as the frictional slip, were obtained by Maxwell. These two effects are termed thermal creep and temperature jump and appear only in nonisothermal flow.

Thermal creep is normally accepted as the mechanism generating low Knudsen number gas and particle motions. Epstein [45] determined the thermal force in an analysis using thermal creep but not accounting for the other gas surface interactions. Brock [46] used the complete set of Maxwell boundary conditions to generate an expression for the low Knudsen number thermal force on a sphere. Here, we will obtain Brock's result; Epstein's result then follows by neglecting temperature jump and frictional slip coefficients.

Thermal creep refers to the center of mass velocity in a gas adjacent to a solid surface and resulting from a temperature gradient in the gas along the surface. The phenomenon is described by Kennard [47] and the resultant velocity calculated. Here, without having developed the requisite background in kinetic theory in the text, we will be content to discuss the mechanism and

cite the result.

Consider a plane solid surface bounding a gas. A temperature gradient in the gas has a component parallel to the surface. The molecules approaching and striking any region of the surface come from all directions above the surface. Those from the hotter gas to one side have a greater mean speed than the molecules coming from the cooler gas on the other side. As the molecules strike the solid surface they transfer momentum to it. The tangential momentum of the molecules is unimpaired only if the reflection is entirely specular.

If the gas were stationary, the net effect would be to produce a tangential stress on the solid surface, the force pushing the wall from the region of the higher temperature. The molecules, however, bounce back from the solid surface with the average tangential velocity altered. The force exerted by the gas on the surface is equal and opposite to the force exerted by the solid on the gas. This force sets the gas in motion in the direction of higher temperature. Steady state is attained when the net force or net momentum transfer vanishes. The analysis described in Kennard shows that this steady state results when the gas moves, relative to the solid, in the direction of increasing temperature with a velocity

$$u = \frac{3}{4} \frac{\mu}{\rho_f T} \frac{dT}{dx} \quad (1)$$

u is the local tangential gas velocity; ρ_f , the gas density and $\frac{dT}{dx}$, the local tangential component of temperature gradient.

A temperature jump at the solid-gas surface results from a normal component of temperature gradient at the surface. The magnitude of this effect is estimated using an argument analogous to that which yielded the slip velocity in the previous section. The plane surface is located at y equal to zero and the temperature varies linearly with the distance from the wall according to

$$T = T_g + \frac{dT}{dy} y$$

T_g , the gas temperature at the surface, is different from T_w , the solid temperature at the surface.

The gas temperature near the wall, just as elsewhere, is related to the distribution of molecular velocities. The molecules near the wall can be separated into those approaching the wall and those receding from the surface. The characteristic length associated with conduction heat transfer is not the $2\phi\lambda$ encountered in the viscosity calculation but a larger distance, $2\phi'\lambda$. ϕ' is also of order unity but is larger than ϕ because of the correlation between a molecule's translational energy and its mean free path. Faster molecules with higher translational energies have longer mean free paths. Molecules approaching the surface have then originated at an effective distance $2\phi'\lambda$ from the surface for heat transfer or temperature calculations.

The ratio, ϕ'/ϕ , is considered at length by Chapman and Cowling [35]. Eucken's formula, which results from assigning different mean free paths to transport of translational energy and molecular internal energy, gives

$$\frac{\phi'}{\phi} = (9\gamma - 5)/4 \quad (2)$$

γ is the ratio of specific heats, C_p/C_v . For a monatomic gas, γ is 5/3 yielding ϕ'/ϕ equal to 5/2. Eucken's formula, though not rigorous, seems to agree quite well with experimental values.

The molecules receding from the wall may, as in the slip argument, be considered to be divided into two groups. In lieu of a detailed examination of gas-surface interaction, the molecules are considered to leave with a fraction, a_T , having come to thermal equilibrium with the surface and the remaining fraction, $1-a_T$, thermally unaffected by the surface. a_T , the thermal accommodation coefficient, is similar in nature to f , the momentum accommodation coefficient. Experimentally, a_T is found to be smaller than f presumably because more extensive interaction is required to produce thermal equilibrium than is necessary to produce diffuse reflection. Paul [36] gives some values of a_T .

In terms of these quantities, an argument analogous to that yielding 5.14-(6) gives the temperature jump.

$$T_g - T_w = 2\phi \cdot \frac{2-a_T}{a_T} \propto \frac{dT}{dy} \quad (3)$$

The coefficients are often combined into a single coefficient, C_t , analogous to 5.14-(17) and the temperature jump written

$$T_g - T_w = C_t \propto \frac{dT}{dy} \quad (4)$$

We turn now to the thermophoresis of a spherical aerosol particle. The gas and particle motion result from thermal creep along the surface. The creep velocity is found from the temperature distribution.

As was found in Section 5.12, the temperature distribution is governed by conduction heat transfer and convection effects are negligible provided that the Peclet number describing the flow is sufficiently small. Since gases have Prandtl numbers of order unity, the low Reynolds number condition implies low Peclet number. The temperature distribution is then uncoupled from the velocity distribution and may be found before considering the fluid flow.

As in Section 5.12, the temperature outside of the particle, T , and the temperature inside of the particle, T' , each obey Laplace's equation.

$$\nabla^2 T = 0 \quad ; \quad \nabla^2 T' = 0$$

Far from the particle, the temperature distribution is unaffected by the presence of the particle

$$T \rightarrow T_0 + \frac{dT}{dz} r \cos\theta \quad \text{as} \quad r \rightarrow \infty$$

dT/dz is the applied temperature gradient. The axis of the coordinate system is aligned with it. At the gas-solid interface, the heat flux is continuous.

$$k \frac{\partial T}{\partial r} = k' \frac{\partial T'}{\partial r} \quad \text{at} \quad r = a$$

but the temperature is not. Instead of the equal temperature boundary condition used in Section 5.12, and used by Epstein, we have a temperature jump.

$$T - T' = C_t \ell \frac{\partial T}{\partial r} \quad \text{at} \quad r = a$$

Using these conditions as well as a boundedness condition, the temperature distribution is found.

$$T = T_0 + \left(r - \frac{a^3}{r^2} \frac{1 - C_t \ell/a - k/k'}{1 + 2C_t \ell/a + 2k/k'} \right) \frac{dT}{dz} \cos \theta$$

and

$$T' = T_0 + \frac{3k/k'}{1 + 2C_t \ell/a + 2k/k'} \frac{dT}{dz} r \cos \theta$$

T_0 is the midplane temperature and changes with time as the particle moves. Only the gas temperature at the surface is of further interest. This temperature is

$$T = T_0 + 3 \frac{C_t \ell/a + k/k'}{1 + 2C_t \ell/a + 2k/k'} a \frac{dT}{dz} \cos \theta \quad (5)$$

Variation of this temperature along the surface generates the gas motion.

The low Reynolds number axisymmetric gas flow obeys

$$E^4 \psi = 0$$

using the continuum approximation. The normal gas velocity vanishes at the surface when the particle frame is used.

$$\psi = 0 \quad \text{at} \quad r = a$$

Here, the gas slips at the surface because of two effects. In addition to the frictional slip of the last section, we have thermal creep. Using (1) and 5.14-(9)

$$v_{\theta} = \frac{C_m \ell \sigma_{r\theta}}{\mu} + \frac{3}{4} \frac{\mu}{\rho_f T_0 a} \frac{\partial T}{\partial \theta} \quad \text{at} \quad r = a \quad (6)$$

which is readily written in terms of the stream function once the stress is expressed in terms of the strain rate. The streaming flow condition far from the sphere is

$$\psi \rightarrow \frac{U}{2} r^2 \sin^2 \theta \quad \text{as} \quad r \rightarrow \infty$$

The solution is of the form

$$\psi = \frac{U}{2} (Ar^2 + Br + \frac{C}{r} + Dr^4) \sin^2 \theta$$

and the coefficients are found from the boundary conditions to be

$$\begin{aligned} A &= 1 \\ B &= -\frac{3a}{2} \frac{1 + 2C_m \ell/a}{1 + 3C_m \ell/a} \\ &\quad - \frac{9 \mu a}{4 \rho_f T_0 U (1 + 3C_m \ell/a)} \frac{C_t \ell/a + k/k'}{1 + 2C_t \ell/a + 2k/k'} \frac{dT}{dz} \end{aligned} \quad (7)$$

$$C = \frac{a^3}{2(1 + 3C_m \ell/a)}$$

$$+ \frac{9 \pi \mu a^3}{4 \rho_f T_0 (1 + 3C_m \ell/a)} \frac{C_t \ell/a + k/k'}{1 + 2C_t \ell/a + 2k/k'} \frac{dT}{dz}$$

$$D = 0$$

The force on the steadily moving isolated particle is given by 5.9-(5)

$$F = 4 \pi \mu U B$$

The force on the particle is

$$F = -6 \pi \mu a U \frac{1 + 2C_m \ell/a}{1 + 3C_m \ell/a} - \frac{9 \pi \mu^2 a}{\rho_f T_0 (1 + 3C_m \ell/a)} \frac{C_t \ell/a + k/k'}{1 + 2C_t \ell/a + 2k/k'} \frac{dT}{dz} \quad (8)$$

The first expression on the right hand side is the drag, 5.14-(17), on a particle moving with small Knudsen number through the gas. The second expression is the thermal force of Brock.

$$F_{th} = - \frac{9 \pi \mu^2 a}{\rho_f T_0 (1 + 3C_m \ell/a)} \frac{C_t \ell/a + k/k'}{1 + 2C_t \ell/a + 2k/k'} \frac{dT}{dz} \quad (9)$$

If frictional slip and temperature jump are ignored, this reduces to the thermal force of Epstein.

$$F_{th} = - \frac{9 \pi \mu^2 a}{\rho_f T_0} \frac{k/k'}{1 + 2k/k'} \frac{dT}{dz} \quad (10)$$

Epstein's expression predicts significantly different forces. Because of the relatively low gas conductivity, k/k' is normally small and $C_t \ell/a$ is significant. Brock's expression was found, [46], to give good agreement with experimental thermal forces on liquid aerosols with normal conductivities. Agreement with high conductivity aerosols [40] is not as good.

Setting the net force on the particle, given by (8), equal to zero, the low Knudsen number thermophoretic velocity results.

$$U = - \frac{3 \mu}{2 \rho_f T_0 (1 + 2 C_m \ell/a)} \frac{C_t /a + k/k'}{1 + 2 C_t \ell/a + 2 k/k'} \quad (11)$$

5.16 HYDRODYNAMIC INTERACTIONS BETWEEN PARTICLES

The motions of neighboring particles are coupled through hydrodynamic interaction. The fluid motion generated by one particle's movement is distorted by the presence of another particle. Motion of the second particle produces fluid flow further affecting the behavior of its neighbor. Effects of particle interaction become important in dealing with clouds of particles that are not excessively dilute. The importance of interactions in the particle dynamics depends on the number density of the particles and on their characteristic dimensions. Very widely separated particles, say ten or more diameters apart, have little effect on one another and their interaction is normally ignored. In less dilute suspensions, these interactions significantly affect behavior. Hydrodynamic interactions are but one mode of interaction among particles. Soo [48] considers interactions over a broad range of volume fraction occupied by particles and treats the transport properties of clouds in detail.

Here, we will examine the interactions of pairs of particles. In dilute suspension, nearest neighbor interactions are the most important and even three particles are seldom close together so that interactions among triplets is of much less importance. We will start with an approximate analysis valid for widely separated particles. This type of analysis is due to Smoluchowski [49]. The approach used here is more closely aligned with the presentation in [50].

First, we consider the fluid flow generated by the motion of a single sphere. A single sphere moving with velocity U through an unbounded fluid at rest causes fluid flow described by the stream function, 5.2-(11)

$$\psi = \frac{Ua^2}{4} \left[-3 \frac{r}{a} + \frac{a}{r} \right] \sin^2 \theta$$

in the Stokes flow regime. The corresponding fluid velocity components are

$$v_r = \frac{U}{2} \left[3 \frac{a}{r} - \left(\frac{a}{r} \right)^3 \right] \cos \theta \quad (1)$$

$$v_{\theta} = -\frac{U}{4} \left[3 \frac{a}{r} + \left(\frac{a}{r} \right)^3 \right] \sin \theta \quad (2)$$

Far away from the particle, the part of the velocity that vanishes as r^{-1} will dominate so that at these distances

$$v_r \rightarrow \frac{3}{2} U \frac{a}{r} \cos \theta \quad (3)$$

$$v_{\theta} \rightarrow -\frac{3}{4} U \frac{a}{r} \sin \theta \quad (4)$$

This velocity distribution is the Stokeslet contribution to the flow and accounts for the particle drag. The same asymptotic velocity distribution is observed far from a non-spherical particle whose drag is the same as that of a sphere of radius a . The radius a can be considered the equivalent radius of such a particle.

One should remember that far from a slowly moving particle, regardless of its Reynolds number, inertial effects will dominate viscous effects and consequently velocities calculated using the creeping flow equations are inapplicable. This behavior is discussed in Section 5.5. If the Reynolds number Ur / ν based on the distance r from the particle is small compared to one, then viscous effects will dominate and (3) and (4) apply. At yet larger separations, Ur / ν is greater than one and inertial effects dominate. (3) and (4) do not then accurately describe the asymptotic behavior and should be used only when the predicted velocities at these very large distances are negligible.

Figure 5.6 schematically shows the distant disturbance velocity produced by a particle's motion. The velocities are calculated from (3) and (4). The disturbance is axisymmetric and also antisymmetric about the mid plane. The disturbance velocity at a large distance in front of a particle is twice as large as the fluid disturbance velocity at the same distance to the side.

Now, let us consider two such particles with equivalent radii, a and b . We will examine their interaction as they move along their line of centers. See Figure 5.7. We then will determine how the analysis changes if we examine motion normal to the line of centers. Finally, superposition yields expressions for the hydrodynamic interaction of widely separated arbitrarily moving particles.

Figure 5.6 Distant Flow Field of Moving Particle

Figure 5.7 Two Particles moving along Line of Centers

Particle a , if it were alone in the fluid, would have a force , $-6\pi\mu a U^a$, exerted on it by the fluid. It would also produce a fluid velocity at the location occupied by the center of b that would be in the z direction and would have approximate magnitude.

$$w_1^b = \frac{3}{2} U^a \frac{a}{L} \quad (5)$$

Similarly, particle b alone in the fluid, would have a force, $-6\pi\mu b U^b$, exerted on it by the fluid and would cause an axial velocity at particle a's location nearly given by

$$w_1^a = \frac{3}{2} U^b \frac{b}{L} \quad (6)$$

The particles, however, are not alone but are each present in the fluid and affected by these flows. To first order in b/L , the effect of particle b on particle a is to produce a velocity, w_1^a , that acts on particle a like a streaming velocity in the z direction. To this order the force on a is

$$F^a = 6\pi\mu a (-U^a + w_1^b) \quad (7)$$

The expression is not exact for several reasons. The velocity produced by a single particle b has more than simply b/r dependence, the velocity produced by b is not uniform, and this superposition fails to satisfy boundary conditions at the particle surfaces.

The expression can be improved by noting that streaming velocities, (5) and (6), moving past particles, b and a , will in turn cause additional disturbance velocities in the fluid. The fluid flowing past particle a with a velocity w_1^a causes fluid motion equal in magnitude and direction to that which would be produced if particle a were moving with velocity $-w_1^a$ through stationary fluid. At the location of particle b , this higher order distur-

bance is

$$w_2^b = \frac{3}{2} (-w_1^a) \frac{a}{L} = - \left(\frac{3}{2}\right)^2 U^b \frac{a}{L} \frac{b}{L}$$

At the location of particle a , the similar velocity is

$$w_2^a = \frac{3}{2} (-w_1^b) \frac{b}{L} = - \left(\frac{3}{2}\right)^2 U^a \frac{a}{L} \frac{b}{L}$$

This procedure may be continued ad infinitum. The accuracy of the resulting expression is not improved thereby because the very next terms in the series are of the order of the terms discarded from (1) and (2) to produce (3) and (4). We continue the series, nonetheless, because no harm is done by doing so and because the infinite series converges to a particularly simple expression. Accordingly,

$$w_3^b = \frac{3}{2} (-w_2^a) \frac{a}{L} = \left(\frac{3}{2}\right)^3 U^a \left(\frac{a}{L}\right)^2 \frac{b}{L}$$

$$w_3^a = \frac{3}{2} (-w_2^b) \frac{b}{L} = \left(\frac{3}{2}\right)^3 U^b \left(\frac{a}{L}\right) \left(\frac{b}{L}\right)^2$$

and so on.

The net force exerted by the fluid on particle a is along the line of centers, in the z direction. Our expression is

$$F_z^a / 6\pi\mu a = -U_z^a + w_1^a + w_2^a + w_3^a + \dots$$

The sign preceeding U_z^a is negative because this velocity is a particle velocity while the w's are fluid velocities. This series is of the form

$$F_z^a / 6\pi\mu a = \left[-U_z^a + \frac{3}{2} U_z^b \frac{b}{L} \right] * \\ \left[1 + \frac{3}{2} \frac{a}{L} \frac{b}{L} + \frac{3}{2} \left(\frac{a}{L} \frac{b}{L} \right)^2 + \dots \right]$$

Since the infinite series converges

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots \quad (x^2 < 1)$$

the force becomes simply

$$\frac{F_z^a}{6\pi\mu a} = \frac{-U_z^a + \frac{3}{2} U_z^b \frac{b}{L}}{1 - \left(\frac{3}{2}\right)^2 \frac{a}{L} \frac{b}{L}} \quad (8)$$

The analogous expression for particle b is

$$\frac{F_z^b}{6\pi\mu b} = \frac{-U_z^b + \frac{3}{2} U_z^a \frac{a}{L}}{1 - \left(\frac{3}{2}\right)^2 \frac{a}{L} \frac{b}{L}} \quad (9)$$

The analysis for the interaction of widely spaced particles moving normal to their line of centers is identical in form. The only difference in the result is that the factor 3/2 is everywhere replaced by 3/4. See Figure 5.6. For motion in the x direction, normal to the line of centers, we have

$$\frac{F_x^a}{6\pi\mu a} = \frac{-U_x^a + \frac{3}{4} U_x^b \frac{b}{L}}{1 - \left(\frac{3}{4}\right)^2 \frac{a}{L} \frac{b}{L}} \quad (10)$$

and

$$\frac{F_x^b}{6\pi\mu b} = \frac{-U_x^b + \frac{3}{4}U_x^a \frac{a}{L}}{1 - \left(\frac{3}{4}\right)^2 \frac{a}{L} \frac{b}{L}} \quad (11)$$

The general expression is simply a vector sum of these expressions

$$\vec{F}^a = F_x^a \hat{i} + F_y^a \hat{j} + F_z^a \hat{k} \quad (12)$$

since there is no coupling between motion in the different directions.

Several special cases remain to be examined. Of particular interest are interactions of equal sized particles

$$a = b \quad (13)$$

In a field where equal forces are applied to both particles, they move with equal velocities. The applied force and resultant velocity components are related by

$$\frac{\vec{F}}{6\pi\mu a} = -\frac{U_x}{1 + \frac{3}{4}\frac{a}{L}} \hat{i} - \frac{U_z}{1 + \frac{3}{2}\frac{a}{L}} \hat{k} \quad (14)$$

This simple form results from (8) and (10) for equal sized spheres with equal velocities.

In a gravitational field, where the force is independent of the proximity of the particles, we find that a pair of vertically aligned equal particles fall with a velocity

$$U = \left(1 + \frac{3}{2}\frac{a}{L}\right) U_{\text{single}}$$

where U_{single} is the settling velocity of a lone particle. For horizontally aligned particles under the same condition the velocity is

$$U = \left(1 + \frac{3}{4} \frac{a}{L} \right) U_{\text{single}}$$

again greater than the velocity of a single particle but less than the velocity of a vertically aligned pair.

To demonstrate that these results are approximate and to determine the range of applicability, it is desirable to compare these predictions with those of more exact analysis. Stimson and Jeffery [51], using a bispherical coordinate system, found an exact solution to the creeping flow past two spheres moving with equal velocities along their line of centers. For equal sized spheres, the force exerted by the fluid on either sphere is

$$F = -6\pi\mu a\lambda \quad (15)$$

λ is the factor modifying Stokes law to account for the presence of the second sphere. In terms of a parameter, α , arising from use of the bispherical coordinate system.

$$\cosh \alpha = \frac{L}{2a}$$

the coefficient λ is

$$\lambda = \frac{4}{3} \sinh \alpha \sum_{n=1}^{\infty} \frac{n(n+1)}{(2n-1)(2n+3)} *$$

$$\left[1 - \frac{4 \sinh^2 \left(\frac{n+1}{2} \right) \alpha - (2n+1)^2 \sinh^2 \alpha}{2 \sinh(2n+1) \alpha + (2n+1) \sinh 2\alpha} \right]$$

The $4/3$ in the expression for λ is misprinted as $2/3$ in [51]. In Table 5.2, the exact creeping flow correction, λ , is compared with the approximate correction, $1/(1 + \frac{3}{2} \frac{a}{L})$, from equation (14).

α	$L/2a$	λ	$1/(1 + \frac{3}{2} \frac{a}{L})$
0	1	0.645*	0.571
0.5	1.128	0.660**	0.600
1.0	1.543	0.702	0.673
1.5	2.352	0.768	0.758
2.0	3.762	0.836	0.834
2.5	6.132	0.892	0.891
3.0	10.068	0.931	0.931
∞	∞	1	1

Table 5.2 Comparison of Stimson and Jeffery Correction with Approximate Expression

Differences between the two calculations are significant at close separations. At large separations, the agreement becomes excellent. At no separation is the error associated with the approximate correction as great as the error involved in neglecting interaction effects altogether.

For equal sized particles moving side by side with equal velocities, the exact solution to the creeping flow equations has been found by O'Neill [53] and by Goldman, Cox and Brenner [54] and by Wakiya [55]. The approximate correction from equation (14) is, in this case, $1/(1 + \frac{3}{4} \frac{a}{L})$.

*Limiting value for touching spheres given by Faxen [52]

**This value is more accurate than given by [51]

Another special case of interest is the case of equal sized spheres moving with velocities equal in magnitude but opposite in direction. The forces exerted by the fluid on these particles are also equal in magnitude but opposite in direction. Expressions (8) and (10) simplify in this case to yield

$$\frac{\vec{F}_a}{6\pi\mu a} = \frac{-U_x^a}{1 - \frac{3}{4}\frac{a}{L}} \hat{i} - \frac{U_z^a}{1 - \frac{3}{2}\frac{a}{L}} \hat{k}$$

Exact solutions of the steady creeping flow equations have been found for the motion of two equal spheres moving with equal and opposite velocities along their line of centers. These analyses, by Brenner [56] and Maude [57], used together with [51], permit calculation of the forces acting on equal spheres moving with arbitrary small velocities along their line of centers.

O'Neill [58] found exact solutions of the creeping flow equations for the motion of two equal spheres normal to their line of centers with equal and opposite velocities. Using superposition, these results and the previously cited work can be combined to yield the exact creeping flow interaction between equal spheres moving with arbitrary orientations and small velocities.

The low Reynolds number interactions of pairs of spherical particles have been the subject of several experimental investigations. Experiments designed to measure interactions in the creeping flow regime must be performed at quite low Reynolds numbers so that the Reynolds number based on separation, Ur/ν , is itself small. Stimson and Jeffery's results for the interaction of two spheres moving along their line of centers are in agreement with the observations of Happel and Pfeffer [59] and of Eveson, Hall and Ward [60]. The results of Eveson, Hall and Ward also agree well with the predicted [53, 54, 55] behavior of separated pairs of spheres falling side by side. The effects of higher Reynolds numbers on the interaction of spheres falling in line have been observed by Steinberger, Pruppacher and Neiburger [61] and by Cataneo, Adam and Semonin [62].

The behavior of a pair of spheres in the limit as they touch is of particular interest. Aerosols frequently form such aggregates. For a pair of equal sized spheres moving along their line of centers, each sphere obeys the modified Stokes law (15) where the factor λ is given by the Faxen limit of the Stimson and Jeffery expression.

$$\lambda = 0.645$$

Using tangent sphere coordinates, Cooley and O'Neill [63] and also Goren [64] found exact solutions of the creeping flow equations for two spheres of arbitrary radii moving in contact along their line of centers. For equal spheres, their results reduce to that of Faxen. The case of equal spheres moving in contact with equal velocities normal to their line of centers was analyzed by O'Neill [65], again using tangent sphere coordinates. For the motion perpendicular to the line of centers, he found

$$\lambda = 0.72426$$

This result agrees well with the experimental value of 0.73 obtained from the data of Eveson, Hall and Ward. The same experimental value results from the investigation of Kunkel [66].

The interaction of spherical particles in the continuum-slip regime has been analyzed by Davis [67] and by Reed and Morrison [68]. Motion of two fluid spheres along their line of centers was treated by Wacholder and Weihs [69]. Particle interactions in electrophoresis and in thermophoresis are analyzed in a forthcoming series of papers by Reed and Morrison. Each of these analyses employs the bispherical coordinate system and yields exact solutions of the creeping flow equations.

As an example of how the bispherical coordinate system has been used to obtain exact solutions of the creeping flow equations in cases of interaction, the next section is devoted to the interaction of a single spherical particle and a plane surface.

17 HYDRODYNAMIC INTERACTION BETWEEN A PARTICLE AND A SURFACE

The presence of a nearby surface can significantly affect the fluid flow about a particle and, as a result, greatly influence the drag on the particle. The surface may be that of another particle or of a much larger body. In general, the effect of a stationary surface near a moving particle is primarily to increase the velocity gradients in the fluid and thus increase the resistance of the fluid to the particle's motion.

In the creeping motion of a particle normal to a surface, the increased resistance is independent of the direction of motion and accounts for viscous adhesion as well as the resistance to deposition of the particle. The presence of increased hydrodynamic resistance of approaching bodies presents a dilemma to cloud physicists who are currently unable to explain how small water droplets collide and coalesce to form larger drops.

As a contrast to the preceeding section where particle interactions were treated using very approximate techniques, we shall here obtain an exact solution to the creeping flow about a spherical particle moving normal to a rigid plane surface. We will then compare this result with experiment and with approximate analysis. This exact solution was obtained by Brenner [56] and by Maude [57] following the pioneering work of Stimson and Jeffery [51].

When analyzing the fluid motion bounded by a plane and by a sphere that is not in contact with that plane, the bispherical coordinate system is naturally considered. The advantage of using the bispherical coordinate system in this instance is that each of the bounding surfaces corresponds to a coordinate surface generated by holding a single variable constant. Considerations relating to the choice of a coordinate system are discussed in the beginning of Chapter 3.

The bispherical coordinate system (η, ξ, ϕ) , the spherical particle and the plane surface are shown in Figure 5.8. The surface of the particle is defined by $\xi = \alpha$ while the rigid plane surface corresponds to $\xi = 0$. Each of the surfaces, ξ constant, is a sphere. These are nonconcentric spheres having centers on the axis of revolution. For positive ξ , each such sphere surrounds the spherical surfaces corresponding to greater values of ξ and is interior to the spheres corresponding to smaller ξ . The point, $z = c$, is the limiting

FIGURE 5.8 - SPHERE APPROACHING SURFACE

sphere of zero radius, $\xi = \infty$. The plane, $z = 0$, $\xi = 0$, bounds all spheres generated by constant positive ξ .

The value of α defining the particle surface depends on the ratio of the sphere radius to the distance of the sphere center from the plane

$$\cosh \alpha = h/a \quad (1)$$

In terms of c , the relation is

$$\coth \alpha = h/c$$

or

$$\operatorname{csch} \alpha = a/c$$

The metric coefficients of the bispherical coordinate system are

$$h_{\xi} = h_{\eta} = \frac{c}{\cosh \xi - \cos \eta} \quad (2)$$

$$h_{\phi} = \rho = \frac{c \sin \eta}{\cosh \xi - \cos \eta} \quad (3)$$

so that the Stokes stream function operator, E^2 , becomes, using the definition 4.6-(1),

$$E^2 = \frac{(\cosh \xi - \mu)}{c^2} \left\{ \frac{\partial}{\partial \xi} \left[(\cosh \xi - \mu) \frac{\partial}{\partial \xi} \right] + (1-\mu^2) \frac{\partial}{\partial \mu} \left[(\cosh \xi - \mu) \frac{\partial}{\partial \mu} \right] \right\} \quad (4)$$

Following the normal notation in these coordinates, we have defined μ by

$$\mu \equiv \cos \eta \quad (5)$$

and will retain this notation throughout the section. In this section, viscosity will be denoted by κ .

We now seek the conditions under which the fluid flow about the particle is governed by the relation for axisymmetric steady creeping flow.

$$E^4 \psi = 0 \quad (6)$$

From before, we know that this relation describes the flow when the effects of both local and convective acceleration are negligible. In steady motion, this occurs when the Reynolds number based on a particle dimension is sufficiently small. The motion of a particle normal to a wall is inherently unsteady, however, so that an additional restriction becomes necessary. A trivial order of magnitude calculation shows that, in the case of relative normal motion, the Reynolds number based on a^2/δ must be sufficiently small as well. δ is the width of the gap between the particle and the plane.

Under these conditions, the fluid flow is governed by the creeping flow equation (6) with the Stokes stream function operator given by (5). It remains to specify the boundary conditions at the bounding surfaces. Both the particle and the plane are rigid and impermeable. We further restrict our attention here to the case where the fluid does not slip along the solid surfaces. These boundary conditions are those normally applied at solid surfaces and have been examined in Section 4.9.

The reference frame is fixed by choice to the rigid plane so that the spherical particle is observed to move with speed U in the negative z direction. The particle approaches the plane. The motion of a particle receding from the plane is of course obtained simply by replacing U everywhere by minus U .

At the plane, the no normal flow boundary condition is

$$\psi = 0 \quad \text{at} \quad \xi = 0 \quad (7)$$

and the no slip condition, 4.9-(5), is

$$\frac{\partial \psi}{\partial \xi} = 0 \quad \text{at} \quad \xi = 0 \quad (8)$$

The sphere is moving in this frame so that the boundary conditions at this surface are expressed in terms of vanishing relative velocity between fluid and solid. The impenetrability condition at the moving sphere surface is following 4.9-(14),

$$\psi - \frac{U_0^2}{2} = 0 \quad \text{at} \quad \xi = \alpha \quad (9)$$

The negative sign results from the direction of the sphere's motion. The cylindrical radial position, ρ , is given by (3) in bispherical coordinates. The no-slip condition at the sphere is correspondingly written

$$\frac{\partial}{\partial \xi} \left(\psi - \frac{U_0^2}{2} \right) = 0 \quad \text{at} \quad \xi = \alpha \quad (10)$$

A solution of the creeping flow equation (6) suitable for satisfying boundary conditions (7) through (10) has been found by Stimson and Jeffery [51] by using separation of variables in bispherical coordinates. Equation (6) is R separable in bispherical coordinates. A sufficiently general solution, bounded on the axis of revolution, is the solution of Stimson and Jeffery written here in Brenner's [56] form

$$(\cosh \xi - \mu)^{3/2} \psi = \sum_{n=0}^{\infty} U_n(\xi) C_{n+1}^{-1/2}(\mu) \quad (11)$$

where

$$U_n(\xi) = a_n \cosh \left[\left(n - \frac{1}{2} \right) \xi \right] + b_n \sinh \left[\left(n - \frac{1}{2} \right) \xi \right] \\ + c_n \cosh \left[\left(n + \frac{3}{2} \right) \xi \right] + d_n \sinh \left[\left(n + \frac{3}{2} \right) \xi \right] \quad (12)$$

and $C_{n+1}^{-1/2}$ is the Gegenbauer polynomial of order $n+1$ and degree $-1/2$, and is related to the Legendre polynomials by

$$C_{n+1}^{-1/2}(\mu) = \frac{P_{n-1}(\mu) - P_{n+1}(\mu)}{2n+1}$$

The coefficients of the solution are determined from the boundary conditions. From the condition of no normal flow at the plane (7), we have

$$a_n = -c_n \quad (13)$$

From (8), the no-slip condition at the surface, we have

$$\left(n - \frac{1}{2} \right) b_n = - \left(n + \frac{3}{2} \right) d_n \quad (14)$$

The boundary conditions imposed on the sphere surface require somewhat more effort. First, following the form of the expansion (11), equations (9) and (10) are written as

$$(\cosh \xi - \mu)^{3/2} \psi = \frac{U}{2} (\cosh \xi - \mu)^{3/2} \rho^2 \text{ at } \xi = \alpha \quad (15)$$

and

$$\frac{\partial}{\partial \xi} \left[(\cosh \xi - \mu)^{3/2} \psi \right] = \frac{U}{2} \frac{\partial}{\partial \xi} \left[(\cosh \xi - \mu)^{3/2} \rho^2 \right] \text{ at } \xi = \alpha \quad (16)$$

respectively.

Expanding the right hand sides of (15) and (16) in terms of the Gegenbauer polynomials as in (10) permits the boundary conditions on the sphere to be readily expressed in terms of the stream function coefficients. The required expansion is given by Stimson and Jeffery [51]

$$\begin{aligned} (\cosh \xi - \mu)^{3/2} \rho^2 &= \frac{c^2 (1 - \mu^2)}{(\cosh \xi - \mu)^{1/2}} \\ &= c^2 \sqrt{2} \sum_{n=0}^{\infty} n(n+1) \left[\frac{e^{-(n - \frac{1}{2}) \xi}}{2n-1} - \frac{e^{-(n + \frac{3}{2}) \xi}}{2n+3} \right] C_{n+1}^{-1/2}(\mu) \quad (17) \end{aligned}$$

Substituting the expansions, (15) and (16) yield

$$\begin{aligned} &a_n \cosh \left[\left(n - \frac{1}{2} \right) \alpha \right] + b_n \sinh \left[\left(n - \frac{1}{2} \right) \alpha \right] \\ &+ c_n \cosh \left[\left(n + \frac{3}{2} \right) \alpha \right] + d_n \sinh \left[\left(n + \frac{3}{2} \right) \alpha \right] \\ &= \frac{c^2 U}{\sqrt{2}} n(n+1) \left[\frac{e^{-(n - \frac{1}{2}) \alpha}}{2n-1} - \frac{e^{-(n + \frac{3}{2}) \alpha}}{2n+3} \right] \end{aligned}$$

and

$$\left(n - \frac{1}{2} \right) \{ a_n \sinh \left[\left(n - \frac{1}{2} \right) \alpha \right] + b_n \cosh \left[\left(n - \frac{1}{2} \right) \alpha \right] \}$$

$$\begin{aligned}
& + (n + \frac{3}{2}) \{c_n \sinh [(n + \frac{3}{2}) \alpha] + d_n \cosh [(n + \frac{3}{2}) \alpha]\} \\
& = - \frac{c^2 U}{2\sqrt{2}} n(n+1) [e^{-(n - \frac{1}{2}) \alpha} - e^{-(n + \frac{3}{2}) \alpha}]
\end{aligned}$$

We now have $4n$ equations in the $4n$ unknowns, a_n , b_n , c_n and d_n . Because the equations are not recursion relations, they are easily solved to yield the coefficients

$$\begin{aligned}
c_n &= -a_n \\
&= \frac{c^2 U n(n+1) (2n+1) \sinh^2 \alpha}{\sqrt{2} \{4 \sinh^2 [(n + \frac{1}{2}) \alpha] - (2n+1)^2 \sinh^2 \alpha\}} \quad (18)
\end{aligned}$$

and

$$\begin{aligned}
d_n &= - \frac{(2n-1)}{(2n+3)} b_n \\
&= - \frac{c^2 U n(n+1)}{\sqrt{2} (2n+3)} \left\{ \frac{2 \sinh [(2n+1) \alpha]}{4 \sinh^2 [(n+\frac{1}{2}) \alpha]} + \frac{(2n+1) \sinh [2\alpha]}{(2n+1)^2 \sinh^2 \alpha} - 1 \right\} \quad (19)
\end{aligned}$$

The stream function describing the fluid flow bounded by the particle and plane has been found. It remains to find the force exerted by the fluid on the particle. The expression of Payne and Pell is inapplicable because the particle is not isolated. Instead, an expression derived by Stimson and Jeffery is well suited to finding this force. The axial force

$$F = \int \sigma_{nz} dA$$

exerted by a fluid on a body in steady axisymmetric flow was shown by Stimson and Jeffery to reduce to

$$F = \pi \kappa \int \frac{\rho^3}{h_1} \frac{\partial}{\partial u_1} \left(\frac{E^2 \psi}{\rho^2} \right) h_2 du_2 \quad (20)$$

when the fluid adheres to the rigid particle surface. The integration is over the surface of the particle. The no-slip condition can be readily removed. The result is applicable to any steady axisymmetric creeping flow independent of the surface conditions. This wider applicability is noted by Morrison [70] who derived the more general expression for accelerating motion.

Substituting (11) into (20), the force reduces to

$$F = \frac{2\pi\kappa\sqrt{2}}{c} \sum_{n=1}^{\infty} (a_n + b_n + c_n + d_n) \quad (21)$$

Again, κ is used here to denote viscosity. For the flow considered here, the coefficients are given by (19) and (20) and the final expression for the force is obtained. Writing the result as Stokes law with a wall correction λ

$$F = 6 \pi \kappa a U \lambda \quad (22)$$

the correction due to the presence of a rigid plane normal to the direction of motion is

$$\lambda = \frac{4}{3} \sinh \alpha \sum_{n=1}^{\infty} \frac{n(n+1)}{(2n-1)(2n+3)} \left\{ \frac{2 \sinh [(2n+1)\alpha] + (2n+1) \sinh [2\alpha]}{4 \sinh^2 [n+\frac{1}{2}\alpha] - (2n+1)^2 \sinh^2 \alpha} - 1 \right\} \quad (23)$$

An approximate expression, due to Lorentz [71] and valid for sufficiently large separations is

$$F = 6\pi\kappa a U \left(1 + \frac{9}{8} \frac{a}{h}\right)$$

A comparison between this approximate expression and the exact correction is seen in Table 5.2. Brenner [56] gives values of λ at these selected values of α .

α	h/a	λ	$(1 + \frac{9}{8} \frac{a}{h})$
0	1	∞	2.1250
0.5	1.1276	9.2518	1.9977
1.0	1.5431	3.0361	1.7291
1.5	2.3524	1.8375	1.4782
2.0	3.7622	1.4129	1.2991
2.5	6.1323	1.2220	1.1835
3.0	10.0677	1.1252	1.1118
∞	∞	1	1

Table 5.2 Comparison of Brenner, Maude Correction with Approximate Expression.

The experiments of Hopper and Grant [72] are in reasonable agreement with the predictions of the exact theory. The more recent experiments [73] cited by MacKay, Suzuki and Mason [74] show excellent agreement with the exact solution even at small separations.

The creeping motion of a spherical particle normal to a rigid wall in the continuum-slip regime was analyzed by Reed and Morrison [68]. Motion of a bubble or a drop normal to a rigid wall was treated by Bart [75] and later by Wacholder and Weihs [69]. Electrophoretic motion normal to an electrode was

analyzed by Morrison and Stukel [76]. O'Neill [77, 78] obtained the solution for the asymmetric motion of a rigid sphere. Bispherical coordinates were used and exact solutions obtained in each of these studies.

5.18 TRANSIENT MOTION - SIMPLE AEROSOL THEORY

The motion of an aerosol particle is normally unsteady. The transient motion of the particle may result from externally applied forces or Brownian fluctuations as well as non-uniform motion of the surrounding gas. The gas about the particle may be turbulent and may have a mean acceleration as well. Most processes of interest to us are in fact unsteady. These include dispersion in a turbulent flow, precipitation and sampling. A knowledge of the types and magnitudes of inertial effects in aerosol motion is, as a result, important in an understanding of these processes.

Unsteady aerosol motion is normally analyzed by assuming that the force, exerted by the gas on a particle moving with an instantaneous velocity $\vec{U}$, is the same as the force exerted on that particle in steady motion with the same velocity. While this assumption is normally adequate, we will determine the extent of its validity when we later develop an exact theory of low Reynolds number transient motion. It should be clear, however, that this assumption corresponds to neglecting the fluid density. A massless fluid responds instantaneously to changes at its boundaries. Additionally, it contributes no added mass to be accelerated. The assumption consequently is less satisfactory for analysis of transient particle motion in liquids.

Consider first the unsteady motion of a particle in a gas that is stationary aside from the fluid motion generated by the particle. With no externally applied forces, particle acceleration results solely from the drag exerted by the gas. At low Reynolds numbers, where the steady drag is the Stokes drag, this yields

$$M \frac{d\vec{U}}{dt} = - 6 \pi \mu a \vec{U} \quad (1)$$

We have simply equated the rate of change of particle momentum to the Stokes drag. The material derivative of the particle velocity is the Lagrangian acceleration of an observer moving with the particle. In terms of the Eulerian, convective, acceleration, this is

$$\frac{D\vec{U}}{Dt} = (\vec{U} \cdot \nabla) \vec{U} \quad (2)$$

Acceleration is discussed in Section 2.6.

For motion in a straight line these relations reduce to scalar equations in the sole velocity component, u_x .

$$M \frac{Du_x}{Dt} = -6 \pi \mu a u_x \quad (3)$$

where, from (2),

$$\frac{Du_x}{Dt} = u_x \frac{\partial u_x}{\partial x} \quad (4)$$

Consider, now, a particle whose velocity relative to the gas is u_{x_0} at some time, t , equal to zero. The subsequent motion of the particle is governed by (3) with the initial condition.

$$u_x = u_{x_0} \quad \text{at} \quad t = 0 \quad (5)$$

The equation of motion is a first order linear ordinary differential equation with a well known solution

$$u_x = u_{x_0} \exp \left(- \frac{6 \pi \mu a}{M} t \right) \quad (6)$$

The particle velocity decays exponentially. The relaxation time for this decay is

$$\tau \equiv \frac{M}{6 \pi \mu a} \quad (7)$$

so that

$$u_x = u_{x_0} e^{-t/\tau} \quad (8)$$

and the velocity has decreased by u_{x_0}/e when t is equal to τ .

For a spherical particle of density ρ_p , the relaxation time is

$$\tau = \frac{2\rho_p a^2}{9\mu} \quad (9)$$

It is of interest to note that the settling velocity of a solid particle in air, given by 5.3-(3), may be expressed in terms of this relaxation time

$$U = \tau g \quad (10)$$

A calculation of the relaxation time of a 10 micron diameter particle with a density of 2g/cm^3 in air at 25C, with a viscosity of $1.84 \times 10^{-4} \text{ g/cm sec}$ yields

$$\tau = 6 \times 10^{-4} \text{ sec}$$

The position of the decelerating particle, whose velocity is given by (8), is found by integrating

$$u_x = \frac{Dx}{Dt} = u_{x_0} e^{-t/\tau}$$

With the initial condition

$$x = 0 \quad \text{at} \quad t = 0$$

the particle position is found as a function of time

$$x = u_{x_0} \tau (1 - e^{-t/\tau}) \quad (11)$$

Note that as time increases without bound, this model predicts that the particle position will approach a finite limit. This stopping distance is $u_{x_0} \tau$.

Further note that this result is more quickly obtained by adopting an Eulerian viewpoint so that (3) and (4) combine to give

$$M \frac{\partial u_x}{\partial x} = -6 \pi \mu a \quad (12)$$

This integrates to give the velocity as a function of position:

$$u_x = u_{x_0} - x/\tau \quad (13)$$

so that the velocity vanishes when x is $u_{x_0} \tau$, the stopping distance.

When an external force is applied to the particle as by gravity or an electric field, the equation of motion (1) is altered simply by equating the net force on the particle to the rate of change of particle momentum. For motion in the x direction, this is

$$F_x - 6 \pi \mu a u_x = M \frac{Du_x}{Dt} \quad (14)$$

The step application of a constant force to a particle initially at rest then produces a transient velocity

$$u_x = U (1 - e^{-t/\tau}) \quad (15)$$

U is the final steady Stokes law velocity of the particle. Further integration yields the particle position as a function of time

$$x = U [t - \tau (1 - e^{-t/\tau})] \quad (16)$$

At long times, the particle position approaches that of a particle which moved with steady velocity U from time τ .

We turn now to the motion of an accelerating particle in a moving fluid. In the absence of an externally applied force, equation (1) is generalized by noting that the drag exerted by the fluid on the particle is proportional to the relative velocity

$$M \frac{D\vec{U}}{Dt} = 6 \pi \mu a (\vec{V} - \vec{U}) \quad (17)$$

$\vec{V}$ is the local fluid velocity, undisturbed by the particle, measured in the same inertial frame as the particle velocity, $\vec{U}$.

In terms of the particle position, $\vec{x}$, the equation of motion is

$$M \frac{D^2\vec{x}}{Dt^2} = 6 \pi \mu a (\vec{V} - \frac{D\vec{x}}{Dt}) \quad (18)$$

A particularity interesting and important case of particle motion in a non-uniform fluid flow is the flow associated with inertial impaction. Consider the flow of a particle laden gas past a large obstacle. The fluid streamlines pass about the obstacle. Because of particle inertia however, particle and fluid streamlines diverge. The particle streamline may not be deflected to an extent sufficient to prevent impaction with the obstacle. The effectiveness of the obstacle in causing such encounters is, in the absence of significant diffusion or interaction, determined by analysis of particle trajectories. The trajectories are normally determined parametrically by finding the scalar components of particle position as functions of time. In a fluid field possessing the velocity distribution, in say two dimensions,

$$\vec{V} = v_x(x, y) \hat{i} + v_y(x, y) \hat{j}$$

the equation of motion (18) becomes the scalar equations.

$$M \frac{D^2x}{Dt^2} = 6 \pi \mu a (v_x - \frac{Dx}{Dt}) \quad (19)$$

$$M \frac{D^2y}{Dt^2} = 6 \pi \mu a (v_y - \frac{Dy}{Dt}) \quad (20)$$

—— FLUID STREAMLINE
---- PARTICLE STREAMLINE

FIGURE 5.9 PARTICLE MOTION IN NONUNIFORM
FLUID FLOW

For definiteness, let us consider particle motion in the potential fluid flow about a circular cylinder of radius R . The assumed potential flow can give a good representation of the flow in front of the cylinder outside of the boundary layer. Figure 5.9 shows fluid streamlines in the potential flow about a cylinder as well as possible particle trajectories. The potential flow about a cylinder is well known. The velocity distribution for flow approaching the cylinder from the positive $\tilde{x}$ direction is

$$v_x = -U \left[1 + R^2 \frac{y^2 - \tilde{x}^2}{(\tilde{x}^2 + y^2)^2} \right] \quad (21)$$

$$v_y = U R^2 \frac{2 \tilde{x} y}{(\tilde{x}^2 + y^2)^2} \quad (22)$$

$\tilde{x}$ and y are coordinates centered on the axis of the cylinder. x and y are coordinates whose origin is at the leading stagnation point on the cylinder

$$\tilde{x} = R + x \quad (23)$$

Examining these expressions for the fluid velocity, we see that in general the equations (19) and (20) for the particle motion are coupled and nonlinear. They must consequently be ordinarily solved using numerical techniques. An important exception is the flow in the forward stagnation region. The forward stagnation point is located at

$$\tilde{x} = R \quad \text{or} \quad x = 0 \quad \text{and} \quad y = 0$$

Fluid streamlines, as well as possible particle streamlines, are shown in the stagnation region in Figure 5.10

——— FLUID STREAMLINE
----- PARTICLE STREAMLINE

FIGURE 5.10 FLOW IN STAGNATION REGION

The particle motion in this region was first analyzed by G.I. Taylor [79] in a study of icing on aircraft and was further considered shortly thereafter by Langmuir and Blodgett [80]. In the stagnation region, the expressions for fluid velocity components may be expanded in terms of the small parameters, x and y . For sufficiently small x and y , equations (21) and (22) reduce to

$$v_x = - 2U x/R \quad (24)$$

and

$$v_y = 2U y/R \quad (25)$$

This same type of dependence is seen in all normal stagnation regions of potential flows. Accordingly we shall use the more general expression

$$v_x = - Cx \quad (26)$$

and

$$v_y = Cy \quad (27)$$

For the circular cylinder C is $2U/R$. C^{-1} is a time constant for the fluid acceleration.

Using the relaxation time, τ , and a stagnation region fluid flow given by (26) and (27), the particle motion near the leading edge of the cylinder is given by

$$\tau \frac{D^2 x}{Dt^2} = - Cx - \frac{Dx}{Dt} \quad (28)$$

$$\tau \frac{D^2 y}{Dt^2} = Cy - \frac{Dy}{Dt} \quad (29)$$

Note that these equations are linear and uncoupled. We may proceed then with their exact solution. This is most conveniently done in dimensionless form. Define a dimensionless time

$$T \equiv t/\tau \quad (30)$$

and a quantity

$$\theta \equiv C \tau \quad (31)$$

that we may describe as a modified Stokes number. The Stokes number or particle parameter is normally defined in terms of the approach velocity, relaxation time and a characteristic length. For a cylinder, this is

$$Stk \equiv U \tau/R \quad (32)$$

and is a ratio of the particle stopping distance, $U\tau$, to a characteristic length R . Alternatively, this is a ratio of the particle's relaxation time, τ , to a characteristic time R/U . By comparison, the modified Stokes number, θ , is $2U\tau/R$ for a circular cylinder. The Stokes number, or modified Stokes number, is an indication of the ability of the particle to respond to the changing fluid velocity. At small values of these parameters, the particle and fluid streamlines should correspond closely. At large Stokes number, the particle continues to move essentially unaffected until any possible impaction.

With these definitions, the equation for motion in the x direction becomes

$$\frac{D^2 x}{DT^2} + \theta x + \frac{Dx}{DT} = 0 \quad (33)$$

This equation governs the motion normal to the front surface of the obstacle. Solution of the equation of motion parallel to the surface, motion in the y direction, is left as an exercise.

The second order equation of motion requires two initial conditions. In addition to specifying some initial position

$$x = x_0 \quad \text{at} \quad t = 0 \quad (34)$$

we follow Taylor by specifying that the particle velocity initially be equal to the fluid velocity

$$\frac{Dx}{Dt} = v_x \quad \text{at} \quad t = 0 \quad (35)$$

This last condition is better written as

$$\frac{Dx}{Dt} = - \theta x_0 \quad \text{at} \quad t = 0 \quad (36)$$

Equations (26), (30), and (31) are used in obtaining this result.

The solution of equation (33), subject to initial conditions (34) and (36) is perhaps most easily obtained using a Laplace transform with respect to dimensionless time. The Laplace transform with respect to time of a function, $f(t)$ is written as $L[f(t)]$ or $\bar{f}$ and is defined

$$L[f(t)] \equiv \int_0^{\infty} f(t) e^{-st} dt \quad (37)$$

Accordingly, we transform equation (33) term by term, using the initial conditions, and obtain

$$L[x] = \bar{x} \quad (38)$$

$$L\left[\frac{Dx}{DT}\right] = s \bar{x} - x_0 \quad (39)$$

$$L\left[\frac{D^2x}{DT^2}\right] = s^2 \bar{x} - sx_0 - \left.\frac{Dx}{DT}\right|_0 = s^2 \bar{x} - sx_0 + \theta x_0 \quad (40)$$

Substituting, the transformed position is

$$\bar{x} = x_0 \frac{s+1-\theta}{s^2+s+\theta} \quad (41)$$

Inversion is simplified by factoring the denominator.

$$\bar{x} = x_0 \frac{s+1-\theta}{(s-\lambda_1)(s-\lambda_2)} \quad (42)$$

The roots are

$$\lambda_1, \lambda_2 = -\frac{1 \pm \sqrt{1-4\theta}}{2} \quad (43)$$

so that if θ is less than $1/4$, the roots are real and negative; if θ is equal to $1/4$, the roots are both equal to $-1/2$; and if θ exceeds $1/4$, then the roots are complex conjugates.

Inverting (42) for the case of unequal roots, θ not equal to $1/4$, we find

$$x = x_0 \frac{(\lambda_1 + 1 - \theta)e^{\lambda_1 T} - (\lambda_2 + 1 - \theta)e^{\lambda_2 T}}{\lambda_1 - \lambda_2} ; \theta \neq \frac{1}{4} \quad (44)$$

In accordance with the discussion of the roots, we find that for θ greater than $1/4$, the exponentials are complex. The motion is equivalent to an oscillating decay or underdamped response.

The particle hits the surface. Conversely, if θ is less than $1/4$, the exponentials are negative and real. The particle approaches the surface, but neglecting the finite particle size effect, the interception effect, the particle does not reach the surface.

This model predicts that a particle, characterized by a modified Stokes number less than $1/4$, no matter how close to the stagnation streamline in its initial approach, will not strike the obstacle. Instead, this particle and all other particles of the same size will be carried about the obstacle without impaction. There exists then a critical Stokes number for impaction. Below this Stokes number, inertial effects do not produce impaction. The collection efficiency of the obstacle is zero. The critical Stokes number, St_k , is $1/8$ for a circular cylinder.

Because of the complex roots associated with large θ , (44) is conveniently written as

$$x = x_0 e^{-T/2} \left\{ \cos \left[\sqrt{\theta - \frac{1}{4}} T \right] + \frac{(\frac{1}{2} - \theta)}{\sqrt{\theta - \frac{1}{4}}} \sin \left[\sqrt{\theta - \frac{1}{4}} T \right] \right\}$$

when θ exceeds $1/4$. Finally, for completeness, the solution for equal roots, θ equal to $1/4$, is

$$x = x_0 e^{-T/2} \left[1 + \frac{T}{4} \right]$$

19 TRANSIENT MOTION - EXACT THEORY

The simple analyses of the last section correspond to a description of particle motion in a massless fluid. Here, we will examine the effects of fluid inertia on particle motion in a continuum. We will continue to restrict our attention to low Reynolds number or creeping flows. The fluid acceleration of interest is the local acceleration. We need not seek solutions to the complete Navier-Stokes equation for an incompressible fluid with constant viscosity.

$$\rho_v \frac{D\vec{V}}{Dt} = - \nabla p + \mu \nabla^2 \vec{V} \quad (1)$$

Instead, convective acceleration is neglected leaving the linear equation

$$\rho_f \frac{\partial \vec{V}}{\partial t} = - \nabla p + \mu \nabla^2 \vec{V} \quad (2)$$

as the equation governing the fluid motion.

Before addressing the more difficult problem of transient fluid motion about a spherical particle, some insight can be obtained by considering the fluid motion bounded by a flat plate moving along its surface with an unsteady velocity. The fluid motion generated by a step change in wall velocity was solved by George Gabriel Stokes and is known as Stokes first problem. Flow near an oscillating plate is Stokes second problem.

An infinite flat plate, whose surface corresponds to y equal to zero bounds an initially motionless fluid. At some time, designated by t equal to zero, the plate begins rectilinear motion along its length with transient velocity, $U(t)$. The direction of motion is the x direction. There is no pressure gradient, and the velocity has a single non-zero component which depends on a single space dimension and time

$$\vec{V} = u(y, t) \hat{i}$$

Consequently, (1) or (2) reduces to the single linear differential equation

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\nu} \frac{\partial u}{\partial t} \quad (3)$$

The initial condition is

$$u = 0 \quad \text{at} \quad t = 0$$

while the boundary conditions express the fluid adherence to the plate

$$u = U(t) \quad \text{at} \quad y = 0$$

and the vanishing of the disturbance far from the plate.

$$u \rightarrow 0 \quad \text{as} \quad y \rightarrow \infty$$

The Laplace transform again provides a convenient method of solution. Transforming the equation of motion (3) with respect to time yields the ordinary linear differential equation

$$\frac{d^2 \bar{u}}{dy^2} = \frac{s}{\nu} \bar{u} \quad (4)$$

Solutions of this equation are well known. The solution satisfying the boundary conditions is

$$\bar{u} = \bar{U} e^{-\sqrt{s/\nu} y} \quad (5)$$

For a prescribed plate velocity distribution, the transformed fluid velocity expression can be inverted to yield the fluid velocity as a function of time.

The stress associated with the fluid velocity gradient

$$\sigma_{yx} = \mu \frac{\partial u}{\partial y} \quad (6)$$

varies, then, according to

$$\bar{\sigma}_{yx} = - \mu \sqrt{\frac{s'}{v}} \bar{U} e^{-\sqrt{s'/v} y}$$

The stress at the plate is of particular interest just as the force on an accelerating particle is of interest. This stress is

$$\bar{\sigma}_{yx} = - \mu \sqrt{\frac{s'}{v}} \bar{U} \quad (7)$$

Using convolution, the transformed stress can be inverted to yield an expression for the stress in terms of the velocity history of the plate

$$\sigma_{yx} = - \sqrt{\frac{\mu \rho}{\pi}} \int_0^t \frac{dU / d\tau}{\sqrt{t - \tau}} d\tau \quad (8)$$

This expression is very similar to a force term, the so-called Basset term, arising in the accelerated motion of a particle. An understanding of the origin and meaning to the result (8) aids in the understanding of the results of analysis of the fluid about a particle.

To help in this understanding, let us consider a particular case. Motion resulting from a step change in the plate velocity

$$\bar{U} = U / s$$

obeys (5) with this substitution.

$$\bar{u} = U e^{-\sqrt{s/\nu} y} / s$$

which inverts to yield

$$u = U \operatorname{erfc} \left(\frac{y}{2 \sqrt{\nu t}} \right) \quad (9)$$

erfc is the complimentary error function.

$$\operatorname{erfc} z = \frac{2}{\sqrt{\pi}} \int_z^{\infty} e^{-x^2} dx \quad (10)$$

As the argument increases from zero, the error function monotonically decreases in value approaching zero as the argument approaches infinity. The fluid velocity given by (9) is close to the plate velocity for small y or large t . At larger distances or smaller times when the disturbance has not yet diffused sufficiently far from the plate, the velocity is small. The corresponding stress at the rigid surface is

$$\sigma_{yx} = - \sqrt{\frac{\mu \rho}{\pi t}} U \quad (11)$$

The stress vanishes as $t^{-1/2}$ as the fluid velocity gradient decreases at the wall.

Return now to a consideration of equation (8). The general plate velocity distribution $U(t)$ can be considered as resulting from a series of infinitesimal step changes. Thus the change, $dU = (dU/d\tau)d\tau$ occurs at time τ , or equivalently a time $(t-\tau)$ ago. Because the relations are linear, the resulting stress is the linear sum of the stresses produced by each of these step changes. Equation (8) is this sum (or integral).

Because the accelerated particle motion that we now examine is a similar type of motion, we may anticipate an expression similar to (8) in the force on the particle. Consider the slow transient motion of a fluid about a rigid spherical particle. The fluid motion is governed by (2). This fluid motion, resulting from rectilinear motion of the sphere, is axisymmetric. It is thus described conveniently by a stream function obeying 4.7-(17).

$$E^2 (E^2 - \frac{1}{\nu} \frac{\partial}{\partial t}) \psi = 0 \quad (12)$$

with the Stokes stream function operator again given by

$$E^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin \theta}{r^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \right) \quad (13)$$

in spherical coordinates.

A frame at rest with respect to the distant fluid is selected. This frame is an inertial frame.

The boundary conditions for the accelerating flow are the same as the conditions applied to a steady flow. The no normal flow condition, 4.9-(14) is

$$\psi + \frac{Ua^2 \sin^2 \theta}{2} = 0 \quad \text{at} \quad r = a \quad (14)$$

and the no-slip condition is 4.9-(17) which we write as

$$\frac{\partial \psi}{\partial r} + Ua \sin^2 \theta = 0 \quad \text{at} \quad r = a \quad (15)$$

The velocity of the fluid vanishes as the distance from the particle increases.

In addition to the boundary conditions, we must also impose an initial condition on this transient motion. Here, we examine the response of a particle

and surrounding fluid that are initially at rest. We write then

$$\psi = 0 \quad \text{at} \quad t = 0 \quad (16)$$

Because the boundary conditions are of the same form as in the steady motion, we are justified in seeking a solution of the form

$$\psi = f(r,t) \sin^2 \theta \quad (17)$$

If this form is correct, the number of independent variables has been reduced to two. Using this form, the Stokes stream function operator (13) reduces to

$$E^2 \psi = \left(\frac{\partial^2}{\partial r^2} - \frac{2}{r^2} \right) f(r,t) \sin^2 \theta$$

so that (12), describing the fluid motion, is

$$\left(\frac{\partial^2}{\partial r^2} - \frac{2}{r^2} \right) \left(\frac{\partial^2}{\partial r^2} - \frac{2}{r^2} - \frac{1}{\nu} \frac{\partial}{\partial t} \right) \psi = 0$$

Use of the Laplace transform reduces this to an ordinary differential equation

$$\left(\frac{d^2}{dr^2} - \frac{2}{r^2} \right) \left(\frac{d^2}{dr^2} - \frac{2}{r^2} - \frac{s}{\nu} \right) \bar{\psi} = 0 \quad (18)$$

when (16) is used.

This fourth order homogenous linear differential equation has four linearly independent solutions. These solutions are most easily found by finding solutions of

$$\left(\frac{d^2}{dr^2} - \frac{2}{r^2}\right) \bar{\psi}_1 = 0 \quad (19)$$

and

$$\left(\frac{d^2}{dr^2} - \frac{2}{r^2} - \frac{s}{v}\right) \bar{\psi}_2 = 0 \quad (20)$$

if these are independent.

Note that any solution of (19) is also a solution of (18) and that any solution of (20) is similarly a solution of (18). A general solution of (18) is

$$\bar{\psi} = \bar{\psi}_1 + \bar{\psi}_2 \quad (21)$$

Equation (19) has been encountered before. It governs steady irrotational flow and has a solution

$$\bar{\psi}_1 = \frac{\bar{U}}{2} \left(A r^2 + \frac{B}{r} \right) \sin^2 \theta \quad (22)$$

in the form of (17). The solution of (20) having this form is

$$\bar{\psi}_2 = \frac{\bar{U}}{2} \left[c \left(\sqrt{\frac{s}{v}} - \frac{1}{r} \right) e^{r \sqrt{\frac{s}{v}}} + D \left(\sqrt{\frac{s}{v}} + \frac{1}{r} \right) e^{-r \sqrt{\frac{s}{v}}} \right] \sin^2 \theta \quad (23)$$

We have for the stream function governing the fluid flow

$$\begin{aligned} \bar{\psi} = \frac{\bar{U}}{2} & \left[A r^2 + \frac{B}{r} + c \left(\sqrt{\frac{s}{v}} - \frac{1}{r} \right) e^{r \sqrt{\frac{s}{v}}} \right. \\ & \left. + D \left(\sqrt{\frac{s}{v}} + \frac{1}{r} \right) e^{-r \sqrt{\frac{s}{v}}} \right] \sin^2 \theta \end{aligned} \quad (24)$$

in accordance with (21).

Vanishing of the flow far from the particle requires that

$$A = C = 0 \quad (25)$$

Use of the remaining boundary conditions, (14) and (15), then yields

$$B = -a^3 - 3 \left(a^2 \sqrt{\frac{s}{v}} + a \frac{v}{s} \right) \quad (26)$$

$$D = 3a \frac{v}{s} e^{a \sqrt{s/v}} \quad (27)$$

so that the assumed form (17) is correct and the transformed stream function is

$$\begin{aligned} \bar{\psi} = \frac{\bar{U}}{2} \left\{ -\frac{a^3}{r} - 3 \left[a \sqrt{\frac{v}{s}} + \frac{v}{s} \right] \frac{a}{r} \right. \\ \left. + 3 \left[a \sqrt{\frac{v}{s}} + \frac{v}{s} \frac{a}{r} \right] e^{(a-r) \sqrt{s/v}} \right\} \sin^2 \theta \end{aligned} \quad (28)$$

Now, as with the steady flow, we can use the stream function distribution to find the force exerted by the fluid on the particle. An expression derived by Morrison [70] is well suited to the ready calculation of this force. Morrison found that the force exerted on an accelerating body in axisymmetric creeping flow is given by

$$F = -\pi \rho_f \frac{\partial}{\partial t} \int \left(\rho \frac{\partial \psi_1}{\partial u_1} + 2\psi_2 \frac{\partial \rho}{\partial u_1} \right) \frac{h_2}{h_1} du_2 \quad (29)$$

ψ_1 and ψ_2 are as used in this section.

u_1 and u_2 are orthogonal curvilinear coordinates and the integration is over the surface of the body. For the spherical particle considered here, the general expression reduces to

$$F = - \pi \rho_f a \frac{\partial}{\partial t} \int_0^\pi \left(a \frac{\partial \psi_1}{\partial r} + 2\psi_2 \right) \sin \theta \, d\theta \quad (30)$$

The Laplace transformed relation is

$$\bar{F} = - \pi \rho_f a s \int_0^\pi \left(a \frac{\partial \bar{\psi}_1}{\partial r} + 2\bar{\psi}_2 \right) \sin \theta \, d\theta \quad (31)$$

Substituting and evaluating, we find

$$\bar{F} = - \frac{2}{3} \pi \rho_f a^3 s \bar{U} - 6\pi \mu \left(a^2 \sqrt{\frac{s}{\nu}} + a \right) \bar{U} \quad (32)$$

The first term on the right hand side represents the force due to acceleration of the added mass. This added mass is half of the fluid mass, m , displaced by the particle

$$m = \frac{4}{3} \pi \rho_f a^3 \quad (33)$$

The second term on the right hand side of (32) is the relaxation term associated with the transient fluid velocity distribution. Compare this term with the stress relation (7) previously considered. The final term of (32) is the Stokes drag.

The net force applied to the particle is the sum of the externally applied force, F^* , and the force exerted by the fluid

$$\bar{F}^* + \bar{F} = M s \bar{U} \quad (34)$$

M is the particle mass.

The externally applied force is related to the particle velocity by

$$\bar{F}^* = (M + \frac{m}{2}) s\bar{U} + 6\pi\mu (a^2 \sqrt{\frac{s}{v}} + a) \bar{U} \quad (35)$$

The sum, $M + \frac{m}{2}$, of the particle mass and the added mass is termed the virtual mass. Using convolution, the Laplace transformed equation (35) can be inverted to yield

$$F^* = (M + \frac{m}{2}) \frac{dU}{dt} + \frac{am}{2a} \sqrt{\frac{v}{\pi}} \int_0^t \frac{dU / d\tau}{\sqrt{t - \tau}} d\tau + 6\pi\mu a U \quad (36)$$

The integral term is frequently designated the "Basset" term although it was first obtained by Boussinesq [81, 82]. Basset [37, 83] obtained this result independently shortly thereafter.

Solutions of (36) yielding the force in case of prescribed velocity, or velocity in the more important case of prescribed force, may be obtained. Laplace transform techniques are again useful. Consider the particularly easy case of a prescribed step force in particle velocity. Using (35) and the transformed velocity for a step change

$$\bar{U} = U / s$$

we invert to find

$$F^* = (M + \frac{m}{2}) U \delta(t) + 6\pi\mu a U \left[1 + \frac{a}{\sqrt{\pi v t}} \right] \quad (37)$$

The leading term on the right hand side is an impulsive force equal to the change in momentum of the particle and added mass. The second term is Stokes drag while the final term is a relaxation term expressing the additional force experienced in the approach to the steady state. Examination of the relaxation term reveals the fluid relaxation time

$$t' = \frac{a^2}{\nu} = \frac{a^2 \rho_f}{\mu}$$

In a gas, the fluid density is normally considerably less than the particle density. This fluid relaxation time is thus ordinarily much smaller than the aerosol relaxation time, $2\rho_p a^2/9\mu$, predicted by simple aerosol theory. This difference is significant in justifying the widespread use of simple aerosol theory. If the fluid relaxation time is much smaller than any other time of interest, it may be neglected altogether.

For a 10 micron diameter particle in air at 20C, the kinematic viscosity of the air being $0.150 \text{ cm}^2/\text{sec}$, the fluid relaxation time, t' , is about $1.67 \times 10^{-6} \text{ sec}$.

The response of a particle to a suddenly applied constant external force is of particular interest and has been studied by many investigators. Basset solved the equation of motion for a step application of a constant force using an approximate technique. Boggio [84] obtained a class of closed form exact solutions. This work is discussed briefly in Fuchs [2]. As Boggio's method of solution would not be considered the most straight forward today, we will show how his solutions may be readily obtained using the Laplace transform.

Similar to the relaxation time τ , 5.18-(9), of simple aerosol theory, but accounting for added mass, we define a time

$$\alpha \equiv \frac{a^2 (2M + m)}{9 m \nu} = \frac{a^2 (2\rho_p + \rho_f)}{9 \mu} \quad (38)$$

and a dimensionless time

$$T = t / \alpha \quad (39)$$

Further, define

$$\beta \equiv \frac{a}{\sqrt{\alpha \nu}} = \left(\frac{9m}{2M + m} \right)^{1/2} \quad (40)$$

a dimensionless parameter indicating the density ratio of particle and fluid.

Substituting these definitions into (36), then taking the Laplace transform with respect to dimensionless time, T , we obtain

$$\frac{\bar{U}}{U_0} = \frac{s+1 - \beta \sqrt{s}}{s(s-\lambda_1)(s-\lambda_2)} \quad (41)$$

$\bar{U}$ is the transformed velocity. U_0 is the limiting velocity given by Stokes law. λ_1 and λ_2 are the roots of

$$s^2 + (2-\beta^2)s + 1 = 0$$

These roots are

$$\lambda_1, \lambda_2 = \frac{\beta^2 - 2 \pm \beta \sqrt{\beta^2 - 4}}{2} \quad (42)$$

For β less than 2, the roots are complex. Aerosol particles invariably have values of β much less than one. For such particles,

$$\lambda_1, \lambda_2 \sim -1 \pm i\beta \quad (43)$$

The inverse transform of (41), for the case of unequal roots, can be found using tabulated transform pairs. The result reduces to

$$\frac{U}{U_0} = 1 + \frac{1}{\lambda_1 - \lambda_2} \left[\frac{\beta}{\sqrt{\lambda_1}} \operatorname{erfc}(\sqrt{\lambda_1} T) e^{\lambda_1 T} - \frac{\beta}{\sqrt{\lambda_2}} \operatorname{erfc}(\sqrt{\lambda_2} T) e^{\lambda_2 T} \right] \quad (44)$$

In terms of the function, $w(z)$,

$$w(i\sqrt{\lambda} T) = \operatorname{erfc}(\sqrt{\lambda} T) e^{\lambda T} \quad (45)$$

this is

$$\frac{U}{U_0} = 1 + \left[\frac{\beta}{\lambda_1 - \lambda_2} w(i\sqrt{\lambda_1} T) / \sqrt{\lambda_1} - w(i\sqrt{\lambda_2} T) / \sqrt{\lambda_2} \right] \quad (46)$$

Figure 5.11 shows the response of a particle governed by (46) for the case of $\beta = .075$. This value is representative for an aerosol particle. The calculations on which Figure 5.11 is based were made using an algorithm of Gautschi [85, 86] to evaluate $w(z)$.

The special case of equal roots, corresponding to $\beta = 2$ or $M/m = 5/8$, remains to be treated. The result, due to Villat and Kravtchenko [87], can be written in the form

$$\frac{U}{U_0} = 1 - 2\sqrt{T/\pi} + (2T-1) w(i\sqrt{T})$$

5-146

FIG. 5.11 RESPONSE TO A SUDDENLY APPLIED STEADY FORCE

and is presented for completeness.

Equation (46) is not easily compared with the corresponding result, 5.18-(15), from simple aerosol theory because of the disparity in the forms of the answers. For this reason, we may profitably expand the results of our analysis in a form suitable for small times.

The small time expansion is most easily found by expanding the equivalent of (41),

$$\frac{\bar{U}}{U_0} = \frac{1}{s(1 + s + \beta\sqrt{s})}$$

in inverse powers of $\sqrt{s}$ and then inverting the result, term by term. We find

$$\frac{\bar{U}}{U_0} = T - \frac{4}{3}\beta T \sqrt{\frac{T}{\pi}} + \frac{\beta^2 - 1}{2} T^2 + \frac{2\beta(2 - \beta^2)}{5} T^2 \sqrt{\frac{T}{\pi}} + \dots$$

Remembering that

$$T = \frac{9 \mu t}{a^2(2\rho_p + \rho_f)}$$

we see that effect of fluid density is to retard the particle's response.

A striking difference between the predictions of the two types of analysis appears when examining the asymptotic approach to rest of a decelerating particle. Unlike the simple aerosol theory, analysis based on (12) does not yield a finite stopping distance

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PROBLEMS

- 1) Show that the pressure distribution in the fluid surrounding a particle moving according to Stokes law is

$$p = \frac{3\mu a U \cos\theta}{2r^2} + p_{\infty}$$

where p_{∞} is the constant pressure far from the particle.

- 2) Plot several streamlines for Stokes flow about a rigid sphere using the derived expression for the stream function in the frame of the
- a) particle, equation 5.2-(6)
 - b) distant fluid, equation 5.2-(11)

Use dimensionless quantities in your plot. Polar graph paper is suggested. A sketch is not a plot.

- 3) Particles diffuse from a finite point source due to Brownian motion. Show that the maximum concentration observed at a distance r from the source occurs at a time $r^2/6D$ following release.
- 4) Smoluchowski (1915) showed the relation governing combined diffusion and settling in a gravitational field is

$$\frac{\partial n}{\partial t} = D \nabla^2 n - U_s \frac{\partial n}{\partial z}$$

U_s is the terminal settling velocity of a particle falling in the negative z direction. Derive this relation. Show that the Boltzmann distribution is a stationary distribution satisfying this equation.

- 5) A torque applied to a rigid spherical particle will cause it to rotate with a constant angular velocity Γ about an axis through its center. The particle radius is a . The azimuthal velocity distribution in the surrounding fluid is governed by

$$E^2 \Omega = 0$$

Ω is the swirl defined by $\Omega \equiv \rho v_\phi$. No slipping is permitted at the surface.

- a) What assumptions are necessary for the given equation of motion to be valid?
 - b) Write the boundary conditions for Ω and explain them.
 - c) Find the azimuthal velocity distribution in the fluid.
 - d) Relate the applied torque to the particle radius, fluid viscosity and Γ , the resulting angular velocity.
 - e) Determine whether your solution (c) is everywhere consistent with your assumptions.
- 6) Generalize the analysis of Section 5.12 by including the internal fluid viscosity. Determine an expression for the velocity of a bubble in a temperature gradient when the gas viscosity is not neglected.
- 7) In Problem 5, replace the no-slip condition by a slip velocity proportional to the tangential stress. Answer all parts.
- 8) We wish to examine the transient rotational motion of spherical particle. Consider a rigid sphere of radius a rotating about its axis with a transient angular velocity $\Gamma(t)$. The motion is governed by

$$E^2 \Omega = \frac{1}{\nu} \frac{\partial \Omega}{\partial t}$$

where Ω is the swirl. No slipping is permitted at the surface.

- a) What assumptions are necessary for the given equation of motion to be valid?
- b) Write the boundary conditions for Ω and explain them.
- c) Using Laplace transform techniques, derive an expression for the transformed swirl, $\bar{\Omega}$, in terms of the transformed angular velocity of the sphere, $\bar{\Gamma}$, and r , θ , a , ν , and s .
- d) Find the final steady distribution of Ω when the angular velocity Γ becomes constant. Verify that this final steady distribution is correct.

- e) Find an expression for the transformed applied torque, $\bar{T}$, in the transient case. Write your answer in terms of $\bar{T}$, μ , a , s , and ν .
- f) Use the final limit theorem to demonstrate that your answer to part e yields the steady torque in your proper limit.

- 9) We wish to examine the transient motion of a gaseous bubble through the surrounding liquid. The bubble may be assumed to have negligible mass and negligible viscosity. The bubble may be assumed to have negligible mass and negligible viscosity. The bubble surface remains spherical. The tangential stress at the interface is negligible. The motion of the liquid is governed by

$$E^2 (E^2 - \frac{1}{\nu} \frac{\partial}{\partial t}) \psi = 0$$

- a) Write the boundary conditions governing this flow.
 - b) Using Laplace transform techniques, find the transformed stream function and determine any constants.
 - c) Use the final limit theorem to check that the stream function approaches the proper limit when the velocity of the bubble becomes steady.
 - d) Determine the transformed force exerted on the bubble by the liquid.
 - e) Use the final limit theorem to check that the force approaches the proper limit when the velocity of the bubble becomes steady.
- 10) A pair of identical spherical particles falls under the influence of gravity. The line of centers of the particles makes an angle β with the vertical. Find the velocity component.
- a) Along the line of centers.
 - b) Normal to the line of centers.
 - c) In the vertical direction.
 - d) In the horizontal direction.

Express all velocities in terms of the velocity with which a single sphere would fall, the distance between the sphere centers, the radius of the sphere, and the angle β .

APPENDIX A

Some Properties of Water

Temperature T °C	Density ρ g/cm ³	Viscosity $\mu \cdot 10^2$ g/cm sec	Kinematic Viscosity $\nu \cdot 10^2$ cm ² /sec
0	1.000	1.79	1.79
20	.998	1.00	1.00
40	.992	0.65	0.66
60	.983	0.47	0.48
80	.971	0.36	0.37
100	.958	0.28	0.30

Some Properties of Dry Air at one Atmosphere

Temperature T °C	Density $\rho \cdot 10^3$ g/cm ³	Viscosity $\mu \cdot 10^4$ g/cm sec	Kinematic Viscosity $\nu \times 10^1$ cm ² /sec
0	1.293	1.71	1.32
20	1.205	1.81	1.50
40	1.127	1.90	1.69
60	1.060	2.00	1.88
80	1.000	2.09	2.09
100	0.946	2.18	2.30

APPENDIX B

Vector operations in orthogonal curvilinear coordinates are given in Section 3.2. These relations are repeated here for convenience as are certain expressions arising from the analysis of fluid flow

Gradient:

$$\nabla \Omega = \frac{1}{h_1} \frac{\partial \Omega}{\partial u_1} \hat{e}_1 + \frac{1}{h_2} \frac{\partial \Omega}{\partial u_2} \hat{e}_2 + \frac{1}{h_3} \frac{\partial \Omega}{\partial u_3} \hat{e}_3$$

Divergence:

$$\nabla \cdot \bar{A} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} (h_2 h_3 A_1) + \frac{\partial}{\partial u_2} (h_1 h_3 A_2) + \frac{\partial}{\partial u_3} (h_1 h_2 A_3) \right]$$

Curl:

$$\nabla \times \bar{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$

Laplacian:

$$\nabla^2 \Omega = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial \Omega}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left(\frac{h_3 h_1}{h_2} \frac{\partial \Omega}{\partial u_2} \right) + \frac{\partial}{\partial u_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial \Omega}{\partial u_3} \right) \right]$$

Stokes stream function operator, (u_1, u_2, ϕ) coordinates:

$$E^2 \Omega = \frac{\rho}{h_1 h_2} \left[\frac{\partial}{\partial u_1} \left(\frac{h_2}{\rho h_1} \frac{\partial \Omega}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left(\frac{h_1}{\rho h_2} \frac{\partial \Omega}{\partial u_2} \right) \right]$$

Strain rate:

$$e_{11} = \frac{1}{h_1} \frac{\partial v_1}{\partial u_1} + \frac{v_2}{h_1 h_2} \frac{\partial h_1}{\partial u_2} + \frac{v_3}{h_3 h_1} \frac{\partial h_1}{\partial u_3}$$

$$e_{12} = \frac{1}{2} \left[\frac{h_2}{h_1} \frac{\partial}{\partial u_1} \left(\frac{v_2}{h_2} \right) + \frac{h_1}{h_2} \frac{\partial}{\partial u_2} \left(\frac{v_1}{h_1} \right) \right]$$

a.) Cylindrical coordinates (ρ, ϕ, z) or (z, ρ, ϕ) :

See Figure 3.1.b.

$$x = \rho \cos \phi, \quad y = \rho \sin \phi, \quad z = z$$

$$h_\rho = 1, \quad h_\phi = \rho, \quad h_z = 1$$

Gradient

$$\nabla \Omega = \frac{\partial \Omega}{\partial \rho} \hat{e}_\rho + \frac{1}{\rho} \frac{\partial \Omega}{\partial \phi} \hat{e}_\phi + \frac{\partial \Omega}{\partial z} \hat{e}_z$$

Divergence:

$$\nabla \cdot \vec{A} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho A_\rho) + \frac{1}{\rho} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

Curl:

$$\nabla \times \vec{A} = \frac{1}{\rho} \begin{vmatrix} \hat{e}_\rho & \rho \hat{e}_\phi & \hat{e}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}$$

Laplacian:

$$\nabla^2 \Omega = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \Omega}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \Omega}{\partial \phi^2} + \frac{\partial^2 \Omega}{\partial z^2}$$

Stokes stream function operator:

$$E^2 \Omega = \rho \frac{\partial}{\partial \rho} \left(\frac{1}{\rho} \frac{\partial \Omega}{\partial \rho} \right) + \frac{\partial^2 \Omega}{\partial z^2}$$

Strain rate:

$$e_{\rho\rho} = \frac{\partial v_\rho}{\partial \rho}, \quad e_{\phi\phi} = \frac{1}{\rho} \frac{\partial v_\phi}{\partial \phi} + \frac{v_\rho}{\rho}, \quad e_{zz} = \frac{\partial v_z}{\partial z}$$

$$e_{\rho\phi} = e_{\phi\rho} = \frac{1}{2} \left[\rho \frac{\partial}{\partial \rho} \left(\frac{v_\phi}{\rho} \right) + \frac{1}{\rho} \frac{\partial v_\rho}{\partial \phi} \right]$$

$$e_{\rho z} = e_{z\rho} = \frac{1}{2} \left[\frac{\partial v_z}{\partial \rho} + \frac{\partial v_\rho}{\partial z} \right]$$

$$e_{\phi z} = e_{z\phi} = \frac{1}{2} \left[\frac{1}{\rho} \frac{\partial v_z}{\partial \phi} + \frac{\partial v_\phi}{\partial z} \right]$$

.) Spherical coordinates (r , θ , ϕ) :

See Figure 3.1.c.

$$x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta$$

$$\rho = r \sin \theta, \quad \phi = \phi, \quad z = r \cos \theta$$

$$h_r = 1, \quad h_\theta = r, \quad h_\phi = \rho = r \sin \theta$$

Commonly $\cos \theta = \mu$

Gradient:

$$\nabla \Omega = \frac{\partial \Omega}{\partial r} \hat{e}_r + \frac{1}{r} \frac{\partial \Omega}{\partial \theta} \hat{e}_\theta + \frac{1}{r \sin \theta} \frac{\partial \Omega}{\partial \phi} \hat{e}_\phi$$

Divergence:

$$\nabla \cdot \vec{A} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta A_\theta) + \frac{1}{r \sin \theta} \frac{\partial A_\phi}{\partial \phi}$$

Curl:

$$\nabla \times \vec{A} = \frac{1}{r^2 \sin \theta} \begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & r\sin \theta \hat{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin \theta A_\phi \end{vmatrix}$$

Laplacian:

$$\nabla^2 \Omega = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \frac{\partial \Omega}{\partial r}) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial \Omega}{\partial \theta}) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \Omega}{\partial \phi^2}$$

Stokes stream function operator:

$$E^2 \Omega = \frac{\partial^2 \Omega}{\partial r^2} + \frac{\sin \theta}{r^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin \theta} \frac{\partial \Omega}{\partial \theta} \right)$$

Strain rate:

$$e_{rr} = \frac{\partial v_r}{\partial r}, \quad e_{\theta\theta} = \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}, \quad e_{\phi\phi} = \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} + \frac{v_\theta \cot \theta}{r}$$

$$e_{r\theta} = e_{\theta r} = \frac{1}{2} \left[r \frac{\partial}{\partial r} \left(\frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} \right]$$

$$e_{r\phi} = e_{\phi r} = \frac{1}{2} \left[r \frac{\partial}{\partial r} \left(\frac{v_\phi}{r} \right) + \frac{1}{r \sin \theta} \frac{\partial v_r}{\partial \phi} \right]$$

$$e_{\theta\phi} = e_{\phi\theta} = \frac{1}{2} \left[\frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \left(\frac{v_\phi}{\sin \theta} \right) + \frac{1}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} \right]$$

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