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SCATTERING PHASE SHIFTS FOR THE COULOMB PLUS SEPARABLE POTENTIAL PROBLEM

by

T. A. Griffy,

W. R. Gibbs, and B. F. Gibson

ABSTRACT

An expression for the phase shift arising in the scattering of two non-relativistic charged particles which interact through a separable potential is derived. A relation between the charged and neutral (Coulomb corrected) scattering lengths is deduced for a simple Yamaguchi potential. It is verified that Coulomb corrections to the p-p scattering length are large and model dependent. It is argued that Coulomb corrections to the π^+ -p scattering length are small and model independent.

I. INTRODUCTION

The scattering of two non-relativistic charged particles has been discussed by Harrington¹ and by Ali, Rehman, and Husain.² The off-shell T-matrix for this same problem has been investigated by Zachary.³ The derivation presented below follows closely from the work of Ref. 2, and the scattering length relations obtained reproduce those quoted in Ref. 1. We apply the results to a calculation of the Coulomb corrections for the s-wave scattering lengths for both the N-N and π -N systems. As is well known, the effects of the Coulomb interaction are quite significant for the N-N problem. We find, however, that the Coulomb corrections to the π -N scattering length are very small.

II. PHASE SHIFT RELATION DERIVATION

The Schrödinger equation for the two-body scattering problem is

$$\left(-\frac{1}{2\mu} \nabla^2 + V \right) \psi = E \psi, \quad (1)$$

where μ is the reduced mass of the system, and we have chosen units such that $\hbar = c = 1$. We assume

that the interaction between the two particles is described by a point Coulomb interaction plus a non-local potential

$$V\psi(\vec{r}) = \frac{e^2}{r} \psi(\vec{r}) + \int V(\vec{r}, \vec{r}') \psi(\vec{r}') d^3r'. \quad (2)$$

Assuming that the non-local potential has the separable form

$$V(\vec{r}, \vec{r}') = \sum_{LM} \lambda_L \frac{g_L(r)}{r} \frac{g_L(r')}{r'} Y_L^M(\hat{r}) Y_L^M(\hat{r}') \quad (3)$$

and writing the wave function as

$$\psi(\vec{r}) = \sum_{\ell m} i^\ell \frac{u_\ell(r)}{r} Y_\ell^m(\hat{r}) \quad (4)$$

we obtain the reduced Schrödinger equation

$$\left(-\frac{d^2}{dr^2} u_\ell(r) - \frac{2\eta k}{r} - \frac{\ell(\ell+1)}{r^2} + k^2 \right) u_\ell(r) = \int_0^\infty K_\ell(r, r') u_\ell(r') dr' \quad (5)$$

$$= \int_0^\infty K_\ell(r, r') u_\ell(r') dr',$$

where $\eta = \mu e^2/k$, $k^2 = 2\mu E$, and

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$$K_\ell(r, r') = 2i\lambda_\ell g_\ell(r) g_\ell(r') .$$

As is indicated in Ref. 2, a solution to Eq. (5) can be written in the form

$$u_\ell(r) = v_\ell(r) + \alpha_\ell Z_\ell(r), \quad (6)$$

where v_ℓ and Z_ℓ are solutions of the following set of uncoupled equations:

$$\left(\frac{d^2}{dr^2} + k^2 - \frac{\ell(\ell+1)}{r^2} - \frac{2\eta k}{r}\right) v_\ell(r) = 0 \quad (7a)$$

$$\left(\frac{d^2}{dr^2} + k^2 - \frac{\ell(\ell+1)}{r^2} - \frac{2\eta k}{r}\right) Z_\ell(r) = 2i\lambda_\ell g_\ell(r), \quad (7b)$$

and the constant α_ℓ is defined by

$$\alpha_\ell = \frac{\int_0^\infty v_\ell(r) g_\ell(r) dr}{1 - \int_0^\infty g_\ell(r) Z_\ell(r) dr}. \quad (8)$$

Since Eq. (7a) is the standard equation for point Coulomb scattering, we can take $v_\ell(r)$ to be the regular Coulomb wave function $F_\ell(kr)$.⁴ The equation for $Z_\ell(r)$ can then be solved using the Green's function solution for

$$\left(\frac{d^2}{dr^2} + k^2 - \frac{\ell(\ell+1)}{r^2} - \frac{2\eta k}{r}\right) G_\ell(r, r') = \delta(r-r'). \quad (9)$$

The Coulomb Green's function $G_\ell(r, r')$ can be written in terms of the regular and irregular Coulomb wave functions as

$$G_\ell(r, r') = -\frac{1}{k} F_\ell(kr_<) G_\ell(kr_>) \quad (10)$$

(It should be noted that we differ from Ref. 2 in that their expression for the Coulomb Green's function omits the minus sign. This difference can be traced to their different convention for the sign of the irregular Coulomb function. We follow the convention $G_\ell \xrightarrow{r \rightarrow \infty} \cos(kr - \frac{\ell\pi}{2} - \eta \ln(2kr) + \sigma_\ell)$.) Using the Green's function defined in Eq. (10), we determine that the asymptotic form of the solution described by Eq. (6) is

$$u_\ell(r) \rightarrow F_\ell(r) - \frac{2i\alpha_\ell \lambda_\ell}{k} G_\ell(kr) \int_0^\infty F_\ell(kr') g_\ell(r') dr'. \quad (11)$$

Comparing this expression with the standard form⁵ (see Appendix A)

$$u_\ell(r) \rightarrow e^{i\delta_\ell} [\cos\delta_\ell F_\ell(kr) + \sin\delta_\ell G_\ell(kr)] \quad (12)$$

we ascertain that the phase shift is given by

$$\tan \delta_\ell = -\frac{2i\alpha_\ell \lambda_\ell}{k} \int_0^\infty F_\ell(kr) g_\ell(r) dr. \quad (13)$$

Using the definition of the constant α_ℓ from Eq. (8), we can reexpress this as

$$\tan \delta_\ell = \frac{-\frac{2i\alpha_\ell \lambda_\ell}{k} \left[\int_0^\infty F_\ell(kr) g_\ell(r) dr \right]^2}{1 + \frac{2i\alpha_\ell \lambda_\ell}{k} \int_0^\infty \int_0^\infty dr dr' F_\ell(kr_<) G_\ell(kr_>) g_\ell(r) g_\ell(r')}. \quad (14)$$

Eq. (14) for a simple, single-term separable potential is the basis for the calculation of the scattering lengths given in the next section. A similar expression for separable potentials with more than one term in a given partial wave can be found in Ref. 1.

III. SCATTERING LENGTH RELATIONS

An expression for the s-wave scattering length can be obtained from Eq. (14) of the previous section by expanding the Coulomb wave functions in powers of the argument $\rho = kr$. From Ref. 4 we find that

$$F_0(\rho) = C_0(\eta) \rho(1+\eta\rho), \quad (15a)$$

$$G_0(\rho) = \frac{1}{C_0(\eta)} [1 + 2\eta\rho(\ln 2\rho + g(\eta))], \quad (15b)$$

where the quantity $C_0^2(\eta)$ is defined by

$$C_0^2(\eta) = \frac{2\pi\eta}{e^{2\pi\eta} - 1}$$

and the function $g(\eta)$ is given by

$$g(\eta) = -1 + \eta^2 \sum_{m=1}^{\infty} \frac{1}{m(m^2 + \eta^2)}.$$

We assume that the separable potential has a Yamaguchi form: $g_0(r) = \exp(-\beta r)$.

Combining these expressions in Eq. (14) we obtain, after considerable algebraic manipulation,

$$\tan \delta_0 = \frac{-\frac{2\mu\lambda_0}{k} \cot^2(\eta) \frac{k^2}{\beta^4} (1 + \frac{4\eta k}{\beta})}{1 + \frac{2\mu\lambda_0}{\beta^3} \left\{ \frac{1}{2} + \frac{2\eta k}{\beta} \left(\ln \frac{4k}{\beta} + g(\eta) - \gamma + 1 \right) \right\}} \quad (16)$$

where $\gamma = 0.5772\dots$, Euler's constant.

In order to compare this result with the corresponding result in the absence of the Coulomb interaction, we define the "neutral" phase shift δ_0 and the "neutral" scattering length $\bar{a}$ by the corresponding set of equations with $\eta = 0$:

$$\tan \delta_0 = \frac{-2\mu\lambda_0 k/\beta^4}{1 + \mu\lambda_0/\beta^3} \quad (17a)$$

$$k \cot \delta_0 = -\frac{1}{\bar{a}} = -\frac{\beta}{2} - \frac{\beta^4}{2\mu\lambda_0} \quad (17b)$$

To conform with standard convention,⁶ we define a function

$$h(\eta) = \eta^2 \sum_{m=1}^{\infty} \frac{1}{m(m^2 + \eta^2)} = \ln \eta - \gamma \quad (18)$$

and we define the scattering length for the charged system as

$$\lim_{k \rightarrow 0} \{k \cot^2(\eta) \cot \delta_0 + 2\eta k h(\eta)\} = -\frac{1}{a} \quad (19)$$

Combining the Eqs. (16-19) we obtain the desired expression for the charged scattering length a in terms of the neutral scattering length $\bar{a}$ and the range of the potential β :

$$\frac{1}{a} = \frac{1}{\bar{a}} \left(1 + \frac{4\eta k}{\beta} \right)^{-1} + 2\eta k \left(\ln \frac{4\eta k}{\beta} + \gamma \right) \quad (20)$$

This result is similar to that of Harrington¹ and to that quoted in the much earlier review by Jackson and Blatt.⁶ Results for the effective range and the s-wave phase shift are discussed briefly in Appendix B.

IV. RESULTS AND DISCUSSION

The relation between "charged" and "neutral" scattering lengths given by Eq. (20) is valid for both the N-N and π -N systems. For the N-N problem there is clearly a large Coulomb correction; model

TABLE I
THE NEUTRAL p-p SCATTERING LENGTH $\bar{a}$ FOR
VARIOUS VALUES OF THE RANGE PARAMETER

($a = -7.823$ fm.)

β (fm ⁻¹)	$\bar{a}$ (fm)
1.0	-16.9
1.1	-18.1
1.2	-19.3
1.5	-23.2

results⁷⁻⁹ indicate a neutral scattering length of approximately -17 fm extracted from a potential fitted to proton-proton scattering data having a scattering length of -7.823 fm. As was pointed out in Ref. 7, extraction of the neutral scattering length is a model-dependent process. We illustrate this with the results shown in Table I, where we have listed $\bar{a}$ for various values of the potential range β . The sensitivity of the extracted neutral scattering length to the value of β is clear. (A value for β that reproduces a neutral scattering length and effective range of -17 fm and 2.84 fm in Eq. (17) is $\beta = 1.13$ fm⁻¹.)

The model dependence, as exhibited here, is obviously due to the large magnitude of the scattering length (i.e., small value of $1/a$) and large value of η ($\eta k = \mu e^2$). Another statement of this is that the N-N singlet system almost supports a zero-energy resonance; hence, the wave function is "large" for small r and very sensitive to the $1/r$ Coulomb interaction.

Because the π^+ -p system is weakly interacting at low energies, and since ηk is considerably smaller than for the p-p system, one might expect that the extraction of the neutral scattering length from the π^+ -p scattering data should not show a strong model dependence. This is the case, as we argue here. Assuming a value¹⁰ of -0.1 fm for the "charged" scattering length, we present in Table II values of the neutral scattering length for a range of values for the parameter β . One notes that $\bar{a}$ is essentially independent of the value of β . In addition, the correction is small, being on the order of 1.5%. Thus the Coulomb corrections to the π^+ -p system are much smaller and more model independent than for the familiar p-p system.

TABLE II
THE NEUTRAL π -p SCATTERING LENGTH
 $\bar{a}$ FOR VARIOUS VALUES OF THE
RANGE PARAMETER ($a = -0.1$ fm.)

$\beta(\text{fm}^{-1})$	$\bar{a}(\text{fm})$
1.0	-0.0985
1.5	-0.0992
2.0	-0.0995
2.5	-0.0997
3.0	-0.0998
4.0	-0.1000
5.0	-0.1001

V. CONCLUSIONS

Using a simple separable potential model to describe the strong interaction in the scattering of two non-relativistic charged particles, we have 1) exhibited the known model dependence in the process of extracting the neutral or Coulomb corrected scattering length from proton-proton scattering data and 2) demonstrated that the Coulomb corrections to the π^+ -p scattering length should be small and reasonably model independent.

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APPENDIX A

COULOMB PLUS NUCLEAR PHASE SHIFT EXPANSION

Although a pure, unscreened Coulomb potential has an infinite range such that the wave function cannot be written as a superposition of plane wave and scattered wave at large distances, the Coulomb field of a charged particle is normally screened. In such cases one may write the asymptotic wave function (for non-identical particles) as

$$\psi(r) \rightarrow e^{ikz} + f_C(\theta) \frac{e^{ikr}}{r}, \quad (\text{A1})$$

where the Coulomb scattering amplitude $f_C(\theta)$ can be expressed as

$$f_C = \frac{1}{2ik} \sum_{\ell=0}^{\infty} (2\ell+1) [e^{2i\sigma_\ell} - 1] P_\ell(\cos\theta) \quad (\text{A2})$$

Here, σ_ℓ is the pure Coulomb phase shift, which appears in the asymptotic form of the scattered radial wave function:

$$\chi_\ell(r) = e^{i\sigma_\ell} F_\ell(kr) \\ \rightarrow e^{i\sigma_\ell} \sin(kr - \frac{\ell\pi}{2} - \eta \ln(2kr) + \sigma_\ell). \quad (\text{A3})$$

In the presence of a short-range strong interaction, the asymptotic form of the radial function is

altered

$$u_\ell(r) \rightarrow e^{i\sigma_\ell + i\delta_\ell} \sin(kr - \frac{\ell\pi}{2} - \eta_\ell \ln(2kr) + \sigma_\ell + \delta_\ell). \quad (A4)$$

The quantity δ_ℓ is referred to as the "nuclear" phase shift. However, it is by no means the same as the phase shift in the absence of a Coulomb interaction between the scattered particles (the Coulomb corrected phase shift $\bar{\delta}_\ell$). It contains all strong and Coulomb scattering effects except those contained in the pure Coulomb phase σ_ℓ .

By expanding the sine function in Eq. (A4) one can rewrite the asymptotic form of the radial function as

$$u_\ell(r) \rightarrow e^{i\sigma_\ell + i\delta_\ell} [\cos\delta_\ell F_\ell(kr) + \sin\delta_\ell G_\ell(kr)]. \quad (A5)$$

This is the form quoted in Eq. (12) of the main text except for the overall Coulomb phase factor $\exp(i\sigma_\ell)$, which was omitted from Eq. (11) when for

convenience both the solution to Eq. (7a) and the Coulomb Green's function in Eq. (10) were written without it.

The scattering amplitude, including the strong interaction, becomes:

$$f(\theta) = \frac{1}{2ik} \sum_{\ell=0}^{\infty} (2\ell+1) [e^{2i(\sigma_\ell + \delta_\ell)} - 1] P_\ell(\cos\theta). \quad (A6)$$

This may be separated as follows:

$$f(\theta) = f_C(\theta) + f_{\text{nucl}}(\theta), \quad (A7)$$

$$f_{\text{nucl}}(\theta) = \frac{1}{2ik} \sum_{\ell=0}^{\infty} (2\ell+1) e^{2i\sigma_\ell} [e^{2i\delta_\ell} - 1] P_\ell(\cos\theta) \quad (A8)$$

It is again emphasized that δ_ℓ in $f_{\text{nucl}}(\theta)$ contains Coulomb effects, and that $f_{\text{nucl}}(\theta)$ is not simply related to the Coulomb corrected amplitude.

APPENDIX B

COULOMB CORRECTIONS FOR $g_0(r) = \exp(-\beta r)$

To order $(\frac{\eta k}{\beta})^2 \ln \frac{\eta k}{\beta}$, Harrington has shown that the charged and neutral s-wave phase shifts for the simple separable potential discussed in the text are related by

$$\begin{aligned} kC_0^2(\eta) \cot \delta_0 + 2\eta k h(\eta) \\ = k \cot \bar{\delta}_0 [1 - 4\eta \tan^{-1}(\frac{k}{\beta})] \\ - 2\eta k [\ln \frac{4\eta k \beta}{k^2 + \beta^2} + \gamma] \end{aligned} \quad (B1)$$

where $C_0^2(\eta) = \frac{2\pi}{e^{2\pi\eta} - 1}$. From this expression one can verify the scattering length relation presented in Eq. (20). In addition, one can deduce from the

effective range relation

$$kC_0^2(\eta) \cot \delta_0 + 2\eta k h(\eta) = -\frac{1}{a} + \frac{1}{2} r_0 k^2 \quad (B2)$$

that

$$r_0 = \bar{r}_0 [1 - \frac{4\eta k}{\beta}] + \frac{2\eta k}{\beta^2} [1 - \frac{2}{3\beta a}], \quad (B3)$$

where r_0 is the charged and $\bar{r}_0$ the neutral effective range. For the π^+p system, the small size of $\bar{a}$ implies that there will be a larger Coulomb correction for the effective range than was shown to be the case for the scattering length in Sec. IV of the main text.