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Argonne National Laboratory

**STOCHASTIC PROBLEMS IN
TRANSPORT THEORY**

by

Peter John Brockwell

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Printed in the United States of America

Available from

Clearinghouse for Federal Scientific and Technical Information
National Bureau of Standards, U. S. Department of Commerce
Springfield, Virginia 22151

Price: Printed Copy \$3.00; Microfiche \$0.65

Millie-

ANL-7131
Physics (TID-4500)
AEC Research and
Development Report

ARGONNE NATIONAL LABORATORY
9700 South Cass Avenue
Argonne, Illinois 60439

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Peter John Brockwell

Applied Mathematics Division

This report is a slightly expanded version of the thesis
submitted by the author for the degree of
Doctor of Philosophy
in the Department of Statistics,
The Australian National University
February 1966

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RELEASED FOR ANNOUNCEMENT
IN NUCLEAR SCIENCE ABSTRACTS

August 1966

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Operated by The University of Chicago
under
Contract W-31-109-eng-38
with the
U. S. Atomic Energy Commission

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STOCHASTIC PROBLEMS IN TRANSPORT THEORY

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ABSTRACT

The application of the theory of discontinuous Markov processes and stochastic population processes to problems in the field of transport theory is investigated.

In Chapter I it is shown how a scattering process (in which there may, at each collision, be a constant probability that the scattered particle is absorbed and lost to the system) can be characterized by transition probabilities $P_k(S; \theta_1, \dots, \theta_k | \omega; s)$ of a transition $\omega \rightarrow S$ in distance s with exactly k scatterings, the angles of scattering $\theta_1, \dots, \theta_k$ (in order of occurrence) being such that $\theta_1 \in \Theta_1, \dots, \theta_k \in \Theta_k$. Such a characterization is of importance when there is a change of state of the particle at each collision and the corresponding transition probabilities depend on the angle of deflection.

For one-dimensional processes the problems simplify considerably owing to the fact that the angular deflections θ_i can take only two values, 0 or π . A detailed investigation of one-dimensional problems in which the probability of a collision in a small element of path length δs is $\lambda \delta s + o(\delta s)$ is made in Chapter II. In order to apply the theory to scattering in three dimensions, an approximation is made in which scattered particles are assumed to move in one of a set of 30 directions in space. This model is considered in detail in Chapters VI and VII.

Chapter III deals with a particular one-dimensional problem which arises in the analysis of bubble chamber tracks and is concerned with the stochastic population process generated by the distances travelled between successive collisions in a fixed length of track. In Chapter V we consider one-dimensional processes in which the distances travelled by a particle between successive collisions have either a gamma distribution or a distribution which is a linear combination of negative exponential distributions. Processes of the latter kind can be used to improve the approximation (used by Fermi and others) to the projection on a fixed axis of the three-dimensional motion of a particle which is scattered isotropically.

Chapter IV is concerned with multiplicative scattering processes in one dimension. Several mathematical models for one-dimensional neutron

multiplication in a rod are considered, with particular reference to the concept of criticality and the distribution of the number of emergent neutrons produced by a single trigger neutron.

CHAPTER I

PROBLEMS TO BE CONSIDERED

1. Introduction

The processes by which particles such as neutrons or photons are transmitted and reflected by scattering media are essentially probabilistic in nature so that the theory of stochastic processes is a natural tool for analysis. This has been recognized for some time; analyses from a statistical viewpoint have been given by Sobolev,^{40,41} Ueno,^{43,44} Pal,³⁵ and others. However, the techniques of stochastic process theory have by no means been fully exploited, and it is the purpose of this thesis to develop a theory which, by characterizing the history of a single particle in a very detailed manner, will lead to direct solutions of many of the problems of transport theory. The new formulation is based on the theory of discontinuous Markov processes as given by Moyal.³¹

The early development of the subject was stimulated by the interest of astronomers in the transfer of radiation through stellar atmospheres; however, with the advent of nuclear reactors, a great deal of attention has since been devoted to the passage of neutrons and gamma radiation through matter. The equations of transfer are essentially the same for all these problems, and many solutions can be translated from one field to another. Radiative-transfer theory has also been applied by a number of authors to investigate the radiation from novae and the optical properties of the earth's atmosphere and the sea. Extensive references to the applications are given in the books by Chandrasekhar,¹⁷ Kourganoff,²⁷ and Sobolev.⁴⁰

For astrophysical and geophysical problems, the greatest interest lies in scattering by a medium which is either

- (1) stratified in plane parallel layers

or

- (2) spherically symmetric.

Such systems are also of interest in reactor physics. We shall restrict attention to problems of type (1), observing that in the diffusion approximation (see Tait,⁴² p. 56) there is a close connection between solutions for plane parallel systems and spherically symmetric systems. Thus we shall be concerned exclusively with the scattering of particles by a plane slab $\{x: a \leq x \leq b\}$, where a and b may be infinite.

A general stochastic formulation of the problem will be given in Sects. 2, 3, and 4. The notation is somewhat different from that used in the classical approach to transport theory, which is thoroughly discussed in the books by Chandrasekhar,¹⁷ Kourganoff,²⁷ and Sobolev.⁴⁰ When numerical solutions are obtained, they will, where possible, be compared with solutions already obtained by classical means. Of course, many of our results, in particular those involving the number of collisions and the angular deflections suffered by a particle, are unobtainable from the classical formulation of the problem. Among the problems considered will be those of determining

- (1) the probability that a particle from the source at x ($a \leq x \leq b$) emerges from a particular face of the slab;
- (2) the distribution of the number of collisions before emergence;
- (3) the state distribution of emergent particles;
- (4) the angular distribution of emergent particles.

Exact solutions of these problems are rare. The usual approach is to consider the Boltzmann integro-differential equation for $N(\underline{x}, \underline{\mu}, v, t)$ where $N(\underline{x}, \underline{\mu}, v, t) d\underline{x} d\underline{\mu} dv$ is the expected number of particles in an element $d\underline{x} d\underline{\mu} dv$ of phase space at time t . An explicit solution in the time-independent case was obtained by Wiener and Hopf⁴⁹ for the famous Milne problem of isotropic scattering in a half-space. Chandrasekhar, extending the invariance principles of Ambarzumian,¹ found exact solutions for a more general class of problems in terms of his H-, X-, and Y-functions, defined as solutions of certain integral equations (see Refs. 14, 15).

For most physical problems in transport theory, some kind of approximation is necessary. In a series of papers Chandrasekhar¹⁰⁻¹³ developed a method of approximation which replaced the integral in the Boltzmann equation by a sum using Gaussian quadrature. This idea was first put forward by Wick.⁴⁸ A second kind of approximation, in which the solution of Boltzmann's equation was expanded in spherical harmonics and the series truncated, was also used by Chandrasekhar⁹ and later by Marshak,³⁰ Mark,²⁹ and others, in neutron transport theory. The close relationship between the Gaussian quadrature and spherical harmonic methods was demonstrated by Wang and Guth.⁴⁵ Further developments of the spherical harmonic method have been made by Yvon⁵³ and by Wilson and Sen.⁵⁰ The method of successive collisions, in which the particle density is replaced by the density of once, twice, ..., and n times scattered particles, has been found useful for scattering by thin slabs (see Peebles and Plesset³⁶). Monte Carlo methods have also been used extensively; these are particularly useful for problems with complicated geometry. The method of Carlson,⁸ which is well suited to large-scale computers, again reduces the integral in the Boltzmann equation to a sum by introducing a finite set of permissible neutron velocities and by assuming a particular functional form for the variation of neutron flux with direction.

Accounts of the various methods at present available for dealing with problems in neutron transport theory are given in the books by Davison,¹⁹ Weinberg and Wigner,⁴⁶ Tait,⁴² and Woods.⁵² The diffusion of gamma radiation is considered in detail in the book by Leipunskii, Novozhilov, and Sakharov,²⁸ and the article of Fano, Spencer, and Berger.²⁰ Extensive numerical calculations of the reflection functions for radiation scattered isotropically by slabs of finite thickness have been carried out by Bellman, Kalaba, and Prestrud.⁶ There appear, however, to be no results available at present pertaining to the distribution of the number of collisions made by a photon before emergence.* This question is studied in detail later in this thesis.

The fundamental equations for a general scattering process will be derived in Sect. IV. It will be noticed that this new formulation in terms of probabilities and generating functionals contains a great deal more information about the history of individual particles than does the usual formulation in terms of the function $N(x, \mu, v, t)$ [or to use the terminology of radiative transfer, in terms of the radiation field intensity $I_\nu(x, \mu, t)$]. In particular, information is available regarding the distribution of the number of collisions suffered by a particle and also the angular deflections resulting from these collisions. This more detailed information leads to a new and direct solution of problems in which there is a change of state of the particle at each collision, the corresponding transition probabilities depending on the angle of deflection. One such example occurs when low-energy gamma radiation undergoes multiple Compton scattering. This problem is examined in Chapter VII, where a new analysis is given which eliminates certain objections to the previous treatments of the problem by Chandrasekhar¹⁶ and O'Rourke.³⁴

Several chapters are devoted to one-dimensional processes. Complete and explicit solutions are obtained for the one-dimensional problem in Chapter II. Some of these results (the probabilities of emergence at boundaries without regard to the number of collisions experienced before emergence) were obtained by Fermi,²¹ who used this process to approximate the projection on the x -axis of a three-dimensional scattering process in the slab $\{x: a \leq x \leq b\}$. All the other results of Chapter II, except where specifically mentioned, are believed to be new. The projected motion is of considerable interest because it can be used to find certain quantities such as the probabilities of emergence and the distribution of the number of collisions before emergence for the corresponding three-dimensional process. It is of interest to observe that two of the earliest workers in the field of radiative transfer, Schuster³⁸ and Schwarzschild,³⁹ used a method which is equivalent to the one-dimensional method of Fermi. Effectively they assumed that the projections on the x -axis of the distances

*More recently a study of the moments of the total number of collisions experienced by a particle before emerging from a slab has been made by I. K. Abu-Shumays.⁵⁷ (See Chapter VII).

travelled by a photon between collisions are independent and have a negative exponential distribution with mean equal to half the mean free path in three dimensions. Fermi made the same assumptions except for the value of the mean projected interval between collisions, which he chose as $1/\sqrt{3}$ times the mean free path in three dimensions. A far better approximation to the exact distribution of projected intervals than either of the two above can be found by using a linear combination of negative exponential distributions. Improved results with this new one-dimensional approximation are derived in Chapter V.

Chapter III deals with a special case of the one-dimensional process of Chapter II which is of interest in the analysis of bubble chamber experiments. Aspects of the process are examined which make possible the estimation of the velocity of a particle by counting the number of distances between bubbles which exceed a fixed value d .

Chapter IV treats several models for neutron multiplication in a rod, with particular reference to the distribution of the total number of emergent particles produced by a single trigger neutron. Models of this kind have been considered previously by Bellman, Kalaba, and Wing,^{3,4,51} Harris,⁵⁸ and Mullikin and Snow.⁶⁰

Chapters VI and VII deal with the application of the ideas of Sect. 4 to three-dimensional processes with scattering and absorption in which the angular, spatial, and state distributions of the scattered particles are required. Special attention is given to the multiple Compton scattering of low-energy gamma radiation.

2. Scattering Processes

Consider a particle whose state ω is specified by its position vector $\underline{x}$, its direction of motion $\underline{\mu}$, and a further state variable v , which will usually be referred to as velocity (although it could equally well, as in the case of photon scattering, refer to frequency or wavelength); ω stands for the set of variables $(\underline{x}, \underline{\mu}, v)$. Further internal state variables such as rotations or vibrations could be included in ω by an obvious generalization of the methods of this thesis.

A scattering process is a stochastic process characterized by the properties that:

(a) Only the position of the scattered particle changes between collisions which the particle makes with the scattering centers in the medium.

(b) The probability that a particle makes a collision in any element δs of path length is given by $\lambda(\omega)\delta s + o(\delta s)$, where $\lambda(\omega)$ depends only on the instantaneous state, ω , of the particle.

(c) The change of state of the particle at a collision is instantaneous.

(d) The probability (given a collision) of a transition $(v, \underline{\mu}) \rightarrow V \times L$ (where V is a set of velocities and L is a set of unit vectors) is given by a transition distribution $\Phi(V \times L | v, \underline{\mu})$ dependent only on the velocity v , and direction of motion, $\underline{\mu}$, immediately prior to the collision and independent of the previous values of the state variables. For many of the problems of scattering theory, the transition probabilities Φ take the form

$$\Phi(d\mathbf{u} \times d\underline{\nu} | v, \underline{\mu}) = \phi_{\theta}(d\mathbf{u} | v) \psi_v(d\underline{\nu} | \underline{\mu}), \quad (2.1)$$

where the probability $\psi_v(d\underline{\nu} | \underline{\mu})$ depends only on the initial velocity v , and the angle θ between $\underline{\mu}$ and $\underline{\nu}$ while the transition probabilities $\phi_{\theta}(d\mathbf{u} | v)$ for the velocities depend only on u , v , and θ . It will generally be assumed in what follows that Φ is of the form (2.1) and furthermore that the dependence of $\psi_v(d\underline{\nu} | \underline{\mu})$ on v can be neglected.

(e) The change of position of the particle between collisions is deterministic: if a particle initially at $\underline{x}$ with direction of motion $\underline{\mu}$ suffers no collisions in a length of path s , its position will then be $\underline{x} + \underline{\mu}s$.

With these assumptions it is clear that a scattering process is a time-homogeneous discontinuous Markov process of the type whose theory is given in Moyal.³¹ The independent variable of the process may be taken as either the path length s , or the time t ; the connection between these two formulations following from the relation $ds = v dt$. Thus, if we use $\lambda(\omega)$ to denote the collision rate per unit path length, then the collision rate per unit time is $v\lambda(\omega)$.

In general, it will be assumed (unless otherwise specified) that the collision rate λ is a function of $\underline{x}$ only. The absence of dependence on v corresponds to the one-velocity group theory of neutron transport processes (see Davison¹⁹); by postulating a finite set of permissible velocities as in multigroup neutron transport theory, this limitation could be overcome to some extent. In many cases, however, the assumption that λ is independent of v is a reasonable one (see, for example, Chapter VII).

3. Note on Absorption and Multiplication

In the most general transport processes a collision may have several possible outcomes. The particle may be absorbed and lost to the system, or may, as in neutron multiplication processes, give rise to several new particles of the same type.

Except where specifically indicated, this thesis will be concerned with pure scattering processes, sometimes known as conservative processes,

in which the probability, p_s , that a particle is scattered when it makes a collision is unity, i.e., the integral over all directions $\underline{\nu}$,

$$p_s = \int_{\Omega} \phi_{\nu}(\underline{\nu} | \underline{\mu}) = 1. \quad (3.1)$$

The generalization to processes in which, at each collision, absorption takes place with probability

$$p_a = 1 - p_s, \quad (3.2)$$

is very simple and will be indicated in the relevant parts of the text.

Some attention will be given to multiplicative processes in one dimension and to the question of criticality for some simple models. This will be discussed in Chapter IV.

4 Characterization of a Scattering Process

Considered as a Markov process in the state variable ω , a scattering process may be characterized by the transition probability function $P(S|\omega; s)$, i.e., the probability of a transition $\omega \rightarrow S$ in path length s , satisfying the Chapman-Kolmogorov relation

$$P(S|\omega; s_1 + s_2) = \int_S P(S|\eta; s_1) P(d\eta|\omega; s_2); \quad s_1, s_2 \geq 0. \quad (4.1)$$

However, we shall seek a more detailed specification provided by the probabilities $P_k(S; \theta_1, \theta_2, \dots, \theta_k | \omega; s)$ of a transition $\omega \rightarrow S$ in s with exactly k scatterings, the angles of scattering $\theta_1, \theta_2, \dots, \theta_k$ (in order of occurrence) being such that $\theta_1 \in \Theta_1, \dots, \theta_k \in \Theta_k$. In terms of these functions the Chapman-Kolmogorov relation becomes

$$P_k(S; \theta_1, \dots, \theta_k | \omega; s_1 + s_2) = \sum_{j=0}^k \int_S P_{k-j}(S; \theta_{j+1}, \dots, \theta_k | \eta; s_1) P_j(d\eta; \theta_1, \dots, \theta_j | \omega; s_2), \quad (4.2)$$

where S denotes the set of all possible states ω . We also introduce the sequence of probability distributions $Q_k(S; \theta_1, \dots, \theta_k | \omega; s)$, where Q_k is the probability that the k^{th} collision occurs in $[0, s)$, the angular deflections being such that $\theta_1 \in \Theta_1, \dots, \theta_k \in \Theta_k$, and the state of the particle immediately after the k^{th} collision belonging to the set S .

The relation between the probabilities P_k and Q_k is

$$P_k(S; \theta_1, \dots, \theta_k | \omega; s) = \int_{\tau=0}^s \int_{\eta \in S} P_0(S|\eta; s-\tau) Q_k(d\eta; \theta_1, \dots, \theta_k | \omega; \tau), \quad (4.3)$$

where conventionally

$$Q_0(S|\omega; s) = \delta(S|\omega) \epsilon(s) = I(S|\omega; s); \quad (4.4)$$

in equation (4.4), $\delta(S|\omega)$ denotes the characteristic function of the set S , i.e., $\delta(S|\omega) = 1$ if $\omega \in S$, 0 otherwise; $\epsilon(s)$ is the unit step function, i.e., $\epsilon(s) = 1$ if $s > 0$, 0 otherwise. The probabilities Q_k are themselves connected by the recurrence relations

$$Q_k(S; \theta_1, \dots, \theta_k | \omega; s) = \int_{\tau=0}^s \int_{\eta \in S} Q_1(S; \theta_k | \eta; s - \tau) Q_{k-1}(d\eta; \theta_1, \dots, \theta_{k-1} | \omega; d\tau). \quad (4.5)$$

Thus, given the functions P_0 and Q_1 , the relations (4.3) and (4.5) determine all the probabilities P_k and Q_k .

For any given S and ω , the angular deflections $\theta_1, \dots, \theta_k$ which the particle experiences in travelling a distance s constitute a finite univariate population process as defined by Moyal.³² Corresponding to k angular deflections $\theta_1, \dots, \theta_k$ we have k individuals with states $\theta_1, \dots, \theta_k$, each belonging to the individual state space $[0, \pi]$. The generating functional for this population process, i.e., the expectation $E\xi(\theta_1), \dots, \xi(\theta_k)$, is given by $G(S; \xi(\cdot) | \omega; s) P(S|\omega; s)^{-1}$, where

$$G(S; \xi(\cdot) | \omega; s) = P_0(S|\omega; s) + \sum_{k=1}^{\infty} \int_{\theta_k=0}^{\pi} \int_{\theta_1=0}^{\pi} \xi(\theta_1) \dots \xi(\theta_k) P_k(S; d\theta_1, \dots, d\theta_k | \omega; s) \quad (4.6)$$

and $\xi(\theta)$ is a complex-valued measurable function of θ such that $\sup_{0 \leq \theta \leq \pi} |\xi(\theta)| \leq 1$. It is convenient also to introduce the functional

$$H(S; \xi(\cdot) | \omega; s) = \sum_{k=1}^{\infty} \int_{\theta_k=0}^{\pi} \int_{\theta_1=0}^{\pi} \xi(\theta_1) \dots \xi(\theta_k) Q_k(S; d\theta_1, \dots, d\theta_k | \omega; s), \quad (4.7)$$

where $\xi(\theta)$ is a complex-valued measurable function of θ such that $\sup_{0 \leq \theta \leq \pi} |\xi(\theta)| < 1$. The transition probability $P(S|\omega; s)$ is obtained by setting

$$\xi(\theta) = 1, \quad 0 \leq \theta \leq \pi,$$

in G .

The concept of population process will again be utilized in Chapter III when we consider a one-dimensional scattering process in which distances between successive collisions take the place of angular deflections as the individuals of the population. This will be discussed more fully in Chapter III.

It follows from equation (4.3) that

$$G = P_0 * (I + H) = P_0 + P_0 * H, \quad (4.8)$$

where the symbol $*$ denotes the type of composition appearing in the right-hand side of (4.3). If we let

$$H_1 = \int_{\theta_1=0}^{\pi} \xi(\theta_1) Q_1(S; d\theta_1 | \omega; s),$$

defining the sequence $\{H_n\}$ by the iteration relation

$$H_{n+1} = H_n * H_1, \quad (4.9)$$

and the sequence $\{G_n\}$ by

$$G_n = P_0 * H_n, \quad (4.10)$$

then it is easily seen to follow from (4.3) and (4.5) that

$$H_n(S; \xi(\cdot) | \omega; s) = \int_{\theta_n=0}^{\pi} \dots \int_{\theta_1=0}^{\pi} \xi(\theta_1) \dots \xi(\theta_n) Q_n(S; d\theta_1, \dots, d\theta_n | \omega; s), \quad (4.11)$$

and

$$G_n(S; \xi(\cdot) | \omega; s) = \int_{\theta_n=0}^{\pi} \int_{\theta_1=0}^{\pi} \xi(\theta_1) \dots \xi(\theta_n) P_n(S; d\theta_1, \dots, d\theta_n | \omega; s), \quad (4.12)$$

and hence (setting $G_0 = P_0$) that

$$H = \sum_{n=1}^{\infty} H_n \quad \text{and} \quad G = \sum_{n=0}^{\infty} G_n. \quad (4.13)$$

It is now easily seen that G and H must satisfy the fundamental integral equations

$$G = P_0 + G * H_1; \quad (4.14)$$

$$H = H_1 + H * H_1. \quad (4.15)$$

The major part of this thesis will be concerned with finding solutions of the fundamental equations (4.14) and (4.15).

The theory of discontinuous Markov processes in Moyal³¹ is easily generalized to show that the expressions (4.6) and (4.7) for G and H are, respectively, solutions of the integral equations (4.14) and (4.15). These solutions are unique under certain not very stringent conditions, which are fulfilled in the present case if the collision rate $\lambda(\underline{\omega})$ is a bounded function of $\underline{\omega}$ (this will be the case throughout this thesis). All transition probabilities obtained from G satisfy the relevant Chapman-Kolmogorov relation.

The foregoing theory applies to general scattering processes which are homogeneous in path length (or time). It remains applicable under the imposition of a variety of boundary conditions, such as absorbing or reflecting boundaries in space. The effect of such boundary conditions is incorporated in the integral equations (4.14) and (4.15) via the definition of the basic distributions P_0 and Q_1 and of the $*$ -operation. It then follows that the distributions P_k and Q_k obtained by $*$ -iterations from P_0 and Q_1 , and hence the generating functionals G and H , automatically satisfy the boundary conditions.

5. The Backward Kolmogorov Equations

We now specialize the theory to processes which have the following properties:

(1) The collision rate $\lambda(\underline{\omega})$ per unit path length depends only on the position of the particle and is written as $\lambda(\underline{x})$.

(2) Given a collision, the probability of a transition $\underline{\mu} \rightarrow d\underline{\nu}$ (where $d\underline{\nu}$ denotes an element of solid angle surrounding the unit vector $\underline{\nu}$) is given by $\phi(d\underline{\nu}|\underline{\mu})$.

(3) If $\psi_\theta(V|\underline{v})$ is the probability of a velocity transition $\underline{v} \rightarrow V$ conditional upon an angular deflection θ , then the function ψ_θ satisfies the commutativity condition

$$\int_0^\infty \psi_{\theta_2}(V|u) \psi_{\theta_1}(du|\underline{v}) = \int_0^\infty \psi_{\theta_1}(V|u) \psi_{\theta_2}(du|\underline{v})$$

or, symbolically,

$$\psi_{\theta_2} \circ \psi_{\theta_1} = \psi_{\theta_1} \circ \psi_{\theta_2}.$$

We introduce the notation $\psi_{\theta}^{(n)}$ for the sequence of distributions defined by the iteration relation

$$\psi_{\theta}^{(n)} = \psi_{\theta} \circ \psi_{\theta}^{(n-1)}.$$

With these postulates the theory is simplified considerably. Postulate (3) implies that the (absolute) velocity distribution conditional on deflections $\theta_1, \dots, \theta_k$ is $\psi_{\theta_1} \circ \psi_{\theta_2} \circ \dots \circ \psi_{\theta_k}$ independently of the order in which the deflections occur. In addition the assumptions (1) and (2) imply that the process is still Markovian, and the theory of Sect. 4 applies if we omit v and take the state of the particle to be $\underline{\omega} = (\underline{\mu}, \underline{x})$. Let P, Q, G, H , etc., be the various functions defined in Sect. 4 in terms of $\underline{\omega} = (\underline{\mu}, \underline{x})$, and let P', Q', G', H' , etc., be the corresponding functions defined in terms of $\underline{\omega}' = (v, \underline{\mu}, \underline{x})$. Then

$$G'(V \times S; \xi(\cdot) | \underline{\omega}'; s) = P_0(S | \underline{\omega}; s) \delta(V | v) + \sum_{k=1}^{\infty} \int_{\theta_k=0}^{\pi} \dots \int_{\theta_1=0}^{\pi} (\psi_{\theta_1} \circ \dots \circ \psi_{\theta_k}) \xi(\theta_1) \dots \xi(\theta_k) P_k(S; d\theta_1, \dots, d\theta_k | \underline{\omega}; s); \quad (5.1)$$

$$H'(V \times S; \xi(\cdot) | \underline{\omega}'; s) = \sum_{k=1}^{\infty} \int_{\theta_k=0}^{\pi} \dots \int_{\theta_1=0}^{\pi} (\psi_{\theta_1} \circ \dots \circ \psi_{\theta_k}) \xi(\theta_1) \dots \xi(\theta_k) Q_k(S; d\theta_1, \dots, d\theta_k | \underline{\omega}; s). \quad (5.2)$$

Thus we can develop the theory initially in terms of P, Q, G, H , etc. (i.e., in terms of $\underline{\mu}$ and $\underline{x}$ only), introducing the dependence on v later.

Consider now the unrestricted process (i.e., the process with no boundaries). The probability of no collisions in s is $\exp \left[- \int_0^s \lambda(\underline{x} + \underline{\mu} \tau) d\tau \right]$; with no collisions the position of the particle initially at $\underline{x}$ becomes $\underline{x} + \underline{\mu} s$. Hence,

$$P_0(X, L | \underline{x}, \underline{\mu}; s) = \delta(X | \underline{x} + \underline{\mu} s) \delta(L | \underline{\mu}) \exp \left[- \int_0^s \lambda(\underline{x} + \underline{\mu} \tau) d\tau \right]; \quad (5.3)$$

$$H_1(X, L; \xi(\cdot) | \underline{x}, \underline{\mu}; s) = \int_L \xi(\theta) \phi(d\underline{\nu} | \underline{\mu}) \int_0^s \delta(X | \underline{x} + \underline{\mu} \tau) \exp \left[- \int_0^{\tau} \lambda(\underline{x} + \underline{\mu} \sigma) d\sigma \right] \lambda(\underline{x} + \underline{\mu} \tau) d\tau, \quad (5.4)$$

where θ is the angle between the vectors $\underline{\mu}$ and $\underline{\nu}$, and L is a subset of the space of all $\underline{\nu}$.

Substituting (5.3) and (5.4) in (4.15), we obtain the "backward" equations for G and H , namely,

$$\left[\underline{\mu} \cdot \nabla - \frac{\partial}{\partial s} - \lambda(\underline{x}) \right] G(\underline{x}, \underline{\mu}; s) = -\lambda(\underline{x}) \int_{\Omega} \xi(\theta) G(\underline{x}, \underline{\nu}; s) \phi(d\underline{\nu} | \underline{\mu}); \quad (5.5)$$

$$\left[\underline{\mu} \cdot \nabla - \frac{\partial}{\partial s} - \lambda(\underline{x}) \right] H(\underline{x}, \underline{\mu}; s) = -\lambda(\underline{x}) \int_{\Omega} \xi(\theta)$$

$$[H(\underline{x}, \underline{\nu}; s) + \delta(X | \underline{x}) \delta(L | \underline{\nu})] \phi(d\underline{\nu} | \underline{\mu}), \quad (5.6)$$

where the arguments X , L of G and H have been suppressed, and Ω denotes the space of all $\underline{\nu}$.

Consider now processes whose motion is restricted by the imposition of boundary conditions such as represented by absorbing or reflecting barriers. For small s , the generating functionals G and H for the restricted process will differ only by terms of order $o(s)$ from the corresponding functionals for the unrestricted process. Hence G and H satisfy the same "backward" equations (5.5) and (5.6). Thus, contrary to what is the case for purely discontinuous Markov processes (cf. Moyal³¹), the "backward" equations do not have a unique solution and are not equivalent to the basic integral equations of the process. However, solutions of (5.5) and (5.6) which satisfy the required boundary conditions will also satisfy the integral equations appropriate to the process with these boundary conditions and therefore give the unique solution of the problem.

CHAPTER II

EXACT SOLUTIONS OF ONE-DIMENSIONAL SCATTERING PROBLEMS

1. Introduction

One-dimensional processes of the type described in Chapter I have been used in radiative transfer theory by Schuster³⁸ and Schwarzschild,³⁹ and in neutron transport theory by Fermi²¹ and Placzek.³⁷ A series of papers by Bellman, Kalaba, and Wing (see Ref. 4 and the references given there) has been devoted to the application of invariance principles to a variety of problems in neutron transport theory and in particular to one-dimensional fission processes. We shall consider fission processes in Chapter IV.

A different type of one-dimensional scattering process in which particles travel a fixed distance $v\tau$ between collisions, and for which the direction of motion is reversed at a collision with probability p and maintained with probability q , has been investigated by Goldstein.²⁴ He determined the characteristic function of the displacement $\nu v\tau$ of a particle after n collisions, and by letting $\tau \rightarrow 0$, $n \rightarrow \infty$ in such a way that $n\tau = t$, $\nu v\tau = y$, found that the limiting distribution of particles on $(-\infty, \infty)$ obeyed the telegraph equation. Generalizations of Goldstein's model have been investigated by Gupta²⁵ and Gillis,²³ and a further generalization will be derived in Chapter V, where it is used to improve on Fermi's analysis of the scattering of thermal neutrons by paraffin.

Mycielski and Paszowski³³ studied the behaviour of particles on a line, which move in such a way that, for any segment of measure x , there are probabilities $p_l(x)$, $q_l(x)$, and $r_l(x)$ of reflection, transmission, and absorption for leftward moving particles, and corresponding probabilities $p_r(x)$, $q_r(x)$, and $r_r(x)$ for particles moving to the right. They showed by invariance arguments that these functions must be of certain restricted types. We shall see later in this chapter (Sect. 6) that our results are consistent with theirs.

We consider a particle which is restricted to move along the x -axis in the manner described in Chapter I, Sect. 2. In the one-dimensional case, the direction of motion, μ , clearly has only two possible values which we shall denote by a scalar $\tilde{\mu}$ having the values $+1$ for motion to the right and -1 for motion to the left. It will be assumed that:

- (1) The collision rate λ per unit path length is constant.
- (2) Given the occurrence of a collision, the probability of a reversal is a constant, p , and that of a continuation is $q = 1 - p$: thus if we

denote by $\phi(\nu|\mu)$ the probability of a change of direction $\mu \rightarrow \nu$ at a collision, then $\phi(-\mu|\mu) = p$ and $\phi(\mu|\mu) = q$.

(3) If $\psi_0(V|v)$ and $\psi_\pi(V|v)$ are the probabilities of a transition $v \rightarrow V$ at a collision conditional, respectively, on a continuation (deflection through 0) or reversal (deflection through π), then ψ_0 and ψ_π satisfy the commutativity condition of Chapter I, Sect. 5, i.e.,

$$\int_0^\infty \psi_\pi(V|u) \psi_0(du|v) = \int_0^\infty \psi_0(V|u) \psi_\pi(du|v),$$

or, symbolically,

$$\psi_\pi \circ \psi_0 = \psi_0 \circ \psi_\pi.$$

We introduce the notation $\psi_\pi^{(n)}, \psi_0^{(n)}$ for the sequences of distributions defined, respectively, by the iteration relations $\psi_\pi^{(n)} = \psi_\pi^{(n-1)} \circ \psi_\pi$ and $\psi_0^{(n)} = \psi_0^{(n-1)} \circ \psi_0$.

With these assumptions we shall show that it is possible to obtain explicit solutions of the problems connected with one-dimensional scattering. The extension of these results to processes with constant nonzero absorption probability per collision is relatively simple and is considered in Sect. 9.

2. Integral Equations for the One-dimensional Process

The fundamental integral equations of Chapter I, Sect. 4, are simplified in one dimension since there are only two possible angular deflections, 0 and π , corresponding to continuations and reversals, respectively. The generating functionals G and H of Chapter I are therefore replaced by the generating functions

$$G(S; z, w | \underline{\omega}; s) = \sum_{r=0}^{\infty} \sum_{k=0}^{\infty} z^r w^k P_{rk}(S | \underline{\omega}; s); \quad (2.1)$$

$$H(S; z, w | \underline{\omega}; s) = \sum_{r=0}^{\infty} \sum_{k=0}^{\infty} z^r w^k Q_{rk}(S | \underline{\omega}; s) - I(S | \underline{\omega}; s), \quad (2.2)$$

where $P_{rk}(S | \underline{\omega}; s)$ is the probability of a transition $\underline{\omega} \rightarrow S$ in distance s involving exactly r reversals and k continuations; similarly, $Q_{rk}(S | \underline{\omega}; s)$ is the probability that the $(r+k)^{\text{th}}$ collision with r reversals and k continuations

occurs in s leaving the particle immediately after the collision in a state belonging to the set S . Clearly G_n and H_n are now given by

$$G_n = \sum_{j=0}^n z^{n-j} w^j P_{n-j,j}; \quad (2.3)$$

$$H_n = \sum_{j=0}^n z^{n-j} w^j Q_{n-j,j}. \quad (2.4)$$

As in Chapter I, Sect. 4, the probabilities P_{rk} and Q_{rk} are related by the equation

$$P_{rk}(S|\underline{\omega}; s) = \int_{\tau=0}^s \int_{\underline{\eta} \in S} P_{00}(S|\underline{\eta}; s-\tau) Q_{rk}(d\underline{\eta}|\underline{\omega}; d\tau); \quad r, k = 0, 1, 2, \dots, \quad (2.5)$$

where, as before, S is the set of all states and

$$Q_{00}(S|\underline{\omega}; s) = \delta(S|\underline{\omega}) \epsilon(s) = I(S|\underline{\omega}; s).$$

If the type of composition in the right-hand side of (2.5) is denoted symbolically as $P_{00} * Q_{rk}$, it follows that

$$G = P_0 * (I + H) = P_0 + P_0 * H, \quad (2.6)$$

$$H_{n+1} = H_n * H_1, \quad (2.7)$$

and

$$G_n = P_0 * H_n. \quad (2.8)$$

With this definition of the $*$ -operation, the fundamental integral equations (4.14) and (4.15) of Chapter I remain the same, i.e.,

$$G = P_0 + G * H_1; \quad (2.9)$$

$$H = H_1 + H * H_1. \quad (2.10)$$

3. The "Backward" Equations

Under the postulates 1), 2), and 3) of Sect. 1, the theory simplifies as already indicated in Chapter I, Sect. 5. First the (absolute) velocity distribution conditional on exactly r reversals and k continuations is $\psi_{\pi}^{(r)} \circ \psi_0^{(k)}$ independently of their order. It also follows that the process

is still Markovian, and the theory of Sect. 2 still applies if we omit v and take the state of the particle to be $\varpi = (\mu, x)$. Let P_{rk} , Q_{rk} , G , H , etc., be the various functions defined in Sect. 2 in terms of $\varpi = (\mu, x)$, and let P'_{rk} , Q'_{rk} , G' , H' , etc., be the corresponding functions in terms of $\varpi = (v, \mu, x)$. Then P'_{rk} and Q'_{rk} split into products of independent distributions:

$$P'_{rk} = P_{rk} \times \left(\psi_{\pi}^{(r)} \circ \psi_0^{(k)} \right); \quad Q'_{rk} = Q_{rk} \times \left(\psi_{\pi}^{(r)} \circ \psi_0^{(k)} \right), \quad (3.1)$$

and hence

$$\begin{aligned} G' &= \sum_{r=0}^{\infty} \sum_{k=0}^{\infty} z^r w^k P_{rk} \times \left(\psi_{\pi}^{(r)} \circ \psi_0^{(k)} \right); \\ H' &= \sum_{r=0}^{\infty} \sum_{k=0}^{\infty} z^r w^k Q_{rk} \times \left(\psi_{\pi}^{(r)} \circ \psi_0^{(k)} \right) - I. \end{aligned} \quad (3.2)$$

Thus we can develop the theory initially in terms of μ and x only, and introduce the dependence on v later. Secondly, we can omit the final direction v from $P_{rk}(X, v | x, \mu; s)$, $Q_{rk}(X, v | x, \mu; s)$, etc., where X denotes a (measurable) subset of the real line, because $v = \mu$ if r is even and $v = -\mu$ if r is odd (i.e., $P_{2r,k}$, $Q_{2r,k}$ vanish if $v = -\mu$, and $P_{2r+1,k}$, $Q_{2r+1,k}$ vanish if $v = \mu$). It will be convenient to express the initial direction μ as a superscript: thus, $P_{rk}^{(\mu)}$, $Q_{rk}^{(\mu)}$, $G^{(\mu)}$, $H^{(\mu)}$, etc.

Consider now the unrestricted process. With constant λ , the probability of no collisions in s is $\exp[-\lambda s]$; with no collisions, the position of the particle initially at x becomes $x + \mu s$. Hence

$$\left. \begin{aligned} P_0^{(\mu)}(X | x; s) &= \delta(X | x + \mu s) \exp[-\lambda s]; \\ H_1^{(\mu)}(X | x; s; z, w) &= (pz + qw) \int_0^s \delta(X | x + \mu \tau) \exp[-\lambda \tau] \lambda d\tau = \\ &= (pz + qw) \int_0^s P_0^{(\mu)}(X | x; \tau) \lambda d\tau, \end{aligned} \right\} (3.3)$$

and therefore

$$\frac{\partial H_1^{(\mu)}}{\partial s} = \lambda (pz + qw) P_0^{(\mu)}.$$

This last relation together with (2.7) implies that

$$\partial H_{n+1}^{(\mu)} / \partial s = \lambda(pz + qw) G_n^{(\mu)},$$

and hence that

$$\frac{\partial H^{(\mu)}}{\partial s} = \lambda(pz + qw) G^{(\mu)}. \quad (3.4)$$

Thus, one need only calculate one of the generating functions G or H . It turns out that H is the more important for our purpose, particularly in the investigation of limiting distributions as $s \rightarrow \infty$; for this reason we concentrate on H .

With only two possible directions of motion, the backward equation for H [see Chapter 1, equation (5.6)] simplifies immediately to the equation

$$\left[\frac{\partial}{\partial s} - \mu \frac{\partial}{\partial x} + \lambda(1 - qw) \right] H^{(\mu)} = \lambda pz H^{(-\mu)} + \lambda(pz + qw) \delta(X|x), \quad (3.5)$$

with initial conditions which from (3.3) are $[H^{(\mu)}]_{s=0} = 0$, $\mu = \pm 1$. By introducing the Laplace transform

$$\overline{H^{(\mu)}} = \int_0^\infty \exp[-\sigma s] H^{(\mu)} ds,$$

equation (3.5) becomes

$$\left[\sigma + \lambda(1 - qw) - \mu \frac{\partial}{\partial x} \right] \overline{H^{(\mu)}} = \lambda pz \overline{H^{(-\mu)}} + \frac{\lambda}{\sigma} (pz + qw) \delta(X|x). \quad (3.6)$$

From (3.6) we obtain the 2nd-order equation

$$\left(\frac{\partial^2}{\partial x^2} - \theta^2 \right) \overline{H^{(\mu)}} = -\frac{\lambda}{\sigma} (pz + qw) [\sigma + \lambda(1 - qw + pz)] \delta(X|x), \quad (3.7)$$

where

$$\theta^2 = [\sigma + \lambda(1 - qw + pz)][\sigma + \lambda(1 - qw - pz)], \quad (3.8)$$

which is the transform of a nonhomogeneous telegraphist's equation with leakage; an equation of the same type was obtained by Goldstein.²⁴ Note that because of (3.4), $G^{(\mu)}$ satisfies the homogeneous part of (3.5), and

hence the homogeneous telegraphist's equation

$$\left[\frac{\partial^2}{\partial s^2} - \frac{\partial^2}{\partial x^2} + 2\lambda(1 - qw) \frac{\partial}{\partial s} + \lambda^2(1 - qw + pz)(1 - qw - pz) \right] G^{(\mu)} = 0. \quad (3.9)$$

It also follows from (3.4) that

$$\lambda(pz + qw) \overline{G^{(\mu)}} = \lambda(pz + qw) \int_0^\infty \exp[-\sigma s] G^{(\mu)} ds = \int_0^\infty \exp[-\sigma s] \frac{\partial H^{(\mu)}}{\partial s} ds = \overline{\sigma H^{(\mu)}}. \quad (3.10)$$

If the motion is restricted by the imposition of boundary conditions such as represented by absorbing or reflecting barriers, for example if there are two such barriers at a_{-1} , a_1 ($a_{-1} < a_1$), then for an initial position $x \in [a_{-1}, a_1]$, one can see that for small s the generating function H appropriate to this process will differ only by terms of order $o(s)$ from the one appropriate to the unrestricted process. Hence H will satisfy the same "backward" equation (3.5). The solution of (3.5) which satisfies the required boundary conditions will also satisfy the integral equation appropriate to the process with these boundary conditions, and is hence the unique solution of the problem.

The general solution of (3.6) or (3.7) is

$$\overline{H^{(\mu)}} = \begin{cases} A_{11}^{(\mu)} \exp[\theta x] + A_{12}^{(\mu)} \exp[-\theta x], & x \notin X; \\ A_{21}^{(\mu)} \exp[\theta x] + A_{22}^{(\mu)} \exp[-\theta x] + \frac{\lambda}{\sigma} (pz + qw) \\ \quad \cdot [\sigma + \lambda(1 - qw - pz)]^{-1}, & x \in X, \end{cases} \quad (3.11)$$

where the coefficients $A_{ij}^{(\mu)}$ are related via equation (3.6) by

$$\lambda pz A_{ij}^{(-1)} = [\sigma + \lambda(1 - qw) + (-1)^j \theta] A_{ij}^{(-1)}, \quad i, j = 1, 2, \quad (3.12)$$

but are otherwise arbitrary functions of σ , z , and w . The solution satisfying required boundary conditions is obtained by appropriately choosing these functions.

4. The Unrestricted Process

In the unrestricted case, it is convenient to express the solutions in terms of cumulative distributions of displacement: i.e., we set

$X = (-\infty, y]^\dagger$ in H . The appropriate boundary conditions for $\overline{H^{(\mu)}}$ are here that $H^{(\mu)}$ is a bounded function of x as $x \rightarrow \pm\infty$; hence

$$\overline{H^{(\mu)}} = \begin{cases} A_{12}^{(\mu)} \exp[-\theta x], & x > y; \\ A_{21}^{(\mu)} \exp[\theta x] + \frac{\lambda}{\sigma} (pz + qw) \phi^{-1}, & x \leq y, \end{cases} \quad (4.1)$$

where

$$\phi = \sigma + \lambda(1 - qw - pz). \quad (4.2)$$

The two further conditions needed to determine $A_{12}^{(\mu)}$ and $A_{21}^{(\mu)}$ are obtained by requiring $\overline{H^{(\mu)}}$ to be continuous at $x = y$: i.e., we set

$$\left[\overline{H^{(\mu)}} \right]_{x=y-} = \left[\overline{H^{(\mu)}} \right]_{x=y+}, \quad \mu = 1, -1. \quad (4.3)$$

Substituting (4.1) in (4.3) and using (3.12), we find that

$$\overline{H^{(1)}} = \begin{cases} -\frac{\lambda(pz + qw)}{2\sigma} \exp[\theta(y - x)](\theta^{-1} - \phi^{-1}), & x > y; \\ -\frac{\lambda(pz + qw)}{2\sigma} \exp[\theta(x - y)](\theta^{-1} + \phi^{-1}) \\ \quad + \frac{\lambda}{\sigma} (pz + qw) \phi^{-1}, & x \leq y; \end{cases} \quad (4.4)$$

we obtain $\overline{H^{(-1)}}$ from (4.4) by using (3.6) and $\overline{G^{(\mu)}}$, $\mu = \pm 1$, by using (3.10). Thus (4.4) yields a complete and explicit solution (in Laplace transform) of the one-dimensional scattering problem in the unrestricted case.

We see from (4.4) and (3.6) that $\overline{H^{(\mu)}}$ is a continuous function of $\xi = y - x$, and that its density, say $h^{(\mu)}(\xi) = \partial \overline{H^{(\mu)}} / \partial \xi$, is continuous apart from a finite saltus at $\xi = 0$; introducing the Fourier-Stieltjes transform

$$\begin{aligned} \phi^{(\mu)}(\alpha) &= \int_{-\infty}^{\infty} \exp[i\alpha\xi] \overline{H^{(\mu)}}(d\xi) = \\ &= \int_0^{\infty} \left[\exp[i\alpha\xi] h^{(\mu)}(\xi) + \exp[-i\alpha\xi] h^{(\mu)}(-\xi) \right] d\xi, \end{aligned}$$

$^\dagger[a, b]$ denotes a closed interval, $(a, b]$ an interval open to the left and closed to the right.

it is easily seen to follow from (4.4) that

$$\overline{\phi^{(1)}}(\alpha) = \frac{\lambda}{\sigma} (pz + qw) \left[\frac{i\alpha}{\theta^2 + \alpha^2} + \frac{\sigma + \lambda(1 - qw + pz)}{\theta^2 + \alpha^2} \right]. \quad (4.5)$$

Clearly, $\overline{\phi^{(\mu)}}(\alpha)$ is the Laplace transform of the Fourier-Stieltjes transform

$$\phi^{(\mu)}(\alpha) = \int_{-\infty}^{\infty} \exp[i\alpha \xi] H^{(\mu)}(d\xi).$$

Inverting (4.5), we find that

$$\begin{aligned} \phi^{(1)}(\alpha) = & \frac{\lambda(pz + qw)[i\alpha + \lambda(1 - qw + pz)]}{\lambda^2(1 - qw)^2 + \eta^2} \\ & \cdot \left[1 - \exp[-\lambda(1 - qw)s] \cos \eta s - \frac{\lambda(1 - qw)}{\eta} \exp[-\lambda(1 - qw)s] \sin \eta s \right] \\ & + \frac{\lambda(pz + qw)}{\eta} \exp[-\lambda(1 - qw)s] \sin \eta s, \end{aligned} \quad (4.6)$$

where $\eta^2 = \alpha^2 - (\lambda zp)^2$.

We now turn our attention to the limiting distributions as $s \rightarrow \infty$: for the unrestricted process, it is easy to see that $G^{(\mu)} \rightarrow 0$; on the other hand, $\lim_{s \rightarrow \infty} Q_{r,n-r}$ will yield the probability distribution of the position at which the n^{th} collision with r reversals occurs, and hence $K^{(\mu)} = \lim_{s \rightarrow \infty} H^{(\mu)}$ will yield the generating function for these distributions. From these distributions we can find in the way outlined above the joint distributions of the n^{th} collision position together with the absolute velocity and direction of motion resulting from this collision. Since $K^{(\mu)} = \lim_{\sigma \rightarrow 0} \sigma H^{(\mu)}$, we find from (4.4) that

$$\begin{aligned} K^{(1)} = & -\frac{1}{2} \lambda(pz + qw) \exp[-\gamma|y - x|] [\gamma^{-1} + \text{sgn}(y - x) \lambda^{-1}(1 - qw - pz)^{-1}] \\ & + (pz + qw)(1 - qw - pz)^{-1} [1 - \epsilon(x - y)], \end{aligned} \quad (4.7)$$

where

$$\gamma^2 = \lambda^2(1 - qw - pz)(1 - qw + pz); \quad \epsilon(s) = \begin{cases} 1 & \text{if } s > 0 \\ 0 & \text{otherwise} \end{cases};$$

$$\text{sgn}(s) = \begin{cases} 1 & \text{if } s \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (4.8)$$

Clearly, the Fourier-Stieltjes transform of $K^{(\mu)}$ is

$$\psi^{(\mu)}(\alpha) = \int_{-\infty}^{\infty} \exp[i\alpha\xi] K^{(\mu)}(d\xi) = \lim_{s \rightarrow \infty} \phi^{(\mu)}(\alpha),$$

and hence

$$\psi^{(1)}(\alpha) = \lambda(pz + qw)[i\alpha + \lambda(1 - qw + pz)][\lambda^2(1 - qw)^2 + \eta^2]^{-1}. \quad (4.9)$$

Equation (4.9) is a special case of a more general result which is derived in Chapter V, Sect. 2.

Placzek³⁷ considers the unrestricted process in which $p = q = \frac{1}{2}$ and derives an expression for $\psi_0(y, u)$, where $\psi_0(y, u) dy du$ is the number of neutrons per unit time scattered into the position-lethargy element $dy \times du$. (Lethargy is defined as $\log E_0/E$, where E is the energy and E_0 the initial energy of the particle.) He assumes that the lethargy increases by a fixed amount, c , per collision and that there is an isotropic source at the origin emitting neutrons of lethargy 0 at the rate of one per second.

In this case $\psi_0(y, u)$ can be deduced from equation (4.4) as follows: assuming neutrons are emitted at the origin (i.e., $x = 0$) with equal probabilities of moving initially to the right or left, it follows that the generating function of the probability density $f_n(y)$ for the point at which a neutron suffers its n^{th} collision is given by

$$\sum_{n=1}^{\infty} f_n(y) z^n = \lim_{s \rightarrow \infty} \frac{1}{2} \left[\frac{\partial H^{(1)}}{\partial y} + \frac{\partial H^{(-1)}}{\partial y} \right]_{w=z} = \lim_{\sigma \rightarrow 0} \frac{\sigma}{2} \left[\frac{\partial \overline{H}^{(1)}}{\partial y} + \frac{\partial \overline{H}^{(-1)}}{\partial y} \right]_{w=z}, \quad (4.10)$$

i.e.,

$$\sum_{n=1}^{\infty} f_n(y) z^n = \frac{z}{2\sqrt{1-z}} \exp\left[-\sqrt{1-z}|y|\right] \quad (4.11)$$

(taking the mean free path as the unit of length).

Since the increase in lethargy per collision is constant for each collision, it follows that

$$\psi_0(y, u) = \delta(u - nc) f_n(y), \quad n = 1, 2, \dots \quad (4.12)$$

From equation (4.11) $f_n(y)$ can be expressed as a contour integral:

$$f_n(y) = \int_C \frac{1}{4\pi i z^n \sqrt{1-z}} \exp[-\sqrt{1-z}|y|] dz, \quad n = 1, 2, \dots, \quad (4.13)$$

where C is the contour illustrated. Changing the variable of integration to w , where $z = 1 + w^2$, and using equation (4.12), we obtain the result

$$\psi_0(y, u) = \delta(u - nc) \frac{1}{\pi} \int_0^\infty \frac{\cos yw}{(1+w^2)^n} dw, \quad (4.14)$$

which was originally obtained by Placzek. For a discussion of the behaviour of the function $\psi_0(y, u)$ we refer the reader to Ref. 37.

5. The Process with Absorbing Boundaries

Consider now the one-dimensional scattering process with absorbing barriers at a_{-1} , a_1 ($a_{-1} < a_1$). As mentioned in Chapter I, Sect. 4, these boundary conditions have to be incorporated in the expressions for the basic probability distributions P_0 , Q_{10} , and Q_{01} , and the $*$ -operation. In the present case, all distributions of interest may be derived from the generating function H , restricted by the condition that all collisions prior to the last one occur in the interval $[a_{-1}, a_1]$. This condition is satisfied if we choose $H_1^{(\mu)}$ as in (3.3) and restrict integration with respect to x in the $*$ -operation to the interval $[a_{-1}, a_1]$. As mentioned in Sect. 3, $H^{(\mu)}$ will then satisfy the "backward" equation (3.6) for initial positions $x \in [a_{-1}, a_1]$. Take $X = (a_{-1}, y]$, where $a_{-1} < y \leq a_1$; then the boundary conditions that must be imposed on the general solution (3.11) are

$$\left[H^{(\mu)} \right]_{x=a_{-1}} = 0; \quad \left[H^{(\mu)} \right]_{x=y+} = \left[H^{(\mu)} \right]_{x=y-}; \quad \mu = 1, -1. \quad (5.1)$$

It is convenient in this case to rewrite (3.11) in the form

$$\overline{H^{(\mu)}} \equiv \begin{cases} B_{11}^{(\mu)} \sinh \theta(a_1 - x) + B_{12}^{(\mu)} \cosh \theta(a_1 - x), & a_{-1} < y \leq x \leq a_1; \\ B_{21}^{(\mu)} \sinh \theta(y - x) + B_{22}^{(\mu)} \cosh \theta(y - x) \\ \quad + \frac{\lambda}{\sigma} (pz + qw) \phi^{-1}, & a_{-1} \leq x \leq y \leq a_1, \end{cases} \quad (5.2)$$

where ϕ was defined in (4.2). Substituting (5.2) in (5.1) and using (3.6) to derive the relations between $B_{ij}^{(1)}$ and $B_{ij}^{(-1)}$, we find the coefficients $B_{ij}^{(\mu)}$ and hence

$$\overline{H^{(1)}} = \begin{cases} \frac{\lambda(pz + qw)}{\sigma \sinh \theta \ell} \frac{\sinh \theta(a_1 - x)}{\sinh \theta(a_1 - y)} [\phi^{-1} \sinh \theta \ell - \alpha \sinh \theta(a_1 - y) \\ - \beta \sinh \theta(y - a_{-1})], & a_{-1} < y \leq x \leq a_1; \\ \frac{\lambda(pz + qw)}{\sigma \sinh \theta \ell} [\phi^{-1} \sinh \theta \ell - \alpha \sinh \theta(a_1 - x) \\ - \beta \sinh \theta(x - a_1)], & a_{-1} \leq x \leq y \leq a_1, \end{cases} \quad (5.3)$$

where

$$\left. \begin{aligned} \ell &= a_1 - a_{-1}; & \alpha &= \lambda p z \phi^{-1} [\theta + \sigma + \lambda(1 - qw)]^{-1}; \\ \beta &= \theta^{-1} \sinh \theta(a_1 - y) - \phi^{-1} \cosh \theta(a_1 - y). \end{aligned} \right\} \quad (5.4)$$

Take $X = (a_1, y]$, where $a_1 < y < \infty$; in this case the boundary conditions are

$$\left[\overline{H^{(1)}} \right]_{x=a_1} = \frac{\lambda(pz + qw)}{\sigma(\sigma + \lambda)} \{1 - \exp[-(\sigma + \lambda)(y - a_1)]\}; \quad \left[\overline{H^{(-1)}} \right]_{x=a_{-1}} = 0. \quad (5.5)$$

The first of these conditions results from the observation that if initially $\mu = 1$, $x = a_1$ and if $X = (a_1, y]$ with $y > a_1$, then the first collision occurs outside $[a_{-1}, a_1]$; hence $H^{(1)} = H_1^{(1)}$, and therefore

$$\begin{aligned} \overline{H^{(1)}} &= \overline{H_1^{(1)}} = (pz + qw) \int_0^\infty \exp[-\sigma s] ds \int_0^s \delta[(a_1, y][a_1 + \tau] \exp[-\lambda \tau] \lambda d\tau = \\ &= \frac{\lambda}{\sigma} (pz + qw) \int_0^{y-a_1} \exp[-(\sigma + \lambda)s] ds. \end{aligned}$$

Substituting the first expression of (3.11) in (5.5) and using (3.12), we find $A_{1i}^{(\mu)}$, $i = 1, 2$, and hence

$$\begin{aligned} \overline{H^{(1)}} &= \frac{\lambda(pz + qw)}{\sigma(\lambda + \sigma)} \frac{[\sigma + \lambda(1 - qw)] \cosh \theta(x - a_{-1}) + \theta \sinh \theta(x - a_{-1})}{[\sigma + \lambda(1 - qw)] \cosh \theta \ell + \theta \sinh \theta \ell} \\ &\cdot \{1 - \exp[-(\sigma + \lambda)(y - a_1)]\}. \end{aligned} \quad (5.6)$$

The expression for $\overline{H^{(\mu)}}$ when $X = [y, a_{-1})$ is obtained by symmetry from (5.6):

$$\overline{H^{(\mu)}}([y, a_{-1}) | x; \sigma, a_1, a_{-1}) = \overline{H^{(-\mu)}}\{(-a_{-1}, -y] | -x; \sigma, -a_{-1}, -a_1\}. \quad (5.7)$$

The expression for $\overline{H^{(\mu)}}$ when $X = (-\infty, y]$ is a linear combination of the solutions (5.3), (5.6), and (5.7).

The generating function for the limiting distribution as $s \rightarrow \infty$,

$$K^{(\mu)} = \lim_{s \rightarrow \infty} H^{(\mu)} = \lim_{\sigma \rightarrow 0} \overline{\sigma H^{(\mu)}}, \quad (5.8)$$

follows directly from (5.3), (5.6), and (5.7), and yields the probability distribution of the position at which the n^{th} collision occurs with the previous $n-1$ collisions all taking place in $[a_{-1}, a_1]$ [see the remarks following equation (4.6)].

If we write $K^{(\mu)} = \sum_{n=1}^{\infty} K_n^{(\mu)}$, where $K_n^{(\mu)} = \lim_{s \rightarrow \infty} H_n^{(\mu)}$, then

the iteration relation (2.7) takes a simple form:

$$\begin{aligned} K_{n+1}^{(\mu)}(X | x; z, w) = & \int_0^{|x-a_{\mu}|} \left\{ pz K_n^{(-\mu)}(X | x + \mu s; z, w) \right. \\ & \left. + qw K_n^{(\mu)}(X | x + \mu s; z, w) \right\} \exp[-\lambda s] \lambda ds; \end{aligned} \quad (5.9)$$

the use of these iterations to calculate an explicit expression for $K^{(\mu)}$ is not prohibitively complicated in this case.

6. First-passage Distributions

In the present section, we wish to evaluate the probability of a first passage at the barriers $a_{\pm 1}$, together with the velocity and direction of motion distribution at this passage. We can do this using the method outlined at the beginning of Sect. 3 if we know the sequence of probabilities $P_{rk}^{(\mu)}(a_{\nu} | x; s)$ that a particle initially at x with direction μ effects a first passage at a_{ν} in a length of path $\leq s$ after suffering exactly r reversals and k continuations in $[a_{-1}, a_1]$. Writing as before P_0 for P_{00} , we can see that $P_0^{(\mu)}(a_{\nu} | x; s)$ is the sum of two terms, namely, the probabilities that in path length s the particle (initially at x with direction μ) effect a first passage at a_{ν} with no collisions, and that it suffer its first collision in path length $\leq s$ but after having passed through a_{ν} ; thus, if we use the notation $(a_{\nu}, \nu\infty)$ to denote (a_1, ∞) when $\nu = +1$ and $(-\infty, a_{-1})$ when $\nu = -1$,

$$P_0^{(\mu)}(a_\nu | x; s) = \delta[(a_\nu, \nu\infty) | x + \mu s] \exp[-\lambda s] + \int_0^s \delta[(a_\nu, \nu\infty) | x + \mu \tau] \cdot \exp[-\lambda \tau] \lambda d\tau = \left(\frac{1}{\lambda} \frac{\partial}{\partial s} + 1 \right) Q_1^{(\mu)}[(a_\nu, \nu\infty) | x; s]. \quad (6.1)$$

Hence, taking Laplace transforms,

$$\overline{P_0^{(\mu)}}(a_\nu | x; \sigma) = \left(1 + \frac{\sigma}{\lambda} \right) \overline{Q_1^{(\mu)}}[(a_\nu, \nu\infty) | x; \sigma] = \left(1 + \frac{\sigma}{\lambda} \right) (pz + qw)^{-1} \overline{H_1^{(\mu)}}[(a_\nu, \nu\infty) | x; \sigma, z, w]. \quad (6.2)$$

In accordance with the theory developed in Sect. 2 and equation (2.6), we see that the generating function for the probabilities $P_{rk}^{(\mu)}(a_\nu | x; s)$ is given by

$$G^{(\mu)} = \sum_{r=0}^{\infty} \sum_{k=0}^{\infty} z^r w^k P_{rk}^{(\mu)} = P_0^{(\mu)} * \left[I + H^{(\mu)} \right] = (pz + qw)^{-1} \left(1 + \frac{1}{\lambda} \frac{\partial}{\partial s} \right) H^{(\mu)}[(a_\nu, \nu\infty) | x; s, z, w], \quad (6.3)$$

where $H^{(\mu)}$ is the generating function $H^{(\mu)}[X | x; s, z, w]$ of Sect. 5 and the *-operation is also defined as in Sect. 5. Taking Laplace transforms of (6.3) with respect to s , we find that

$$\overline{G^{(\mu)}}(a_\nu | x; \sigma, z, w) = (pz + qw)^{-1} \left(1 + \frac{\sigma}{\lambda} \right) \overline{H^{(\mu)}}[(a_\nu, \nu\infty) | x; \sigma, z, w]. \quad (6.4)$$

In this type of problem, one is mainly interested in the limiting first-passage distributions as $s \rightarrow \infty$. Let $M^{(\mu)} = \lim_{s \rightarrow \infty} G^{(\mu)} = \lim_{\sigma \rightarrow 0} \sigma \overline{G^{(\mu)}}$; then clearly

$$M^{(\mu)}(a_\nu | x; z, w) = (pz + qw)^{-1} K^{(\mu)}[(a_\nu, \nu\infty) | x; z, w], \quad (6.5)$$

where $K^{(\mu)}$ is the limiting generating function defined in (5.8). Explicitly,

$$\left. \begin{aligned} M^{(1)}(a_1 | x; z, w) &= \frac{\lambda(1 - qw) \sinh \gamma(x - a_{-1}) + \gamma \cosh \gamma(x - a_{-1})}{\lambda(1 - qw) \sinh \gamma \ell + \gamma \cosh \gamma \ell}, \\ M^{(-1)}(a_1 | x; z, w) &= \frac{\lambda p z \sinh \gamma(x - a_{-1})}{\lambda(1 - qw) \sinh \gamma \ell + \gamma \cosh \gamma \ell}, \end{aligned} \right\} \quad (6.6)$$

where $\ell = a_1 - a_{-1}$ and γ was defined in (4.8). The expressions for $M^{(\mu)}(a_{-1})$ may be obtained by symmetry from (6.6), as in (5.7).

Letting z and w tend to 1 in (6.6), we obtain the probability $M^{(\mu)}(a_1 | x)$ that the particle first emerges from $[a_{-1}, a_1]$ at a_1 (regardless of the number of collisions which occur before emergence). Thus,

$$\left. \begin{aligned} M^{(1)}(a_1 | x) &= [1 + (x - a_{-1}) \lambda p][1 + (a_1 - a_{-1}) \lambda p]^{-1}; \\ M^{(-1)}(a_1 | x) &= (x - a_{-1}) \lambda p[1 + (a_1 - a_{-1}) \lambda p]^{-1}. \end{aligned} \right\} \quad (6.7)$$

We notice that these are consistent with the general results found by Mycielski and Paszkowski³³ using invariance arguments.

Under postulate (3) of Sect. 1, i.e., that the velocity transition probabilities satisfy the commutativity condition $\psi_\pi \circ \psi_0 = \psi_0 \circ \psi_\pi$, we can use equation (3.2) together with the first-passage distributions given by (6.6) to determine the distribution of the velocity with which the particle first emerges from the interval $[a_{-1}, a_1]$.

The probability of emerging first at a_ν (after any number of previous collisions) is given by $M^{(\mu)}(a_\nu | x)$. Hence, by means of (3.2), the distribution of emergent velocity conditional on the initial state (u, μ, x) and first emergence at a_ν is given by

$$F_{a_\nu}(V | u, \mu, x) = \left[M^{(\mu)}(a_\nu | x) \right]^{-1} \sum_{i+j \geq 0} P_{ij}^{(\mu)}(a_\nu | x; \infty) \psi_\pi^{(i)} \circ \psi_0^{(j)}. \quad (6.8)$$

In principle equation (6.8) gives the required distribution of emergent velocity. For functions ψ_π and ψ_0 of certain forms the equation simplifies considerably, leading to more manageable solutions. We shall now give two such examples.

Example 1.

$$\psi_\pi(dv | u) = \frac{1}{u} f_p\left(\frac{v}{u}\right) dv, \quad 0 \leq v \leq u;$$

$$\psi_0(dv | u) = \frac{1}{u} f_q\left(\frac{v}{u}\right) dv, \quad 0 \leq v \leq u.$$

In this case it is convenient to introduce the Mellin transforms

$$\bar{\psi}_{\pi}(\alpha) = \int_{v=0}^u f_p \left(\frac{v}{u} \right) \left(\frac{v}{u} \right)^{\alpha-1} \frac{dv}{u};$$

$$\bar{\psi}_0(\alpha) = \int_{v=0}^u f_q \left(\frac{v}{u} \right) \left(\frac{v}{u} \right)^{\alpha-1} \frac{dv}{u};$$

$$F_{a_{\nu}}(\alpha | u, \mu, x) = \int_{v=0}^u \left(\frac{v}{u} \right)^{\alpha-1} F_{a_{\nu}}(dv | u, \mu, x).$$

Then, taking Mellin transforms in equation (6.8), we have

$$F_{a_{\nu}}(\alpha | u, \mu, x) = \left[M^{(\mu)}(a_{\nu} | x) \right]^{-1} M^{(\mu)}(a_{\nu} | x; \bar{\psi}_{\pi}(\alpha), \bar{\psi}_0(\alpha)), \quad (6.9)$$

where the generating functions $M^{(\mu)}(a_{\nu} | x; z, w)$ are obtained from equation (6.6). The moments of the distribution of emergent velocity are easily found from equation (6.9); for example, the n^{th} moment is given by

$$E_{a_{\nu}}(v^n | u, \mu, x) = u^n \left[M^{(\mu)}(a_{\nu} | x) \right]^{-1} M^{(\mu)}(a_{\nu} | x; \bar{\psi}_{\pi}(n+1), \bar{\psi}_0(n+1)). \quad (6.10)$$

Example 2.

$$\psi_{\pi}(dv | u) = \delta(v - c_1 u) dv, \quad 0 \leq c_1 \leq 1;$$

$$\psi_0(dv | u) = \delta(v - c_2 u) dv, \quad 0 \leq c_2 \leq 1.$$

This is the case in which, as a result of a collision, the logarithmic energy of the particle decreases by a fixed amount depending only on the angle of deflection. The emergent distribution, as distinct from the previous example, is now discrete. If the particle, starting with velocity u , makes i collisions with reversals and j collisions without reversals, then the resulting velocity is given by

$$v = u c_1^i c_2^j.$$

Taking Mellin transforms as in Example 1, again defining

$$F_{a_{\nu}}(\alpha | u, \mu, x) = \int_{v=0}^u \left(\frac{v}{u} \right)^{\alpha-1} F_{a_{\nu}}(dv | u, \mu, x),$$

we find that

$$F_{a_{\nu}}(\alpha | u, \mu, x) = \left[M^{(\mu)}(a_{\nu} | x) \right]^{-1} M^{(\mu)}(a_{\nu} | x; c_1^{\alpha-1}, c_2^{\alpha-1}). \quad (6.11)$$

7. Higher-order Passage Distributions and Further Boundary Conditions

The results of Sect. 6 can be generalized to give passage distributions for any prescribed sequence of passages through points on $(-\infty, \infty)$. Let us take two points a_1, a_{-1} ($-\infty < a_{-1} < a_1 < \infty$). These divide the real line into three regions: $(-\infty, a_{-1})$, $[a_{-1}, a_1]$, and (a_1, ∞) , which we shall denote by L_{-1} , L_0 , and L_1 , respectively. An arbitrary sequence of passages through the points a_1, a_{-1} can then be specified by a sequence of numbers $\{\gamma_0, \gamma_1, \dots, \gamma_n\}$ where $\gamma_j = -1, 0$, or 1 . This sequence indicates that the particle starts in L_{γ_0} , makes a first passage from L_{γ_0} to L_{γ_1} , thence to L_{γ_2} , thence to L_{γ_3} and eventually to L_{γ_n} . Clearly $\{\gamma_0, \dots, \gamma_n\}$ must be a nearest-neighbor sequence, i.e., $|\gamma_j - \gamma_{j-1}| = 1, j \geq 1$. For example, the sequence $\{0, 1, 0, 1, 0, -1, 0\}$ is of the required type.

We introduce the following generalization of the function $M^{(\mu)}(a_1 | x; z, w)$ defined by equation (6.5). Let $P_{rk}^{\gamma_0 \gamma_1 \dots \gamma_n}(x, \mu)$ be the probability that a particle which starts at x with direction μ ($-\infty < x < \infty, \mu = \pm 1$) makes the sequence of first passages corresponding to the sequence $\{\gamma_0, \dots, \gamma_n\}$, and experiences exactly r reversals and k continuations before making the last of these from $L_{\gamma_{n-1}}$ to L_{γ_n} . Then we define

$$M_{\gamma_0 \dots \gamma_n}^{(\mu)}(x; z, w) = \sum_{r=0}^{\infty} \sum_{k=0}^{\infty} P_{rk}^{\gamma_0 \dots \gamma_n}(x, \mu) z^r w^k, \quad |z| \leq 1, |w| \leq 1.$$

Clearly the passage $L_{\gamma_{n-1}} \rightarrow L_{\gamma_n}$ must occur through $a_{\gamma_{n-1} + \gamma_n}$. Furthermore

$$\left. \begin{aligned} M_{0,1}^{(\mu)}(x; z, w) &= M^{(\mu)}(a_1 | x; z, w), \\ M_{0,-1}^{(\mu)}(x; z, w) &= M^{(\mu)}(a_{-1} | x; z, w), \end{aligned} \right\} \quad (7.1)$$

where the expressions on the right-hand side are found directly from equations (6.6). The functions $M_{\pm 1,0}^{(\mu)}(x; z, w)$ can also be found from (6.6) by letting $a_1 \rightarrow \infty$ or $a_{-1} \rightarrow -\infty$ as required. Thus for example, letting a_{-1} tend to $-\infty$ in the first of equations (6.6) and then replacing a_1 by a_{-1} , we find that

$$M_{-1,0}^{(1)}(x; z, w) = \exp[-\gamma(a_{-1} - x)]. \quad (7.2)$$

The higher-order passage probabilities are then determined recursively from the relation

$$M_{\gamma_0 \dots \gamma_n}^{(\mu)}(x; z, w) = M_{\gamma_{n-1} \gamma_n}^{(\gamma_{n-1} - \gamma_{n-2})} \left(a_{\gamma_{n-1} + \gamma_{n-2}}; z, w \right) M_{\gamma_0 \dots \gamma_{n-1}}^{(\mu)}(x; z, w). \quad (7.3)$$

Suppose now that the particle moves in the interval $(-\infty, a_{-1}]$ and that there is a reflecting barrier at a_{-1} , i.e., when the particle strikes the barrier at a_{-1} its direction of motion is reversed (reflection is not counted as a collision) and the process continues. The argument we are about to use extends in an obvious way to the problem of scattering with two reflecting barriers. Let X be a subinterval of $(-\infty, a_1]$ and let $R_{rk}^{(\mu)}(X|x)$ be the probability, conditional on initial position and direction (x, μ) , that the particle makes its $(r+k)^{\text{th}}$ collision (r of these being reversals and k being continuations) in the interval X . Then the generating function,

$$R^{(\mu)}(X|x) = \sum_{r+k \geq 1} R_{rk}^{(\mu)}(X|x) z^r w^k,$$

is given by

$$R^{(\mu)}(X|x) = K_{\infty}^{(\mu)}(X|x) + \sum_{n=0}^{\infty} M_{\infty}^{(\mu)}(a_{-1}|x; z, w) \cdot K_{\infty}^{(-1)}(X|a_{-1}) \left[M_{\infty}^{(-1)}(a_{-1}|a_{-1}; z, w) \right]^n, \quad (7.4)$$

where $K_{\infty}^{(\mu)}(X|x)$ is the analogue for the half-line $(-\infty, a_{-1}]$ of the generating function defined by equation (5.8) for the interval $[a_{-1}, a_1]$. Thus

$$K_{\infty}^{(\mu)}(X|x) = \lim_{\sigma \rightarrow 0} \overline{H}^{(\mu)}(X|x; \sigma, a_{-1}, -\infty), \quad (7.5)$$

where $\overline{H}^{(\mu)}$ is defined by equations (5.6) and (5.7). Similarly, $M_{\infty}^{(\mu)}(a_{-1}|x; z, w)$ is obtained from (6.6) upon letting a_{-1} tend to $-\infty$ and then replacing a_1 by a_{-1} . The n^{th} term on the right-hand side of (7.4) corresponds to the particle being reflected $(n-1)$ times at the barrier before making its $(r+k)^{\text{th}}$ collision in X . Equation (7.4) can be written more simply as

$$R^{(\mu)}(X|x) = M_{\infty}^{(\mu)}(a_{-1}|x; z, w) K_{\infty}^{(-1)}(X|a_{-1}) \left[1 - M_{\infty}^{(-1)}(a_{-1}|a_{-1}; z, w) \right]^{-1} + K_{\infty}^{(\mu)}(X|x). \quad (7.6)$$

We observe finally that the same argument can also be applied to scattering problems on a line which is made up of a number of adjacent homogeneous segments. At each passage through an interface the particle, instead of being reflected, passes into a different segment in which it may have a different mean free path.

8. The Steady-state Distribution of Particles in a Finite Interval

For an interval of length ℓ we define the following probabilities:

p_ℓ = Prob {a particle incident at one end emerges for the first time from the opposite end};

p'_ℓ = Prob {a particle incident at one end emerges for the first time from the same end}.

From equation (6.7) we have immediately that

$$p_\ell = [1 + \lambda p \ell]^{-1}; \quad p'_\ell = \lambda p [1 + \lambda p \ell]^{-1}. \quad (8.1)$$

Suppose that N particles per second are incident upon the left-hand end of the interval $[a, b]$ and suppose that the system is in a steady state. Then the expected number of particles per second passing to the right through any point x in the interval for the k^{th} time is given by

$$E_{n_{kR}} = N p_{x-a} [p'_{b-x} p'_{x-a}]^k. \quad (8.2)$$

Hence the total expected number of particles per second passing to the right through x is given by

$$E_{n_R} = \sum_{k=1}^{\infty} E_{n_{kR}} = N p_{x-a} [1 - p'_{b-x} p'_{x-a}]^{-1}. \quad (8.3)$$

Similarly, the expected number of particles per second passing to the left through x is given by

$$E_{n_L} = N p_{x-a} p'_{b-x} [1 - p'_{b-x} p'_{x-a}]^{-1}. \quad (8.4)$$

Substitution for p_ℓ and p'_ℓ from equation (8.1) gives

$$\left. \begin{aligned} E_{n_L} &= N(b-x) \lambda p [1 + (b-a) \lambda p]^{-1}; \\ E_{n_R} &= N[1 + (b-x) \lambda p][1 + (b-a) \lambda p]^{-1}. \end{aligned} \right\} \quad (8.5)$$

9. The Effect of Absorption

Consider now the effect of introducing a third possible outcome at each collision, namely, that the particle is absorbed. The probabilities of reversal and continuation will still be denoted by p and q , respectively; however, $(p+q)$ is now less than 1. The probability of absorption, given a collision, will be denoted by r where $r = 1 - p - q$.

Once a particle is absorbed, its motion ceases completely. To the set of all possible states S in the pure scattering process, we must therefore adjoin the family of states $A(\underline{\omega})$; a particle being in state $A(\underline{\omega})$ if it has been absorbed and if its state immediately before absorption was $\underline{\omega}$ where $\underline{\omega} \in S$. The enlarged state space we shall denote by S^* .

The transition probabilities $P_{rk}(S|\underline{\omega}; s)$ and $Q_{rk}(S|\underline{\omega}; s)$ are then defined exactly as in Sect. 2; it should be observed that $Q_{rk}(S^*|\underline{\omega}; s)$ is zero for all sets S^* exterior to S since it is impossible for the particle to be absorbed at the $(r+k)^{\text{th}}$ collision if r of the collisions were reversals and the remaining k were continuations. We observe also that

$$\left[1 - \sum_{r=0}^n \int_S Q_{r,n-r}(d\underline{\eta}|\underline{\omega}; \infty) \right]$$

is the probability that a particle initially in state $\underline{\omega}$ is absorbed before making n collisions.

The fundamental equations (2.9) and (2.10) are unchanged by the possibility of absorption at a collision. In fact, the only results which are no longer valid are those in whose derivation the fact that $p + q = 1$ has been used. These are equations (4.11) to (4.14), which can easily be modified if $p + q < 1$ [the density function $f_n(y)$ will then no longer be normalized to unity], and equations (6.7), which become

$$\left. \begin{aligned} M^{(1)}(a_1|x) &= \frac{(1-q) \sinh(x-a_{-1}) \lambda \sqrt{rs} + \sqrt{rs} \cosh \lambda(x-a_{-1}) \sqrt{rs}}{(1-q) \sinh \lambda t \sqrt{rs} + \sqrt{rs} \cosh \lambda t \sqrt{rs}}; \\ M^{(-1)}(a_1|x) &= \frac{p \sinh(x-a_{-1}) \lambda \sqrt{rs}}{(1-q) \sinh \lambda t \sqrt{rs} + \sqrt{rs} \cosh \lambda t \sqrt{rs}}, \end{aligned} \right\} \quad (9.1)$$

where $t = a_1 - a_{-1}$, $r = 1 - p - q$, and $s = 1 - q + p$.

Equation (8.1), which was obtained directly from (6.7), must also be changed and the results of Sect. 8 modified accordingly.

It will sometimes be of interest to know the probabilities $P_{mk}(S|\underline{\omega}; s)$ and $Q_{mk}(S|\underline{\omega}; s)$ conditional upon no absorption in the first $(m+k)$ collisions. These conditional probabilities are simply $P_{mk}(1-r)^{-m-k}$ and $Q_{mk}(1-r)^{-m-k}$; the corresponding generating functions are therefore obtained by replacing z by $z/(1-r)$ and w by $w/(1-r)$ in the generating functions G and H , respectively.

CHAPTER III

THE APPLICATION OF A SIMPLE ONE-DIMENSIONAL PROCESS IN BUBBLE CHAMBER EXPERIMENTS

1. Introduction

The material of this chapter is concerned with the process of Chapter II in the special case in which there is no absorption or reversal, i.e., $p = r = 0$ and $q = 1$. Nevertheless, this process is of considerable interest in physics partly because it occurs in the formation of bubble chamber tracks; the theory of such processes can be used to estimate the velocity of particles passing through the chamber.

Consider a particle which moves along the half-line $[0, \infty)$ in such a way that it has probability $\lambda \delta u + o(\delta u)$ of forming a bubble in any small element $(u, u + \delta u)$ of its track (λ is constant). Following bubble formation, the particle continues moving in the same direction and generates further bubbles in the same way. The particle is observed only in the line segment $0 \leq u \leq x$, and successive bubbles are formed at $X_1, X_2, \dots, X_n$ ($0 \leq X_1 \leq X_2 \leq \dots \leq X_n \leq x$), where $X_1, \dots, X_n$ are random variables and the number n of bubbles in $[0, x]$ is also random (having a Poisson distribution). The distances

$$Y_1 = X_1; \quad Y_i = X_i - X_{i-1}, \quad 2 \leq i \leq n,$$

constitute a finite univariate population process as defined by Moyal,³² the individuals being the distances Y_i , each with state space $[0, x]$.

These distances Y_i between successive bubbles are used to estimate the mean free path, λ^{-1} , and hence the velocity of the particle whose track is observed. Owing to the nonzero diameter d of the bubbles, only those distances Y_i which are greater than the bubble diameter are observed. These are then used to estimate the mean free path by equating the total number of observed distances to the expected number. Such a procedure requires knowledge of the distribution of the number of distances Y_i which are greater than the bubble diameter d .

It is the purpose of this chapter to give a complete description of the population process generated by the Y_i . We shall determine the probability-generating functional for the process and use this to find explicit expressions for the k^{th} order factorial moment densities and the distribution of the number of distances Y_i such that $Y_i \geq d$.

We shall assume throughout that the mean free path λ^{-1} for bubble formation is unity. This implies no loss of generality, being equivalent to measuring all distances in units of λ^{-1} .

2. The Probability-generating Functional and Factorial Moment Densities

The process may be described as a population process in the following way: the state space of each individual is $[0, x]$, and corresponding to n bubbles with successive mutual distances $Y_1, \dots, Y_n$ we have n individuals with states $Y_1, \dots, Y_n$ in $[0, x]$; the process may then be characterized by the integral-valued random function N on a class of subsets of $[0, x]$, where $N(A|Y_1, \dots, Y_n)$ is the number of individuals in the subset A corresponding to a total of n individuals with states $Y_1, \dots, Y_n$, i.e.,

$$N(A|Y_1, \dots, Y_n) = \sum_{i=1}^n \delta(A|Y_i),$$

where $\delta(A|Y) = 1$ if $Y \in A$ and $\delta(A|Y) = 0$ if $Y \notin A$.

We define the integral with respect to N of any measurable function θ on the individual state space $X(=[0, x])$ to be the random variable

$$\begin{aligned} \int_X \theta(u) N(du|Y_1, \dots, Y_n) &= \sum_{i=1}^n \int_X \theta(u) \delta(du|Y_i) \\ &= \sum_{i=1}^n \theta(Y_i). \end{aligned}$$

The probability-generating functional for the process is defined, for any bounded measurable function ξ on X such that $\sup_{x \in X} |\xi(x)| \leq 1$, as the expectation

$$\begin{aligned} G[\xi] &= E \xi(Y_1) \xi(Y_2), \dots, \xi(Y_n) \\ &= E \exp \int_X \log \xi(u) N(du|Y_1, \dots, Y_n). \end{aligned}$$

Consideration of the point at which the first bubble in $[0, x]$ occurs leads to the following equation for G :

$$G[\xi, x] = e^{-x} + \int_0^x \xi(y) G[\xi, x-y] e^{-y} dy. \quad (2.1)$$

Taking Laplace transforms with respect to x and defining

$$H[\xi, s] = \int_0^{\infty} e^{-su} G[\xi, u] du,$$

$$\phi(s) = \int_0^{\infty} e^{-su} \xi(u) du,$$

we find from equation (2.1) that

$$H[\xi, s] = (s+1)^{-1} [1 - \phi(s+1)]^{-1}. \quad (2.2)$$

It has been assumed here that $\xi(u)$ is defined independently of x for all nonnegative values of u and in such a way that $\phi(s)$ exists for some s with positive real part. These restrictions are not severe, since for a given x any bounded measurable function ξ on $[0, x]$ can be extended to a function on $[0, \infty)$ having the required properties.

The Laplace transform $H[\xi, s]$ of equation (2.2) can be inverted to give the solution

$$\begin{aligned} G[\xi, x] &= \sum_{n=0}^{\infty} [f * g^{(n)}](x) \\ &= e^{-x} \left[1 + \sum_{n=1}^{\infty} \int_{u_n=0}^x \int_{u_{n-1}=0}^{u_n} \dots \int_{u_1=0}^{u_2} \xi(u_n - u_{n-1}) \dots \xi(u_2 - u_1) \xi(u_1) du_1 \dots du_n \right], \end{aligned} \quad (2.3)$$

where

$$f(x) = e^{-x};$$

$$g(x) = e^{-x} \xi(x);$$

$$[\phi * \psi](x) = \int_0^x \phi(x-y) \psi(y) dy;$$

$$g^{(n)}(x) = [g * g^{(n-1)}](x), \quad n = 1, 2, \dots;$$

$$g^{(0)}(x) = \delta(x), \text{ the Dirac delta function.}$$

The solution (2.3) clearly converges for all functions ξ which are bounded and measurable on X .

The k^{th} order factorial moment distribution $M_{(k)}(A_1, \dots, A_k; x)$ is defined for any family of measurable disjoint subsets $A_1, \dots, A_k$ of the individual state space X by

$$M_{(k)}(A_1, \dots, A_k; x) = EN(A_1) N(A_2) \dots N(A_k). \quad (2.4)$$

It is shown by Moyal³² that, provided the factorial moment distributions exist, they are related to the probability-generating functional by

$$\int_{X^k} \xi_1(u_1) \dots \xi_k(u_k) M_{(k)}(du_1, du_2, \dots, du_k; x) = \lim_{\eta \rightarrow 1} \delta_{\xi_1 \dots \xi_k}^k G[\eta, x], \quad (2.5)$$

where $\delta_{\xi_1 \dots \xi_k}^k G[\eta]$ is the k^{th} order variation of G defined by

$$\delta_{\xi_1 \dots \xi_k}^k G[\eta] = \left\{ \frac{\partial^k}{\partial \lambda_1 \dots \partial \lambda_k} G \left[\eta + \sum_{i=1}^k \lambda_i \xi_i \right] \right\}_{\lambda_1 = \dots = \lambda_k = 0},$$

where $\sup_{x \in X} |\eta(x)| < 1$ and $\lambda_1, \dots, \lambda_k$ are complex variables. The corresponding factorial moment densities are defined by

$$m_{(k)}(u_1, \dots, u_k; x) du_1 \dots du_k = M_{(k)}(du_1, \dots, du_k; x). \quad (2.6)$$

Taking Laplace transforms of equation (2.5) with respect to x , choosing

$$\xi_i(u) = \begin{cases} \mu \exp[-\mu(u - u_i)], & u \geq u_i, \\ 0, & u < u_i; \end{cases}$$

$$\eta(u) = c < 1,$$

and letting $c \rightarrow 1$, $\mu \rightarrow \infty$, we find that the Laplace transform of $m_{(k)}(u_1, \dots, u_k; x)$ is given by

$$\int_0^\infty m_{(k)}(u_1, \dots, u_k; x) e^{-sx} dx = \frac{k!}{s} \left(1 + \frac{1}{s}\right)^k \exp \left[-(s+1) \sum_{i=1}^k u_i \right]. \quad (2.7)$$

Hence

$$m_{(k)}(u_1, \dots, u_k; x) = (k!)^2 \exp \left\{ - \left[\sum_{i=1}^k u_i \right] \right\} \sum_{i=0}^k \frac{\left(x - \sum_{i=1}^k u_i \right)^i}{(k-i)! (i!)^2}. \quad (2.8)$$

3. The Number of Distances Exceeding the Bubble Diameter d

Denote by N the number of individuals Y such that $Y > d$ and let p_k be the probability that $N = k$, $k = 0, 1, 2, \dots$. Then the probability-generating function

$$g(z, s) = \sum_{k=0}^{\infty} p_k z^k, \quad |z| \leq 1, \quad (3.1)$$

is obtained from the generating functional $G[\xi, x]$ upon defining $\xi(u)$ as

$$\xi(u) = \begin{cases} z; & u > d; \\ 1; & u \leq d. \end{cases}$$

From equation (2.2) we find that the Laplace transform of $g(z, x)$ with respect to x is given by

$$h(z, s) = \left[s + e^{-(s+1)d} - ze^{-(s+1)d} \right]^{-1}. \quad (3.2)$$

This can be inverted to give

$$g(z, x) = \sum_{0 \leq n \leq x/d} \frac{(z-1)^n}{n!} (x-nd)^n e^{-nd}, \quad (3.3)$$

and hence

$$p_k = \frac{1}{k!} \left[\frac{\partial^k g(z, x)}{\partial z^k} \right]_{z=0} = \sum_{k \leq n \leq x/d} \frac{(-1)^{n-k} e^{-nd}}{k!(n-k)!} (x-nd)^n. \quad (3.4)$$

The factorial moments of N are obtained immediately from equation (3.3) by use of the relation

$$EN(N-1)\dots(N-k+1) = \left[\frac{\partial^k g(z, x)}{\partial z^k} \right]_{z=1}.$$

Thus

$$EN(N-1)\dots(N-k+1) = \begin{cases} (x - kd)^k e^{-kd}, & 1 \leq k \leq x/d; \\ 0, & k > x/d. \end{cases}$$

The first two moments agree well with a Monte Carlo computation which was performed by Dr. L. G. Hyman of Argonne National Laboratory, who raised this problem in connection with its bubble chamber applications.

CHAPTER IV

MULTIPLICATIVE TRANSPORT PROCESSES IN ONE DIMENSION

1. Introduction

The following simple model for the multiplication of neutrons in a one-dimensional rod has been studied by Bellman, Kalaba, and Wing^{3,4,51} in order to clarify the concepts of criticality and critical size. It is assumed, as in the scattering model of Chapter II, that a neutron travels with constant direction between collisions, the probability of a collision in any small interval δs of path length being $\lambda \delta s + o(\delta s)$. As a result of a collision, the parent neutron is replaced by two neutrons which leave the point of collision in opposite directions and continue to move and multiply in the same manner as the parent neutron. Bellman, Kalaba, and Wing, using the principle of invariant imbedding (in which the process is considered as a function of the length of the rod), determined the expected number of transmitted and reflected neutrons generated over all time by a single incident neutron. They used as a criterion for criticality the condition that these expected values become infinite, in this way showing that the critical length of the rod is given by $\pi/2\lambda$. They also derived equations for the probabilities

$$p_n(t) = \text{the probability that a single neutron incident on a rod of thickness } t \text{ eventually gives rise to a total of } n \text{ reflected neutrons,} \quad (1.1)$$

and for the corresponding generating function,

$$u(t, z) = \sum_{n=0}^{\infty} p_n(t) z^n, \quad (1.2)$$

and made the following conjectures regarding the behaviour of the probabilities $p_n(t)$:

$$p_n(t) < p_{n-1}(t), \quad n = 1, 2, \dots; \quad t > 0; \quad (1.3)$$

$$p_n(t) \text{ is unimodal and has a maximum at } t = y_n,$$

where

$$\frac{\pi}{2\lambda} > y_n > y_{n-1} > 0 \quad (1.4)$$

and

$$\lim_{n \rightarrow \infty} y_n = \frac{\pi}{2\lambda} = \text{critical length.}$$

In this chapter we shall consider a modified model in which a random number, ν , of neutron pairs emerges from a collision (ν neutrons moving in each direction) in addition to the parent neutron which continues to move with its original direction of motion. We show that for this model explicit solutions can be obtained for the probabilities $p_n(t)$; in particular

we show that there is a certain length of rod, t_c , for which $\sum_{n=0}^{\infty} p_n(t) < 1$

when $t > t_c$. This critical length, t_c , is the same as is obtained by applying the condition that the mean number of particles produced become infinite.

However, the property that $\sum_{n=0}^{\infty} p_n(t) < 1$ for lengths greater than t_c is

clearly a stronger property of criticality than is implied by the behaviour of the mean number of emergent particles. Following the detailed analysis of the model just described, we shall return to the model of Bellman, Kalaba, and Wing, and show that it also exhibits the property of criticality in this stronger sense.

In the particular case when the random variable ν is equal to 1 with probability 1, i.e., exactly two additional particles are produced at each collision, the conjectures (1.3) and (1.4) of Bellman, Kalaba, and Wing are established.

Finally, a process is examined in which a random number of particles emerges from each collision, each particle moving to the left or right with probability $\frac{1}{2}$ independently of the others. A particular case of this model in which exactly two particles emerge from a collision has been studied by Harris.⁵⁸

The behaviour of the functions $\sum_{n=0}^{\infty} p_n(t)$ for the models of Harris⁵⁸

and Bellman, Kalaba, and Wing³ has previously been investigated by T. W. Mullikin and R. N. Snow⁶⁰ using arguments of a different kind. In particular they derive the function illustrated in Fig. 1, Sect. 5, for the latter model.

2. The Distribution of the Number of Emergent Particles

Consider a particle which originates at a point x in the rod ($-a \leq x \leq a$). The initial direction of motion is μ ($\mu = 1$ for motion to the right and -1 for motion to the left). Motion is in the manner described in the introduction, i.e., the probability of a collision in a small element of path length δs is $\lambda \delta s + o(\delta s)$, and at each collision it gives rise to ν new particles (in addition to itself) moving in the same direction and ν new particles moving in the opposite direction; ν is a random variable taking

the values 0, 1, 2, ..., with associated probabilities $\pi_0, \pi_1, \pi_2, \dots$, respectively. Without loss of generality we shall assume throughout that the collision rate λ is equal to 1.

We introduce the probabilities $P_n(\mu, x)$ that a single particle originating at x with direction μ ultimately gives rise to a total of n particles emerging from the right-hand end of the rod. The corresponding generating function is defined by

$$G(\mu, x) = \sum_{n=0}^{\infty} P_n(\mu, x) z^n, \quad |z| \leq 1. \quad (2.1)$$

Consideration of the point at which the first collision occurs gives the following integral equation for $G(\mu, x)$:

$$G(\mu, x) = z^{\delta_{1,\mu}} e^{\mu(x-a)} + \mu \int_x^{\mu a} G(\mu, y) \sum_{n=0}^{\infty} \pi_n G(\mu, y)^n G(-\mu, y)^n e^{\mu(x-y)} dy. \quad (2.2)$$

Differentiating this equation with respect to x , we find that $G(\mu, x)$ must satisfy the differential equation

$$\frac{d}{dx} G(\mu, x) = \mu G(\mu, x) \left[1 - \sum_{n=0}^{\infty} \pi_n G(\mu, x)^n G(-\mu, x)^n \right] \quad (2.3)$$

with boundary conditions

$$G(\mu, \mu a) = z^{\delta_{1,\mu}}. \quad (2.4)$$

From equation (2.3) it follows that $G(1, x) G(-1, x)$ is independent of x , i.e.,

$$G(1, x) G(-1, x) = K(z), \quad (2.5)$$

where K is a function of z with the properties

$$\left. \begin{aligned} |K(z)| &\leq 1, & |z| &\leq 1; \\ |K(z)| &< 1, & |z| &< 1. \end{aligned} \right\} \quad (2.6)$$

Substituting $K(z)$ for $G(1, x) G(-1, x)$ in equation (2.3) and using the boundary conditions (2.4), we find that

$$G(\mu, x) = z^{\delta_{1,\mu}} \exp \left[(\mu x - a) \left(1 - \sum_{n=0}^{\infty} \pi_n K(z)^n \right) \right], \quad (2.7)$$

where $K(z)$ is a root of the equation

$$\theta = z \exp \left[-2a \left(1 - \sum_{n=0}^{\infty} \pi_n \theta^n \right) \right], \quad (2.8)$$

having the properties (2.6).

By Lagrange's Theorem,⁵⁴ if $|z| < 1$, equation (2.8) has only one root θ such that $|\theta| < 1$. From conditions (2.6) this must therefore be the required root $K(z)$ appearing in equation (2.7). Making use of Lagrange's expansion, we will derive an explicit solution for $G(\mu, x)$.

We introduce functions $f(\theta)$ and $\phi(\theta)$, in which z , x , and a are regarded as fixed, defined by the following equations.

$$f(\theta) = z^{\delta_{1,\mu}} \exp \left[(\mu x - a) \left(1 - \sum_{n=0}^{\infty} \pi_n \theta^n \right) \right]; \quad (2.9)$$

$$\phi(\theta) = \exp \left[-2a \left(1 - \sum_{n=0}^{\infty} \pi_n \theta^n \right) \right]. \quad (2.10)$$

For $|z| < 1$ the generating function $G(\mu, x)$ is obtained from $f(\theta)$ upon replacing θ by the root $K(z)$ of (2.8) in the interior of the unit circle. By Lagrange's expansion, therefore,

$$G(\mu, x) = f(0) + \sum_{n=1}^{\infty} \frac{z^n}{n!} \left\{ \frac{d^{n-1}}{d\theta^{n-1}} \left[\phi(\theta)^n \frac{d}{d\theta} f(\theta) \right] \right\}_{\theta=0}. \quad (2.11)$$

From equation (2.11) it can be shown directly that the coefficient of z^n in $G(1, x)$, i.e., $P_n(1, x)$, is given by

$$\left. \begin{aligned} P_1(1, x) &= \exp[(x - a)(1 - \pi_0)]; \\ P_n(1, x) &= \frac{a - x}{n - 1} \exp[(1 - \pi_0)(x - 2na + a)] \sum_{j=0}^{n-2} (n - j - 1) \gamma_j^{(n-1)} \pi_{n-j-1}, \quad n \geq 2, \end{aligned} \right\} \quad (2.12)$$

where

$$\left. \begin{aligned} \gamma_0^{(n)} &= 1; \\ \gamma_j^{(n)} &= \sum_{m=1}^j \frac{(2na + a - x)^m}{m!} \sum_{i_1 + \dots + i_m = j} \pi_{i_1} \pi_{i_2} \dots \pi_{i_m}, \quad j \geq 1, \end{aligned} \right\} \quad (2.13)$$

the inner summation being over all sets of positive integers $(i_1, i_2, \dots, i_m)$ such that $i_1 + i_2 + \dots + i_m = j$.

The probabilities $P_n(-1, x)$ can be found immediately from (2.12) by use of the relation

$$P_n(-1, x) = P_{n+1}(1, -x), \quad n = 0, 1, 2, \dots, \quad (2.14)$$

which is a direct consequence of equation (2.7).

The reflection probabilities $p_n(t)$ defined by (1.1) are obtained from $P_n(-1, x)$ on setting $a = x = t/2$. Thus

$$\left. \begin{aligned} p_n(t) &= \frac{t}{n} \exp[-(1 - \pi_0)(n+1)t] * \\ &\left\{ n\pi_n + \sum_{j=1}^{n-1} (n-j) \pi_{n-j} \sum_{m=1}^j \frac{[(n+1)t]^m}{m!} \sum_{i_1 + \dots + i_m = j} \pi_{i_1} \dots \pi_{i_m} \right\}, \quad n \geq 2; \end{aligned} \right\} \quad (2.15)$$

$$p_0(t) = \exp[-t(1 - \pi_0)]; \quad p_1(t) = t\pi_1 \exp[-2t(1 - \pi_0)].$$

In the particular case when $\nu = 1$ with probability 1, i.e., if

$$\pi_n = \delta_{1,n}, \quad n = 0, 1, 2, \dots, \quad (2.16)$$

then $p_n(t)$ takes the very simple form

$$p_n(t) = \frac{(n+1)^{n-1}}{n!} t^n \exp[-(n+1)t], \quad n = 0, 1, 2, \dots \quad (2.17)$$

The maximum with respect to t of this function occurs when $t = n/(n+1)$ and it will be shown in Sect. 3 that the critical length for this particular model is $t_c = 1$. This proves conjecture (1.4). The proof of conjecture (1.3) follows trivially from equation (2.17).

3. Criticality in the Model of Sect. 2

The results of Sect. 2 may be summarized as follows: for $|z| < 1$ the generating functions $G(\mu, x)$ are defined by

$$G(\mu, x) = z^{\delta_{1,\mu}} \exp \left[(\mu x - a) \left(1 - \sum_{n=0}^{\infty} \pi_n K(z)^n \right) \right], \quad (3.1)$$

where $K(z)$ is the unique root of the equation

$$\theta = z \exp \left[- 2a \left(1 - \sum_{n=0}^{\infty} \pi_n \theta^n \right) \right] \quad (3.2)$$

in the interior of the unit circle, $|\theta| = 1$. We shall now examine the limiting behaviour of $G(\mu, x)$ as z tends to 1 and show that

$$\lim_{z \rightarrow 1} G(\mu, x) < 1 \text{ if } 2a > 1/\bar{\nu}, x \neq \mu a, \quad (3.3)$$

$$\lim_{z \rightarrow 1} G(\mu, x) = 1 \text{ otherwise,} \quad (3.4)$$

where $\bar{\nu} = \sum_{n=0}^{\infty} n\pi_n$ is the mean number of additional particle pairs emitted per collision. Thus there is a certain "critical" length of rod, $t_c = 1/\bar{\nu}$, beyond which there is a nonzero probability $[1 - \lim_{z \rightarrow 1} G(\mu, x)]$ that a single initial particle at x with direction μ will eventually produce an infinite number of descendants. The critical length is usually derived (see Bellman, Kalaba, and Wing³ by considering the equation for the mean number of particles produced [derived on the assumption that $\lim_{z \rightarrow 1} G(\mu, x) = 1$] and taking as the critical length that length for which the solution first becomes infinite. As we shall see, this procedure leads to the same value, $1/\bar{\nu}$, of t_c . The fact that the mean number of particles produced by a single ancestor is infinite for rod lengths greater than t_c is clearly implied by (3.3) but not conversely.

To prove (3.3) and (3.4) we first observe that if $x = \mu a$, then $G(\mu, x) = z^{\delta_{1,\mu}}$, so that $\lim_{z \rightarrow 1} G(\mu, x) = 1$ for all values of a . It will be assumed in what follows that this trivial case is excluded, i.e., $x \neq \mu a$.

Now if $2a > 1/\sqrt{\nu}$, then by Lagrange's theorem, equation (3.2) has a unique root $\theta = \psi(z)$ such that $|\psi(z)| < 1$ for $|z| \leq 1$. By the uniqueness of $K(z)$ and $\psi(z)$ we must have

$$\psi(z) = K(z) \text{ for all } z \text{ such that } |z| < 1. \quad (3.5)$$

Hence, $\lim_{z \rightarrow 1} K(z) = \lim_{z \rightarrow 1} \psi(z)$ so that

$$\left| \lim_{z \rightarrow 1} K(z) \right| = \left| \lim_{z \rightarrow 1} \psi(z) \right| < 1. \quad (3.6)$$

From this last result, it follows from equation (3.1) that

$$\lim_{z \rightarrow 1} G(\mu, x) < 1 \text{ if } 2a > 1/\sqrt{\nu}.$$

This completes the proof of (3.3).

To prove (3.4) we observe from (2.5) that $K(1)$ must be real and less than or equal to 1; $K(1)$ must also satisfy the equation

$$K(1) = \exp \left[-2a \left\{ 1 - \sum_{n=0}^{\infty} \pi_n K(1)^n \right\} \right]. \quad (3.7)$$

However, if $2a \leq 1/\sqrt{\nu}$, the only real root of this equation which is less than or equal to 1 is $K(1) = 1$. Substituting in equation (3.1) gives

$$\lim_{z \rightarrow 1} G(\mu, x) = 1 \text{ if } 2a \leq 1/\sqrt{\nu},$$

thus completing the proof of the result (3.4)

Now let us consider the behaviour of the mean number, $M(\mu, x)$, of particles ultimately emerging from the right-hand end of the rod conditional upon a single particle originating at x with direction μ , and conditional upon the total number of particles emerging from the right-hand end of the rod being finite. In terms of the generating function $G(\mu, x)$ we have

$$M(\mu, x) = \left\{ [G(\mu, x)]^{-1} \frac{\partial}{\partial z} G(\mu, x) \right\}_{z=1}. \quad (3.8)$$

Hence

$$M(\mu, x) = \delta_{1,\mu} + \frac{(a - \mu x) \sum n \pi_n \gamma^n}{1 - 2a \sum n \pi_n \gamma^n}. \quad (3.9)$$

where $\gamma = \lim_{z \rightarrow 1} K(z)$. Setting $\gamma = 1$ in (3.9) we get

$$M(\mu, x) = \delta_{1, \mu} + \frac{(a - \mu x) \bar{\nu}}{1 - 2a\bar{\nu}}, \quad (3.10)$$

which is the solution of the equation for the mean:

$$\frac{d}{dx} M(\mu, x) = -\mu \bar{\nu} [M(-\mu, x) + M(\mu, x)], \quad (3.11)$$

with boundary conditions

$$M(\mu, a\mu) = \delta_{1, \mu}. \quad (3.12)$$

The reason for the discrepancy between (3.9) and (3.10) is that equation (3.11) is obtained by differentiating (2.3) with respect to z and setting $z = 1$ and $G(\mu, x) = 1$. It therefore ceases to be valid in the supercritical case when $\lim_{z \rightarrow 1} G(\mu, x) < 1$; this is indicated by the fact that when $2a > 1/\bar{\nu}$, $M(\mu, x)$ as given by (3.10) becomes negative (for further discussion see also Moyal⁵⁵).

In the model of Bellman, Kalaba, and Wing³ only two particles emerge from each collision (one moving in each direction), and the equation corresponding to (3.11) for the mean number of emergent particles [based on the assumption that $\lim_{z \rightarrow 1} G(\mu, x) = 1$] is

$$\frac{d}{dx} M(\mu, x) = -\mu M(-\mu, x), \quad (3.13)$$

with the boundary conditions

$$M(\mu, a\mu) = \delta_{1, \mu}. \quad (3.14)$$

The solution of equations (3.13) and (3.14) can be written

$$M(\mu, x) = \delta_{1, \mu} \frac{\cos(a+x)}{\cos 2a} + \delta_{-1, \mu} \frac{\sin(a+x)}{\cos 2a}. \quad (3.15)$$

Bellman, Kalaba, and Wing took the critical length for their model to be the smallest rod length for which the mean number of particles produced by a single ancestor [calculated on the assumption that $\lim_{z \rightarrow 1} G(\mu, x) = 1$] becomes

infinite. Hence, from (3.15) this criterion gives a critical length $t_c = \pi/2$. Applying the same argument to the model considered previously, we find

from (3.10) that the critical length is $t_c = 1/\bar{\nu}$. Thus we arrive at the same value as was previously derived by considering the behaviour of $\lim_{z \rightarrow 1} G(\mu, x)$.

4. Criticality in the Model of Bellman, Kalaba, and Wing

It has been shown that for the model of Sect. 2 there is a critical length $t_c = 1/\bar{\nu}$ such that

$$\lim_{z \rightarrow 1} G(\mu, x) < 1 \text{ if } 2a < t_c, x \neq \mu a; \quad (4.1)$$

$$\lim_{z \rightarrow 1} G(\mu, x) = 1 \text{ otherwise.} \quad (4.2)$$

The same kind of behaviour will now be established for the model of Bellman, Kalaba, and Wing in which, at each collision, the incident particle is lost and two new particles emerge (one in each direction) from the point of collision.

Instead of considering, as in Sect. 2, the probability $P_n(\mu, x)$ that a single particle originating at x ($-a \leq x \leq a$) with direction μ ultimately generates n particles emerging from the right-hand end of the rod, it is more convenient now to consider the probability $Q_n(\mu, x)$ that the total number of particles ultimately emerging from the rod is n . The corresponding generating function is defined by

$$H(\mu, x) = \sum_{n=0}^{\infty} Q_n(\mu, x) z^n, \quad |z| \leq 1.$$

Clearly

$$\sum_{n=0}^{\infty} P_n(\mu, x) = \sum_{n=0}^{\infty} Q_n(\mu, x) = \lim_{z \rightarrow 1} H(\mu, x)$$

so that it will be sufficient to prove (4.1) and (4.2) with G replaced by H . In fact, we shall prove the following:

$$\lim_{z \rightarrow 1} H(\mu, x) < 1 \text{ if } 2a > \pi/2, x \neq \mu a; \quad (4.3)$$

$$\lim_{z \rightarrow 1} H(\mu, x) = 1 \text{ otherwise.} \quad (4.4)$$

The critical length t_c is thus $\pi/2$, as already found by Bellman, Kalaba, and Wing from their study of the mean number of particles produced by a single ancestor (see Sect. 3).

Proceeding as in Sect. 2, we find that the equations for the generating functions $H(\mu, x)$ are

$$\frac{d}{dx} H(\mu, x) = \mu H(\mu, x) [1 - H(-\mu, x)]; \quad (4.5)$$

$$H(\mu, \mu a) = z. \quad (4.6)$$

Introducing new functions $u(x, z)$ and $v(x, z)$ defined by

$$u(x, z) = \frac{1}{2} [H(1, x) + H(-1, x)]; \quad (4.7)$$

$$v(x, z) = \frac{1}{2} [H(1, x) - H(-1, x)], \quad (4.8)$$

we find from (4.5) that

$$\frac{du}{dx} = v; \quad (4.9)$$

$$\frac{dv}{dx} = u - u^2 + v^2. \quad (4.10)$$

Inspection of equation (4.5) shows that

$$\log H(1, x) - H(1, x) + \log H(-1, x) - H(-1, x) = \chi(z), \quad (4.11)$$

where $\chi(z)$ is independent of x .

Suppose that $H(1, 0) = \alpha(z)$. Then by symmetry $H(-1, 0)$ is also equal to $\alpha(z)$. Since a particle which starts at $x = a$ moving to the right can have no collisions, we must have $H(1, a) = z$. We denote by $\beta(z)$ the corresponding function $H(-1, a)$. For real nonnegative values of z less than or equal to 1, equation (4.5) shows that

$$\beta(z) \leq H(-1, x) \leq \alpha(z) \leq H(1, x) \leq z \text{ for all } x \geq 0. \quad (4.12)$$

Thus in order to prove (4.3), it will be sufficient [using (4.5) and (4.12)] to show that

$$\lim_{z \rightarrow 1} \alpha(z) < 1 \text{ for } 2a \leq \pi/2. \quad (4.13)$$

To prove (4.4) it will be sufficient [using (4.5) and (4.12)] to show that

$$\lim_{z \rightarrow 1} \alpha(z) = 1 \text{ for } 2a \leq \pi/2. \quad (4.14)$$

Substituting the values of $H(1, x)$ and $H(-1, x)$ at $x = 0$ and $x = a$ in equation (4.11), we find that

$$2[\log \alpha(z) - \alpha(z)] = \log [z\beta(z)] - \beta(z) - z = \chi(z). \quad (4.15)$$

Rewriting equation (4.11) in terms of $u(x, z)$ and $v(x, z)$ and using the relation between $\alpha(z)$ and $\chi(z)$, we obtain

$$u^2 - v^2 = \alpha^2 e^{-2(\alpha-u)}, \quad (4.16)$$

and hence from (4.9),

$$\frac{du}{dx} = \sqrt{u^2 - \alpha^2 e^{-2(\alpha-u)}}, \quad (4.17)$$

the positive value of the square root being appropriate if z is real, positive, and less than or equal to 1, and if $x \geq 0$.

Integration of equation (4.17) between the limits $x = 0$ and $x = a$ leads to the relationship (for $0 < z < 1$)

$$a = \int_{\alpha(z)}^{\frac{1}{2}[z+\beta(z)]} \frac{dy}{\sqrt{y^2 - \alpha^2 e^{2(y-\alpha)}}}, \quad (4.18)$$

where

$$z\beta e^{-(z+\beta)} = \alpha^2 e^{-2\alpha}.$$

We shall now use equation (4.18) to deduce the limiting behaviour of $\alpha(z)$ as z tends to 1 (from below).

Suppose that z is real and positive and $z \leq 1$, so that $\alpha(z)$ and $\beta(z)$ are both real and positive and $\alpha \leq 1$, $\beta \leq 1$. Define $\gamma(z)$ to be the unique real root of the equation.

$$\gamma e^{-(1+\gamma)} = \alpha^2 e^{-2\alpha}$$

such that $\gamma \leq 1$ for $z \leq 1$. It then follows from the relations

$$\beta z e^{-(z+\beta)} = \alpha^2 e^{-2\alpha} \text{ and } \beta(z) \leq z$$

that $z + \beta(z) \leq 1 + \gamma(z)$.

We wish to show that $\lim_{z \rightarrow 1} \alpha(z) < 1$ if $2a > \pi/2$ and that $\lim_{z \rightarrow 1} \alpha(z) = 1$ if $2a \leq \pi/2$. Suppose, first, that $\lim_{z \rightarrow 1} \alpha(z) = 1$. Then from equation (4.18)

$$\begin{aligned} a &= \lim_{z \rightarrow 1} \int_{\alpha(z)}^{\frac{1}{2}[z+\beta(z)]} \frac{dy}{\sqrt{y^2 - \alpha^2 e^{2(y-\alpha)}}} \\ &\leq \lim_{z \rightarrow 1} \int_{\alpha(z)}^{\frac{1}{2}[1+\gamma(z)]} \frac{dy}{\sqrt{y^2 - \alpha^2 e^{2(y-\alpha)}}} \\ &= \lim_{\delta \rightarrow 0} \int_{1-\delta}^{1-\delta/\sqrt{2}} \frac{dy}{\sqrt{y^2 - (1-\delta)^2 e^{2(y-1+\delta)}}} \\ &= \lim_{\delta \rightarrow 0} \int_{\delta/\sqrt{2}}^{\delta} \frac{dv}{\sqrt{\delta^2 - v^2}} = \frac{\pi}{4}. \end{aligned}$$

Thus, if $\lim_{z \rightarrow 1} \alpha(z) = 1$, then $a \leq \pi/4$, i.e., if $2a > \pi/2$, then $\lim_{z \rightarrow 1} \alpha(z) < 1$.

Conversely, if $\lim_{z \rightarrow 1} \alpha(z) = \alpha(1) < 1$, then

$$a = \int_{\alpha(1)}^{\frac{1}{2}[1+\beta(1)]} \frac{dy}{\sqrt{y^2 - \alpha(1)^2 e^{2[y-\alpha(1)]}}} > \frac{\pi}{4};$$

hence $\lim_{z \rightarrow 1} \alpha(z) = 1$ if $2a \leq \pi/2$. This completes the proof of the properties (4.1) and (4.2) for the model of Bellman, Kalaba, and Wing.

5. A Model Which Includes Scattering and Absorption

Finally, we consider a model which is a generalization of a model previously considered by Harris.⁵⁸ We suppose that as a result of each

collision, ν particles are emitted, where ν is a random variable taking the values $0, 1, 2, \dots$, with respective probabilities $\pi_0, \pi_1, \pi_2, \dots$. Each particle emerging after a collision moves to the right or left with probability $\frac{1}{2}$ independently of the other particles. We observe that π_0 is the probability that the colliding particle is absorbed, π_1 is the probability that it is scat-

tered, and $\sum_{n=2}^{\infty} \pi_n$ is the probability that multiplication occurs. It is also of

interest to notice that the process can be considered as the projection on the x -axis of the corresponding process in the three-dimensional slab $\{x: -a \leq x \leq a\}$ when scattering is isotropic and the projected distances between collisions are assumed to be independently distributed with a negative exponential distribution. Such an approximation was used by Fermi²¹ and will be improved upon in Chapter V in the case where there is no multiplication of particles.

Defining the generating functions $H(\mu, x)$, $u(x, z)$, and $v(x, z)$ exactly as in Sect. 4, we find that they satisfy the equations

$$\mu \frac{d}{dx} H(\mu, x) = H(\mu, x) - \sum_{n=0}^{\infty} 2^{-n} \pi_n [H(\mu, x) + H(-\mu, x)]^n; \quad (5.1)$$

$$H(\mu, \mu a) = z, \quad (5.2)$$

Or, in terms of u and v ,

$$\frac{d}{dx} u(x, z) = v(x, z); \quad (5.3)$$

$$\frac{d}{dx} v(x, z) = u(x, z) - \sum_{n=0}^{\infty} \pi_n u(x, z)^n, \quad (5.4)$$

with boundary conditions

$$u(a, z) + v(a, z) = z; \quad (5.5)$$

$$v(0, z) = 0. \quad (5.6)$$

We shall suppose throughout that the distribution defined by the probabilities π_n has a mean,

$$m = \sum_{n=0}^{\infty} n \pi_n > 1,$$

and that z is real, positive, and less than or equal to 1. If we consider a particle which starts at a point x and moves and multiplies on the real line $(-\infty, \infty)$ in the manner described, then a well-known result in the theory of branching processes gives the probability that it ultimately produces only a finite number of descendants as the unique real root, p , in $[0, 1)$ of the equation

$$p = \sum_{n=0}^{\infty} \pi_n p^n. \quad (5.7)$$

For the process in the finite rod $[-a, a]$ it follows that $u(x, 1) > p$, $-a \leq x \leq a$. Thus there exists $\epsilon (0 < \epsilon \leq 1)$ such that $p < u(0, z) < 1$ provided $1 - \epsilon < z < 1$. Noting that

$$y - \sum_{n=0}^{\infty} \pi_n y^n > 0 \text{ if } p < y < 1, \quad (5.8)$$

it can easily be shown from (5.3) and (5.4) that for all x in $(0, a]$

$$v(x, z) > 0 \text{ provided } 1 - \epsilon < z < 1. \quad (5.9)$$

We shall assume in all that follows that z belongs to the interval $(1 - \epsilon, 1)$. Then, writing $u(0, z) = \tau(z)$ and $u(a, z) = \rho(z)$, it follows from (5.3) and (5.4) that

$$v(x, z) = \left\{ u(x, z)^2 - \tau(z)^2 - 2 \sum_{n=0}^{\infty} \frac{\pi_n}{n+1} [u(x, z)^{n+1} - \tau(z)^{n+1}] \right\}^{1/2}; \quad (5.10)$$

$$x = \int_{\tau(z)}^{u(x, z)} \left\{ y^2 - \tau(z)^2 - 2 \sum_{n=0}^{\infty} \frac{\pi_n}{n+1} [y^{n+1} - \tau(z)^{n+1}] \right\}^{-1/2} dy, \quad (5.11)$$

the positive value of the square root being appropriate for $x \geq 0$. The unknown functions $\tau(z)$ and $\rho(z)$ are uniquely determined (in the range $1 - \epsilon < z < 1$) by the boundary conditions (5.5) and (5.6), which can be written

$$z = \rho(z) + \left\{ \rho(z)^2 - \tau(z)^2 - 2 \sum_{n=0}^{\infty} \frac{\pi_n}{n+1} [\rho(z)^{n+1} - \tau(z)^{n+1}] \right\}^{1/2}; \quad (5.12)$$

$$a = \int_{\tau(z)}^{\rho(z)} \left\{ y^2 - \tau(z)^2 - 2 \sum_{n=0}^{\infty} \frac{\pi_n}{n+1} [y^{n+1} - \tau(z)^{n+1}] \right\}^{-1/2} dy, \quad (5.13)$$

and the additional condition, from (5.5) and (5.9), that $z > \rho(z) > \tau(z)$. Equations (5.12) and (5.13) correspond to the equations (4.18) which were derived in Sect. 4 for the model of Bellman, Kalaba, and Wing. Notice that if $\pi_0 = 0$ then equations (5.10)-(5.13) are valid for all real z in $0 < z < 1$.

The arguments of Sect. 4 can be applied to equations (5.12) and (5.13) to deduce the limiting behaviour of $\tau(z)$ and $H(\mu, x)$ as z tends to 1. Proceeding in this way (recalling that $m > 1$) it is found that

$$\lim_{z \rightarrow 1} H(\mu, x) < 1 \text{ if } 2a > \frac{2}{\sqrt{m-1}} \arcsin m^{-1/2}, x \neq \mu a; \quad (5.14)$$

$$\lim_{z \rightarrow 1} H(\mu, x) = 1 \text{ otherwise.} \quad (5.15)$$

(If $m \leq 1$, then $\lim_{z \rightarrow 1} H(\mu, x) = 1$ for all values of a .)

Furthermore, we can use equations (5.12) and (5.13) to determine $\lim_{z \rightarrow 1} H(\mu, x)$ numerically for rods of any length. If a value of $\tau(1)$ [such that

$p < \tau(1) < 1$] is assumed, the corresponding value of $\rho(1)$ is determined by (5.12) and the corresponding rod length $2a$ is then found from (5.13) by numerical integration. Knowing the value of $\tau(1)$ corresponding to a given rod length, we can use (5.10) and (5.11) to determine $u(x, 1)$, $v(x, 1)$, and hence $\lim_{z \rightarrow 1} H(\mu, x)$ for a rod of this length. We observe in particular that as

the rod length, $2a$, tends to infinity, $\tau(1)$ tends to p and $\rho(1)$ tends to the unique root ρ (such that $p < \rho < 1$) of the equation

$$1 = \rho + \left\{ \rho^2 - p^2 - 2 \sum_{n=0}^{\infty} \frac{\pi_n}{n+1} [\rho^{n+1} - p^{n+1}] \right\}^{1/2}. \quad (5.16)$$

Figure 1 shows the graph of $\lim_{z \rightarrow 1} H(-1, a)$ determined as described above for the particular case in which exactly two particles are emitted at each collision. This is the model considered by Harris.⁵⁸ The corresponding function for the model of Sect. 4 is also shown. This was derived by applying exactly the same technique to equations (4.16), (4.17), and (4.18).

Fig. 1

The Probability That a Single Particle Incident at One End of a Rod of Length $2a$ Produces a Finite Number of Emergent Particles. The upper curve, which is asymptotic to the value 0.11575, refers to the model of Harris; the lower curve, which is asymptotic to zero, refers to the model of Bellman, Kalaba, and Wing.

and hence, from (5.13),

$$a = \frac{1}{\sqrt{m-1}} \arcsin m^{-1/2} + \delta \frac{(s-m)(3m-1)}{6\sqrt{m^3(m-1)^3}} + o(\delta). \quad (5.17)$$

This generalizes a result which was obtained by Harris⁵⁸ for the particular case in which exactly two particles are emitted at each collision.

Let us denote by $1 - \delta$ the value of $\tau(1)$ for a rod whose length, $2a$, is slightly greater than the critical length, $\frac{2}{\sqrt{m-1}} \arcsin m^{-1/2}$.

If the distribution defined by the probabilities π_n possesses a second moment

$$s = \sum_{n=0}^{\infty} n^2 \pi_n,$$

we can use equation (5.12) to derive the asymptotic expression

$$\rho(1) = 1 + k_1 \delta + k_2 \delta^2 + o(\delta^2),$$

where $k_1 = -\sqrt{1-m^{-1}}$ and

$$k_2 = \frac{s-m}{6\sqrt{m^2-m}} [1 - (1-m^{-1})^{3/2}],$$

CHAPTER V

GENERALIZED ONE-DIMENSIONAL SCATTERING PROCESSES
WITH APPLICATIONS IN THREE DIMENSIONS1. Introduction

Consider a particle which moves along the x -axis starting from a point x in the interval $[a_{-1}, a_1]$. It continues to move with constant direction until it makes a collision, as a result of which its direction of motion is reversed with probability p and unchanged with probability q . The probability that the particle is absorbed at any collision is $r = 1 - p - q$. The distances travelled between successive collisions are assumed to be independently distributed with density function $f(x)$ and characteristic function

$$\phi(\alpha) = \int_0^{\infty} f(x) \exp(i\alpha x) dx.$$

Results for the special case in which the distances between successive collisions have a negative exponential distribution, i.e., $f(x) = \lambda \exp(-\lambda x)$; $\phi(\alpha) = \lambda(\lambda - i\alpha)^{-1}$, have already been derived in Chapter II. It is the purpose of the present chapter to extend these results to include more general distributions for the distances travelled between collisions. In Sect. 2 we determine the characteristic function [for arbitrary $\phi(\alpha)$ and no barriers] of the displacement of the particle at its n^{th} collision, given that it makes at least n collisions before being absorbed. In Sects. 4 and 5 we generalize the results of Chapter II for scattering processes with absorbing boundaries at a_{-1} and a_1 to processes where the distances between collisions have probability density functions belonging to one of the following two classes:

$$f(x) = \frac{\lambda^n x^{n-1}}{(n-1)!} \exp(-\lambda x); \quad (1.1)$$

$$f(x) = \sum_{i=1}^n p_i \lambda_i \exp(-\lambda_i x), \quad 0 \leq p_i \leq 1, \quad \sum_{i=1}^n p_i = 1. \quad (1.2)$$

As an application of these generalized results we shall discuss the method used by Fermi²¹ to study the scattering of thermal neutrons in a three-dimensional plane slab $\{x: a_{-1} \leq x \leq a_1\}$ of paraffin. Fermi supposed that the motion of the neutrons is as follows: between collisions each neutron moves in a straight line and in any element δs of its path it has probability $\lambda \delta s + o(\delta s)$ of making a collision, where λ^{-1} is its (constant) mean free path; given that it makes a collision, the neutron is absorbed

with probability r and scattered isotropically with probability $1 - r$. Fermi obtained approximate results for this process by supposing that the projection of the motion on the x -axis is a one-dimensional scattering process of the type studied in Chapter II with collision rate $\lambda\sqrt{3}$ per unit path length (as opposed to λ for the three-dimensional motion) and probabilities r , $\frac{1}{2}(1 - r)$, $\frac{1}{2}(1 - r)$ of absorption, reversal, and continuation, respectively.

This is an approximate procedure because, as Fermi pointed out, the projections of the distances travelled between collisions do not in fact have a negative exponential distribution as do the distances themselves.

For a more precise study of the motion of the projection of the particle on the x -axis, we shall determine in Sect. 6 the true characteristic function, $\phi(\alpha)$, of the projections of the distances travelled between collisions. In the absence of barriers we can then use the results of Sect. 2 to derive the exact characteristic function of the x -displacement at the n th collision. When there are barriers present (at $x = a_{-1}$ and $x = a_1$) we shall use the generalized results of Sect. 5 to improve on Fermi's results for the scattering of particles by a slab of finite thickness.

Fermi's method and the method to be described both apply only to isotropic scattering. For anisotropic scattering, the projected displacements between successive collisions are no longer independent.

2. The General One-dimensional Process with No Boundaries

Consider the following generalization of the process studied in Chapter II. A particle starting from the origin moves in steps X_i , $i = 1, 2, \dots$, along the x -axis, where $\{|X_i|\}$ is a sequence of identically and independently distributed random variables with characteristic function $\phi(\alpha)$. At the end of each step the particle suffers a collision, the probabilities of reversal, continuation, and absorption at each collision being p , q , and r , respectively. For the moment we assume that $r = 0$ (we show at the end of this Section how to adapt the results when $r > 0$) and determine the characteristic function $\sigma_n^{(\mu)}(\alpha)$ of the displacement at the n th collision, given that the initial direction of motion was μ ($\mu = +1$ for motion to the right and -1 for motion to the left); thus

$$\sigma_n^{(\mu)}(\alpha) = E \left\{ \exp \left[i\alpha \sum_{j=1}^n X_j \right] \middle| \mu \right\}. \quad (2.1)$$

Given that m of the first n steps $X_1, \dots, X_n$ have the sign of the initial direction μ , we have

$$\sigma_n^{(\mu)}(\alpha) = [\phi(\mu\alpha)]^m [\phi(-\mu\alpha)]^{n-m}. \quad (2.2)$$

Hence, defining

$$P_{mn}^{(\mu)} = \text{Prob} \{m \text{ of the first } n \text{ steps have the sign of the initial direction, } \mu\} \quad (2.3)$$

(observing that these probabilities are independent of μ), we have

$$\sigma_n^{(\mu)}(\alpha) = \sum_{k=1}^n P_{kn}^{(\mu)} [\phi(\mu\alpha)]^k [\phi(-\mu\alpha)]^{n-k}, \quad (2.4)$$

i.e.,

$$\sigma_n^{(\mu)}(\alpha) = [\phi(-\mu\alpha)]^n P_n^{(\mu)}([\phi(\mu\alpha)][\phi(-\mu\alpha)]^{-1}), \quad n \geq 1, \quad (2.5)$$

where $P_n^{(\mu)}(z) = \sum_{k=0}^n P_{kn}^{(\mu)} z^k$ and $P_{0n}^{(\mu)}$ is defined for convenience as $p^{-1}\delta_{0,n}$.

Consideration of the collision at which the first reversal occurs yields the recurrence relation

$$P_{mn}^{(\mu)} = \sum_{k=1}^m (1-p)^{k-1} p P_{(n-m)(n-k)}^{(\mu)}, \quad 1 \leq m \leq n, \quad (2.6)$$

from which it follows that, if $n \geq 1$,

$$\begin{aligned} P_n^{(\mu)}(z) &= z^n \sum_{m=1}^n \sum_{k=1}^m p(1-p)^{k-1} z^{m-n} P_{(n-m)(n-k)}^{(\mu)} \\ &= z^n \sum_{k=1}^n p(1-p)^{k-1} P_{n-k}^{(\mu)}(z^{-1}), \quad n \geq 1. \end{aligned} \quad (2.7)$$

This equation can be written in the more convenient form

$$P_{n+1}^{(\mu)}(z) - z(1-p) P_n^{(\mu)}(z) = pz^{n+1} P_n^{(\mu)}(z^{-1}), \quad (2.8)$$

together with a second equation which is obtained from (2.8) on replacing z by z^{-1} , namely,

$$P_{n+1}^{(\mu)}(z^{-1}) - z^{-1}(1-p) P_n^{(\mu)}(z^{-1}) = pz^{-n-1} P_n^{(\mu)}(z). \quad (2.9)$$

Elimination of $P_n^{(\mu)}(z^{-1})$ from (2.8) and (2.9) leads to a second-order difference equation

$$P_{n+2}^{(\mu)}(z) - (1-p)(1+z) P_{n+1}^{(\mu)}(z) + z(1-2p) P_n^{(\mu)}(z) = 0. \quad (2.10)$$

Introducing the generating function

$$P^{(\mu)}(z, w) = \sum_{n=1}^{\infty} P_n^{(\mu)}(z) w^n, \quad |w| < 1, \quad |z| \leq 1; \quad (2.11)$$

we find from (2.10) that

$$P^{(\mu)}(z, w) = wz \frac{1 - w(1-2p)}{1 - (1-p)(1+z)w + (1-2p)zw^2}. \quad (2.12)$$

Hence, from equation (2.5), we obtain for the generating function of $\sigma_n^{(\mu)}(\alpha)$,

$$\sum_{n=1}^{\infty} \sigma_n^{(\mu)}(\alpha) t^n = t\phi(\mu\alpha) \frac{1 - (1-2p)t\phi(-\mu\alpha)}{1 - (1-p)[\phi(\mu\alpha) + \phi(-\mu\alpha)]t + (1-2p)\phi(\mu\alpha)\phi(-\mu\alpha)t^2}. \quad (2.13)$$

Notice that if we replace $\phi(\alpha)$ by $\lambda[\lambda - i\alpha]^{-1}$, then $\sigma_n^{(\mu)}(\alpha)$ becomes the characteristic function of the displacement at the n^{th} collision for the model considered in Chapter II. The same generating function could be obtained from equation (4.9), Chapter II, by taking $z = w = t$ and $x = 0$.

If there is a positive probability ($r > 0$) of absorption at each collision, then we define $\tau_n^{(\mu)}(\alpha)$ to be the characteristic function of the displacement at the n^{th} collision, given that the particle makes at least n collisions before being absorbed. Then the generating function

$$\sum_{n=1}^{\infty} \tau_n^{(\mu)}(\alpha) t^n$$

is obtained directly from (2.13) upon replacing p by $p^* = p(1-r)^{-1}$.

3. The One-dimensional First-passage Problem

Suppose that the initial coordinate of the particle in the interval $[a_{-1}, a_1]$ is x and that its initial direction of motion is μ ; $\mu = +1$ for motion to the right and -1 for motion to the left. For distances between successive collisions having probability density function $f(x)$ and for probabilities p , q , and r of reversal, continuation, and absorption, respectively, at a collision, we wish to find the probabilities $P_{rk}^{(\mu)}(a_{\pm 1} | x)$ of first emergence at $a_{\pm 1}$ after exactly r reversals and k continuations.

The solution when $f(x) = \lambda \exp(-\lambda x)$ is given in Chapter II as a generating function

$$M^{(\mu)}(a_\nu | x; z, w) \equiv \sum_{r+k \geq 0} p_{rk}^{(\mu)}(a_\nu | x) z^r w^k. \quad (3.1)$$

For arbitrary $f(x)$ the one-dimensional process is not in general a scattering process as defined in Chapter I, Sect. 2. However, in spite of the non-Markovian nature of the process, the integral equations (4.14) and (4.15) of Chapter I are still satisfied, and it can easily be shown that the generating functions $M^{(\mu)}(a_\nu | x; z, w)$ satisfy

$$\begin{aligned} M^{(\mu)}(a_\nu | x) &= \mu \delta_{\nu\mu} \int_{a_\mu}^{\mu\infty} f(\mu y - \mu x) dy \\ &+ \mu \int_x^{a_\mu} f(\mu y - \mu x) [pz M^{(-\mu)}(a_\nu | y) + qw M^{(\mu)}(a_\nu | y)] dy. \end{aligned} \quad (3.2)$$

Here $\delta_{\nu\mu}$ denotes Kronecker's delta, i.e.,

$$\delta_{\nu\mu} = \begin{cases} 1, & \nu = \mu \\ 0, & \nu \neq \mu. \end{cases}$$

A large number of distributions encountered in practice can be closely approximated by a gamma distribution with suitably chosen parameters or by a linear combination of negative exponential distributions. If $f(x)$ belongs to either of these two families, it is possible to solve equation (3.2). In Sect. 4 we solve (3.2) when $f(x)$ is of the form

$$f(x) = \frac{\lambda^n x^{n-1}}{(n-1)!} \exp(-\lambda x), \quad 0 \leq x < \infty, \quad (3.3)$$

and in Sect. 5 we do the same when $f(x)$ is of the form

$$f(x) = \sum_{i=1}^n p_i \lambda_i \exp(-\lambda_i x), \quad 0 \leq x < \infty, \quad 0 \leq p_i \leq 1, \quad \sum_{i=1}^n p_i = 1. \quad (3.4)$$

Both families include the negative exponential distribution as a special case.

Although further generalizations of the results of Chapter II can be derived in a similar manner, we shall restrict ourselves to the determination of $M^{(\mu)}(a_\nu | x)$, which is sufficient for the discussion of Fermi's model in Sect. 6.

4. First-passage Probabilities When the Distances between Collisions Have a Gamma Distribution

When $f(x)$ is given by equation (3.3), the integral equation (3.2) takes the form

$$\begin{aligned} M^{(\mu)}(a_\nu | x) = & \mu \delta_{\mu\nu} \int_{a_\mu}^{\mu\infty} \frac{\lambda^n \mu^{n-1} (y-x)^{n-1}}{(n-1)!} \exp\{-\lambda\mu(y-x)\} dy \\ & + \mu \int_x^{a_\mu} \frac{\lambda^n \mu^{n-1} (y-x)^{n-1}}{(n-1)!} \exp\{-\lambda\mu(y-x)\} \\ & [pzM^{(-\mu)}(a_\nu | y) + qwM^{(\mu)}(a_\nu | y)] dy. \end{aligned} \quad (4.1)$$

If the symbol D is used to denote the operator $\partial/\partial x$, this equation can be rewritten as

$$[(\lambda - \mu D)^n - qw\lambda^n] M^{(\mu)} - \lambda^n pz M^{(-\mu)} = 0; \quad \mu = \pm 1, \quad (4.2)$$

with boundary conditions

$$M^{(\mu)}(a_\nu | a_\mu) = \delta_{\nu\mu}; \quad (4.3)$$

$$(\lambda - \mu D)^i M^{(\mu)}(a_\nu | x) = \delta_{\nu\mu} \lambda^i \text{ at } x = a_\mu, \quad i = 1, \dots, n-1. \quad (4.4)$$

The general solution of the equations (4.2) is given by

$$M^{(\mu)}(a_\nu | x) = \sum_{i=1}^{2n} A_{\nu i}^{(\mu)}(z, w) \exp(\theta_i x), \quad (4.5)$$

where θ_i , $i = 1, \dots, 2n$, are the roots (assumed distinct) of the equation

$$[(\lambda - \theta)^n - qw\lambda^n][(\lambda + \theta)^n - qw\lambda^n] + \lambda^{2n} p^2 z^2 = 0. \quad (4.6)$$

The terms in the solution (4.5) corresponding to a k -tuple root θ of (4.6) are, of course, of the form $[A_{\nu 1}^{(\mu)} + A_{\nu 2}^{(\mu)}x + \dots + A_{\nu k}^{(\mu)}x^{k-1}] \exp(\theta x)$.

Equation (4.2) can be used to express $A_{\nu i}^{(\mu)}$ in terms of $A_{\nu 1}^{(\mu)}$, $i = 1, \dots, 2n$, and the $2n$ boundary conditions (4.3) and (4.4) then determine the coefficients $A_{\nu i}^{(\mu)}(z, w)$ completely.

Although this solves the first-passage problem in principle, it will generally be difficult, especially for a gamma distribution with large parameter n , to obtain an explicit solution in terms of z and w . However, the probabilities of emergence at either boundary,

$$P^{(\mu)}(a_\nu | x) = \sum_{r+k \geq 0} P_{rk}^{(\mu)}(a_\nu | x) = [M^{(\mu)}(a_\nu | x)]_{z=w=1}, \quad (4.7)$$

are easily found by setting $z = w = 1$ in equations (4.2), (4.3), and (4.4), and solving numerically. Thus $P^{(\mu)}(a_\nu | x)$ satisfies

$$[(\lambda - \mu D)^n - q\lambda^n] P^{(\mu)} - p\lambda^n P^{(-\mu)} = 0, \quad (4.8)$$

with boundary conditions

$$P^{(\mu)}(a_\nu | a_\mu) = \delta_{\nu\mu}; \quad (4.9)$$

$$(\lambda - \mu D)^i P^{(\mu)}(a_\nu | x) = \delta_{\nu\mu} \lambda^i \text{ at } x = a_\mu, \quad i = 1, \dots, n-1. \quad (4.10)$$

The solution of these equations is given, for $p + q < 1$, by

$$P^{(\mu)}(a_\nu | x) = \sum_{i=1}^{2n} A_{\nu i}^{(\mu)} \exp(\theta_i x), \quad (4.11)$$

where θ_i , $i = 1, \dots, 2n$ are the roots of (4.6) with $z = w = 1$. If $p + q = 1$, then $\theta = 0$ is a double root of (4.6) and the solution is of the form

$$P^{(\mu)}(a_\nu | x) = A_{\nu 1}^{(\mu)} + A_{\nu 2}^{(\mu)} x + \sum_{i=3}^{2n} A_{\nu i}^{(\mu)} \exp(\theta_i x). \quad (4.12)$$

The equations for the coefficients $A_{\nu i}^{(\mu)}$ are obtained as above and can be solved numerically.

The moments of the joint distribution of the number of reversals and continuations before emergence can be obtained by finding the appropriate partial derivatives of (4.2), (4.3), and (4.4) with respect to z and w , setting $z = w = 1$, and solving numerically. For example, the mean number of reversals before emergence, conditional on emergence at a_ν , is given by

$$m = [P^{(\mu)}(a_\nu | x)]^{-1} \left[\frac{\partial}{\partial z} M^{(\mu)}(a_\nu | x) \right]_{z=w=1} \quad (4.13)$$

5. Linear Combinations of Negative Exponential Distributions

In this section we determine the generating functions $M^{(\mu)}(a_\nu | x)$ for density functions $f(x)$ of the form

$$f(x) = \sum_{i=1}^n p_i \lambda_i \exp(-\lambda_i x), \quad 0 \leq p_i \leq 1, \quad \sum_{i=1}^n p_i = 1. \quad (5.1)$$

For such a function $f(x)$, equation (3.2) becomes

$$\begin{aligned} M^{(\mu)}(A_\nu | x) = & \mu \delta_{\mu\nu} \int_{a_\mu}^{\mu\infty} \sum_{i=1}^n p_i \lambda_i \exp\{\lambda_i \mu(x-y)\} dy \\ & + \mu \int_x^{a_\mu} \sum_{i=1}^n p_i \lambda_i \exp\{\lambda_i \mu(x-y)\} [pz M^{(-\mu)}(a_\nu | y) \\ & + qw M^{(\mu)}(a_\nu | y)] dy. \end{aligned} \quad (5.2)$$

To solve (5.2) we first determine the constants $c_1, c_2, \dots, c_n$ satisfying the equations

$$c_n \lambda_i^n + c_{n-1} \lambda_i^{n-1} + \dots + c_1 \lambda_i + 1 = 0, \quad i = 1, 2, \dots, n. \quad (5.3)$$

With this set of constants c_k we deduce from (5.2) that $M^{(\mu)}(a_\nu | x)$ satisfies the differential equation

$$[c_n (\mu D)^n + c_{n-1} (\mu D)^{n-1} + \dots + c_1 \mu D + 1 + \phi(\mu D) qw] M^{(\mu)} + \phi(\mu D) pz M^{(-\mu)} = 0, \quad (5.4)$$

where D again denotes the operator $\partial/\partial x$ and

$$\phi(\theta) = c_1 \sum_1^n p_i \lambda_i + c_2 \left[\sum_1^n p_i \lambda_i^2 + \sum_1^n p_i \lambda_i \theta \right] + \dots + c_n \left[\sum_1^n p_i \lambda_i^n + \dots + \sum_1^n p_i \lambda_i \theta^{n-1} \right].$$

The boundary conditions to be satisfied by $M^{(\mu)}(a_\nu | x)$ are

$$M^{(\mu)}(a_\nu | a_\mu) = \delta_{\nu\mu}; \quad (5.5)$$

$$\begin{aligned} (\mu D)^j M^{(\mu)}(a_\nu | x) = & \delta_{\nu\mu} \sum_{i=1}^n \lambda_i^j p_i - \sum_{r=1}^j \left\{ (\mu D)^{j-r} [qw M^{(\mu)}(a_\nu | x) \right. \\ & \left. + pz M^{(-\mu)}(a_\nu | x)] \sum_{i=1}^n \lambda_i^r p_i \right\} \text{ at } x = a_\mu, \quad j = 1, \dots, n-1. \end{aligned} \quad (5.6)$$

The general solution of equation (5.4) is given by

$$M^{(\mu)}(a_\nu | x) = \sum_{i=1}^{2n} A_{\nu i}^{(\mu)}(z, w) \exp(\theta_i x), \quad (5.7)$$

where $\theta_1, \dots, \theta_{2n}$ are the roots (assumed distinct) of the equation

$$[c_n \theta^n + \dots + c_1 \theta + 1 + qw\phi(\theta)][c_n(-\theta)^n + \dots + c_1(-\theta) + 1 + qw\phi(-\theta)] - p^2 z^2 \phi(\theta) \phi(-\theta) = 0. \quad (5.8)$$

Multiple roots of equation (5.8) are treated in the same way as in Sect. 4. From equation (5.4), $A_{\nu i}^{(-1)}$ can be expressed in terms of $A_{\nu i}^{(1)}$ for each i and the $2n$ boundary conditions (5.5) and (5.6) then determine the coefficients $A_{\nu i}^{(\mu)}$ completely.

As in Sect. 4 an explicit solution for $M^{(\mu)}(a_\nu | x)$ in terms of z and w will in general be extremely complicated; however, the probabilities of emergence, $P^{(\mu)}(a_\nu | x)$, and the moments of the joint distribution of the numbers of reversals and continuations before emergence can easily be found numerically as before.

An illustration of the use of equation (5.4) and its solution will be given in Sect. 7, where we discuss Fermi's method for determining the probabilities of reflection and transmission of thermal neutrons by a slab of paraffin.

6. Three-dimensional Scattering with No Boundaries

We now consider Fermi's model for three-dimensional scattering, which was described in the introduction. As indicated in Sect. 1, Fermi's approximation for the projected motion of the particle on the x -axis supposes that the projected displacements between successive collisions are independently distributed with probability density function

$$f_1^*(x) = \sqrt{3} \lambda \exp(-\sqrt{3} \lambda x), \quad 0 \leq x < \infty, \quad (6.1)$$

where λ^{-1} is the mean free path of the particle in three dimensions. The characteristic function corresponding to $f_1^*(x)$ is

$$\phi_1^*(\alpha) = \int_0^\infty \exp(i\alpha x) f_1^*(x) dx = \sqrt{3} \lambda (\sqrt{3} \lambda - i\alpha)^{-1}. \quad (6.2)$$

The exact density function $f(x)$ for the projected displacements can, in fact, be found as follows. Suppose that the particle has just suffered a collision and moves off in a direction whose angle with the positive x -axis has cosine μ , $-1 \leq \mu \leq 1$. The probability that the projection on the x -axis of the distance travelled to the next collision lies in $(x, x + \delta x)$ is $(\lambda/|\mu|) \exp\{-\lambda x/|\mu|\} \delta x + o(\delta x)$. Since scattering is assumed to be isotropic, $|\mu|$ is a random variable uniformly distributed on $[0, 1]$. Hence,

$$f(x) = \lambda \int_0^1 \theta^{-1} \exp\left(-\frac{\lambda x}{\theta}\right) d\theta, \quad 0 \leq x < \infty, \quad (6.3)$$

and the corresponding characteristic function is

$$\phi(\alpha) = \int_0^\infty \exp(i\alpha x) f(x) dx = -\frac{\lambda}{i\alpha} \log\left(1 - \frac{i\alpha}{\lambda}\right). \quad (6.4)$$

Expanding (6.2) and (6.4) in powers of $i\alpha$, we see that the exact distribution and Fermi's approximating distribution have the same mean square, but their means and all the other moments about the origin are different. In fact, for isotropic scattering, as we shall see later, it is the mean square which determines the asymptotic behaviour of the particle after a large number of collisions.

When there are no boundaries, the exact characteristic function $\tau_n^{(\mu)}(\alpha)$ for the x -displacement at the n^{th} collision, given that the particle experiences at least n collisions before being absorbed, can be found immediately from equation (2.13). Since we are assuming isotropic scattering, we replace p in equation (2.13) by $p^* = \frac{1}{2}$. For an isotropic source of particles at $x = 0$, the appropriate characteristic function is

$$\tau_n(\alpha) = \frac{1}{2}[\tau_n^{(1)}(\alpha) + \tau_n^{(-1)}(\alpha)],$$

which is found to be [expanding (2.13) as a power series in t]

$$\tau_n(\alpha) = \left[\frac{\lambda}{2i\alpha} \log \frac{\lambda + i\alpha}{\lambda - i\alpha} \right]^n. \quad (6.5)$$

The corresponding Fermi approximation is obtained from (2.13) by using, instead of the exact characteristic function $\phi(\alpha)$ given by (6.4), the corresponding approximation $\phi_1^*(\alpha)$ given by (6.2). This leads to the approximation of $\tau_n(\alpha)$ by the function

$$\tau_n^*(\alpha) = \left[\frac{3\lambda^2}{3\lambda^2 + \alpha^2} \right]^n. \quad (6.6)$$

The distributions corresponding to $\tau_n(\alpha)$ and $\tau_n^*(\alpha)$ are compared most conveniently by considering their moments. Thus, if S_n is a random variable with characteristic function $\tau_n(\alpha)$ and S_n^* is a random variable with characteristic function $\tau_n^*(\alpha)$, then

$$E\{S_n^{2k+1}\} = 0; \quad E\{S_n^{*2k+1}\} = 0, \quad k = 0, 1, 2, \dots;$$

$$E\{S_n^2\} = \frac{2n}{3} \lambda^{-2}; \quad E\{S_n^{*2}\} = \frac{2n}{3} \lambda^{-2};$$

$$E\{S_n^4\} = \left[\frac{4n^2}{3} + \frac{52n}{15} \right] \lambda^{-4}; \quad E\{S_n^{*4}\} = \left[\frac{4n^2}{3} + \frac{4n}{3} \right] \lambda^{-4},$$

and for large n the moments differ only by terms which are $O(n^{k-1})$, i.e.,

$$E\{S_n^{2k}\} = \frac{2k!}{k!} \left(\frac{n}{3\lambda^2} \right)^k + O(n^{k-1}); \quad E\{S_n^{*2k}\} = \frac{2k!}{k!} \left(\frac{n}{3\lambda^2} \right)^k + O(n^{k-1}),$$

$$k = 1, 2, \dots$$

From equations (5.9) and (5.10) we see that the displacements S_n and S_n^* are both distributed as sums of independent random variables. We can therefore apply the central limit theorem to deduce that, as $n \rightarrow \infty$, $n^{-1/2}S_n$ and $n^{-1/2}S_n^*$ are asymptotically normally distributed. In fact, the asymptotic distributions are identical, both having mean zero and variance $\frac{2}{3}\lambda^{-2}$. This suggests that Fermi's approximation is a good one when the number of collisions is large. The parameter $\lambda\sqrt{3}$ of his approximating negative exponential distribution is, in fact, determined uniquely by the condition that $S_n^*/\sqrt{n}$ and $S_n/\sqrt{n}$ have the same asymptotic behaviour.

7. Boundary-value Problems in Three Dimensions

Let us consider again the three-dimensional model described in the introduction, this time, however, with the planes $x = a_{-1}$ and $x = a_1$ as absorbing boundaries. The particle starts at a point with x -coordinate $x \in [a_{-1}, a_1]$ and continues to move as described in the introduction until it is either absorbed at a collision or crosses one of the boundaries, after which it is lost to the system. The problem is to determine the probabilities that the particle eventually crosses one or another of the boundaries, and to investigate the distribution of the number of collisions made before doing so.

As in Sect. 6, we consider the projection of the motion on the x -axis; this reduces the problem to the one-dimensional first-passage problem of Sect. 3, in which the density function $f(x)$ is given by equation (6.3), i.e.,

$$f(x) = \lambda \int_0^1 \theta^{-1} \exp(-\lambda x \theta^{-1}) d\theta, \quad 0 \leq x < \infty. \quad (7.1)$$

The solution of the basic equations (3.2) for this particular density function is very difficult; the approach used by Fermi was to replace $f(x)$ by an approximating function

$$f_1^*(x) = \sqrt{3} \lambda \exp(-\sqrt{3} \lambda x), \quad (7.2)$$

in which case the equations (3.2) are relatively simple to solve.

It is possible to improve on Fermi's procedure by using the results of Sect. 5. We seek an improved approximating function

$$f_n(x) = \sum_{i=1}^n \lambda_i p_i \exp(-\lambda_i x) \quad (7.3)$$

for the exact density function $f(x)$, choosing the p_i and λ_i by fitting moments about the origin of $f_n(x)$ to those of $f(x)$. Taking $n = 2$, we can fit the first three moments of $f_2(x)$ to those of $f(x)$ by choosing

$$f_2(x) = \frac{1}{2} \lambda_1 \exp(-\lambda_1 x) + \frac{1}{2} \lambda_2 \exp(-\lambda_2 x), \quad 0 \leq x < \infty, \quad (7.4)$$

where $\lambda_1 = \lambda(3 - \sqrt{3})$ and $\lambda_2 = \lambda(3 + \sqrt{3})$.

Table 1 shows for comparison the first five moments of the density functions $f(x)$, $f_1^*(x)$ and $f_2(x)$ about the origin; Fig. 2 shows the graphs of the density functions themselves. Better approximations could be derived by taking $n > 2$, but even $f_2(x)$ gives substantially better approximations than $f_1^*(x)$.

TABLE 1. Moments of the Density Functions
 $f(x)$, $f_1^*(x)$ and $f_2(x)$.

k	k th Moment about the Origin		
	$f(x)$	$f_1^*(x)$	$f_2(x)$
1	$\frac{1}{2} \lambda^{-1}$	$\frac{1}{3} \sqrt{3} \lambda^{-1}$	$\frac{1}{2} \lambda^{-1}$
2	$\frac{2}{3} \lambda^{-2}$	$\frac{2}{3} \lambda^{-2}$	$\frac{2}{3} \lambda^{-2}$
3	$\frac{3}{2} \lambda^{-3}$	$\frac{2}{3} \sqrt{3} \lambda^{-3}$	$\frac{3}{2} \lambda^{-3}$
4	$\frac{24}{5} \lambda^{-4}$	$\frac{8}{3} \lambda^{-4}$	$\frac{14}{3} \lambda^{-4}$
5	$20 \lambda^{-5}$	$\frac{40}{9} \sqrt{3} \lambda^{-5}$	$\frac{55}{3} \lambda^{-5}$

Fig. 2. The Density Functions $f(x)$, $f_1^*(x)$, and $f_2(x)$

The generating functions $M^{(\mu)}(a_\nu | x; z, w)$ for the projected motion on the x -axis are given by equations (3.2) which become, upon replacing the density function $f(x)$ by the approximating function $f_2(x)$, equations of the form (5.2) with $n = 2$, $p_1 = p_2 = 1/2$, $\lambda_1 = 3 - \sqrt{3}$, and $\lambda_2 = 3 + \sqrt{3}$. (The mean free path for the three-dimensional motion, i.e., λ^{-1} , has here been used as the unit of length.)

As already shown in Sect. 5, the solution for $M^{(\mu)}(a_\nu | x; z, w)$ is of the form

$$M^{(\mu)}(a_\nu | x) = \sum_{i=1}^4 A_{\nu i}^{(\mu)}(z, w) \exp(\theta_i x), \quad (7.5)$$

where the functions $\theta_i(z, w)$ and $A_{\nu i}^{(\mu)}(z, w)$ are to be determined from equations (5.3), (5.4), (5.5), and (5.6).

Substituting the above numerical values for λ_1 , λ_2 , p_1 , and p_2 in (5.3), and solving for c_1 and c_2 , we find

$$c_1 = -1; \quad c_2 = \frac{1}{6}. \quad (7.6)$$

Therefore $\theta_1, \dots, \theta_4$ are the roots of the equation

$$\theta^4 - 3\theta^2[8(1 - qw) + 3(q^2w^2 - p^2z^2)] + 36[(1 - qw)^2 - p^2z^2] = 0. \quad (7.7)$$

Since scattering is assumed to be isotropic, $p = q = (1 - r)/2$, where r is the probability of absorption at a collision.

The coefficients $A_{\nu i}^{(1)}$ and $A_{\nu i}^{(-1)}$ are related by equation (5.4), which in this case takes the form

$$[D^2 - D(6 - 3qw) + 6(1 - qw)] M^{(1)} - 3pz[2 - D] M^{(-1)} = 0. \quad (7.8)$$

Thus

$$A_{\nu i}^{(-1)} = \frac{\theta_i^2 - (6 - 3qw)\theta_i + 6(1 - qw)}{3pz(2 - \theta_i)} A_{\nu i}^{(1)} = k_i(z, w) A_{\nu i}^{(1)}; \quad i = 1, \dots, 4. \quad (7.9)$$

The coefficients $A_{\nu 1}^{(1)}$ are found from the boundary conditions (5.5) and (5.6), which can now be written

$$\left. \begin{aligned} \sum_{i=1}^4 A_{\nu 1}^{(1)} \exp(\theta_i a_1) &= \delta_{1\nu}; \\ \sum_{i=1}^4 k_i A_{\nu 1}^{(1)} \exp(\theta_i a_{-1}) &= \delta_{-1\nu}; \\ \sum_{i=1}^4 (\theta_i + 3qw + 3pz k_i) A_{\nu 1}^{(1)} \exp(\theta_i a_1) &= 3\delta_{1\nu}; \\ \sum_{i=1}^4 (-k_i \theta_i + 3qwk_i + 3pz) A_{\nu 1}^{(1)} \exp(\theta_i a_{-1}) &= 3\delta_{-1\nu}. \end{aligned} \right\} \quad (7.10)$$

Equations (7.8), (7.9), and (7.10) provide a complete solution to the first-passage problem in generating-function form. The probabilities of emergence at the two boundaries regardless of the number of collisions before emergence, i.e.,

$$P^{(\mu)}(a_\nu | x) = \sum_{r+k \geq 0} P_{rk}^{(\mu)}(a_\nu | x),$$

are obtained when z and w are replaced throughout by unity. Moments of the distribution of the number of reversals and continuations before emergence are obtained by differentiating partially with respect to z and w and setting $z = w = 1$.

8. The Albedo for a Semi-infinite Slab

To illustrate the application of these results, we now consider a stream of particles incident on the face $x = 0$ of the half-space $\{x: 0 \leq x < \infty\}$. We shall use (7.5), (7.9), and the appropriate boundary conditions to determine the albedo, i.e., the proportion of incident particles which ultimately emerge again from the half-space. The boundary conditions at a_1 will now be replaced by the condition that $P^{(\mu)}(a_\nu | x)$ is bounded as $x \rightarrow \infty$.

Denote by $p(x)$ the probability that a particle from an isotropic source with coordinate x ($x > 0$) emerges from the half-space without being absorbed, i.e.,

$$p(x) = \frac{1}{2}[P^{(1)}(0 | x) + P^{(-1)}(0 | x)].$$

Then the probability that a particle incident on the half-space at an angle θ to the positive x-axis ultimately emerges again is given by

$$\beta(\theta) = (1-r) \int_0^{\infty} \exp(-x \sec \theta) p(x) \sec \theta dx. \quad (8.1)$$

If there is a stream of particles incident on the half-space in such a way that the angle of incidence θ has probability density function $\phi(\theta)$, $0 \leq \theta \leq \pi/2$, then the albedo is given by

$$\beta = \int_0^{\pi/2} \beta(\theta) \phi(\theta) d\theta. \quad (8.2)$$

In the particular case $\phi(\theta) = 2 \cos \theta \sin \theta$, i.e., for incident particles obeying the cosine law, Wick⁴⁷ obtained an exact asymptotic expression for β when r is small, namely,

$$\beta = 1 - \frac{4}{\sqrt{3}} r^{1/2} + \dots; \quad (8.3)$$

Fermi,²¹ using the approximation $f_1^*(x)$ to derive $p(x)$, evaluated the integrals (8.1) and (8.2) to find

$$\beta = 1 - \left(1 + \frac{2}{\sqrt{3}}\right) r^{1/2} + \dots \quad (8.4)$$

From the approximation $f_2(x)$, proceeding as in Sect. 7, we find that

$$p(x) = A \exp(\theta_1 x) + B \exp(\theta_2 x), \quad (8.5)$$

where

$$\theta_1 = -\sqrt{6(1+r+\sqrt{1+r+r^2})};$$

$$\theta_2 = -\sqrt{6(1+r-\sqrt{1+r+r^2})};$$

$$A = -\frac{1}{2}(1+k_1) \frac{k_2(2\theta_2+3+3r) - 3+3r}{2k_1k_2(\theta_1-\theta_2) + 3(1-r)(k_1-k_2)};$$

$$B = \frac{1}{2}(1+k_2) \frac{k_1(2\theta_1+3+3r) - 3+3r}{2k_1k_2(\theta_1-\theta_2) + 3(1-r)(k_1-k_2)};$$

k_1 and k_2 are defined by equation (7.9) with $z = w = 1$. From (8.1) we obtain

$$\beta(\theta) = (1-r)[A(1-\theta_1 \cos \theta)^{-1} + B(1-\theta_2 \cos \theta)^{-1}], \quad (8.6)$$

and in the particular case $\phi(\theta) = 2 \cos \theta \sin \theta$, the integral (8.2) can be evaluated explicitly to give

$$\beta = -2(1-r)[A\theta_1^2(\theta_1 + \log[1-\theta_1]) + B\theta_2^2(\theta_2 + \log[1-\theta_2])], \quad (8.7)$$

i.e.,

$$\beta = 1 - \left[\frac{2\sqrt{3}-3}{12} \log(1+2\sqrt{3}) + \frac{13\sqrt{3}}{6} - \frac{3}{2} \right] r^{1/2} + \dots \quad (8.8)$$

Comparing the coefficients of $r^{1/2}$ in (8.3), (8.4), and (8.8), we see that Fermi's approximation gives 2.1547, the improved approximation gives 2.3106, and the exact value is 2.3094. This gives an idea of the accuracy of the approximation.

For a particle incident isotropically on the half-space $\{x: 0 \leq x < \infty\}$, i.e., for $\phi(\theta) = \sin \theta$, $0 \leq \theta \leq \pi/2$, it is shown by Chandrasekhar¹⁷ that the albedo is given by

$$\int_0^1 H(\mu) d\mu - 1 = \frac{1 - \sqrt{r}}{1 + \sqrt{r}}.$$

It can be shown from equation (8.5) that the application of the approximation (7.4) to this problem yields the exact solution. This property is also exhibited by Fermi's approximation. In fact, it can be shown from equation (3.2) that a more remarkable result is true, namely, that the distribution of the total number of collisions experienced by reflected particles is exact in both the Fermi approximation and the approximation (7.4). This distribution has the generating function

$$G(z) = \frac{1 - \sqrt{1 - z(1-r)}}{1 + \sqrt{1 - z(1-r)}} \frac{1 + \sqrt{r}}{1 - \sqrt{r}}. \quad (8.9)$$

9. Transmission of Particles through Slabs of Finite Thickness

In Table 2 are given the probabilities of transmission through a slab of thickness t mean free paths and the mean number of collisions experienced by transmitted particles when the particles are incident isotropically on one face of the slab. The results both of Fermi's and of the improved approximation are given. In Chapter VII these results will be compared with those obtained by a new method of solution which also takes into account directions of motion in space and the angular deflections experienced by scattered particles. The tabulated results refer to the case of zero absorption, i.e., $r = 0$; however, corresponding results could easily be obtained for any nonzero value of r .

TABLE 2. Solution for a Slab with No Absorption

Thickness, t	Probability of Transmission		Mean Number of Collisions	
	Fermi's Approximation	Improved Approximation	Fermi's Approximation	Improved Approximation
0.5	0.69783	0.62083	0.6016	0.9184
1.0	0.53590	0.47929	1.5000	1.9973
2.0	0.36603	0.33743	4.0981	4.8266
4.0	0.22401	0.21294	12.360	13.543
8.0	0.12613	0.12255	46.856	43.142

When there is no absorption (i.e., $r = 0$), the application of Fermi's method (and the improved approximation) to the scattering of particles in the interior of a thick slab can be justified asymptotically using Wald's identity (Bartlett,² p. 17). Suppose there is an isotropic source of particles at $x = 0$ and that scattering is isotropic with no absorption, i.e., for the projected motion on the x -axis $r = 0$ and $p = q = \frac{1}{2}$. Let N be a random variable denoting the number of collisions made by the particle before emerging from $[-a, b]$, $-a \leq 0 \leq b$. P_a and P_b will denote the probabilities of first departures from $-a$ and b , respectively.

The Fermi approximations to P_a , P_b and EN (the expected value of N) can be deduced from equation (6.6) of Chapter II. They are

$$P_a = \frac{\sqrt{3}b + 1}{\sqrt{3}(a+b) + 2}; \quad P_b = \frac{\sqrt{3}a + 1}{\sqrt{3}(a+b) + 2}; \quad (9.1)$$

$$EN = \frac{\sqrt{3}(\ell^3 - a^3 - b^3) - 3(2\ell^2 - a^2 - b^2)}{2\sqrt{3}\ell + 4}, \quad (9.2)$$

where $\ell = a + b$.

As we have already seen in the unrestricted case, the x -displacement at the n^{th} collision is distributed as a sum of n independent random variables, each with mean zero and variance $\frac{2}{3}$. Using Wald's identity (Bartlett, p. 20), we can deduce from this that for a and b large, P_a , P_b , and EN are given asymptotically by

$$P_a = \frac{b}{a+b}; \quad P_b = \frac{a}{a+b}; \quad (9.3)$$

$$EN = \frac{3}{2}ab. \quad (9.4)$$

Examination of (9.3) and (9.4) shows that Fermi's approximations are, in fact, asymptotic to these values. The approximations we derived earlier also have this property.

We remark finally that for scattering by very thick slabs with $r > 0$, the penetration of particles is largely determined by the tail of the exact density function $f(x)$ and not by its second moment about the origin.

10. The Relationship with "Double Gaussian" Quadrature

It is of interest to observe that the approximation derived in Sect. 7 by fitting a distribution of type (7.3) to the function $f(x)$ of equation (6.3) is equivalent to assuming that between collisions the particle moves with probability p_i in a direction the magnitude of whose cosine with the x -axis is $\mu_i = \lambda_i^{-1}$. This suggests (for slab geometry with isotropic scattering) use of the quadrature formula

$$\frac{(1-r)z}{2} \sum_{i=1}^n p_i [G(x, \mu_i; s) + G(x, -\mu_i; s)]$$

for the integral

$$\frac{1}{2} z(1-r) \int_{-1}^1 G(x, \mu; s) d\mu$$

which appears on the right-hand side of equation (5.5), Chapter I, when $\xi(\theta) \equiv z$.

Now in the n^{th} approximation the equations for p_i and μ_i , $i = 1, \dots, n$, are

$$\sum_{i=1}^n p_i \mu_i^k = (k+1)^{-1}; \quad k = 0, 1, \dots, 2n-1.$$

From these it follows that the cosines μ_i , $i = 1, \dots, n$, are the zeros of the Legendre polynomial $P_n(1-2\mu)$, so that the above quadrature scheme becomes identical with the "double Gaussian" scheme used by J. B. Sykes⁵⁹ to derive very accurate approximations to the angular distribution of emergent radiation in the Milne problem.

CHAPTER VI

A THREE-DIMENSIONAL SCATTERING MODEL

1. Introduction

In Chapter V we developed a simple method for determining the probabilities of transmission and reflection, and the distribution of the number of collisions experienced by a particle which undergoes scattering in a three-dimensional plane slab. Unfortunately, this procedure is limited to problems with isotropic scattering and clearly gives no indication of the direction of motion of emergent particles or the angular deflections which they experience as a result of collisions in the scattering material.

For a more general treatment of scattering processes we return to the formulation of Chapter I in terms of the generating functionals $G(S; \xi(\cdot) | \varpi; s)$ and $H(S; \xi(\cdot) | \varpi; s)$. To solve the basic equations (4.14) and (4.15) of Chapter I, in closed form would be an extremely difficult, if not impossible, task. However, by choosing a certain finite set of directions in which the particle is restricted to move, the backward Kolmogorov equations (5.5) and (5.6) of Chapter I can be reduced as in the one-dimensional case to a finite set of ordinary linear differential equations.

It is desirable to choose a set of directions of motion with the following property: if $\{\underline{a}_1, \underline{a}_2, \dots, \underline{a}_n\}$ are unit vectors in the permissible directions of motion, then the set of scalar products $\{\underline{a}_i \cdot \underline{a}_1, \underline{a}_i \cdot \underline{a}_2, \dots, \underline{a}_i \cdot \underline{a}_n\}$ is the same for each i ($i = 1, \dots, n$). This means that for any direction of motion prior to a collision, the set of all possible angular deflections resulting from the collision is the same independently of the initial direction of motion; that is to say, we can define a single set of n possible angular deflections $\{\epsilon_1, \epsilon_2, \dots, \epsilon_n\}$ which take an arbitrary initial direction of motion into each member of the set; we also define a set of transition probabilities $\{p_1, p_2, \dots, p_m\}$ corresponding to the distinct angular deflections $\{\theta_1, \theta_2, \dots, \theta_m\}$ which constitute a subset of $\{\epsilon_1, \epsilon_2, \dots, \epsilon_n\}$.

Thus the probabilities p_i , $i = 1, \dots, m$, replace the transition function $\phi(d\underline{\nu} | \underline{\mu})$ of Chapter I, Sect. 5. Similarly, the generating functionals G and H are replaced by generating functions

$$G(S; z_1, \dots, z_m | \varpi; s) = \sum_{i_1 + \dots + i_m \geq 0} z_1^{i_1} \dots z_m^{i_m} P^{i_1 \dots i_m}(S | \varpi; s); \quad (1.1)$$

$$H(S; z_1, \dots, z_m | \varpi; s) = \sum_{i_1 + \dots + i_m \geq 0} z_1^{i_1} \dots z_m^{i_m} Q^{i_1 \dots i_m}(S | \varpi; s) - I(S | \varpi; s), \quad (1.2)$$

where $P^{i_1 \dots i_m}(S|\omega; s)$ is the probability of a transition $\omega \rightarrow S$ in distance s with exactly i_1 deflections through angle $\theta_1, \dots, i_m$ deflections through θ_m ; similarly, $Q^{i_1 \dots i_m}(S|\omega; s)$ is the probability that the first $(i_1 + \dots + i_m)$ collisions occur in distance s , that i_k of these collisions result in deflections through $\theta_k, k = 1, \dots, m$, and that the particle is left immediately after the $(i_1 + \dots + i_m)^{\text{th}}$ collision in a state belonging to the set S .

In two dimensions the obvious choice for the permissible directions of motion is the set M_n of directions making angles $2k\pi/n, k = 0, 1, \dots, n-1$, with the unit vector $\underline{i}$ in the direction of the x -axis. The transition probabilities $p_j, j = 0, 1, \dots, n-1$, are the probabilities that the particle is deflected through an angle $2j\pi/n$, given that it makes a collision.

In three dimensions, suitable sets of directions can be chosen by taking the lines joining the center of a regular polyhedron to its vertices, face centers, or edge centers. The largest set obtainable in this way is the set of 30 directions corresponding to midpoints of the edges of a regular icosahedron. In fact, each finite subgroup of the three-dimensional real orthogonal group generates a suitable set of directions of motion; consequently, sets of at least 120 directions can be constructed (the degenerate sets obtained from rotations about a fixed axis are excluded). For the purpose of developing the theory, we shall consider the sets of 30 directions corresponding to the midpoints of the edges of a regular icosahedron which, as we shall see, leads to good numerical approximations. In Sect. 9 the construction and use of a set of 120 directions will be described. The two-dimensional problem will not be examined in detail since it is of limited interest and since the analysis is exactly parallel to the three-dimensional case.

2. Geometry of the Icosahedron

The cosines of the angles between the faces of a regular icosahedron are $\pm 1, \pm \frac{1}{3}\sqrt{5}$, and $\pm \frac{1}{3}$ (see for example Cundy and Rollett¹⁸). From these we can readily determine the coordinates of the face centers. An icosahedron of unit inradius can be oriented (see Fig. 3) so that they are given by

Fig. 3
Orientation of
the Icosahedron

$$\begin{aligned} & \pm[1, 0, 0]; \quad \pm\left[\frac{\sqrt{5}}{3}, \frac{2}{3}, 0\right]; \quad \pm\left[\frac{\sqrt{5}}{3}, -\frac{1}{3}, \frac{\sqrt{3}}{3}\right]; \quad \pm\left[\frac{\sqrt{5}}{3}, -\frac{1}{3}, -\frac{\sqrt{3}}{3}\right]; \\ & \pm\left[\frac{1}{3}, \frac{\sqrt{5}}{3}, \frac{\sqrt{3}}{3}\right]; \quad \pm\left[\frac{1}{3}, \frac{3-\sqrt{5}}{6}, \frac{\sqrt{15}+\sqrt{3}}{6}\right]; \quad \pm\left[\frac{1}{3}, -\frac{3+\sqrt{5}}{6}, \frac{\sqrt{15}-\sqrt{3}}{6}\right]; \\ & \pm\left[\frac{1}{3}, -\frac{3+\sqrt{5}}{6}, -\frac{\sqrt{15}-\sqrt{3}}{6}\right]; \quad \pm\left[\frac{1}{3}, \frac{3-\sqrt{5}}{6}, -\frac{\sqrt{15}+\sqrt{3}}{6}\right]; \quad \pm\left[\frac{1}{3}, \frac{\sqrt{5}}{3}, -\frac{\sqrt{3}}{3}\right]. \end{aligned}$$

From these coordinates it is a matter of elementary algebra to determine the vertices and the midpoints of the edges. It is then found that the cosines of the angles between the lines joining the center of the icosahedron to the midpoints of its edges are $\mu_1, \mu_2, \mu_3, \dots, \mu_{10}$, where

$$\left. \begin{aligned} \mu_1 &= 1 & &= -\mu_{10} \\ \mu_2 &= \frac{\sqrt{5} + 1}{4} & &= -\mu_9 \\ \mu_3 &= \frac{1}{2} & &= -\mu_8 \\ \mu_4 &= \frac{\sqrt{5} - 1}{4} & &= -\mu_7 \\ \mu_5 &= 0 & &= -\mu_6 \end{aligned} \right\} \quad (2.1)$$

Fig. 4
Alternative
Orientation of
Icosahedron

Throughout the remainder of this chapter we shall assume that the icosahedron is oriented as in Fig. 4 with the 30 possible directions of motion along the lines joining the center of the icosahedron to the midpoints of its edges.

In the notation of Sect. 1 the number of directions, n , is equal to 30 and the unit vectors $\{\underline{a}_1, \dots, \underline{a}_{30}\}$ are the unit vectors in the directions just indicated. The set of angular deflections $\{\epsilon_1, \dots, \epsilon_{30}\}$ contains 9 distinct angles $\theta_1, \theta_2, \dots, \theta_9$, where

$$\left. \begin{aligned} \cos \theta_1 &= \mu_1; & \cos \theta_9 &= \mu_{10} = -\mu_1; \\ \cos \theta_2 &= \mu_2; & \cos \theta_8 &= \mu_9 = -\mu_2; \\ \cos \theta_3 &= \mu_3; & \cos \theta_7 &= \mu_8 = -\mu_3; \\ \cos \theta_4 &= \mu_4; & \cos \theta_6 &= \mu_7 = -\mu_4; \\ \cos \theta_5 &= \mu_5; \end{aligned} \right\} \quad (2.2)$$

and the angles $\theta_2, \theta_3, \dots, \theta_8$ each occur four times in the set $\{\epsilon_1, \dots, \epsilon_{30}\}$. Assuming that the probability of a deflection through angle θ depends only on θ , we need therefore define only 9 distinct transition probabilities $\{p_1, 4p_2, \dots, 4p_8, p_9\}$ corresponding to angular deflections $\{\theta_1, \theta_2, \dots, \theta_9\}$. Collisions which result in deflections through $\theta_1, \dots, \theta_9$ are said to be of types 1, ..., 9, respectively.

We can achieve a further simplification by grouping the directions of motion into classes as follows. The number α , $\alpha = 1, \dots, 10$, will be used to denote the set of all directions of motion whose cosine with the

positive x-axis is μ_α . The definition must be modified slightly for the directions 5 and 6 perpendicular to the x-axis. Direction 5 consists (see Fig. 4) of the two unit vectors $\pm \underline{k}$, and direction 6 consists of the unit vectors $\pm \underline{j}$. This splitting of the directions perpendicular to the x-axis is necessary for the process, when defined in terms of the state variables (x, α) , to be Markovian.

In practice, of course, the particles can move in any direction in space, the probabilities $\phi(d\underline{\nu}|\underline{\mu})$ of a change of direction $\underline{\mu} \rightarrow \underline{\nu}$ at a collision being given by

$$\phi(d\underline{\nu}|\underline{\mu}) = \sigma(\theta) d\underline{\nu}, \quad (2.3)$$

where θ is the angle between $\underline{\mu}$ and $\underline{\nu}$, and $d\underline{\nu}$ is an element of solid angle surrounding the unit vector $\underline{\nu}$. The first step in any physical problem is, therefore, to determine the probabilities $p_1, \dots, p_9$ corresponding to the continuous function $\sigma(\theta)$. These probabilities are chosen so that

$$p_i = k\sigma(\theta_i), \quad i = 1, \dots, 9, \quad (2.4)$$

the constant k being uniquely determined by the normalization condition

$$p_1 + 4 \sum_{i=2}^8 p_i + p_9 = 1 - r, \quad (2.5)$$

where r is the probability of absorption at a collision.

3. The General Solution

We shall be concerned primarily with first-passage problems for a scattered particle when it emerges from the slab $\{x: 0 \leq x \leq t\}$. Once the particle leaves the slab it continues to move in a straight line so that it never returns. In the notation of Chapter I, Sect. 4, we seek the functional $G(S; \xi(\cdot)|\underline{\omega}; \infty)$ in which the initial state $\underline{\omega}$ is represented by an x-coordinate ($0 \leq x \leq t$) and a direction β ($\beta = 1, \dots, 10$), and the set S is the Cartesian product $X^c \times \Omega$ where X^c is the exterior of the slab and Ω is a set of directions. Since there are only a finite number of directions of motion, it will suffice to find G when S is of the form $X^c \times \{\alpha\}$, with $\alpha = 1, \dots, 10$. As already explained in Sect. 1, the functional G reduces to a multivariate generating function when the set of directions of motion is finite.

It is convenient to introduce the notation

$$P_{\alpha\beta}^{i_1 \dots i_9}(x) = \text{Prob} \{ \text{the first passage of the particle from the slab occurs after } i_j \text{ collisions of type } j, j = 1, \dots, 9, \text{ and the direction of emergence is } \alpha, \text{ given that the initial position was } x \text{ and the initial direction of motion was } \beta \};$$

$$P_{\alpha\beta}(x) = \sum_{i_1 + \dots + i_9 \geq 0} P_{\alpha\beta}^{i_1 \dots i_9}(x).$$

The functional $G(X^c x \{ \alpha \}; \xi(\cdot) | x, \beta; \infty)$ then reduces to the generating function

$$G_{\alpha\beta}(x) = \sum_{i_1 + \dots + i_9 \geq 0} P_{\alpha\beta}^{i_1 \dots i_9}(x) z_1^{i_1} \dots z_9^{i_9}. \quad (3.1)$$

We also define the column vectors $P_{\alpha}^{i_1 \dots i_9}(x)$, $G_{\alpha}(x)$, and $P_{\alpha}(x)$, whose i^{th} components ($i = 1, \dots, 10$) are $P_{\alpha 1}^{i_1 \dots i_9}(x)$, $G_{\alpha i}(x)$, and $P_{\alpha i}(x)$, respectively.

The backward Kolmogorov equation [equation (5.5) of Chapter I] can now be written

$$A_{\theta}(z_1, \dots, z_9) G_{\alpha}(x) = 0, \quad (3.2)$$

where $A_{\theta}(z_1, \dots, z_9)$ is the 10×10 matrix

$$\begin{bmatrix} \mu_1 \theta + (1) - 1 & (2222) & (3333) & (4444) & (55) & (55) & (6666) & (7777) & (8888) & (9) \\ (2) & \mu_2 \theta + (1234) - 1 & (2345) & (2356) & (46) & (37) & (4578) & (5678) & (6789) & (8) \\ (3) & (2345) & \mu_3 \theta + (1267) - 1 & (2457) & (28) & (46) & (3568) & (3489) & (5678) & (7) \\ (4) & (2356) & (2457) & \mu_4 \theta + (1368) - 1 & (37) & (28) & (2479) & (3568) & (4578) & (6) \\ (5) & (4466) & (2288) & (3377) & (19) - 1 & (55) & (3377) & (2288) & (4466) & (5) \\ (5) & (3377) & (4466) & (2288) & (55) & (19) - 1 & (2208) & (4466) & (3377) & (5) \\ (6) & (4578) & (3568) & (2479) & (37) & (28) & -\mu_5 \theta + (1368) - 1 & (2457) & (2356) & (4) \\ (7) & (5678) & (3489) & (3568) & (28) & (46) & (2457) & -\mu_3 \theta + (1267) - 1 & (2345) & (3) \\ (8) & (6789) & (5678) & (4578) & (46) & (37) & (2356) & (2345) & -\mu_2 \theta + (1234) - 1 & (2) \\ (9) & (8888) & (7777) & (6666) & (55) & (55) & (4444) & (3333) & (2222) & -\mu_1 \theta + (1) - 1 \end{bmatrix}$$

The symbol $(i_1 \dots i_n)$ is here used to denote the expression $(p_{i_1} z_{i_1} + \dots + p_{i_n} z_{i_n})$ and θ denotes the operator $\lambda^{-1} \partial / \partial x$.

The boundary conditions for equation (3.2) are

$$\left. \begin{aligned} G_{\alpha\beta}(0) &= \delta_{\alpha\beta}; & \beta &= 7, 8, 9, 10; \\ G_{\alpha\beta}(t) &= \delta_{\alpha\beta}; & \beta &= 1, 2, 3, 4. \end{aligned} \right\} \quad (3.3)$$

If the slab is semi-infinite, i.e., if $t = \infty$, the second of these is replaced by the condition that $G_{\alpha\beta}(x)$ be a bounded function of x as $x \rightarrow \infty$.

In solving equation (3.2) it will be assumed that the mean free path λ^{-1} is unity. If λ is any function of x , the problem can be reduced to the

case of unit λ by the simple coordinate transformation

$$x' = \int_0^x \lambda(u) du. \quad (3.4)$$

It is apparent from equation (3.2) that $G_{\alpha 5}(x)$ and $G_{\alpha 6}(x)$ can be expressed as linear combinations of $G_{\alpha 1}(x), \dots, G_{\alpha 4}(x), G_{\alpha 7}(x), \dots, G_{\alpha 10}(x)$; equation (3.2) can therefore be rewritten in the form

$$\frac{\partial}{\partial x} \underline{g}_{\alpha}(x) = R(z_1, \dots, z_9) \underline{g}_{\alpha}(x), \quad (3.5)$$

where $R(z_1, \dots, z_9)$ is an easily determined 8×8 matrix and $\underline{g}_{\alpha}(x)$ is the column vector whose transpose is

$$\underline{g}_{\alpha}^t(x) = [G_{\alpha 1}(x) \dots G_{\alpha 4}(x) G_{\alpha 7}(x) \dots G_{\alpha 10}(x)]. \quad (3.6)$$

The solution of equation (3.5) is

$$\underline{g}_{\alpha}(x) = \exp(xR) \underline{g}_{\alpha}(0), \quad (3.7)$$

where $\underline{g}_{\alpha}(0)$ is uniquely determined by the boundary conditions. In principle this result completely solves the scattering problem for the icosahedral model. The evaluation of the components of the matrix $\exp(xR)$ is extremely difficult for arbitrary $z_1, \dots, z_9$; however, the angular distribution, which is obtained when all the z_i are set equal to 1, can be determined numerically, as will now be shown. The determination of the moments of the distribution of the number of collisions of each type experienced by the particle before emergence will be considered in Sect. 4.

The emergent angular distribution of particles for fixed initial position x and direction β is given by $P_{\alpha\beta}(x)$, $\alpha = 1, \dots, 10$, which is obtained from $G_{\alpha\beta}(x)$ when $z_1, \dots, z_9$ are each replaced by unity. From equation (3.7), defining $\underline{p}_{\alpha}(x) = [\underline{g}_{\alpha}(x)]_{z_1=\dots=z_9=1}$, we see immediately that the solution for $\underline{p}_{\alpha}(x)$ is

$$\underline{p}_{\alpha}(x) = \exp[xR(1, \dots, 1)] \underline{p}_{\alpha}(0), \quad (3.8)$$

where $\underline{p}_{\alpha}(0)$ is uniquely determined by the boundary conditions

$$\left. \begin{aligned} P_{\alpha\beta}(0) &= \delta_{\alpha\beta}; & \beta &= 7, 8, 9, 10; \\ P_{\alpha\beta}(t) &= \delta_{\alpha\beta}; & \beta &= 1, 2, 3, 4. \end{aligned} \right\} \quad (3.9)$$

Explicit numerical determinations of the probabilities $P_{\alpha\beta}(x)$ are most conveniently carried out by reducing $R(1, \dots, 1)$ to canonical form,

i.e., by finding a matrix T such that $T^{-1}RT = \Lambda$, where Λ is the canonical form of the matrix R . Then

$$\exp[\mathbf{x}R(1, \dots, 1)] = T \exp(\mathbf{x}\Lambda) T^{-1}. \quad (3.10)$$

Details of the numerical calculations are given in Appendix 1. It is worth noticing that the solution for isotropic scattering is very much simplified owing to the fact that the eigenvectors of the matrix R , and hence the components of the matrix T , take a simple form. Thus the eigenvalues γ_i of R are the roots of the equation

$$\theta^8[13+2r] - \theta^6[156+99r] + \theta^4[448+752r] - \theta^2[320+1600r] + 960r = 0, \quad (3.11)$$

where r is the probability of absorption at a collision; the right eigenvector of R corresponding to γ_i is simply

$$\underline{e}_i = \begin{bmatrix} (1 - \mu_1 \gamma_i)^{-1} \\ \dots \\ (1 - \mu_4 \gamma_i)^{-1} \\ (1 - \mu_7 \gamma_i)^{-1} \\ \dots \\ (1 - \mu_{10} \gamma_i)^{-1} \end{bmatrix}. \quad (3.12)$$

4. Moments of the Distribution of the Number of Collisions

Let N_i , $i = 1, \dots, 9$, be the (random) number of collisions of type i experienced by a particle which originates at $\mathbf{x}$ with direction β and emerges from the slab $\{\underline{x}: 0 \leq x \leq t\}$ in direction α . We seek the moments of the distribution of the random variable

$$N = \sum_{i=1}^9 a_i N_i, \quad (4.1)$$

where $a_1, \dots, a_9$ constitute a set of arbitrary real constants. In particular, we shall consider the determination of the first two moments EN and EN^2 , third and higher-order moments being found in the same way.

In terms of the probabilities $P_{\alpha\beta}^{i_1 \dots i_9}(\mathbf{x})$, the mean EN can be written

$$EN = [P_{\alpha\beta}(\mathbf{x})]^{-1} M_{\alpha\beta}(\mathbf{x}), \quad (4.2)$$

where

$$M_{\alpha\beta}(x) = \sum_{i_1 + \dots + i_9 \geq 0} (a_1 i_1 + \dots + a_9 i_9) P_{\alpha\beta}^{i_1 \dots i_9}(x),$$

i.e.,

$$M_{\alpha\beta}(x) = \left\{ \sum_{i=1}^9 a_i \frac{\partial}{\partial z_i} G_{\alpha\beta}(x) \right\}_{z_1 = \dots = z_9 = 1} \quad (4.3)$$

In the notation of Sect. 3, we require to determine the vectors

$$\tilde{M}_\alpha(x) = \left\{ \sum_{k=1}^9 a_k \frac{\partial}{\partial z_k} G_\alpha(x) \right\}_{z_1 = \dots = z_9 = 1}$$

From equation (3.2) the equation for $\tilde{M}_\alpha(x)$ is

$$A_\theta(1, \dots, 1) \tilde{M}_\alpha(x) + B \tilde{P}_\alpha(x) = 0, \quad (4.4)$$

where $\tilde{P}_\alpha(x)$ is found as described in Sect. 3 and B is the matrix

$$B = \left\{ \sum_{k=1}^9 a_k \frac{\partial}{\partial z_k} A_\theta(z_1, \dots, z_9) \right\}_{z_1 = \dots = z_9 = 1} \quad (4.5)$$

It suffices to determine the vectors

$$\tilde{m}_\alpha(x) = \left\{ \sum_{k=1}^9 a_k \frac{\partial}{\partial z_k} g_\alpha(x) \right\}_{z_1 = \dots = z_9 = 1}$$

since the components $M_{\alpha_5}(x)$ and $M_{\alpha_6}(x)$ of $\tilde{M}_\alpha(x)$ can be expressed, by means of (4.4), as linear combinations of the components of $\tilde{m}_\alpha(x)$. Equation (4.4) can thus be rewritten

$$\frac{\partial}{\partial x} \tilde{m}_\alpha(x) = R(1, \dots, 1) \tilde{m}_\alpha(x) + \tilde{V}_\alpha(x), \quad (4.6)$$

where $R(z_1, \dots, z_9)$ is the same matrix appearing in equation (3.5) and

$$\tilde{V}_\alpha(x) = \left\{ \sum_{k=1}^9 a_k \frac{\partial}{\partial z_k} R(z_1, \dots, z_9) \tilde{P}_\alpha(x) \right\}_{z_1 = \dots = z_9 = 1}$$

The solution of equation (4.6) is given by

$$\underline{m}_\alpha(x) = \exp[xR(1, \dots, 1)] \underline{m}_\alpha(0) + \int_0^x \exp[(x-y)R(1, \dots, 1)] \underline{V}_\alpha(y) dy, \quad (4.7)$$

where $\underline{m}_\alpha(0)$ is uniquely determined by the boundary conditions

$$\left. \begin{aligned} M_{\alpha\beta}(t) &= 0; & \beta &= 1, 2, 3, 4; \\ M_{\alpha\beta}(0) &= 0; & \beta &= 7, 8, 9, 10. \end{aligned} \right\} \quad (4.8)$$

The second moment, EN^2 , is found in a similar manner:

$$EN^2 = [P_{\alpha\beta}(x)]^{-1} \sum_{i_1 + \dots + i_9 \geq 0} (a_1 i_1 + \dots + a_9 i_9)^2 P_{\alpha\beta}^{i_1 \dots i_9}(x),$$

i.e.,

$$EN^2 = [P_{\alpha\beta}(x)]^{-1} \left[S_{\alpha\beta}(x) + \left\{ \sum_{k=1}^9 a_k^2 \frac{\partial}{\partial z_k} G_{\alpha\beta}(x) \right\}_{z_1=\dots=z_9=1} \right], \quad (4.9)$$

where

$$S_{\alpha\beta}(x) = \left\{ \sum_{j=1}^9 \sum_{k=1}^9 a_j a_k \frac{\partial^2}{\partial z_j \partial z_k} G_{\alpha\beta}(x) \right\}_{z_1=\dots=z_9=1} \quad (4.10)$$

Defining, as before, the vector $\underline{S}_\alpha(x)$ and the corresponding eight-component vector $\underline{s}_\alpha(x)$, we find from (3.2) that the equation for $\underline{S}_\alpha(x)$ is

$$A_\theta(1, \dots, 1) \underline{S}_\alpha(x) + 2B \underline{M}_\alpha(x) = \underline{0}, \quad (4.11)$$

which can be rewritten as

$$\frac{\partial}{\partial x} \underline{s}_\alpha(x) = R(1, \dots, 1) \underline{s}_\alpha(x) + \underline{W}_\alpha(x), \quad (4.12)$$

where

$$\underline{W}_\alpha(x) = \left\{ \sum_{j=1}^9 \sum_{k=1}^9 a_j a_k \frac{\partial^2}{\partial z_j \partial z_k} R(z_1, \dots, z_9) \underline{p}_\alpha(x) + 2 \sum_{k=1}^9 a_k \frac{\partial}{\partial z_k} R(z_1, \dots, z_9) \underline{m}_\alpha(x) \right\}_{z_1=\dots=z_9=1}$$

The solution of (4.12) is given by

$$\underline{s}_\alpha(x) = \exp[xR(1, \dots, 1)] \underline{s}_\alpha(0) + \int_0^x \exp[(x-y)R(1, \dots, 1)] \underline{W}_\alpha(y) dy, \quad (4.13)$$

where $\underline{s}_\alpha(0)$ is uniquely determined by the boundary conditions

$$\left. \begin{aligned} S_{\alpha\beta}(0) &= 0; & \beta &= 7, 8, 9, 10; \\ S_{\alpha\beta}(t) &= 0; & \beta &= 1, 2, 3, 4. \end{aligned} \right\} \quad (4.14)$$

The only term in equation (4.9) not yet accounted for is the sum

$$\left\{ \sum_{k=1}^9 a_k^2 \frac{\partial}{\partial z_k} G_{\alpha\beta}(x) \right\}_{z_1=\dots=z_9=1}$$

which is simply $M_{\alpha\beta}(x)$ with a new set of constants $a_1^2, \dots, a_9^2$ replacing the constants $a_1, \dots, a_9$. (If each a_i is either 1 or 0, as will often be the case, then $a_i^2 = a_i$, and no additional computation is required.)

The determination of each of the moments of N requires the solution of a two-point boundary-value problem. However, it is sufficient to solve the problem for a small number of slab thicknesses owing to an iteration relation which holds for the generating functions $G_{\alpha\beta}(0)$. This will be considered in Sect. 5.

5. An Iteration Relation for the Generating Functions $G_{\alpha\beta}(0)$

Consider a particle which is incident on the left face of the slab $\{x: 0 \leq x \leq t\}$ in direction β ($\beta = 1, \dots, 6$). If $t > 0$, the possible directions of emergence, α , are 1, 2, 3, 4, in which case the particle is transmitted, or 7, 8, 9, 10, in which case the particle is reflected. We define the two 4×6 matrices of generating functions

$$G_T^*(t) = [G_{ij}(0)]; \quad i = 1, \dots, 4; \quad j = 1, \dots, 6; \quad (5.1)$$

$$G_R^*(t) = [G_{(11-i)j}(0)]; \quad i = 1, \dots, 4; \quad j = 1, \dots, 6, \quad (5.2)$$

and the analogous 4×4 matrices $G_T(t)$ and $G_R(t)$ obtained by deleting the last two columns of $G_T^*(t)$ and $G_R^*(t)$, respectively. Clearly the matrices $G_T^*(t)$ and $G_T(t)$ refer to transmitted particles, and $G_R^*(t)$ and $G_R(t)$ to reflected particles. The following theorem relates the transmission and reflection matrices for slab thickness $(t_1 + t_2)$ to the corresponding matrices for thicknesses t_1 and t_2 . Given the matrices G_T^* and G_R^* for a set of slab thicknesses $t_1, \dots, t_n$, the result of the theorem can be used to generate

G_T^* and G_R^* for any linear combination of the thicknesses $t_1, \dots, t_n$, with positive integral coefficients. Corollaries 1, 2, and 3 show how the results can also be applied to the matrices

$$P_R^*(T) = [G_R^*(T)]_{z_1=\dots=z_9=1}, \quad (5.3)$$

$$M_R^*(T) = \left[\sum_{k=1}^9 a_k \frac{\partial}{\partial z_k} G_R^*(T) \right]_{z_1=\dots=z_9=1}, \quad (5.4)$$

$$S_R^*(T) = \left[\sum_{j=1}^9 \sum_{k=1}^9 a_j a_k \frac{\partial^2}{\partial z_j \partial z_k} G_R^*(T) \right]_{z_1=\dots=z_9=1}, \quad (5.5)$$

and thus to the determination of the moments of the distribution of the random variable $N = \sum_{i=1}^9 a_i N_i$ defined in Sect. 4. Matrices $P_R(T)$, $M_R(T)$, and $S_R(T)$ are defined from $P_R^*(T)$, $M_R^*(T)$, and $S_R^*(T)$ as before, by deleting the fifth and sixth columns.

Theorem

If $G_T^*(t)$, $G_T(t)$, $G_R^*(t)$, and $G_R(t)$ are the transmission and reflection matrices defined above, then

$$G_T^*(t_1 + t_2) = G_T(t_2)[I - G_R(t_1) G_R(t_2)]^{-1} G_T^*(t_1); \quad (5.6)$$

$$G_R^*(t_1 + t_2) = G_R^*(t_1) + G_T(t_1) G_R(t_2)[I - G_R(t_1) G_R(t_2)]^{-1} G_T^*(t_1). \quad (5.7)$$

Proof

Let $\pi_{\alpha\beta}^{i_1 \dots i_9}$ be the probability of emergence in direction α from a slab of thickness t_1 after i_k collisions of type k , $k = 1, \dots, 9$, conditional on incident direction β . Let $\psi_{\alpha\beta}^{i_1 \dots i_9}$ be the corresponding probability for a slab of thickness t_2 , and let $P_{\alpha\beta}^{i_1 \dots i_9}$ be the corresponding probability for the slab of thickness $(t_1 + t_2)$ obtained by placing the two smaller slabs together.

Then the transmission probabilities $P_{\alpha\beta}^{i_1 \dots i_9}$, $\alpha = 1, \dots, 4$, $\beta = 1, \dots, 6$, are given by

$$P_{\alpha\beta}^{i_1 \dots i_9} = \sum_{n=0}^{\infty} \sum_{\substack{(1) \\ m_j}} \sum_{\substack{(2n+2) \\ m_j}} \sum_{\gamma_1=1}^4 \dots \sum_{\gamma_{2k}=1}^4 \sum_{\gamma_{2k+1}=1}^4 \dots \sum_{\gamma_{2n+1}=1}^4 \psi_{\alpha\gamma_{2n+1}}^{m_1(2n+2) \dots m_9(2n+2)} \dots \pi_{\gamma_1\beta}^{m_1(1) \dots m_9(1)}, \quad (5.8)$$

where the k^{th} factor on the right-hand side ($2 \leq k \leq 2n+1$) is

$$d_k = \begin{cases} \psi \gamma_{2n+3-k}^{m_1^{(2n+3-k)} \dots m_9^{(2n+3-k)}} & \text{if } k \text{ is odd;} \\ \pi \gamma_{2n+3-k}^{m_1^{(2n+3-k)} \dots m_9^{(2n+3-k)}} & \text{if } k \text{ is even.} \end{cases}$$

The n^{th} term in the outer summation arises from particles which make $(n+1)$ passages through the interface between the two slabs before emerging from the extreme right-hand face. The second summation is over all non-negative sets of integers $m_j^{(h)}$, $j = 1, \dots, 9$; $h = 1, \dots, 2n+2$, such that

$$\sum_{h=1}^{2n+2} m_j^{(h)} = i_j; \quad j = 1, \dots, 9.$$

Multiplying equation (5.8) by $z_1^{i_1} \dots z_9^{i_9}$ and summing over all non-negative sets of integers $i_1, \dots, i_9$, we find immediately that

$$G_T^*(t_1+t_2) = G_T(t_2) \sum_{n=0}^{\infty} [G_R(t_1) G_R(t_2)]^n G_T^*(t_1). \quad (5.9)$$

Since the sum of the moduli of the elements in any column of the matrix $G_R(t_1) G_R(t_2)$ is strictly less than 1, it follows (cf. Frechet,²² p. 105) that the spectral radius of $G_R(t_1) G_R(t_2)$ is also less than 1. The summation in equation (5.9) can therefore be replaced by $[I - G_R(t_1) G_R(t_2)]^{-1}$, thus establishing the result (5.6) for $G_T^*(t_1+t_2)$.

Equation (5.7) for the reflection matrix $G_R^*(t_1+t_2)$ is established in an exactly parallel manner.

Corollary 1

Setting $z_1 = \dots = z_9 = 1$ in equations (5.6) and (5.7), we find immediately that

$$P_T^*(t_1+t_2) = P_T(t_2) [I - P_R(t_1) P_R(t_2)]^{-1} P_T^*(t_1); \quad (5.10)$$

$$P_R^*(t_1+t_2) = P_R^*(t_1) + P_T(t_1) P_R(t_2) [I - P_R(t_1) P_R(t_2)]^{-1} P_T^*(t_1). \quad (5.11)$$

Corollary 2

If we define $M_T^*(t)$ and $M_R^*(t)$ as in equation (5.4), it follows, on differentiating (5.6) and (5.7) with respect to z_k , $k = 1, \dots, 9$, that

$$M_T^*(t_1 + t_2) = M_T(t_2) V^{-1} P_T^*(t_1) + P_T(t_2) V^{-1} W V^{-1} P_T^*(t_1) + P_T(t_2) V^{-1} M_T^*(t_1); \quad (5.12)$$

$$M_R^*(t_1 + t_2) = M_R^*(t_1) + X V^{-1} P_T^*(t_1) + P_T(t_1) P_R(t_2) V^{-1} W V^{-1} P_T^*(t_1) + P_T(t_1) P_R(t_2) V^{-1} M_T^*(t_1), \quad (5.13)$$

where

$$V = I - P_R(t_1) P_R(t_2);$$

$$W = M_R(t_1) P_R(t_2) + P_R(t_1) M_R(t_2);$$

$$X = M_T(t_1) P_R(t_2) + P_T(t_1) M_R(t_2).$$

Corollary 3

Differentiating equations (5.6) and (5.7) twice in order to form the derivatives appearing in equation (5.5) for S_T^* and S_R^* , we find that

$$S_T^*(t_1 + t_2) = S_T(t_2) V^{-1} P_T^*(t_1) + P_T(t_2) V^{-1} S_T^*(t_1) + P_T(t_2) V^{-1} Z V^{-1} P_T^*(t_1) + 2M_T(t_2) V^{-1} W V^{-1} P_T^*(t_1) + 2P_T(t_2) V^{-1} W V^{-1} M_T^*(t_1) + 2M_T(t_2) V^{-1} M_T^*(t_1) + 2P_T(t_2) V^{-1} W V^{-1} W V^{-1} P_T^*(t_1); \quad (5.14)$$

$$S_R^*(t_1 + t_2) = S_R^*(t_1) + Y V^{-1} P_T^*(t_1) + P_T(t_1) P_R(t_2) V^{-1} S_T^*(t_1) + 2X V^{-1} M_T^*(t_1) + 2P_T(t_1) P_R(t_2) V^{-1} W V^{-1} W V^{-1} P_T^*(t_1) + 2P_T(t_1) P_R(t_2) V^{-1} W V^{-1} M_T^*(t_1) + 2X V^{-1} W V^{-1} P_T^*(t_1) + P_T(t_1) P_R(t_2) V^{-1} Z V^{-1} P_T^*(t_1), \quad (5.15)$$

where

$$Y = S_T(t_1) P_R(t_2) + 2M_T(t_1) M_R(t_2) + P_T(t_1) S_R(t_2);$$

$$Z = S_R(t_1) P_R(t_2) + 2M_R(t_1) M_R(t_2) + P_R(t_1) S_R(t_2).$$

The application of this theorem and its corollaries is very simple, involving as it does only elementary operations on 4×4 and 4×6 matrices.

6. The Distribution of Emergent Velocity

Consider again a slab $[0, t]$ and a particle that originates at a point x inside the slab moving in direction β . We wish to find the distribution of the particle's emergent velocity given that it emerges from the slab in direction α . In the notation of the preceding sections $G_{\alpha\beta}(x)[P_{\alpha\beta}(x)]^{-1}$ is the generating function for the joint distribution, conditional on emergence in direction α , of the number of collisions of each type before emergence.

Let $\psi_i(dv|u)$ be the transition probability for the absolute velocities u and v before and after a collision of type i . If we assume, as in Chapter I, Sect. 5, that

$$\int_w \psi_i(dv|w) \psi_j(dw|u) = \int_w \psi_j(dv|w) \psi_i(dw|u), \quad i, j = 1, 2, \dots, 9, \quad (6.1)$$

then the probability $F_{\alpha\beta}(V|u, x)$ that the emergent velocity belongs to the set V , conditional on initial state (β, u, x) and emergent direction α , is given by $G_{\alpha\beta}(x)[P_{\alpha\beta}(x)]^{-1}$ with terms such as $z_1^{i_1} \dots z_9^{i_9}$ replaced by $\psi_1^{(i_1)} \circ \psi_2^{(i_2)} \circ \dots \circ \psi_9^{(i_9)}$. The functions $\psi^{(n)}$ and $\psi \circ \phi$ are defined as follows:

$$\left. \begin{aligned} \psi^{(n)} &= \int \psi(V|w) \psi^{(n-1)}(dw|u); \quad n = 1, 2, \dots, \\ \psi^{(0)} &= \delta(V|u) = \begin{cases} 1, & u \in V, \\ 0, & u \notin V; \end{cases} \\ \psi \circ \phi &= \int \psi(V|w) \phi(dw|u). \end{aligned} \right\} \quad (6.2)$$

Further simplification occurs if we assume particular forms for $\psi_i(dv|u)$, as in the following examples.

Example 1

$$\psi_i(dv|u) = \begin{cases} \frac{1}{u} f_i\left(\frac{v}{u}\right) dv, & 0 \leq v \leq u, \\ 0, & \text{otherwise.} \end{cases}$$

It is convenient in this case to introduce the Mellin transforms

$$\left. \begin{aligned} \psi_i(\omega) &= \int_{v=0}^u f_i\left(\frac{v}{u}\right) \left(\frac{v}{u}\right)^{\omega-1} \frac{dv}{u}; \\ F_{\alpha\beta}(\omega|u, x) &= \int_{v=0}^u \left(\frac{v}{u}\right)^{\omega-1} F_{\alpha\beta}(dv|u, x), \end{aligned} \right\} \quad (6.3)$$

from which, using the preceding remarks, we find

$$F_{\alpha\beta}(\omega|u, \vec{x}) = [P_{\alpha\beta}(x)]^{-1} \sum_{i_1 + \dots + i_9 \geq 0} P_{\alpha\beta}^{i_1 \dots i_9}(x) \psi_1(\omega)^{i_1} \dots \psi_9(\omega)^{i_9}. \quad (6.4)$$

Clearly the right side of (6.4) is just $[P_{\alpha\beta}(x)]^{-1} G_{\alpha\beta}(x)$ with z_k replaced by $\psi_k(\omega)$, $k = 1, \dots, 9$.

If u is the initial velocity of a particle and v its emergent velocity, then we denote by $E_{\alpha\beta}[(v/u)^n|u, x]$ the expectation of the random variable $(v/u)^n$ conditional upon initial state (β, u, x) and emergent direction α . To determine $E_{\alpha\beta}[(v/u)^n|u, x]$ we put $\omega = n + 1$ in $F_{\alpha\beta}(\omega|u, x)$. $P_{\alpha\beta}(x)$ is determined as in Sect. 3 and $G_{\alpha\beta}(x)$ is determined from the equation

$$A_\theta(\psi_1(n+1), \dots, \psi_9(n+1)) G_\alpha(x) = Q, \quad (6.5)$$

with boundary conditions $G_{\alpha\beta}(0) = \delta_{\alpha\beta}$, $\beta = 7, 8, 9, 10$, and $G_{\alpha\beta}(t) = \delta_{\alpha\beta}$, $\beta = 1, 2, 3, 4$.

Example 2

$$\psi_i(dv|u) = \delta(v - c_i u) dv, \quad 0 \leq c_i \leq 1, i = 1, \dots, 9.$$

This model corresponds to a physical situation in which the logarithmic energy of the particle decreases by a fixed amount for each particular angular deflection. In this case the distribution of emergent velocity is discrete since a particle which starts with velocity u and makes i_1 collisions of type 1, i_2 of type 2, ..., and i_9 of type 9 has a resultant velocity

$$v = u c_1^{i_1} \dots c_9^{i_9}.$$

If we again take Mellin transforms as in Example 1, we find that

$$F_{\alpha\beta}(\omega|u, x) = [P_{\alpha\beta}(x)]^{-1} \sum_{i_1 + \dots + i_9 \geq 0} P_{\alpha\beta}^{i_1 \dots i_9}(x) c_1^{(\omega-1)i_1} \dots c_9^{(\omega-1)i_9}, \quad (6.6)$$

which is simply $[P_{\alpha\beta}(x)]^{-1} G_{\alpha\beta}(x)$ with z_k replaced by $c_k^{\omega-1}$, $k = 1, \dots, 9$.

To find moments of the distribution, say $E_{\alpha\beta}[(v/u)^n | u, x]$, we put $\omega = n + 1$ as before, determining $P_{\alpha\beta}(x)$ as in Sect. 3 and $G_{\alpha\beta}(x)$ from the equation

$$A_{\theta}(c_1^n, \dots, c_9^n) G_{\alpha}(x) = Q, \quad (6.7)$$

with boundary conditions $G_{\alpha\beta}(0) = \delta_{\alpha\beta}$, $\beta = 7, 8, 9, 10$, and $G_{\alpha\beta}(t) = \delta_{\alpha\beta}$, $\beta = 1, 2, 3, 4$.

7. The Steady-state Distribution of Particles in a Finite Slab

Suppose now that there is a steady stream of particles incident on one face of the slab $\{x: 0 \leq x \leq t\}$ and that the time-independent solution for the mean number of particles per second crossing the plane $x = \xi$ in a given direction is required.

The incident particles can be represented by a column vector $\underline{n} = [n_i]$, where n_i is the number of particles incident per second in direction i , $i = 1, \dots, 6$. The particles crossing the plane $x = \xi$ can be represented by two 4-component vectors, $\underline{n}^L$, the vector of particles crossing $x = \xi$ in directions 1, 2, 3, 4, and $\underline{n}^R$, the vector of particles crossing $x = \xi$ in directions 10, 9, 8, 7. In terms of the matrices $P_T^*(t)$, $P_R^*(t)$, $P_T(t)$, and $P_R(t)$, defined by equation (5.3), the mean number of particles per second crossing the plane $x = \xi$ from the left for the $(k+1)$ th time is

$$\underline{n}_k^L = [P_R(\xi) P_R(t - \xi)]^k P_T^*(\xi) \underline{n}, \quad k = 0, 1, 2, \dots \quad (7.1)$$

Hence the total number of particles per second crossing the plane $x = \xi$ from the left is given by

$$\underline{n}^L = \sum_{k=0}^{\infty} \underline{n}_k^L = [I - P_R(\xi) P_R(t - \xi)]^{-1} P_T^*(\xi) \underline{n}. \quad (7.2)$$

Similarly, the total number of particles per second crossing the plane $x = \xi$ from the right is given by

$$\underline{n}^R = \sum_{k=0}^{\infty} \underline{n}_k^R = [I - P_R(t - \xi) P_R(\xi)]^{-1} P_R(t - \xi) P_T^*(\xi) \underline{n}. \quad (7.3)$$

We remark that the argument of this section can also be used when there is a distribution of sources in the interior of the scattering medium.

8. The Effect of Absorption

Consider the modified process in which there is at each collision a constant probability of absorption,

$$r = 1 - p_1 - p_9 - 4 \sum_{j=2}^8 p_j.$$

The Kolmogorov equations (3.2) for the generating functions $G_{\alpha\beta}(x)$ remain the same and so also do the boundary conditions. Thus the behaviour of the particles when there is absorption (i.e., $r > 0$) is determined by exactly the same equations, the only difference being that the probabilities $p_1, 4p_2, \dots, 4p_8, p_9$ will no longer add to 1. All the results of this chapter therefore apply without modification to the process with absorption. The numerical calculations are, in fact, simplified when there is a nonzero probability of absorption, owing to the fact that the matrix $R(1, \dots, 1)$ [see equation (3.5)] no longer has zero as a double eigenvalue.

9. A Model Which Uses 120 Directions of Motion

As already indicated in Sect. 1, it is possible to choose a set of 120 unit vectors $\{\underline{a}_1, \dots, \underline{a}_{120}\}$ in three dimensions having the property that the set of scalar products $\{\underline{a}_i \cdot \underline{a}_1, \dots, \underline{a}_i \cdot \underline{a}_{120}\}$ is the same for each i , $i = 1, \dots, 120$. This can be done as follows: each triangular face of the icosahedron is subdivided by its altitudes into 6 congruent triangles. If we consider the set of 120 triangles formed by subdividing each of the 20 faces in this way and consider the unit vectors along the lines joining the centre of the icosahedron to a representative point on each small triangle (for example, the centroid), we arrive at a set of 120 unit vectors $\underline{a}_i$ having the above property.

A numerical study using the set of 120 directions obtained using the centroids of the small triangles has been carried out by Brockwell and Landwehr.⁷ The icosahedron is oriented as in Fig. 4, Sect. 2, so that the directions of motion fall into classes 1, ..., 30 where class i consists of the 4 directions of motion making an angle with the positive x -axis whose cosine is μ_i . The numerical values of μ_i are as follows:

$$\mu_1 = 0.971400229 = -\mu_{30}$$

$$\mu_2 = 0.906819049 = -\mu_{29}$$

$$\mu_3 = 0.794101899 = -\mu_{28}$$

$$\mu_4 = 0.777656688 = -\mu_{27}$$

$$\mu_5 = 0.689607354 = -\mu_{26}$$

$$\mu_6 = 0.664939538 = -\mu_{25}$$

$$\mu_7 = 0.619944324 = -\mu_{24}$$

$$\mu_8 = 0.495863814 = -\mu_{23}$$

$$\mu_9 = 0.351455904 = -\mu_{22}$$

$$\mu_{10} = 0.313483534 = -\mu_{21}$$

$$\mu_{11} = 0.286874724 = -\mu_{20}$$

$$\mu_{12} = 0.281792875 = -\mu_{19}$$

$$\mu_{13} = 0.208989089 = -\mu_{18}$$

$$\mu_{14} = 0.112717150 = -\mu_{17}$$

$$\mu_{15} = 0.104494545 = -\mu_{16}$$

The set of angular deflections $\{\epsilon_1, \dots, \epsilon_{120}\}$ for this model contains 76 distinct angles $\{\theta_1, \dots, \theta_{76}\}$ (see Ref. 7). The analogue of the generating function $G_{\alpha\beta}(x)$ of Sect. 3 is therefore of the form

$$G'_{\alpha\beta}(x) = \sum_{i_1 + \dots + i_{76} \geq 0} P_{\alpha\beta}^{i_1 \dots i_{76}}(x) z_1^{i_1} \dots z_{76}^{i_{76}}; \quad \alpha = 1, \dots, 30;$$

$$\beta = 1, \dots, 30, \quad (9.1)$$

and the Kolmogorov equation (3.2) is replaced by

$$A'_\theta(z_1, \dots, z_{76}) G'_\alpha(x) = Q, \quad (9.2)$$

with boundary conditions

$$G'_{\alpha\beta}(0) = \delta_{\alpha\beta}; \quad \beta = 16, \dots, 30,$$

$$G'_{\alpha\beta}(t) = \delta_{\alpha\beta}; \quad \beta = 1, \dots, 15. \quad (9.3)$$

The matrix $A'_\theta(z_1, \dots, z_{76})$ is a 30×30 matrix as compared with the 10×10 matrix $A_\theta(z_1, \dots, z_9)$ of Sect. 3. Equation (9.2) can be rewritten in the more convenient form

$$\frac{\partial}{\partial x} G'_\alpha(x) = R'(z_1, \dots, z_{76}) G'_\alpha(x), \quad (9.4)$$

and the analysis then proceeds exactly as for the case of 30 directions already described. (Since there are no directions of motion perpendicular to the x -axis, there is no analogue of the vector $\underline{g}_\alpha(x)$ introduced in Sect. 3.)

Numerical results derived for this model are given in detail in Ref. 7. For the purpose of this thesis we shall consider only the model which employs 30 directions. In Chapter VII some of the results will be compared with the corresponding results obtained from the 120-direction approximation.

CHAPTER VII

PHYSICAL APPLICATIONS OF THE THREE-DIMENSIONAL MODEL

1. Introduction

In this chapter we shall examine a number of physical applications of the three-dimensional model developed in Chapter VI. The earliest problems to receive attention from astronomers were concerned with scattering in a three-dimensional half-space (this being the idealization of a stellar atmosphere). The Milne problem is concerned with the determination of the angular distribution of emergent radiation from a half-space when the source of radiation is a very large distance below the surface. For isotropic scattering the exact solution of this problem was obtained by Wiener and Hopf⁴⁹ in closed form and is given in the book by Hopf.²⁶

Explicit solutions of the problems of transport theory are, however, very rare. Various approximate methods of analysis have been devised; for example, the spherical harmonics method, in which the distribution of particles is expanded in terms of Legendre polynomials and the series truncated (see for example Chandrasekhar⁹), and the Gaussian quadrature method devised by Wick⁴⁸ and extensively developed by Chandrasekhar,¹⁰⁻¹³ in which the integral in the transport equation is replaced by an approximating sum. Wang and Guth⁴⁵ have formulated the spherical harmonic method in such a way as to demonstrate its close relationship with the Gaussian quadrature method. Chandrasekhar,^{14,15} making use of invariance principles introduced by Ambarzumian,¹ has found exact solutions to many of the problems of radiative transfer in slab geometry by expressing the solution in terms of his X-, Y-, and H-functions, defined as solutions of certain integral equations.

We shall be concerned throughout this chapter with the scattering of particles in the plane slab $\{x: 0 \leq x \leq t\}$. The direction of motion of a particle will be represented by the cosine, μ , of the angle which it makes with the positive x-axis and its position by an x-coordinate only.

In the notation of Chapter I we seek the functional $G(X^c \times L; \xi(\cdot)|x, \mu; \infty)$, where X^c is the exterior of the slab, L is a set of directions ($L \in [-1, 1]$), and (x, μ) is the initial state of the particle ($0 \leq x \leq t$, $-1 \leq \mu \leq 1$). From equation (5.5) of Chapter I, the equation satisfied by G is

$$\left[\mu \frac{\partial}{\partial x} - \lambda(x) \right] G(x, \mu) = -\lambda(x) \int_{\Omega} \xi(\theta) G(x, \nu) \phi(d\nu|\mu), \quad (1.1)$$

where $d\nu$ is an element of solid angle surrounding the unit vector ν , Ω is the space of all ν , ν is the cosine of the angle made by ν with the positive x-axis, and θ is the angle between ν and μ . For isotropic scattering,

$$\phi(d\nu|_{\mu}) = \frac{(1-r)}{4\pi} d\nu, \quad (1.2)$$

whereas for Rayleigh scattering,

$$\phi(d\nu|_{\mu}) = \frac{3(1-r)}{16\pi} (1 + \cos^2 \theta) d\nu. \quad (1.3)$$

The probability of absorption at any collision is given by r in each case.

If the arbitrary function $\xi(\theta)$ is chosen to be independent of θ , i.e., $\xi(\theta) \equiv z$, then equation (1.1) reduces to the usual (backward) form of the transport equation with absorption probability $\left[1 - z \int_{\Omega} \phi(d\nu|_{\mu})\right]$ per collision. The solution can be expressed in terms of the X- and Y-functions of Chandrasekhar which, for isotropic scattering, have been extensively tabulated by Carlstedt and Mullikin.⁵⁶ The case where $\xi(\theta) \equiv z$ has been studied by Abu-Shumays.⁵⁷

We have already seen how the discrete approximation of Chapter VI can be used to solve (1.1). In this chapter we shall use this approximation to derive numerical results for the particular scattering functions (1.2) and (1.3). We shall use the special case in which $\xi(\theta) \equiv z$ to examine the accuracy of the approximation. In particular, if $\xi(\theta) \equiv z \equiv 1$, then $G(x, \mu)$ reduces to the first-passage probability $P(X^c \times L | x, \mu; \infty)$, and the derivatives of G with respect to z at $z = 1$ give the moments of the total number of collisions experienced before emergence.

In Sect. 7 it will be shown how the icosahedral approximation leads to a particularly direct method of dealing with the multiple Compton scattering of low-energy gamma radiation. This method eliminates certain objections to the previous treatments of the problem by Chandrasekhar¹⁶ and O'Rourke.³⁴ A similar approximation could be used for the elastic scattering of neutrons, for which the lethargy of a neutron increases by a certain amount $\delta(\theta)$ each time it is scattered through an angle θ (in the centre of mass coordinate system).

We shall assume (making use of the transformation (3.4) of Chapter VI) that $\lambda(x) \equiv 1$ throughout.

2. Application of the Icosahedral Approximation to Astrophysical Problems

It has already been shown in Chapter VI, Sect. 2, how the icosahedral approximation $\{p_1, \dots, p_9\}$ to the scattering function $\sigma(\theta)$ is derived. The probabilities $P_{\alpha\beta}(x)$ are then found from equation (3.2) of Chapter VI, i.e.,

$$A_{\theta}(1, \dots, 1) P_{\alpha}(x) = 0 \quad (2.1)$$

with boundary conditions

$$P_{\alpha\beta}(0) = \delta_{\alpha\beta}, \quad \beta = 7, 8, 9, 10; \quad (2.2)$$

$$P_{\alpha\beta}(t) = \delta_{\alpha\beta}, \quad \beta = 1, 2, 3, 4. \quad (2.3)$$

[If $t = \infty$, then (2.3) is replaced by the condition that $P_{\alpha\beta}(x)$ be a bounded function of x as x tends to ∞ .]

The next step is to use the icosahedral probabilities $P_{\alpha\beta}(x)$, $\alpha = 1, \dots, 10$, to estimate the exact probability density $f(\mu|\mu_\beta, x)$ of emergence in direction μ after at least one collision. This can be done as follows.

Since the probabilities $P_{\alpha\beta}(x)$, $\alpha = 2, 3, 4, 7, 8, 9$, each correspond to areas $8\pi/15$ of the unit sphere, while $P_{1\beta}(x)$ and $P_{10\beta}(x)$ each correspond to areas $2\pi/15$, we can estimate the probability density per unit solid angle of emergence in directions $\mu_1, \dots, \mu_{10}$ by

$$\left. \begin{aligned} \pi(\mu_\alpha|\mu_\beta, x) &= \frac{15}{8\pi} [P_{\alpha\beta}(x) - \delta_{\alpha\beta} \exp(-t\mu_\beta^{-1})], & \alpha &= 2, 3, 4, 7, 8, 9, \\ \pi(\mu_\alpha|\mu_\beta, x) &= \frac{15}{2\pi} [P_{\alpha\beta}(x) - \delta_{\alpha\beta} \exp(-t\mu_\beta^{-1})], & \alpha &= 1, 10, \\ \pi(0|\mu_\beta, x) &= 0. \end{aligned} \right\} (2.4)$$

Hence, if $f(\mu|\mu_\beta, x)$ denotes the corresponding density per unit μ -interval, it is estimated (at $\mu = \mu_1, \dots, \mu_{10}$) by

$$\left. \begin{aligned} f(\mu_\alpha|\mu_\beta, x) &= 3.75 [P_{\alpha\beta}(x) - \delta_{\alpha\beta} \exp(-t\mu_\beta^{-1})], & \alpha &= 2, 3, 4, 7, 8, 9, \\ f(\mu_\alpha|\mu_\beta, x) &= 15 [P_{\alpha\beta}(x) - \delta_{\alpha\beta} \exp(-t\mu_\beta^{-1})], & \alpha &= 1, 10, \\ f(0|\mu_\beta, x) &= 0. \end{aligned} \right\} (2.5)$$

We remark that for glancing incidence (i.e., $\mu_\beta = 0+$, $x = 0$) the exact density $f(0|\mu_\beta, x)$ is nonzero, and equations (2.4) and (2.5) must be modified accordingly. This special case will be considered in Sect. 3. In all other cases

$f(0|\mu_\beta, x)$ is zero. We remark also that $\int_{-1}^1 f(\mu|\mu', x) d\mu$ is the probability

that a particle initially at x with direction μ' eventually leaves the slab after making at least one collision.

Let us now relate the density $f(\mu|\mu', x)$ to the conventional astrophysical notation as used, for example, by Chandrasekhar.¹⁷

In Chandrasekhar's account of the scattering of radiation by plane slabs he considers an incident, parallel beam of particles of flux πF per unit area normal to the beam, i.e., $\mu' \pi F$ particles per unit area of slab face enter the slab $\{x: 0 \leq x \leq t\}$ per second (μ' is the cosine of the angle made by the incident beam with the positive x -axis; for convenience of notation we shall always consider incidence on the face $x = 0$ so that $0 \leq \mu' \leq 1$).

The diffusely reflected and transmitted particles (i.e., reflected and transmitted particles which are scattered at least once in the slab) are represented by intensities $I(-\mu, 0|\mu', 0)$ and $I(+\mu, t|\mu', 0)$, respectively, where $0 \leq \mu \leq 1$. In terms of these intensities we can write the number of particles diffusely reflected and transmitted per unit area of slab face per second in directions $(\mu, \mu + d\mu)$ as

$$2\pi\mu I(-\mu, 0|\mu', 0) d\mu, \quad 0 \leq \mu \leq 1; \quad (2.6)$$

and

$$2\pi\mu I(+\mu, t|\mu', 0) d\mu, \quad 0 \leq \mu \leq 1. \quad (2.7)$$

The probability density $f(\mu|\mu', 0)$ is therefore related to the intensities by

$$f(-\mu|\mu', 0) = \frac{2\mu}{\mu'F} I(-\mu, 0|\mu', 0), \quad 0 \leq \mu \leq 1; \quad (2.8)$$

$$f(+\mu|\mu', 0) = \frac{2\mu}{\mu'F} I(+\mu, t|\mu', 0), \quad 0 \leq \mu \leq 1. \quad (2.9)$$

Chandrasekhar expresses the diffusely reflected and transmitted radiation in terms of a scattering function, $S(t; \mu|\mu')$, and a transmission function, $T(t; \mu|\mu')$, defined as

$$S(t; \mu|\mu') = \frac{4\mu}{F} I(-\mu, 0|\mu', 0) = 2\mu' f(-\mu|\mu', 0); \quad (2.10)$$

$$T(t; \mu|\mu') = \frac{4\mu}{F} I(+\mu, t|\mu', 0) = 2\mu' f(+\mu|\mu', 0); \quad (2.11)$$

where $0 \leq \mu, \mu' \leq 1$. The icosahedral approximations to $f(\mu|\mu', x)$ defined by equation (2.5) thus lead to an approximation for the scattering and transmission functions S and T .

3. Glancing Incidence

We first observe that in the icosahedral approximation as formulated in Chapter VI the particles are restricted to leave the slab only in directions $\mu_1, \dots, \mu_4, \mu_7, \dots, \mu_{10}$, i.e., the probability of emergence in directions $\mu = \pm 0$ is zero. This leads to an estimate of zero for the probability densities $f(\pm 0 | \mu', x)$. This estimate is exact except in the case of glancing incidence (i.e., $x = 0$ and $\mu = +0$) when the exact value $f(-0 | +0, 0)$ is in general nonzero. The application of the icosahedral approximation in this case therefore requires special consideration. For $x = 0$ and $\mu' = +0$ we therefore use the following method of approximation.

Consider a particle incident with direction $\mu' = +0$ on the face $x = 0$ of $\{x: 0 \leq x \leq t\}$. It experiences its first collision on the face $x = 0$ and is scattered in accordance with the icosahedral approximation to the scattering law. The probability that it remains in the plane $x = 0$ is given by $p_1 + p_9 + 2p_5$. We now suppose that, instead of remaining in the plane $x = 0$ with this probability, the particle

- (i) leaves the slab in direction $\mu = -0$ with probability $0.5(p_1 + p_9 + 2p_5)$,
- (ii) enters the slab in the icosahedral direction 5 with probability $0.25(p_1 + p_9 + 2p_5)$,

or

- (iii) enters the slab in the icosahedral direction 6 with probability $0.25(p_1 + p_9 + 2p_5)$.

It is assumed in cases (ii) and (iii) that, as in Chapter VI, the particle is subsequently restricted to leave the slab in directions $\mu_1, \dots, \mu_4, \mu_7, \dots, \mu_{10}$. This is consistent with the exact model inasmuch as only the singly scattered particles contribute to the density $f(-0 | +0, 0)$.

The device described in the preceeding paragraph gives the following approximations for the probability densities $f(\mu | +0, 0)$:

$$\left. \begin{aligned} f(\mu_\alpha | +0, 0) &= 7.5k(P_{\alpha 5} + P_{\alpha 6}), & \alpha &= 1, 10; \\ f(\mu_\alpha | +0, 0) &= 1.875k(P_{\alpha 5} + P_{\alpha 6}), & \alpha &= 2, 3, 4, 7, 8, 9; \\ f(-0 | +0, 0) &= 3.75(p_1 + p_9 + 2p_5), \end{aligned} \right\} \quad (3.1)$$

where

$$k = 1 - [p_1 + p_9 + 2p_5] \left[\sum_{\alpha=1}^{10} (P_{\alpha 5} + P_{\alpha 6}) \right]^{-1},$$

and the probabilities $P_{\alpha 5}$ and $P_{\alpha 6}$ are the probabilities defined in Chapter VI ($P_{\alpha 5}$ and $P_{\alpha 6}$ are both zero if $\alpha = 5, 6$).

4. Emergence of Particles Parallel to the Slab Face

As we have already remarked, the densities $f(+0|\mu', 0)$ and $f(-0|\mu', 0)$ are always zero provided $\mu' > 0$. However, the corresponding intensities are, in general, nonzero. In order to determine the emergent intensities parallel to the slab face, we make use of the results of Sect. 3 and Helmholtz's Reciprocity Principle, which states that

$$S(t; \mu|\mu') = S(t; \mu'|\mu); \quad (4.1)$$

$$T(t; \mu|\mu') = T(t; \mu'|\mu), \quad (4.2)$$

or, in terms of the probability density of scattered particles,

$$\mu' f(-\mu|\mu', 0) = \mu f(-\mu'|\mu, 0); \quad (4.3)$$

$$\mu' f(\mu|\mu', 0) = \mu f(\mu'|\mu, 0), \quad (4.4)$$

where $0 \leq \mu, \mu' \leq 1$.

From (2.8), (2.9), (4.3), and (4.4), we have

$$\frac{I(-\mu|\mu', 0)}{\mu' F} = \frac{f(-\mu|\mu', 0)}{2\mu} = \frac{f(-\mu'|\mu, 0)}{2\mu'}; \quad (4.5)$$

$$\frac{I(\mu|\mu', 0)}{\mu' F} = \frac{f(\mu|\mu', 0)}{2\mu} = \frac{f(\mu'|\mu, 0)}{2\mu'}. \quad (4.6)$$

In the icosahedral approximation the emergent intensities parallel to the slab faces are therefore estimated by

$$\frac{I(-0|\mu', 0)}{\mu' F} = \frac{f(-\mu'|+0, 0)}{2\mu'}; \quad (4.7)$$

$$\frac{I(+0|\mu', 0)}{\mu' F} = \frac{f(\mu'|+0, 0)}{2\mu'}, \quad (4.8)$$

where the probability densities $f(\mu'|+0, 0)$ are given by equations (3.1).

5. Numerical Results for Isotropic Scattering

For isotropic scattering the discrete approximation to the scattering function is found, as described in Chapter VI, Sect. 2, to be given by

$$p_i = (1 - r)/30, \quad i = 1, \dots, 9, \quad (5.1)$$

where r is the probability of absorption at each collision.

The numerical determination of the matrices $P_{T(R)}^*$, $M_{T(R)}^*$, and $S_{T(R)}^*$ of Chapter VI is described in the Appendix, and results are presented for several slab thicknesses in the particular case in which $a_1 = a_2 = \dots = a_9 = 1$.

To indicate the accuracy of the icosahedral approximation we shall apply it in this section to some problems for which the exact solution is known.

We begin with the Milne problem. A particle which originates in the deep interior of the three-dimensional half-space $\{x: 0 \leq x < \infty\}$ is scattered isotropically and conservatively (i.e., $r = 0$), and it is required to find the distribution of its direction of emergence from the face $x = 0$. Applying the icosahedral approximation, we find that the probabilities $P_{\alpha\beta}(\infty)$ are independent of the initial direction β and are given by

$$\begin{bmatrix} P_{7\beta}(\infty) \\ P_{8\beta}(\infty) \\ P_{9\beta}(\infty) \\ P_{10\beta}(\infty) \end{bmatrix} = \begin{bmatrix} 0.11906 \\ 0.23289 \\ 0.48016 \\ 0.16790 \end{bmatrix}$$

Using the argument of Sect. 2 to determine the probability densities $f(\mu_\alpha | \mu_\beta, \infty)$, we arrive at the results given in Table 3. The icosahedral approximation to f is compared with the exact solution, the 4th-order Gaussian quadrature

TABLE 3. Approximations for the Milne Problem with Isotropic Scattering.

μ	Exact Solution	Icosahedral Approximation	Gaussian Quadrature Approximation, $n = 4$	Wang and Guth's Spherical Harmonics Method, $n = 3$, r odd	Chandrasekhar's Spherical Harmonics Method, $n = 3$
1	2.5182	2.5185	2.5096	2.5128	2.4837
0.8090	1.7997	1.8006	1.7922	1.7948	1.7702
0.5000	0.8716	0.8733	0.8660	0.8680	0.8520
0.3090	0.4441	0.4465	0.4384	0.4417	0.4324
0	0	0	0	0	0

approximation of Chandrasekhar (which, like the icosahedral approximation, involves the solution of a system of eight first-order differential equations), Chandrasekhar's spherical harmonic approximation,⁹ and the approximation of Wang and Guth.⁴⁵

As can be seen from Table 3 the icosahedral approximation is very good in comparison with the other standard approximations. Densities at values of μ other than those given could be obtained by interpolation.

Let us consider now isotropic scattering in a slab of finite thickness (whose exterior we denote by X^c). It follows from the results of Chandrasekhar¹⁷ that the generating functional $G(X^c \times d\mu; \xi(\cdot) 0, |\mu'; \infty)$ can be written, when $\xi(\theta) \equiv z$, and μ is not equal to μ' , as

$$G(X^c \times d\mu; z|0, \mu'; \infty) = \begin{cases} d\mu \frac{(1-r) z \mu}{2(\mu - \mu')} [Y(\mu) X(\mu') - X(\mu) Y(\mu')], & 0 \leq \mu \leq 1; \\ d\mu \frac{(1-r) z \mu}{2(\mu - \mu')} [X(-\mu) X(\mu') - Y(-\mu) Y(\mu')], & -1 \leq \mu \leq 0, \end{cases}$$

where $X(\mu)$ and $Y(\mu)$ are the X- and Y-functions as tabulated by Carlstedt and Mullikin⁵⁶ with parameter $\bar{\omega}_0 = (1-r)z$. In particular, setting $z = 1$, the tables can be used to give the first-passage probabilities

$$P(X^c \times d\mu|0, \mu'; \infty) = G(X^c \times d\mu; 1|0, \mu'; \infty).$$

By numerical differentiation of G with respect to z at $z = 1$, the moments of the total number of collisions experienced by a particle before emergence can also be found. Determinations of this kind have been carried out by Abu-Shumays.⁵⁷

To facilitate comparison with Fermi's approximation and the improved approximation derived in Chapter V, we shall consider isotropic incidence of particles on the face $x = 0$ of $\{x: 0 \leq x \leq t\}$, i.e., we shall assume that the incident direction μ' is uniformly distributed on $[0, 1]$ or, in the icosahedral approximation, that the incident direction is μ_β with probability r_β , where $r_1 = \frac{1}{15}$; $r_2 = r_3 = r_4 = \frac{4}{15}$; $r_5 = r_6 = \frac{1}{15}$.

For isotropic scattering with zero absorption, the transmission probability and the mean and variance of the total number of collisions experienced by transmitted particles are shown in Table 4. The icosahedral approximation is compared with the corresponding results derived from the tables of Carlstedt and Mullikin.⁵⁶ Tables 5 and 6 compare the transmission probabilities and the mean number of collisions with the Fermi approximation and the approximation of Chapter V. The accuracies of both the icosahedral approximation and the approximation of Chapter V are seen to be very good.

TABLE 4. Transmission of Particles Isotropically Incident on a Slab of Thickness t (mean free paths)

t	Transmission Probability		Mean Number of Collisions		Variance of Number of Collisions	
	Exact	Icosahedral Approximation	Exact	Icosahedral Approximation	Exact	Icosahedral Approximation
0.5	0.6254	0.6301	0.8856	0.8712	1.5	1.588
1.0	0.4837	0.4864	1.948	1.940	4.9	4.802
2.0	0.3384	0.3396	4.811	4.811	19.6	19.30
3.5	0.2347	0.2354	11.00	10.99	76	75.68

TABLE 5. Comparison of Transmission Probabilities

	Thickness, t				
	0.5	1.0	2.0	4.0	8.0
Fermi's approximation	0.69783	0.53590	0.36603	0.22401	0.12613
Method of Chapter V	0.62083	0.47929	0.33743	0.21294	0.12255
30-direction approximation	0.63011	0.48638	0.33958	0.21366	0.12290
120-direction approximation	0.62940	0.48512	0.33904	0.21331	0.12266

TABLE 6. Comparison of Mean Numbers of Collisions

	Thickness, t				
	0.5	1.0	2.0	4.0	8.0
Fermi's approximation	0.6016	1.5000	4.0981	12.360	46.856
Method of Chapter V	0.9184	1.9973	4.8266	13.543	43.142
30-direction approximation	0.8712	1.9398	4.8106	13.550	43.121
120-direction approximation	0.8733	1.9394	4.7901	13.512	43.064

One point of interest which arises from the results is the large mean and variance of the distribution of the number of collisions, even for slab thicknesses as small as one mean free path. This implies that the expansion of the solution by number of collisions [i.e., the Neumann series solution of the integral equation for $G(z)$] requires the evaluation of a large number of terms to give good results.

6. Numerical Results for Rayleigh Scattering

An important feature of the icosahedral approximation of Chapter VI is the fact that it can be applied without essential modification to a wide class of scattering laws. We consider now the same problems as in Sect. 5 with the isotropic scattering law (1.2) replaced by Rayleigh's law (1.3).

The icosahedral approximation to this scattering law can be determined as in Chapter VI, Sect. 2, and is found to be

$$\left. \begin{aligned} p_1 &= p_9 = \frac{1-r}{20}; \\ p_2 &= p_8 = \frac{1-r}{320} (11 + \sqrt{5}); \\ p_3 &= p_7 = \frac{1-r}{32}; \\ p_4 &= p_6 = \frac{1-r}{320} (11 - \sqrt{5}); \\ p_5 &= \frac{1-r}{40}. \end{aligned} \right\} \quad (6.1)$$

With these probabilities p_i the analysis then proceeds exactly as for isotropic scattering.

Thus, when $r = 0$, we find the following results for the probability density $f(\mu | \mu', \infty)$ of emergent radiation in the Milne problem. Table 7 shows the results obtained from the icosahedral approximation, the 4th-order Gaussian quadrature approximation, the 2nd-order double Gaussian quadrature approximation (c.f. Sykes⁵⁹), and the exact solution as given by Chandrasekhar.¹⁷

TABLE 7. The Milne Problem with Rayleigh's Law

μ	Exact Solution	Icosahedral Approximation	Gaussian Quadrature $n = 4$	Double Gaussian Quadrature $n = 2$
1	2.5292	2.5287	2.5214	2.5328
0.8090	1.8036	1.8042	1.7966	1.8063
0.5000	0.8689	0.8706	0.8633	0.8692
0.3090	0.4404	0.4430	0.4361	0.4391
0	0	0	0	0

As in Sect. 5, we again consider isotropic incidence on the face $x = 0$ of the slab $\{x: 0 \leq x \leq t\}$. Assuming Rayleigh scattering with no absorption, the icosahedral approximations to the transmission probability, and the mean and variance of the total number of collisions experienced by transmitted particles, are given in Table 8. In Table 9 the transmission probabilities and means are compared with the results of Brockwell and Landwehr⁷ obtained using the model of Chapter VI, Sect. 9, with 120 directions.

TABLE 8. Transmission of Particles Isotropically Incident on a Slab of Thickness t (mean free paths)

Thickness, t	Transmission Probability	Mean Number of Collisions	Variance of Number of Collisions
0.5	0.62931	0.8793	1.6364
1.0	0.48529	1.9557	4.9358
2.0	0.33850	4.8374	19.734
4.0	0.21293	13.584	110.21
8.0	0.12250	43.158	881.45

TABLE 9. Transmission of Particles Isotropically Incident on a Slab of Thickness t (mean free paths) Considering 30 and 120 Directions

Thickness, t	Transmission Probability		Mean Number of Collisions Experienced	
	30 Directions	120 Directions	30 Directions	120 Directions
0.5	0.62931	0.62856	0.8793	0.8810
1.0	0.48529	0.48384	1.9557	1.9561
2.0	0.33850	0.33784	4.8374	4.8177
4.0	0.21293	0.21253	13.584	13.546
8.0	0.12250	0.12222	43.158	43.103

The angular distribution of transmitted particles which are incident isotropically on a slab of thickness t is shown for several thicknesses in Fig. 5. The emergent density is normalized so that $\int_0^1 f(\mu) d\mu = 1$. Thus, in the notation of Sect. 2,

Fig. 5

Fig. 5

The Angular Distribution of Emergent Photons Transmitted through a Slab of Thickness ℓ

$$f(\mu) = P_T^{-1} \int_0^1 f(\mu|\mu', 0) d\mu',$$

where P_T is the probability of transmission of an isotropically incident particle, i.e.,

$$P_T = \int_{\mu=0}^1 \int_{\mu'=0}^1 f(\mu|\mu', 0) d\mu' d\mu.$$

The icosahedral approximations to P_T for the slab thicknesses $2/3$, $4/3$, and 2 were found to be 0.57154 , 0.42321 , and 0.33850 , respectively.

7. The Multiple Compton Scattering of Low-energy Gamma Radiation

The multiple Compton scattering of low-energy ($E \ll 0.5$ MeV) photons in slabs of finite thickness has been studied analytically by Chandrasekhar¹⁶ and O'Rourke³⁴ who used the Gaussian quadrature approximation to Boltzmann's equation. Provided the momenta of the scattering electrons and the energy of the scattered photons are small, the problem is somewhat simplified owing to the fact that the Klein-Nishina differential cross section (i.e., the cross section for scattering through a given angle per unit solid angle) is then effectively energy independent and is given, in Thomson units per electron, by

$$\sigma(\theta) = \frac{3}{16\pi} (1 + \cos^2 \theta), \quad (7.1)$$

where θ is the angle of deflection resulting from a collision and

$$1 \text{ Thomson unit} = \frac{8\pi}{3} \left(\frac{e^2}{mc^2} \right)^2 = 0.665 \text{ barn.}$$

The mean free path d of the photons is related to $\sigma(\theta)$ by the equation

$$d^{-1} = N \int \sigma(\theta) d\Omega, \quad (7.2)$$

where N is the number of electrons per unit volume. Thus d is independent of the photon energy and, by the transformation (3.4) of Chapter VI, may be taken as unity. Assuming the momenta of the scattering electrons are sufficiently small for Compton's relation to be applicable, then the wavelength of each scattered photon increases according to the equation

$$\lambda' = \lambda + 1 - \cos \theta, \quad (7.3)$$

where λ and λ' are the wavelengths in Compton units ($1 \text{ Compton unit} = h/mc = 0.02426 \text{ \AA}$) before and after a collision, and θ is the angle of scattering.

For scattering by plane slabs of finite thickness both Chandrasekhar and O'Rourke derived the spectral distribution of forward scattered radiation integrated over all forward directions of emergence. They considered only the isotropic approximation to (7.1), i.e., $\sigma(\theta) = 1/4\pi$. In this section we shall make use of the more accurate cross section (7.1) (which is in fact the Rayleigh scattering law considered in Sect. 6). The angular distribution of photons transmitted by a finite slab has already been determined in Sect. 6. We shall now show how to derive the spectral distribution of the emergent radiation for each angle of emergence.

Chandrasekhar's method for the Compton-scattering problem depends on the replacement of the radiation field intensity $I(x, \mu, \lambda')$ by the first two terms of a power series in $(\lambda' - \lambda)$, i.e.,

$$I(x, \mu, \lambda') = I(x, \mu, \lambda) + (\lambda' - \lambda) \frac{\partial I}{\partial \lambda}(x, \mu, \lambda).$$

The resulting approximation to the Boltzmann equation is then solved by Gaussian quadrature. As a consequence of this power-series expansion, Chandrasekhar's results (and also those of O'Rourke) are of physical significance only when integrated with respect to a relatively smooth spectral distribution of incident radiation. The method to be used does not suffer from this limitation. Assuming a monochromatic source of radiation of wavelength λ_0 , we are led quite naturally to determine the distribution function of the emergent wavelength λ as the sum of two components, an absolutely continuous component due to the scattered radiation and a simple discontinuity at λ_0 whose magnitude is equal to the probability π_0 that an incident photon passes through the slab without being scattered.

Like Chandrasekhar and O'Rourke we shall neglect photoelectric absorption and polarization; however, if we assume a constant probability of absorption at any collision the problem can be solved in exactly the same way as explained in Chapter VI, Sect. 8

The discrete approximation to the continuous cross section (7.1) has already been derived in Sect. 6 and is given by equations (6.1) with $r = 0$. In addition, we must introduce the Compton increments δ_i ($i = 1, \dots, 9$), δ_i being the increase in wavelength of a photon resulting from a collision of type i . If θ_i is the angle of deflection resulting from a collision of type i , then

$$\delta_i = 1 - \cos \theta_i. \quad (7.4)$$

Thus

$$\begin{aligned} \delta_1 &= 0; & \delta_2 &= \frac{1}{4}(3 - \sqrt{5}); & \delta_3 &= \frac{1}{2}; & \delta_4 &= \frac{1}{4}(5 - \sqrt{5}); & \delta_5 &= 1; \\ \delta_6 &= \frac{1}{4}(3 + \sqrt{5}); & \delta_7 &= \frac{3}{2}; & \delta_8 &= \frac{1}{4}(5 + \sqrt{5}); & \delta_9 &= 2. \end{aligned} \quad (7.5)$$

We now consider a photon which originates at a point x in the slab $[x: 0 \leq x \leq t]$ with initial direction β and initial wavelength λ_0 . Let $\Phi_{\alpha\beta}(\lambda|x)$ denote the distribution of its increase in wavelength due to multiple scattering when it emerges from the slab, given that it emerges with direction α . The Mellin-Stieltjes transform

$$\bar{\Phi}_{\alpha\beta}(s|x) = \int_0^\infty \lambda^{s-1} d\Phi_{\alpha\beta}(\lambda|x) \quad (7.6)$$

is clearly given by

$$\bar{\Phi}_{\alpha\beta}(s|x) = [P_{\alpha\beta}(x)]^{-1} K_{\alpha\beta}(s, x), \quad (7.7)$$

where

$$K_{\alpha\beta}(s, x) = \sum_{i_1 + \dots + i_9 \geq 0} P_{\alpha\beta}^{i_1 \dots i_9}(x) (i_1 \delta_1 + \dots + i_9 \delta_9) s^{-1}, \quad (7.8)$$

and $\delta_1, \dots, \delta_9$ are defined by equation (7.5). If n is a positive integer, then $\bar{\Phi}_{\alpha\beta}(n+1|x)$ is the n th moment about the origin of the wavelength increment λ . These moments can be determined exactly as described in Chapter VI, Sect. 4.

For radiation incident isotropically on the face $x = 0$ of the slab, the direction of motion of an incident photon is a random variable β which takes the values 1, 2, 3, 4, 5, 6 with associated probabilities $r_1 = \frac{1}{15}$, $r_2 = \frac{4}{15}$, $r_3 = \frac{4}{15}$, $r_4 = \frac{4}{15}$, and $r_5 = r_6 = \frac{1}{15}$. The mean and mean square of the increase in wavelength of the transmitted radiation are given by

$$E(\lambda) = \frac{1}{\pi} \left[\sum_{\alpha=1}^4 \sum_{\beta=1}^6 r_{\beta} K_{\alpha\beta}(2, 0) \right]; \quad (7.9)$$

$$E(\lambda^2) = \frac{1}{\pi} \left[\sum_{\alpha=1}^4 \sum_{\beta=1}^6 r_{\beta} K_{\alpha\beta}(3, 0) \right], \quad (7.10)$$

where $\pi = \sum_{\alpha=1}^4 \sum_{\beta=1}^6 r_{\beta} P_{\alpha\beta}(0)$ is the probability that an incident photon is transmitted through the slab and the functions $K_{\alpha\beta}(2, x)$ and $K_{\alpha\beta}(3, x)$ are found as described in Chapter VI, Sect. 4.

Numerical determinations of π , $E(\lambda)$, and $E(\lambda^2)$ for the slab thicknesses $t = 2/3$, $4/3$, and 2 lead to the following results:

$t =$	$2/3$	$4/3$	2
π	0.57154	0.42321	0.33850
$E(\lambda)$	1.0794	2.6124	4.6021
$E(\lambda^2)$	4.1550	16.898	44.233

We shall use these moments together with certain exact parameters which will now be determined to estimate the transmitted spectral intensity for the continuous model when particles are incident isotropically on one face of a slab of thickness t . It will be assumed that the incident radiation

is monochromatic so that the spectral intensity (rate of incidence of photons per unit wavelength interval) is given by

$$I(\lambda) = \delta(\lambda - \lambda_0), \quad (7.11)$$

where $\delta(x)$ is the Dirac delta function and λ_0 is the incident wavelength.

If $\eta(\lambda)$ is the spectral intensity corresponding to a transmitted wavelength $(\lambda + \lambda_0)$, i.e., if the rate of transmission of photons with wavelength in the small interval $(\lambda_0 + \lambda, \lambda_0 + \lambda + \delta\lambda)$ is $\eta(\lambda) \delta\lambda + o(\delta\lambda)$, then $\eta(\lambda)$ is clearly independent of λ_0 and can be written:

$$\eta(\lambda) = \pi_0 \delta(\lambda) + \gamma(\lambda), \quad 0 \leq \lambda < \infty, \quad (7.12)$$

where π_0 is the probability that an incident photon is transmitted through the slab with no collisions and $\gamma(\lambda)$ is the intensity of the transmitted radiation, which is scattered at least once in its passage through the slab. It is worth noticing at this point that the spectral intensities found by Chandrasekhar and O'Rourke (Fig. 7) are not of the form (7.12); in fact, they differ in two respects: first, in the complete absence of a delta function component (although their analysis could be modified to allow for this) and secondly, in the existence of a nonzero probability of a negative wavelength increment which for Compton scattering is physically impossible.

By making use of the moments derived above from the icosahedral approximation it is possible, as we shall now show, to fit a transmitted spectral distribution of the correct form as given by equation (7.12).

We first observe that the values π_0 and $\gamma(0)$ appearing in (7.12) can be determined exactly as follows. If μ is the cosine of the angle made by an incident photon with the positive x-axis, then for isotropically incident radiation μ is a random variable uniformly distributed on $[0, 1]$. The probability π_0 is therefore given by

$$\pi_0 = \int_0^1 \exp(-t\mu^{-1}) d\mu. \quad (7.13)$$

For a single collision the probability density $f(\lambda)$ corresponding to an increase in wavelength λ is readily found from equations (7.1) and (7.3) to be

$$f(\lambda) = \frac{3}{8}(2 - 2\lambda + \lambda^2), \quad 0 \leq \lambda \leq 2. \quad (7.14)$$

The only contribution to the intensity $\gamma(0)$ comes from photons which are scattered exactly once. Hence,

$$\gamma(0) = \frac{3}{4} \int_0^1 t \mu^{-1} \exp(-t \mu^{-1}) d\mu. \quad (7.15)$$

For any given slab thickness, π_0 and $\gamma(0)$ are easily found from equations (7.14) and (7.15) using tables of the exponential integral. Thus for thicknesses $t = 2/3, 4/3$, and 2 they are given by the following table:

$t =$	$2/3$	$4/3$	2
π_0	0.2478	0.09204	0.03753
$\gamma(0)$	0.1992	0.1287	0.07335

From these exact values of π_0 , and $\gamma(0)$, and from the icosahedral approximations to π , $E(\lambda)$, and $E(\lambda^2)$, we now fit a spectral-intensity function having the required form (7.12). Expressing $E(\lambda)$ and $E(\lambda^2)$ in terms of the transmitted spectral intensity $\eta(\lambda)$ we have

$$E(\lambda) = \frac{1}{\pi} \int_0^\infty \lambda \eta(\lambda) d\lambda = \frac{1}{\pi} \int_0^\infty \lambda \gamma(\lambda) d\lambda; \quad (7.16)$$

$$E(\lambda^2) = \frac{1}{\pi} \int_0^\infty \lambda^2 \eta(\lambda) d\lambda = \frac{1}{\pi} \int_0^\infty \lambda^2 \gamma(\lambda) d\lambda. \quad (7.17)$$

There are two further conditions to be satisfied by $\gamma(\lambda)$, namely,

$$\int_0^\infty \gamma(\lambda) d\lambda = \pi - \pi_0 \quad (7.18)$$

and

$$\gamma(0) = \frac{3}{4} \int_0^1 t \mu^{-1} \exp(-t \mu^{-1}) d\mu. \quad (7.19)$$

Using the numerically determined values of π_0 , $\gamma(0)$, π , $E(\lambda)$, and $E(\lambda^2)$, we obtain the parameters a , b , α , and β by fitting a curve of the form

$$\gamma^*(\lambda) = a \exp(-\alpha \lambda) + b \lambda \exp(-\beta \lambda) \quad (7.20)$$

to the scattered component $\gamma(\lambda)$ of the transmitted spectral intensity. The parameters a , α , b , and β are given, for each slab thickness, in the table below. Figure 6 shows the graphs of the fitted functions $\gamma^*(\lambda)$.

$t =$	$2/3$	$4/3$	2
a	0.1992	0.1287	0.07335
α	1.2931	1.1269	0.8790
b	0.07886	0.04053	0.01924
β	0.6817	0.4322	0.2974

Fig. 6. The Spectral Intensity of Scattered Radiation Transmitted through a Slab of Thickness l . In addition the intensity of the transmitted radiation will have a component $\pi_0 \delta(\lambda)$ due to unscattered photons. The wavelength increase is measured in Compton units.

Fig. 7. The Results of Chandrasekhar and O'Rourke Showing the Spectral Intensity of Emergent Radiation Measured in Units of the Integrated Incident Intensity. The wavelength increments are again measured in Compton units.

We remark finally that the assumption of an isotropically incident stream of radiation is not necessary for the analysis. Any incident angular distribution which can be reasonably well approximated by a distribution on the set of 30 directions in space which we have used can be dealt with in the same way. Furthermore, although we have here considered only the overall emergent spectral distribution for transmitted radiation the same method can be used to determine the spectral distribution for particular directions of emergence.

8. A Monte Carlo Study of Multiple Compton Scattering

As a check on the accuracy of the approximation used in Sect. 7, a Monte Carlo study was carried out in which the random paths of 40,000 photons isotropically incident on a plane slab were generated, using the CDC-3600 computer. Scattering was assumed to be isotropic and conservative (as in the analyses of Chandrasekhar and O'Rourke).

Denote by N_0 the number of photons which are transmitted by the slab without scattering, and by N_i , $i = 1, \dots, 12$, the number of particles which are transmitted after at least one scattering collision and emerge from the slab with an increase in wavelength λ such that $\frac{i-1}{2} \leq \lambda < \frac{i}{2}$. N_{13} denotes the number of diffusely transmitted particles for which $\lambda \geq 6$.

The spectral intensity of scattered radiation, $\gamma(\lambda)$, is estimated (for $0 \leq \lambda \leq 6$) by the histogram

$$\hat{\gamma}(\lambda) = 5N_i \times 10^{-5}, \quad \frac{i-1}{2} \leq \lambda \leq \frac{i}{2}, \quad i = 1, \dots, 12. \quad (8.1)$$

This is plotted in Fig. 8 for slab thicknesses t equal to $2/3$ and 2 . We see from Figs. 6 and 8 that the agreement between the Monte Carlo results and the analysis of Sect. 7 is very good except in the vicinity of $\lambda = 0$. The latter discrepancy is due to the fact that in the Monte Carlo experiment isotropic scattering was assumed while in the analysis of Sect. 7 the more accurate Rayleigh scattering law was used. The spectral intensity at the origin, i.e., $\gamma(0)$, can be shown analytically to be 1.5 times as great for Rayleigh scattering as for isotropic scattering, a result which is also confirmed by inspection of Figs. 6 and 8.

Fig. 8

Histograms for the Spectral Intensity of Scattered Radiation Transmitted through a Slab of Thickness t . The wavelength increment is measured in Compton units.

APPENDIX

Numerical Calculations

1. Introduction

In this appendix we shall first outline the method used to solve equations (3.2), (4.4), and (4.11) of Chapter VI for $P_{\alpha\beta}(x)$, $M_{\alpha\beta}(x)$, and $S_{\alpha\beta}(x)$, respectively. The results are printed out by a computer program as the matrices P_R^* , P_T^* , M_R^* , M_T^* , S_R^* , and S_T^* defined in Chapter VI by equations (5.3), (5.4), and (5.5). A brief description is given in Sect. 3 of a second program which uses the matrices for thickness t to generate the corresponding matrices for thicknesses 2^nt , $n = 1, 2, \dots$. In Sect. 4 the matrices are then tabulated for a number of slab thicknesses and several scattering cross sections. For all the tabulated results the coefficients a_i in the linear combination $N = \sum a_i N_i$ are taken as unity.

2. Program Transport

Program Transport makes use of the backward Kolmogorov equation

$$A_\theta(z_1, \dots, z_9) G_\alpha(x) = Q,$$

with boundary conditions

$$\left. \begin{aligned} G_{\alpha\beta}(0) &= \delta_{\alpha\beta}, & \beta &= 7, 8, 9, 10; \\ G_{\alpha\beta}(t) &= \delta_{\alpha\beta}, & \beta &= 1, 2, 3, 4, \end{aligned} \right\}$$

to solve for

$$\begin{aligned} P_{\alpha\beta}(x) &= [G_{\alpha\beta}(x)]_{z_1=\dots=z_9=1}; \\ M_{\alpha\beta}(x) &= \left[\sum_{k=1}^9 a_k \frac{\partial}{\partial z_k} G_{\alpha\beta}(x) \right]_{z_1=\dots=z_9=1}; \\ S_{\alpha\beta}(x) &= \left[\sum_{j=1}^9 \sum_{k=1}^9 a_j a_k \frac{\partial^2}{\partial z_j \partial z_k} G_{\alpha\beta}(x) \right]_{z_1=\dots=z_9=1} \end{aligned}$$

As already described in Chapter VI, Sects. 3 and 4, this requires the solution of three two-point boundary-value problems.

The input data consist of the discrete approximation $\{p_1, \dots, p_9\}$ to the scattering function, $\sigma(\theta)$, and the coefficients $\{a_1, \dots, a_9\}$ of the particular linear combination under consideration. The probability of absorption at each collision is given by

$$r = 1 - p_1 - p_9 - 4 \sum_{i=2}^8 p_i.$$

The first step in the program is to determine the matrices B and $R(1, \dots, 1)$ (denoted for simplicity by R), which have already been defined in Chapter VI, Sects. 3 and 4. The eigenvalues $\lambda_1, \dots, \lambda_8$ of R (assumed to be ordered so that $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4 \leq \lambda_5 < \lambda_6 < \lambda_7 < \lambda_8$) and the corresponding eigenvectors $\underline{e}_1, \dots, \underline{e}_8$ are then determined. We observe that the eigenvalues have the property $\lambda_i = -\lambda_{9-i}$, $i = 1, \dots, 4$, and that $\lambda_4 = \lambda_5 = 0$, if there is zero probability of absorption at each collision. Let $T = [\underline{t}_1 \underline{t}_2 \dots \underline{t}_8]$ be a matrix such that $T^{-1}RT$ is in Jordan canonical form. If all the eigenvalues of R are distinct, then such a matrix is obtained by choosing $\underline{t}_i = \underline{e}_i$, $i = 1, \dots, 8$; if, on the other hand, the probability of absorption, r , is zero, then $\lambda_4 = \lambda_5 = 0$, and we obtain a suitable matrix T by choosing $\underline{t}_5$ such that $R\underline{t}_5 = \underline{e}_4$ and the remaining $\underline{t}_i$ such that $\underline{t}_i = \underline{e}_i$, $i \neq 5$.

If $\Lambda = T^{-1}RT$, then $\exp[xR] = T \exp[x\Lambda] T^{-1}$. The evaluation of $\exp[x\Lambda]$ is trivial. Thus, if the eigenvalues of R are all distinct, then $\exp[x\Lambda]$ is a diagonal matrix whose i th diagonal element is $\exp[\lambda_i x]$; if $r = 0$, implying that $\lambda_4 = \lambda_5 = 0$, then $\exp[x\Lambda]$ is the same as before except for an additional element x immediately above the 5th diagonal element.

From equation (3.8), Chapter VI, $\underline{p}_\alpha(x)$ can be written in one of the forms:

$$\underline{p}_\alpha(x) = \sum_{k=1}^8 \underline{p}_\alpha^{(k)} \exp[\lambda_k x] \text{ if } r \neq 0;$$

$$\underline{p}_\alpha(x) = \sum_{\substack{k=1 \\ k \neq 5}}^8 \underline{p}_\alpha^{(k)} \exp[\lambda_k x] + x \underline{p}_\alpha^{(5)} \text{ if } r = 0,$$

where $\underline{p}_\alpha^{(k)} = c_{\alpha k} \underline{t}_k$, $k \neq 4, 5$, and

$$\underline{p}_\alpha^{(4)} = c_{\alpha 4} \underline{t}_4; \quad \underline{p}_\alpha^{(5)} = c_{\alpha 5} \underline{t}_5 \text{ if } r \neq 0,$$

$$\underline{p}_\alpha^{(4)} = c_{\alpha 4} \underline{t}_4 + c_{\alpha 5} \underline{t}_5; \quad \underline{p}_\alpha^{(5)} = c_{\alpha 5} \underline{t}_4 \text{ if } r = 0.$$

The unknown constants $c_{\alpha k}$, $k = 1, \dots, 8$, are determined by the boundary conditions (3.9) of Chapter 6.

The 10-component vectors $\underline{P}_\alpha(x)$ corresponding to the 8-component vectors $\underline{p}_\alpha(x)$ are easily determined by use of equation (3.2), Chapter VI,

to find the additional components $P_{\alpha 5}(x)$ and $P_{\alpha 6}(x)$. Proceeding in this way we can express the vectors $\underline{P}_{\alpha}(x)$ as follows:

$$\underline{P}_{\alpha}(x) = \sum_{k=1}^8 \underline{P}_{\alpha}^{(k)} \exp[\lambda_k x] \text{ if } r \neq 0;$$

$$\underline{P}_{\alpha}(x) = \sum_{\substack{k=1 \\ k \neq 5}}^8 \underline{P}_{\alpha}^{(k)} \exp[\lambda_k x] + x \underline{P}_{\alpha}^{(5)} \text{ if } r = 0.$$

The next step is to form the vectors $B \underline{P}_{\alpha}^{(k)}$, $k = 1, \dots, 8$, and thence [from equation (4.4), Chapter VI] to determine the 8-component vector $\underline{V}_{\alpha}(x)$ of equation (4.6), Chapter VI, in the form

$$\underline{V}_{\alpha}(x) = \sum_{k=1}^8 \underline{V}_{\alpha}^{(k)} \exp[\lambda_k x] \text{ if } r \neq 0;$$

$$\underline{V}_{\alpha}(x) = \sum_{\substack{k=1 \\ k \neq 5}}^8 \underline{V}_{\alpha}^{(k)} \exp[\lambda_k x] + x \underline{V}_{\alpha}^{(5)} \text{ if } r = 0.$$

The integral $\int_0^t \exp[-yR] \underline{V}_{\alpha}(y) dy$ is evaluated analytically by expanding the integrand as a sum of the form

$$\underline{I}_{\alpha}(y) = \sum_{i=1}^8 \sum_{j=1}^8 \exp[(\lambda_j - \lambda_i) y] \underline{I}_{\alpha}^{(ij)} \text{ if } r \neq 0;$$

$$\begin{aligned} \underline{I}_{\alpha}(y) = & \sum_{\substack{i=1 \\ i \neq 5}}^8 \sum_{\substack{j=1 \\ j \neq 5}}^8 \exp[(\lambda_j - \lambda_i) y] \underline{I}_{\alpha}^{(ij)} \\ & + \sum_{i=1}^8 y [\exp(-\lambda_i y) \underline{I}_{\alpha}^{(i5)} - \exp(\lambda_i y) \underline{I}_{\alpha}^{(5i)}] - y^2 \underline{I}_{\alpha}^{(55)} \text{ if } r = 0. \end{aligned}$$

The boundary conditions (4.8) of Chapter VI define four components of each of the vectors $\underline{m}_{\alpha}(t)$ and $\underline{m}_{\alpha}(0)$. The remaining eight unknown components are determined from the equation

$$\exp[-tR] \tilde{m}_\alpha(t) = \tilde{m}_\alpha(0) + \int_0^t \exp[-yR] \tilde{V}_\alpha(y) dy,$$

which is a direct consequence of (4.7), Chapter VI.

Having found $\tilde{m}_\alpha(0)$, it is now possible to expand $\tilde{m}_\alpha(x)$ as follows:

$$\tilde{m}_\alpha(x) = \sum_{k=1}^8 \tilde{m}_\alpha^{(k)} \exp[\lambda_k x] + \sum_{k=1}^8 \tilde{m}_\alpha^{(k+8)} x \exp[\lambda_k x] \text{ if } r \neq 0;$$

$$\begin{aligned} \tilde{m}_\alpha(x) = & \sum_{\substack{k=1 \\ k \neq 5}}^8 \tilde{m}_\alpha^{(k)} \exp[\lambda_k x] + \sum_{\substack{k=1 \\ k \neq 4,5}}^8 \tilde{m}_\alpha^{(k+8)} x \exp[\lambda_k x] \\ & + \tilde{m}_\alpha^{(5)} x + \tilde{m}_\alpha^{(12)} x^2 + \tilde{m}_\alpha^{(13)} x^3 \text{ if } r = 0. \end{aligned}$$

Equation (4.4), Chapter VI, is then used to determine $M_{\alpha 5}(x)$ and $M_{\alpha 6}(x)$, thus giving the 10-component vectors $\tilde{M}_\alpha^{(k)}$ such that

$$\tilde{M}_\alpha(x) = \sum_{k=1}^8 \tilde{M}_\alpha^{(k)} \exp[\lambda_k x] + \sum_{k=1}^8 \tilde{M}_\alpha^{(k+8)} x \exp[\lambda_k x] \text{ if } r \neq 0;$$

$$\begin{aligned} \tilde{M}_\alpha(x) = & \sum_{\substack{k=1 \\ k \neq 5}}^8 \tilde{M}_\alpha^{(k)} \exp[\lambda_k x] + \sum_{\substack{k=1 \\ k \neq 4,5}}^8 \tilde{M}_\alpha^{(k+8)} x \exp[\lambda_k x] \\ & + \tilde{M}_\alpha^{(5)} x + \tilde{M}_\alpha^{(12)} x^2 + \tilde{M}_\alpha^{(13)} x^3 \text{ if } r = 0. \end{aligned}$$

The determination of the vectors $\tilde{S}_\alpha(x)$ proceeds in much the same way as the determination of $\tilde{M}_\alpha(x)$. Thus we begin [see Chapter VI, equation (4.11)] by forming the vectors $2B\tilde{M}_\alpha^{(k)}$, $k = 1, \dots, 16$, and thence [from equation (4.11), Chapter VI] we derive the 8-component vector $\tilde{W}_\alpha(x)$ of equation (4.12), Chapter VI, in the form

$$\tilde{W}_\alpha(x) = \sum_{k=1}^8 \tilde{W}_\alpha^{(k)} \exp[\lambda_k x] + \sum_{k=1}^8 \tilde{W}_\alpha^{(k+8)} x \exp[\lambda_k x] \text{ if } r \neq 0;$$

$$\begin{aligned} \tilde{W}_\alpha(x) = & \sum_{\substack{k=1 \\ k \neq 5}}^8 \tilde{W}_\alpha^{(k)} \exp[\lambda_k x] + \sum_{\substack{k=1 \\ k \neq 4,5}}^8 \tilde{W}_\alpha^{(k+8)} x \exp[\lambda_k x] \\ & + \tilde{W}_\alpha^{(5)} x + \tilde{W}_\alpha^{(12)} x^2 + \tilde{W}_\alpha^{(13)} x^3 \text{ if } r = 0. \end{aligned}$$

The integral $\int_0^t \exp[-yR] \underline{W}_\alpha(y) dy$ is again evaluated analytically by expanding the integrand as a sum of the form

$$\begin{aligned} \underline{I}_\alpha(y) &= \sum_{i=1}^8 \left\{ \sum_{\substack{j=1 \\ j \neq 5}}^8 \exp[(\lambda_j - \lambda_i) y] \underline{I}_\alpha^{(ij)} + \sum_{j=1}^8 y \exp[(\lambda_j - \lambda_i) y] \underline{I}_\alpha^{(i,j+8)} \right\} \text{ if } r \neq 0; \\ \underline{I}_\alpha(y) &= \sum_{\substack{i=1 \\ i \neq 5}}^8 \left\{ \sum_{\substack{j=1 \\ j \neq 5}}^8 \exp[(\lambda_j - \lambda_i) y] \underline{I}_\alpha^{(ij)} + \sum_{\substack{j=1 \\ j \neq 4,5}}^8 y \exp[(\lambda_j - \lambda_i) y] \underline{I}_\alpha^{(i,j+8)} \right. \\ &\quad \left. + y \exp(-\lambda_i y) \underline{I}_\alpha^{(i5)} + y^2 \exp(-\lambda_i y) \underline{I}_\alpha^{(i12)} + y^3 \exp(-\lambda_i y) \underline{I}_\alpha^{(i13)} \right\} \\ &\quad - \sum_{\substack{j=1 \\ j \neq 5}}^8 y \exp(\lambda_j y) \underline{I}_\alpha^{(5j)} - \sum_{\substack{j=1 \\ j \neq 4,5}}^8 y^2 \exp(\lambda_j y) \underline{I}_\alpha^{(5,j+8)} - y^2 \underline{I}_\alpha^{(55)} - y^3 \underline{I}_\alpha^{(5,12)} \\ &\quad - y^4 \underline{I}_\alpha^{(5,13)} \text{ if } r = 0. \end{aligned}$$

The boundary conditions (4.14) of Chapter VI define four components of each of the vectors $\underline{s}_\alpha(t)$ and $\underline{s}_\alpha(0)$. The remaining eight unknown components are determined from the equation

$$\exp[-tR] \underline{s}_\alpha(t) = \underline{s}_\alpha(0) + \int_0^t \exp[-yR] \underline{W}_\alpha(y) dy,$$

which is a direct consequence of equation (4.13), Chapter VI.

It would be possible now to proceed in the same way as before, obtaining an expansion for $\underline{s}_\alpha(x)$ analogous to the one above for $\underline{m}_\alpha(x)$, and proceeding to find third- and higher-order moments of $N = \sum_{i=1}^9 a_i N_i$. However the program, as written, stops at the determination of $\underline{s}_\alpha(0)$ and $\underline{s}_\alpha(t)$, which are easily found from $\underline{s}_\alpha(0)$, $\underline{s}_\alpha(t)$, and equation (4.11) of Chapter VI.

The output of Program Transport consists of the matrices P_T^* , P_R^* , M_T^* , M_R^* , S_T^* , and S_R^* defined in Chapter VI. A block diagram of the program is given on the next page.

3. Program Iterate

The evaluation of the integrals $\int_0^t \exp[-yR] \underline{V}_\alpha(y) dy$ and $\int_0^t \exp[-yR] \underline{W}_\alpha(y) dy$ becomes numerically unstable for large values of t since sums of large positive and negative numbers are involved. This difficulty can be overcome by using the method of iteration described in Chapter VI, Sect. 5; to determine the matrices $P_{T(R)}^*$, $M_{T(R)}^*$, and $S_{T(R)}^*$ for thicker slabs.

Program Iterate uses as input data the six matrices $P_{T(R)}^*(t)$, $M_{T(R)}^*(t)$, and $S_{T(R)}^*(t)$; these are then used to generate the corresponding matrices for thicknesses $2t$, $4t$, ..., 2^nt by a direct application of equations (5.10), ..., (5.15) of Chapter VI. Each new matrix element is found as a sum of positive terms so that the question of instability does not arise. The only errors are rounding errors and those introduced in the inversion of the 4×4 matrix $(I - P_R^2)$ (see Sect. 5, Chapter VI).

In Sect. 4 transmission and reflection matrices are tabulated for isotropic and Rayleigh scattering laws with absorption probabilities 0 and 0.05. Program Transport was used to derive the results for thickness $t = 0.5$, and Program Iterate was then used to generate the corresponding results for thicknesses 1, 2, 4, 8, and 16. Only the results for $t = 0.5$, $t = 2$, and $t = 16$ are tabulated.

4. Numerical Results

We recall from Chapter VI that the 4×6 matrices G_T^* and G_R^* are defined as follows: denoting the $(i, j)^{th}$ element of G_T^* by $\{G_T^*\}_{ij}$, we have

$$\{G_T^*\}_{ij} = G_{ij}(0); \quad \{G_R^*\}_{ij} = G_{(11-i)j}(0).$$

The matrices $P_R^*(T)$, $M_R^*(T)$ and $S_R^*(T)$ are defined in terms of $G_R^*(T)$ by equations (5.3), (5.4), and (5.5) of Chapter VI.

Consider a particle incident on the face $x = 0$ of the slab $\{x: 0 \leq x \leq t\}$ with direction j , $j = 1, \dots, 6$, and suppose that it makes a random number, N_k , of collisions of type k ($k = 1, \dots, 9$) before emerging from the slab. Let N denote the random variable, $N = \sum_{k=1}^9 a_k N_k$. Then the $(i, j)^{th}$ element of P_T^* , namely, $\{P_T^*\}_{ij}$, denotes the probability that the particle emerges from the slab with direction i , and $\{P_R^*\}_{ij}$ denotes the probability that it emerges with direction $(11-i)$. Similarly, $\{M_T^*\}_{ij} / \{P_T^*\}_{ij}$ is the

expectation EN conditional on emergent direction i , and $\{M_R^*\}_{ij}/\{P_R^*\}_{ij}$ is the expectation EN conditional on emergent direction $(11-i)$. In the same way $[\{S_T^*\}_{ij} + \{\bar{M}_T^*\}_{ij}]/\{P_T^*\}_{ij}$ and $[\{S_R^*\}_{ij} + \{\bar{M}_R^*\}_{ij}]/\{P_R^*\}_{ij}$ are the expectations EN^2 conditional on emergent directions i and $(11-i)$, respectively; the matrix $\bar{M}_T^*(R)$ is defined as the matrix $M_T^*(R)$ in which the set of constants $\{a_1, \dots, a_9\}$ is replaced by a new set $\{a_1^2, \dots, a_9^2\}$.

The tabulated results which follow were all obtained with $a_1 = a_2 = \dots = a_9 = 1$. Isotropic and Rayleigh cross sections were used with absorption probabilities 0 and 0.05. Although numerical results were obtained for thicknesses $t = 0.5, 1, 2, 4, 8$, and 16, only the results for $t = 0.5, 2$, and 16 are listed.

In addition to the results tabulated below, the matrices $P_R^*(T)$, $M_R^*(T)$, $\bar{M}_R^*(T)$, and $S_R^*(T)$ have been calculated using the Rayleigh cross section with no absorption, thicknesses $t = 2/3, 4/3$, and 2, and $a_1, \dots, a_9$ as defined by equation (4.5), Chapter VII. These results were used to derive the results in Chapter VII, Sect. 4.

I. - Isotropic scattering with no absorption.

$$p_1 = p_2 = \dots = p_9 = 1/30,$$

$$a_1 = a_2 = \dots = a_9 = 1.$$

$$\text{Slab thickness, } t = 0.5.$$

P*_T

.624971202	.021530545	.029192636	.036365861	.037435668	.037435668
.069674307	.620288379	.109934548	.136409600	.136007862	.136007862
.058385273	.067943288	.459027078	.111668018	.100497647	.100497647
.044950676	.052103831	.069014631	.281150850	.063616459	.063616459

P*_R

.018972398	.022299095	.030880567	.039741847	.053486863	.053486863
.072161387	.084879214	.117827773	.152196999	.211222205	.211222205
.061761133	.072821568	.101862115	.133100531	.203573000	.203573000
.049123625	.058134080	.082260652	.109366294	.194160297	.194160297

M*_T

.033597669	.039265675	.053399305	.066809008	.069021679	.069021679
.127066392	.148423161	.201502601	.251436676	.254336616	.254336616
.106798611	.124535456	.168149797	.208046242	.197102076	.197102076
.082580476	.096040264	.128579648	.157002910	.135290982	.135290982

M*_R

.034261009	.040224207	.055504325	.071018387	.088686714	.088686714
.130168268	.152905408	.211346083	.271120970	.346474010	.346474010
.111008650	.130619063	.181510550	.234766775	.323299392	.323299392
.087783554	.103558996	.145093846	.190038472	.294898374	.294898374

S*_T

.055124232	.064519237	.088143475	.111001193	.115574636	.115574636
.208788637	.244333843	.333628046	.419816398	.434601039	.434601039
.176286951	.206193473	.281092772	.352833147	.358962704	.358962704
.137205020	.160355595	.218062877	.272686762	.270849898	.270849898

S*_R

.055452441	.064993465	.089184546	.113081093	.124479998	.124479998
.210323270	.246551232	.338495902	.429541772	.476299508	.476299508
.178369091	.209201972	.287697534	.366029539	.415910339	.415910339
.139775918	.164070357	.226218696	.288985008	.342352184	.342352184

$$p_1 = p_2 = \dots = p_9 = 1/30,$$

$$a_1 = a_2 = \dots = a_9 = 1.$$

Slab thickness, $t = 2.0$.

P*
T

.182533895	.048218653	.046484408	.041583132	.027987641	.027987641
.156038837	.242881246	.150367030	.132394947	.086869682	.086869682
.092968816	.092931936	.102830694	.071495969	.044692431	.044692431
.051399578	.050570370	.044186939	.037773969	.022366334	.022366334

P*
R

.061232207	.066011295	.073627113	.076820537	.074117999	.074117999
.213617037	.231500226	.261906509	.277561741	.280672729	.280672729
.147254225	.161867124	.190372710	.210707824	.246117110	.246117110
.094955406	.106019151	.130224597	.151661882	.217176074	.217176074

M*
T

.229110606	.237260342	.236605735	.217747438	.147507458	.147507458
.767790594	.793301634	.786429935	.719621467	.483511277	.483511277
.473211471	.486040430	.474586170	.428367545	.283460038	.283460038
.269150636	.274870942	.264745702	.236461389	.155555877	.155555877

M*
R

.256283092	.271655027	.288715219	.284478997	.225354617	.225354617
.879094132	.934218640	1.00011737	.993761247	.808140967	.808140967
.577430438	.618106529	.675611118	.688209531	.611102018	.611102018
.351635379	.379583031	.425336881	.447140810	.456736942	.456736942

S*
T

1.69650142	1.76948614	1.79644856	1.67904727	1.14798532	1.14798532
5.72617742	5.96892124	6.05028631	5.64620638	3.85332034	3.85332034
3.59289713	3.73928259	3.77477410	3.50915594	2.38506939	2.38506939
2.07541657	2.15665893	2.16877764	2.00939255	1.36177330	1.36177330

S*
R

1.74830214	1.83490326	1.89458789	1.80224121	1.26728572	1.26728572
5.93787167	6.23628139	6.45150786	6.15017696	4.34376273	4.34376273
3.78917579	3.98725114	4.14738962	3.97845239	2.85156319	2.85156319
2.22769265	2.34915856	2.45881880	2.37667599	1.74384455	1.74384455

$$p_1 = p_2 = \dots = p_9 = 1/30,$$

$$a_1 = a_2 = \dots = a_9 = 1.$$

Slab thickness, $t = 16.0$.

P^*_T

.016184136	.014301916	.011223908	.009284038	.005977830	.005977830
.046281973	.040899556	.032097275	.026549779	.017094939	.017094939
.022447817	.019837207	.015567904	.012877243	.008291432	.008291432
.011475702	.010141113	.007958574	.006583063	.004238719	.004238719

P^*_R

.124772705	.123412562	.119116976	.114259316	.098151082	.098151082
.399371441	.399410458	.395023363	.387100745	.350981143	.350981143
.238233952	.244137865	.255612214	.264386412	.280568459	.280568459
.141232281	.147859328	.163399789	.178959408	.234696399	.234696399

M^*_T

2.38504214	2.12226806	1.67872073	1.39254241	0.89763469	0.89763469
6.86780372	6.11084845	4.83344071	4.00938718	2.58443682	2.58443682
3.35744147	2.98723065	2.36263779	1.95978865	1.26326203	1.26326203
1.72127709	1.53144963	1.21121600	1.00468583	0.64760940	0.64760940

M^*_R

3.93394810	3.58363928	2.94340261	2.49552779	1.65264459	1.65264459
11.5969003	10.5888147	8.73999966	7.44004973	4.96929966	4.96929966
5.88680523	5.40161686	4.51024442	3.88142587	2.67219260	2.67219260
3.08464200	2.84184613	2.39885312	2.09034099	1.51729596	1.51729596

S^*_T

503.700965	449.390360	356.592509	296.150771	190.989806	190.989806
1454.25775	1297.42335	1029.47895	854.974962	551.377205	551.377205
713.185020	636.252984	504.836185	419.257444	270.379604	270.379604
366.062486	326.571377	259.115351	215.189642	138.775777	138.775777

S^*_R

565.954882	506.958724	404.443276	336.662711	217.364744	217.364744
1640.55289	1469.80370	1172.92980	976.517008	630.563617	630.563617
808.886554	724.910482	578.790112	482.023041	311.356868	311.356868
416.137997	372.996308	297.906623	248.159777	160.359416	160.359416

II. - Isotropic scattering with absorption probability, $r = 0.05$.

$$p_1 = p_2 = \dots = p_9 = 19/600,$$

$$a_1 = a_2 = \dots = a_9 = 1.$$

$$\text{Slab thickness, } t = 0.5.$$

P*
T

.623357516	.019644740	.026628514	.033158689	.034123322	.034123322
.063571715	.613160627	.100260024	.124341809	.123812613	.123812613
.053257027	.061964102	.450957072	.101689248	.091073045	.091073045
.040986394	.047494345	.062847411	.273627944	.057176410	.057176410

P*
R

.017325951	.020365949	.028212478	.036326775	.049202268	.049202268
.065905595	.077530087	.107667084	.139157001	.194471610	.194471610
.056424956	.066541917	.093132226	.121802027	.187908899	.187908899
.044902363	.053153245	.075277793	.100211734	.179828236	.179828236

M*
T

.029451676	.034416054	.046786486	.058503848	.060403168	.060403168
.111372691	.130071846	.176504384	.220089552	.222195131	.222195131
.093572972	.109085708	.147171328	.181868775	.171219991	.171219991
.072314733	.084066728	.112401084	.136966180	.116451926	.116451926

M*
R

.030066545	.035304547	.048737709	.062405770	.078669227	.078669227
.114247913	.134226582	.185628701	.238336128	.307779052	.307779052
.097475418	.114724847	.159556059	.206637917	.288448861	.288448861
.077137774	.091036300	.127709256	.167589946	.264740567	.264740567

S*
T

.044112951	.051628341	.070520076	.088784842	.092375744	.092375744
.167072820	.195502300	.266891704	.335730360	.347119955	.347119955
.141040152	.164948144	.224783677	.281999348	.286083333	.286083333
.109744100	.128237586	.174285182	.217751047	.215184289	.215184289

S*
R

.044398400	.052040786	.071425523	.090593829	.100138342	.100138342
.168407520	.197430809	.271125415	.344188989	.383468149	.383468149
.142851046	.167564722	.230528079	.293477007	.335727918	.335727918
.111980130	.131468495	.181378765	.231926937	.277529121	.277529121

$$p_1 = p_2 = \dots = p_9 = 19/600,$$

$$a_1 = a_2 = \dots = a_9 = 1.$$

$$\text{Slab thickness, } t = 2.0.$$

P*
T

.172873753	.038227604	.036552885	.032468919	.021823581	.021823581
.123707126	.209528084	.117436387	.102371636	.066755686	.066755686
.073105770	.072579679	.083082127	.053772515	.033026752	.033026752
.040133792	.039102485	.033233242	.028062499	.016016808	.016016808

P*
R

.050275177	.054378479	.061207100	.064516645	.064204194	.064204194
.175972454	.191419877	.218769784	.234432421	.244913655	.244913655
.122414200	.135207162	.161017179	.180552067	.218627627	.218627627
.079746959	.089545217	.111587315	.131854809	.196225630	.196225630

M*
T

.154667978	.159735492	.158205007	.144720661	.097700550	.097700550
.516914912	.532286385	.523118762	.474943083	.317139810	.317139810
.316410015	.323305174	.311455822	.277660380	.181625698	.181625698
.178884575	.181412115	.171603552	.150826830	.097902521	.097902521

M*
R

.178190478	.189516607	.203366620	.202662013	.166354148	.166354148
.613288624	.654329707	.708367552	.713063490	.603737583	.603737583
.406733241	.437795224	.485935880	.503723878	.472120395	.472120395
.250504024	.272366017	.311318477	.334689988	.366968349	.366968349

S*
T

0.92855470	0.96734687	0.97905800	0.91242562	0.62207256	0.62207256
3.13040022	3.25836440	3.29032902	3.05962894	2.08047911	2.08047911
1.95811601	2.03353517	2.04143172	1.88781583	1.27613440	1.27613440
1.12782010	1.16867426	1.16673434	1.07372716	0.72287358	0.72287358

S*
R

0.96907587	1.01852542	1.05587430	1.00894882	0.71627266	0.71627266
3.29601750	3.46755718	3.60442404	3.45457712	2.46799901	2.46799901
2.11174861	2.22765656	2.33332197	2.25592270	1.64579969	1.64579969
1.24712932	1.31953104	1.39423690	1.36233890	1.02742527	1.02742527

$$p_1 = p_2 = \dots = p_9 = 19/600,$$

$$a_1 = a_2 = \dots = a_9 = 1.$$

$$\text{Slab thickness, } t = 16.0.$$

a

I

.)

P*
T

.000352489	.000294581	.000217640	.000176298	.000112643	.000112643
.000953282	.000796840	.000588695	.000476868	.000304685	.000304685
.000435279	.000363833	.000268797	.000217737	.000139119	.000139119
.000217916	.000182147	.000134569	.000109006	.000069648	.000069648

P*
R

.068348175	.070535578	.073480093	.074306334	.070402839	.070402839
.228257927	.238218095	.254347267	.262801275	.262871689	.262871689
.146960186	.157195256	.177743842	.193886231	.227066734	.227066734
.091847680	.100381155	.119828281	.138425033	.200384209	.200384209

M*
T

.016690976	.014290396	.010813682	.008828099	.005655648	.005655648
.046244693	.039571355	.029927590	.024428056	.015648687	.015648687
.021627365	.018496268	.013981129	.011409979	.007308839	.007308839
.010912131	.009330687	.007051755	.005754610	.003686135	.003686135

M*
R

.449191468	.436655775	.401691425	.366586494	.271662790	.271662790
1.41304778	1.38381535	1.29338155	1.19623701	.914030340	.914030340
.803382851	.799353759	.775167154	.742177527	.625140068	.625140068
.453125827	.456921879	.458690937	.456062003	.444822905	.444822905

S*
T

.954305460	.829061352	.637416624	.523235959	.335881048	.335881048
2.68289891	2.32956135	1.79004227	1.46910484	.942995655	.942995655
1.27483325	1.10630696	.849571438	.697106729	.447428178	.447428178
.646755216	.561148114	.430835652	.353492686	.226878520	.226878520

S*
R

7.39116311	6.88876570	5.83650318	5.00792254	3.30089043	3.30089043
22.2925142	20.8385857	17.7516760	15.2857691	10.1136002	10.1136002
11.6730064	10.9711391	9.44786135	8.20072810	5.48582918	5.48582918
6.19013270	5.83864423	5.06832869	4.43053729	3.00875903	3.00875903

III. - Rayleigh scattering with no absorption.

$$p_1 = p_9 = 1/20, p_2 = p_8 = (11+\sqrt{5})/320, p_3 = p_7 = 1/32,$$

$$p_4 = p_6 = (11-\sqrt{5})/320, p_5 = 1/40.$$

$$a_1 = a_2 = \dots = a_9 = 1.$$

Slab thickness, $t = 0.5$.

P^*_T

.628839256	.023313157	.027204354	.031966379	.031504321	.031643801
.075442962	.623433936	.104846582	.132072485	.122962859	.132556464
.054408709	.064798752	.466708908	.110531046	.115773526	.094070843
.039512618	.050447201	.068311943	.290203564	.064176301	.073333858

P^*_R

.023019048	.024192050	.028751844	.034776891	.043928168	.044110552
.078287118	.088274313	.112239761	.147061788	.187055815	.204412807
.057503689	.069367987	.110792733	.131293547	.242703358	.184058654
.042986601	.056172605	.081143874	.122094299	.191895653	.235813021

M^*_T

.035290514	.039219186	.049882204	.061261607	.061901956	.062359886
.126915952	.147786304	.195256791	.249281046	.242791171	.254417984
.099764409	.120675334	.179444645	.213446857	.219694112	.197341164
.075723511	.095216887	.131917412	.174024434	.141907055	.152742600

M^*_R

.036090400	.040242835	.051783488	.064817286	.077572637	.078131901
.130228550	.152393169	.204506915	.268233412	.323974012	.344110413
.103566975	.126392225	.194313639	.240133695	.374424135	.313539202
.080118572	.102456046	.148410785	.212145018	.304241373	.352436565

S^*_T

.047472230	.058423158	.083683403	.108472520	.112767757	.114082862
.189061310	.233270538	.334238356	.434065765	.446871097	.453760025
.167366806	.206570665	.297749970	.383051197	.390398091	.391178280
.134079408	.165798369	.236738659	.306321941	.299650004	.304991678

S^*_R

.047698798	.058790694	.084582949	.110369368	.120924668	.122396592
.190250683	.235210600	.338967155	.444146661	.489873693	.498516106
.169165897	.209493223	.305113978	.398163823	.459713977	.457404570
.136424041	.169648929	.246078775	.326502044	.386791765	.397979061

$$p_1 = p_9 = 1/20, p_2 = p_8 = (11 + \sqrt{5})/320, p_3 = p_7 = 1/32,$$

$$p_4 = p_6 = (11 - \sqrt{5})/320, p_5 = 1/40.$$

$$a_1 = a_2 = \dots = a_9 = 1.$$

Slab thickness, $t = 2.0$.

P^*_T

.185347454	.049103081	.045264344	.039932841	.026453279	.026288698
.158900909	.244122232	.148462489	.131392281	.084405453	.085234421
.090528688	.091754865	.105000536	.071751614	.045382007	.043750845
.049359706	.050187386	.044344936	.038125194	.022148287	.021978473

P^*_R

.066675461	.068156044	.070506066	.070884496	.064303610	.064281503
.220557592	.235153902	.254943468	.272167572	.256305107	.272980699
.141012133	.157563728	.201741448	.209918413	.285758245	.226544019
.087618056	.103958762	.129736714	.165827586	.215244012	.258941341

M^*_T

.223034918	.232219466	.232833294	.214933432	.144482909	.143537509
.751477978	.786151675	.787479427	.725100007	.482530631	.480725528
.465666590	.486689053	.485552021	.438044734	.288523014	.284801641
.265672333	.276963558	.270726534	.242647034	.157871042	.156598109

M^*_R

.250866389	.266130003	.281510488	.276306547	.212728592	.212233589
.861214779	.925818070	.995468551	.998424088	.787752357	.804298659
.563020978	.615233401	.701196613	.706810369	.668732288	.605242448
.341533676	.381364067	.436832831	.479451630	.470719725	.517629469

S^*_T

1.62326857	1.72664329	1.78754495	1.68337027	1.14307281	1.13600273
5.58753505	5.93957313	6.13752412	5.76893289	3.90819546	3.88373601
3.57508992	3.79319853	3.90186348	3.64976238	2.46134442	2.44444221
2.08076008	2.20353629	2.25567720	2.10140622	1.41206184	1.40232263

S^*_R

1.66787254	1.78615051	1.88144801	1.80490481	1.25979775	1.25399306
5.78010448	6.19705269	6.54426017	6.29713149	4.41806084	4.40161895
3.76289603	4.04457523	4.30217360	4.16835967	2.98235185	2.96346318
2.23098504	2.40529020	2.57618795	2.52308095	1.84932619	1.85104840

$$p_1 = p_9 = 1/20, p_2 = p_8 = (11 + \sqrt{5})/320, p_3 = p_7 = 1/32,$$

$$p_4 = p_6 = (11 - \sqrt{5})/320, p_5 = 1/40.$$

$$a_1 = a_2 = \dots = a_9 = 1.$$

Slab thickness, $t = 16.0$.

P^*_T

.016315134	.014387962	.011234061	.009247615	.005878870	.005845583
.046560424	.041060834	.032060108	.026391124	.016777297	.016682301
.022468122	.019814236	.015470865	.012735251	.008096021	.008050180
.011430680	.010080512	.007870818	.006479071	.004118859	.004095537

P^*_R

.130573011	.125769089	.115991869	.108142569	.087932043	.087756838
.406997323	.403529316	.387984911	.381107701	.325379384	.341603419
.231983739	.239787862	.266758038	.263139785	.319499983	.260063383
.133671567	.145570188	.162629331	.192756883	.232317544	.275902758

M^*_T

2.40007734	2.13277352	1.67988752	1.38720499	0.88306728	0.87793563
6.90180011	6.13277273	4.83018590	3.98854123	2.53900091	2.52424915
3.35977506	2.98521907	2.35098978	1.94128531	1.23575720	1.22857893
1.71467967	1.52348719	1.19978031	0.99068640	0.63063518	0.62697223

M^*_R

3.94029950	3.58804096	2.93679090	2.47885062	1.61741491	1.60832342
11.6111445	10.6079656	8.73628520	7.41916674	4.88252363	4.87395997
5.87358182	5.39932121	4.52595981	3.87871663	2.69140948	2.61548952
3.06402789	2.83386954	2.39717871	2.10494661	1.50720793	1.54774627

S^*_T

506.961234	451.811653	357.112750	295.276509	188.074101	186.970578
1462.09322	1303.00103	1029.85760	851.520198	542.366085	539.184069
714.225504	636.487001	503.041095	415.924381	264.916461	263.362400
364.981839	325.251775	257.055404	212.537275	135.371911	134.577823

S^*_R

569.113015	509.441140	405.048085	335.759180	214.138260	212.867032
1648.58616	1476.05939	1174.01404	973.382419	620.894459	617.211892
810.096174	725.580585	577.472126	478.974896	305.655392	303.835537
415.021173	371.799001	296.022766	245.610967	156.809575	155.889062

IV. - Rayleigh scattering with absorption probability, $r = 0.05$.

$$p_1 = p_9 = 19/400, p_2 = p_8 = 19(11 + \sqrt{5})/6400, p_3 = p_7 = 19/640,$$

$$p_4 = p_6 = 19(11 - \sqrt{5})/6400, p_5 = 19/800.$$

$$a_1 = a_2 = \dots = a_9 = 1.$$

Slab thickness, $t = 0.5$.

P*
T

.627131693	.021422285	.024810609	.029033370	.028544426	.028662577
.069323971	.616324426	.095484568	.120128917	.111358904	.120379468
.049621219	.059012709	.458093755	.100317878	.105256502	.084672193
.035887219	.045885164	.061999858	.281869476	.057439367	.066061604

P*
R

.021271771	.022250448	.026264145	.031668437	.040194806	.040350988
.072003964	.080936853	.102421072	.134183034	.171445785	.187806427
.052528290	.063299703	.101443212	.119764671	.224535412	.168931810
.039144341	.051253358	.074018637	.111879070	.177149538	.218670863

M*
T

.031404727	.034648553	.043651484	.053356448	.053774072	.054150921
.112125072	.129979190	.170571996	.217441446	.210717630	.221453424
.087302968	.105419291	.157178846	.185683862	.191307754	.170049327
.065952197	.083055242	.114758937	.151659544	.121485839	.131538235

M*
R

.032154070	.035603909	.045415829	.056646053	.068281112	.068746958
.115216670	.134265399	.179139517	.234976949	.285837679	.304575426
.090831659	.110714311	.170961510	.210333053	.335069210	.277352730
.070018373	.089753208	.129992976	.186934676	.271643259	.316912389

S*
T

.037900257	.046609124	.066709719	.086418972	.089803552	.090832157
.150830293	.186024415	.266356864	.345793903	.355513233	.361081681
.133419439	.164617596	.237302605	.304869665	.310104242	.310294644
.106819724	.132081518	.188419815	.243672009	.236891781	.241255685

S*
R

.038096731	.046927734	.067489356	.088062677	.096888425	.098051801
.151861337	.187705987	.270454307	.354530468	.392854874	.399963021
.134978712	.167149954	.243688380	.317961055	.370384720	.367765258
.108851454	.135418589	.196510739	.261164054	.312558998	.322095374

$$p_1 = p_9 = 19/400, p_2 = p_8 = 19(11 + \sqrt{5})/6400, p_3 = p_7 = 19/640,$$

$$p_4 = p_6 = 19(11 - \sqrt{5})/6400, p_5 = 19/800.$$

$$a_1 = a_2 = \dots = a_9 = 1.$$

$$\text{Slab thickness, } t = 2.0.$$

P*
T

.175910654	.039315458	.035508463	.030960421	.020432986	.020308163
.127227495	.211083656	.115558506	.101211691	.064389615	.065282868
.071016927	.071419085	.084828796	.053683101	.033536991	.032073780
.038269185	.038659426	.033177981	.028195110	.015731471	.015614910

P*
R

.055900165	.056743695	.058428054	.058988058	.055010305	.055006941
.183626456	.195438345	.212123979	.228949483	.221636712	.237468695
.116856109	.131099829	.171263982	.179029054	.255525624	.199464989
.072913250	.087450921	.110646040	.144559727	.193708298	.235063768

M*
T

.151764230	.156704068	.155132998	.142064546	.095124702	.094490623
.507105019	.527187616	.521550628	.476268463	.314628439	.313826210
.310265998	.322336016	.317635437	.282147819	.183958929	.181063451
.175601437	.181918365	.174376942	.153633229	.098480153	.097637096

M*
R

.176235603	.186293365	.197232809	.195004401	.154785787	.154523771
.602857995	.648510155	.701192481	.712605849	.581986195	.598256317
.394465620	.433360788	.504822390	.514569675	.522003202	.462438899
.241038697	.272191214	.318021549	.359949904	.376209511	.420612655

S*
T

0.88403769	0.93891202	0.96874114	0.90944016	0.61593975	0.61201829
3.03838314	3.22415145	3.31770943	3.10624321	2.09665032	2.08311696
1.93748229	2.05045720	2.09636261	1.94898065	1.30698416	1.29732328
1.12412987	1.18647933	1.20453628	1.11337867	0.74280430	0.73732286

S*
R

0.91877040	0.98522975	1.04183495	1.00410826	0.70759005	0.70465258
3.18827045	3.42454084	3.63432406	3.51787876	2.49734128	2.49033049
2.08366991	2.24613580	2.40838166	2.35345647	1.71853798	1.70646500
1.24114608	1.34371012	1.45451609	1.44310118	1.08980695	1.09403818

$$p_1 = p_9 = 19/400, p_2 = p_8 = 19(11 + \sqrt{5})/6400, p_3 = p_7 = 19/640,$$

$$p_4 = p_6 = 19(11 - \sqrt{5})/6400, p_5 = 19/800.$$

$$a_1 = a_2 = \dots = a_9 = 1.$$

Slab thickness, $t = 16.0$.

P^*_T

.000364025	.000302034	.000220603	.000177450	.000111722	.000111236
.000977401	.000811136	.000592430	.000476540	.000300028	.000298723
.000441206	.000366142	.000267420	.000215108	.000135431	.000134842
.000219340	.000182022	.000132944	.000106938	.000067328	.000067035

P^*_R

.074298286	.073069097	.070728837	.068734869	.061103446	.061059065
.236456565	.242519798	.247675353	.257094356	.239221454	.254932306
.141457674	.153071786	.187891326	.192179327	.263739323	.207620769
.084960979	.098201306	.118773356	.150990571	.197725456	.239052906

M^*_T

.017033474	.014511792	.010885487	.008832227	.005580197	.005552731
.046961146	.039982124	.029971162	.024312724	.015359457	.015284070
.021770974	.018523197	.013876101	.011253979	.007109052	.007074258
.010917233	.009286634	.006955342	.005640634	.003563049	.003545626

M^*_R

.447105739	.433409438	.395185088	.357885005	.258187190	.257224783
1.40254241	1.37832101	1.28535178	1.19277874	.886672955	.900865141
.790370182	.794391090	.793015350	.750956521	.671868200	.611275434
.442370198	.455600938	.464116654	.479640966	.452055755	.495939247

S^*_T

.967187137	.837232720	.639149655	.521736488	.330465115	.328723986
2.70934202	2.34381957	1.78805387	1.45924066	.924184736	.919327834
1.27829933	1.10507807	.842413462	.687324210	.435259144	.432978035
.644901772	.557380338	.424789723	.346555779	.219454255	.218305224

S^*_R

7.32483786	6.84869281	5.81257717	4.98042893	3.24780942	3.22827590
22.1628356	20.7996755	17.7718756	15.2973974	10.0211894	9.96482783
11.6251545	10.9836232	9.51086570	8.26544693	5.49283749	5.45589242
6.15614877	5.84308588	5.10832713	4.48232416	3.02945340	3.02088738

ACKNOWLEDGMENTS

It is a pleasure to express my indebtedness to my supervisor, Dr. J. E. Moyal, for suggesting the area of this research, for his invaluable guidance and encouragement throughout the investigation, and in particular for his collaboration in the writing of Chapters II and III.

I am grateful for the financial support and stimulating conditions provided by both the Australian National University, where the major part of the work was carried out, and by Argonne National Laboratory, where I was fortunate in having the opportunity to spend my third year. I also wish to thank General Motors--Holden's Pty. Ltd. for their Postgraduate Fellowship with which this work was partly supported.

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