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WANL-TME-300

STABILITY OF CORE LATERAL SUPPORT

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ABSTRACT

The combination of the core flow leakage area, the rate of lateral support spring acting to reduce the leakage area, and the pressure differential between the inside and the periphery of the core may provide an unstable dynamic system which if excited will sustain and contribute energy to the excitation.

The analysis in this report considers three cases of leakage at the flow inlet and exit of the core and derives the required conditions for static and dynamic stability of the core. Values of lateral support spring constants required to statically stabilize the core are presented parametrically. The results indicate that, for the assumptions of this analysis, the NRX-A design has sufficient lateral support spring force to insure that the core is statically stable within a considerable range of pressure drops and leakage areas in the core.

I. INTRODUCTION

The core structure consists of fuel element clusters suspended from the support plate and bundled together by a lateral spring support system extending the length of the core.

The lateral support arrangement provides a means to absorb lateral static, vibrational, and shock loads during handling, shipping, booster, and nuclear operation.

Various transient and steady state flow disturbances, such as chugging, wave motion of cylindrical and longitudinal modes, cavity resonances, eddies, and others may occur under certain propellant flow conditions. These disturbances may originate oscillations of the core or parts of the core. Insufficient positive damping, coupled with a source of energy provided by the combined interaction of the geometry, the pressures and the flow existing in the system, may not suppress the initial oscillation, but will sustain and increase them to the extent of causing failure of parts.

This analysis establishes the lateral support spring system which, insures a stable design.

II. DEFINITION OF PROBLEM

The lateral support spring rate will be established on the basis of minimum stability of the core-lateral support system against self-excited vibrations. The parameters affecting the stability and some insights on the core performance under possible disturbances will be obtained. The spring rate values obtained from the analysis will be compared to the NRX-A spring rates. From this data, the degree of stability will be apparent. The analysis will be predicated on the existence of leakage paths between fuel elements and on the assumed behavior of the core during flow conditions. In system one, it is assumed that the fluid, leaving the plenum at the dome end of the core, flows through interface leakage paths which by virtue of the core opening and closing, provides a variable inlet flow passage area. The flow continues through these variable passages and exits through a non-variable exit flow area. The second system assumes that both the inlet flow area and the exit flow area may experience variation. The variation of the flow area is made possible by the deflection of the lateral support spring.

A third system assumes that the inlet flow passage is fixed and the outlet passage is variable. These three systems are modeled by a spring loaded piston moving in a cylinder through which flow enters and exits from diametrically opposite ends. The piston, during its movement, uncovers or obstructs either inlet, inlet and exit flow or exit flow area. The three systems are analyzed for both compressible and incompressible flows.

NOMENCLATURE

System I.

x	=	upward displacement of piston, measured from the equilibrium position, in
A	=	area of piston, in ²
p'	=	pressure in cylinder above static equilibrium pressure, psi
$C_1 x$	=	volume or mass rate of fluid entering cylinder per second (above amount entering at equilibrium), in ³ /sec, lb/sec
$C_2 p'$	=	volume or mass rate of fluid leaking out per second (above normal leak at equilibrium), in ³ /sec, lb/sec

System II & III.

x	=	upward displacement of piston from static equilibrium position, in
P_o, ρ_o	=	pressure and density of gas in cylinder at equilibrium, psi, lb/in ³
V_o	=	volume of cylinder at equilibrium, in ³
A	=	area of piston, in ²
P, ρ	=	pressure and density in cylinder at any time, psi, lb/in ³
$P' = P - P_o$	=	pressure in cylinder above P_o , psi
$W_m = W_1 = C_1 x - C_2 P'$	=	mass rate of gas into the cylinder above amount entering at static equilibrium, lb/sec
$W_o = W_2 = C_3 x + C_4 P'$	=	mass rate of gas out of cylinder above amount leaving at static equilibrium, lb/sec

General for Evaluation of Coefficients

C	=	orifice flow coefficient to account for, temperature, entrance losses and flow/less than sonic
$\Delta P = P'$	=	pressure differential, psi
P	=	general equilibrium pressure across piston, psi
R	=	equivalent radius of leakage area at dome or nozzle end of core, in
K	=	spring constant of lateral support, lb/in
δ	=	preload deflection of lateral support spring, in
A	=	surface area of core on which support spring acts, in ²
ΔR	=	small change of equivalent radius of leakage area, in
W	=	flow through an orifice, lb/sec

Subscript

1	=	inlet location, in, dome end of core.
2	=	outlet location, out, nozzle end of core.
0	=	outlet, out, leaving flow
in	=	in, entering flow
ss	=	steady state

III. THEORY

System I.

System I, shown on the enclosed Figure 1, consists of a piston of mass, m , attached to a spring, K , and a dashpot, viscous damping. The piston acts as a valve for the fluid flow which enters the tank through the inlet. The fluid exits through a leak hole. Under normal conditions, the fluid coming in is equal to the flow out of the cylinder. The gas obeys the relation $P/\rho^\gamma = \text{constant}$. This problem is given on Page 413 of the Fourth Edition of Den Hartog's "Mechanical Vibration". The condition for stability for the system is given in the same book on Page 426. The derivation of the stability condition is included in the appendix to this report. The condition for stability is given by:

$$K - \frac{AC_1}{C_2} > 0$$

or

$$K > \frac{AC_1}{C_2}$$

The result indicates that the compressibility of the gas and the viscous damping in the system do not enter in the condition for the stability. (Coloumb friction, such as sliding fuel elements, is not included in this analysis). Even though the dashpot is not necessary for stability, it affects the amplitude of oscillations. The spring rate for stability is directly dependent on the piston area and the coefficient related to the inlet flow pressure differential, while it is inversely related to the coefficient of the geometry of the area.

System II.

The mathematical model for System II, is shown in the Figure 2. The piston acts as a valve for the inward and outward flow of the gas. Under normal steady state conditions, the flow in equals the flow out. The gas again obeys the law $P/\rho^\delta = \text{constant}$. This problem is also given in Den Hartog's "Mechanical Vibrations" on Page 413 with the solution on Page 426. As for the other systems, the derivation of this solution is shown in the appendix to the report. The condition for stability is given by:

$$K - \frac{A(C_1 - C_3)}{(C_2 + C_4)} > 0$$

or

$$K > \frac{A(C_1 - C_3)}{(C_2 + C_4)}$$

For this system, the compressibility of the gas and the viscous damping do not enter in the stability equation. The spring rate for stability is directly dependent on the piston area and the coefficient difference at the inlet and outlet openings. The stability is also inversely proportional to the sum of the flow coefficients relating the flow areas at exit and inlet.

System III.

For this arrangement we consider the inlet passage in the tank fixed and the outlet area variable with the movement of the piston, i. e. the reverse of System I. It can be deduced immediately that if the flow-out increases due to the enlarged outlet passage while the inlet flow is unchanged, there will be a pressure drop in the tank. The piston returns to the original position to re-establish equilibrium. The system is therefore stable. This is shown by the relation

$$K + A \frac{C_3}{C_4} > 0$$

Evaluation of Coefficients

The equation of the maximum flow-through an orifice properly modified by a coefficient C to account for flow less than the maximum sonic flow, for orifice coefficient and temperature effect can be used to express the flow in and out of the tank. This equation for a circular orifice area A and a pressure P is

$$W = CAP = \pi R^2 CP$$

If the normal flows in and out of the tank are known, expressions for the coefficients C_1, C_2, C_3, C_4 , can be obtained.

a. For the System I, it was found that for stability

$$K > \frac{AC_1}{C_2}$$

The coefficients of this inequality appear in the following relationships expressed in terms of ΔR and ΔP at the appropriate location.

$$\Delta W_{in} = C_1 x = C_1 \Delta R_1 \quad \text{increase in mass rate of flow-in due to enlarged port.}$$

$$\Delta W_o = C_2 P' = C_2 \Delta P \quad \text{increase in mass rate of flow-out due to increase in pressure.}$$

For C_1 : The original mass rate through an assumed circular orifice is expressed by:

$$W_{in} = CP\pi R^2$$

a small change in R_1 by ΔR_1 will produce a new mass rate

$$W_{in} = CP\pi (R_1 + \Delta R_1)^2$$

The increase in mass rate will be the difference of these two which when expanded and neglecting the second order expression of ΔR^2 give

$$\Delta W_{in} = C_1 \Delta R_1 = CP\pi (R_1 + \Delta R_1)^2 - CP\pi R^2 = CP (2\pi R_1 \Delta R_1)$$

From the expression of the mass rate through the inlet region, we obtain

$$CP = \frac{W_{in}}{\pi R_1^2}$$

which substituted in ΔW_{in} gives

$$\Delta W_{in} = \frac{2W_{in}}{R_1} \Delta R_1 = C_1 \Delta R_1$$

from which

$$C_1 = \frac{2W_{in}}{R_1}$$

For C_2 : The original mass rate at the outlet using the indicated flow expression can be expressed as

$$W_o = CPA$$

a small change in P by ΔP produces a new mass rate

$$W_o' = CA (P + \Delta P)$$

so the net increase in mass rate is

$$\Delta W_o = C_2 \Delta P = W_o' - W_o = CA (P + \Delta P) - CAP = CA \Delta P$$

and from the initial flow expression

$$CA = \frac{W_o}{P}$$

which substituted into the net mass rate increase gives:

$$\Delta W_o = C_2 \Delta P = \frac{W_o}{P} \Delta P$$

so that

$$C_2 = \frac{W_o}{P}$$

b. System II

The stability equation for this system is given by:

$$K > \frac{A (C_1 - C_3)}{(C_2 + C_4)}$$

The coefficients of this inequality appear in the following relationship expressed in terms of ΔR and ΔP for the appropriate locations:

$\Delta W_{in} = C_1 x = C \Delta R_1$	increased flow in tank due to area increase.
$\Delta W_{in} = C_2 P' = C_2 \Delta P$	decrease flow at inlet of tank due to pressure increase.
$\Delta W_o = C_3 x = C_3 \Delta R_2$	increased flow out of tank due to area increase.
$\Delta W_o = C_4 P' = C_4 \Delta P$	increased flow out of tank due to pressure increase.

Following the same reasoning as that used for System I, and considering the location at which the mass rate applies, we obtain:

$$C_1 = \frac{2W_{in}}{R_1}$$

$$C_2 = \frac{2W_o}{P}$$

$$C_3 = \frac{2W_o}{R_2}$$

$$C_4 = \frac{W_o}{P}$$

IV. REDUCTION OF THEORY

Having established the coefficients, the stability inequality for the Systems I and II can be written in terms of the known parameters of flow and geometry.

a. System I

The stability expression is:

$$K > \frac{AC_1}{C_2}$$

Replacing the coefficients

$$K > \frac{2AP}{R_1} \frac{W_{in}}{W_o}$$

In this expression, R_1 is the equivalent flow radius at inlet, the pressure is the pressure difference across the piston, the area is the piston area, and the mass rates of the flow are at inlet and outlet. If the $W_{in} = W_o$, the expression reduces to:

$$K > 2A \frac{P}{R_1}$$

indicating that the spring rate is directly related to the pressure acting on the piston and the size of the opening.

b. System II

The stability relation for this system is given by:

$$K > \frac{A (C_1 - C_3)}{(C_2 + C_4)}$$

Replacing the coefficients with the appropriate subscripts, referring to inlet and outlet flow conditions:

$$K > A \frac{\frac{2W_{in}}{R_1} - \frac{2W_o}{R_2}}{\frac{W_{in}}{P} + \frac{W_o}{P}}$$

Where A is the piston area on which the spring acts.

For $W_{in} = W_o$

$$K > A \frac{\frac{1}{R_1} - \frac{1}{R_2}}{\frac{1}{P}} > A \frac{R_2 - R_1}{R_1 R_2} P$$

Indicating that the spring rate is a function of the leakage radii and pressure across the piston. The inequality has been plotted for $\frac{K}{A} \delta$ i.e., the spring force expressed in equivalent psi on the core, versus the dimension-less parameter

$\frac{R_2 - R_1}{R_2 R_1} \delta$ and the pressure differential ΔP . The δ in these expressions is the preload deflection of the spring which is .25 inches.

V. DISCUSSION OF RESULTS

Before we analyze the results, it is necessary to distinguish between the two types of instabilities, a static and a dynamic instability, and to establish the criteria to recognize them. A system is statically unstable if the forces acting on the system are not balanced.

The system under such conditions is also dynamically unstable, since an unbalanced force on the system produces an accelerating motion. The stability criteria is provided from the static balance of forces. A system is dynamically unstable if, when disturbed, the system itself feeds energy to the disturbance, increasing its effect. A statically stable system may or may not be dynamically stable. The stability criteria is obtained by noting the signs of the direction in the dynamic equilibrium equation of the forces acting on the system. A negative force will indicate a dynamic unstable system which will add to an external disturbance.

The analysis of three separate but similar systems of the core behavior under flow conditions has provided:

- a. Values of the lateral support spring rate which insures static stability of the core.
- b. The parameters which affect stability and those which do not enter in the stability.
- c. The conditions necessary for static and dynamic stability of the core.

For the first systems which assumes the core expanding at the dome end and providing a variable leakage path to the gas which would emerge at the nozzle end from a fixed leakage area, the following expression relating the lateral support spring rate for stable conditions of the core has been obtained:

$$K - \frac{2AP}{R_1} > 0$$

This result indicates the following:

- a. The compressibility of the gas does not affect the stability.
- b. The viscous damping of the system has no effect on the stability of the core. The damping, however, affects the amplitude of oscillation.
- c. The spring force of the lateral support must be greater than the differential pressure acting on the piston thus implying that the core should not open up due to leakage between elements.

- d. The spring rate of the support is inversely related to the equivalent leakage radius at the dome end of the core. This implies that for no leakage or $R_1 = 0$ an infinite spring is required. However, this violates the model and this is not a valid representation. We may conclude from the results that the instability is initiated by a leakage path; and that if the spring force is not sufficient to compensate for the pressure differential across the path, a static and dynamic instability exists.

The second system, which assumes both the dome end and the nozzle end has a variation of leakage area, provides the following relation for a stability of the core:

$$K - \frac{R_2 - R_1}{R_2 R_1} AP > 0$$

This relation indicates the following:

- The fluid being gas or liquid does not alter the requirement of the spring rate for stability.
- The viscous damping does not affect the condition for stability, but affects the magnitude of the oscillation as indicated by the dynamic equation.
- The condition necessary for the dynamic and static instability to exist is that a leakage passage and a pressure differential between this passage and the outside of the core be present.

Insufficient spring force to balance the pressure inside the leakage results in a statically unstable system. Static stability can be obtained by applying sufficient force to balance the internal pressure as related by the equation.

The dynamic instability results from the relation between the exit and the inlet passages. This is apparent from the following:

- If the radii of the leakage openings at inlet and outlet are equal, the system is stable and a bundling force is required to hold the system statically balanced.
- If the opening at the outlet is greater than at the inlet, the system is dynamically stable. An external force is required to maintain a static balance.
- If the opening at the outlet is less than the inlet, the system is dynamically unstable. An external force sufficiently large to overcome internal pressure holds the system statically balanced, but the system feeds energy to a disturbance.

The dynamically stable system, occurring when $R_2 = R_1$, i. e., the nozzle opening is greater than the dome end opening, will require less spring force to make the system statically stable if R_1 is made closer to the values of R_2 . If R_1 increases in respect to R_2 , the opposite is true.

Graph, Figure 3, has been made to show the amount of lateral spring needed for different pressure differential between the inside and the outside of the core and for various leakage areas (radii) at the dome end and nozzle end of the core. The graph shows requirement for static stability which if not satisfied degenerates in a dynamic unstable condition. The graph in effect presumes a dynamic stable system, i.e., the leakage at the nozzle end is greater than at the dome end. A range of presumed leakage areas at the inlet and the outlet has been indicated. These are estimated values obtained from a study in progress to obtain clearance between elements at the dome and nozzle ends of the core. Also a range of estimated accuracy in the calculation for the leakage drop between the inside of the core and the periphery is indicated. The plot shows that for these elements:

The NRX-A is provided with a lateral support spring which insures static stability for the estimated leakage areas and pressure drops. The system is presumed dynamically stable, i.e., the leakage area at nozzle end is greater than at the dome end. This does not take into account the possibility of gap variation due to operating temperature.

VI. CONCLUSIONS AND RECOMMENDATIONS

The following conclusions within the limits of the assumptions can be drawn from the analysis and the discussion of the results:

- a. The presence of leakage paths between elements is the condition necessary for the existence of a stability problem. If there is an absence of leakage paths the core stability does not require investigation. However, geometric tolerance studies in progress indicate that leakage paths do exist and thus stability problem may in fact occur.
- b. Two types of instabilities can be present because of the leakage: a static and a dynamic instability. The static instability produces also a dynamic instability. The converse is not true.

- c. To insure static stability, it is necessary to provide a lateral support spring which keeps the core together against the pressure acting inside the passage. The relation to insure against this type of instability has been determined and a graph of the values is shown on the enclosed Figure 3.
- d. To insure dynamic stability, the leakage area at the dome end must be equal or less than the leakage area at the nozzle end.
- e. The condition for dynamic instability is therefore predicated on the design of the core. If the core can open more at the nozzle than at the dome end, the core design is dynamically stable.
- f. The NRX-A core design is provided with sufficient lateral support spring to make it statically stable within a considerable range of leakage areas and pressure differentials between the inside and outside of the core. The enclosed graph shows the combination of pressures and leakages which the lateral support spring can accommodate without making the core statically unstable.

The results of the planned test, A-11, will indicate the validity of this analysis.

APPENDIX

DERIVATION OF STABILITY CONDITIONS

System I

Assuming the fluid to be compressible and the flow to be sonic, with the nomenclature used by Den Hartog and listed previously in the analysis, we can write the following:

The mass of gas entering the tank per unit time through the inlet opening

$$\text{at Steady State} \quad \left| \frac{\Delta M}{\Delta t} \right|_{in, ss} = C_1 x_o \quad (\text{pounds/second})$$

$$\text{at } t = \Delta t \quad \left| \frac{\Delta M}{\Delta t} \right|_{in, \Delta t} = C_1 (x_o + x) \quad (\text{pounds/second})$$

The increase in mass entering above the steady state value at time = ΔT .

$$\left| \frac{\Delta M}{\Delta t} \right|_{in, \Delta t} - \left| \frac{\Delta M}{\Delta t} \right|_{in, ss} = C_1 (x_o + x) - C_1 x_o = C_1 x \quad (\text{pounds/second})$$

The mass of gas leaving the tank per unit time through the non-variable exit area:

$$\text{at Steady State} \quad \left| \frac{\Delta M}{\Delta t} \right|_{o, ss} = C_2 p_o \quad (\text{pounds/second})$$

$$\text{at } t = \Delta t \quad \left| \frac{\Delta M}{\Delta t} \right|_{o, \Delta t} = C_2 (p_o + p') \quad (\text{pounds/second})$$

and the net change in mass leaving at time Δt

$$\left| \frac{\Delta M}{\Delta t} \right|_{o, \Delta t} - \left| \frac{\Delta M}{\Delta t} \right|_{o, ss} = C_2 (p_o + p') - C_2 p_o = C_2 p' \quad (\text{pounds/second})$$

If we subtract the net mass leaving from the net mass entering, the increase in mass in the tank at Δt time

$$\frac{\Delta M}{\Delta t} = C_1 x - C_2 p' \quad (\text{pounds/second}) \text{-----(1)}$$

Differentiating the gas law, $pv = MRT$,

$$p \frac{dv}{dt} + v \frac{dp}{dt} = RT \frac{dM}{dt} + RM \frac{dT}{dt}$$

Since

$$V = A(x + x_0), \quad \frac{dv}{dt} = \frac{A dx}{dt}$$

We have

$$pA \frac{dx}{dt} + Ax \frac{dp}{dt} = RT \frac{dM}{dt} + RM \frac{dT}{dt}$$

If the gas follows the adiabatic law $pv^\gamma = C$, the differentiation of this gives

$$\gamma pv^{\gamma-1} dv + v^\gamma dp = 0$$

and
$$dp = - \frac{\gamma pv^{\gamma-1} dv}{v^\gamma} = -\gamma p \frac{dv}{v}$$

Replacing this in the above differentiated characteristic gas law

$$pA \frac{dx}{dt} - \frac{Ax \gamma p}{v} \cdot \frac{dv}{dt} = RT \frac{dM}{dt} + RM \frac{dT}{dt}$$

which with the substitution for v , and rearrangement and neglecting $\frac{dT}{dt}$, i. e., a small temperature variation, results in

$$\frac{pA}{RT} (1 - \gamma) \frac{dx}{dt} = \frac{dM}{dt}$$

or as a finite difference

$$\frac{pA}{RT} (1 - \gamma) \frac{\Delta x}{\Delta t} = \frac{\Delta M}{\Delta t} \quad \text{-----} \quad (2)$$

Substituting 2 in 1 and letting

$$\frac{p}{RT} (1-\gamma) = \beta$$

we have

$$\frac{\beta A \Delta x}{\Delta t} = C_1 x - C_2 p'$$

solving for p' and replacing $\frac{\Delta x}{\Delta t} = \dot{x}$

$$p' = \frac{C_1 x - \beta A \dot{x}}{C_2} \text{----- (3)}$$

Considering the free body of the piston, the forces acting on it when displaced x inches

$F_s = F_o + Kx$ = spring force and preload
due to pressure on outside

$F_D = r\dot{x}$ = damping force

$F_p = p_o A + p' A$

$F_i = m\ddot{x}$ = inertia force

For dynamic equilibrium, the inertia must be equal to the external forces

$$F_i = -F_s - F_D + F_p$$

Substituting:

$$m\ddot{x} + r\dot{x} + Kx + F_o - p_o A - p' A = 0$$

and since $F_o = p_o A$ when the piston is in equilibrium position, we obtain

$$\ddot{x} + \frac{r}{m} \dot{x} + \frac{K}{m} x - p' \frac{A}{m} = 0 \text{----- (4)}$$

Replacing p' with equation (3,) and rearranging we get

$$\ddot{x} + \frac{\dot{x}}{m} \left[r + \frac{\beta A^2}{C_2} \right] + \frac{x}{m} \left[K - \frac{AC_1}{C_2} \right] = 0$$

By inspection of this equation, if the term in x is negative, a spring force is present which will accelerate the mass when it is displaced. Therefore, if the term in x is negative, the system is unstable. This can be seen if one solves the equation with a negative term in x -- one of the roots s_1 will be positive and the other s_2 will be negative. The solution of the equation will be of the form:

$$x = ae^{+s_1 t} + be^{-s_2 t}$$

which for large t provides an increasing displacement indicating an instability..

In view of the above, the condition for stability of the system is that the term in x in the equation must be positive, or

$$K - \frac{AC_1}{C_2} > 0$$

Or for stability of the system, the spring constant must be

$$K > \frac{AC_1}{C_2}$$

This inequality conforms to the expression used in the analysis and given by Den Hartog.

System II

The assumptions used in System I are valid also for this system.

If we assume temporarily that the exit opening does not change during movement of the piston, we can obtain as done in System I, the net mass entering in tank during time Δt , while piston is displaced x and pressure is increased an amount p' .

For a fixed exit:

$$\left| \frac{\Delta M}{\Delta t} \right|_{in, \Delta t} = C_1 x - C_2 p' \quad (\text{pounds/second})$$

Now if we hold fixed the inlet opening and let the exit vary during movement of the piston, we can reason as follows:

The mass of gas leaving the tank per unit time through the outlet opening is the sum of the flow due to the size of the opening and the pressure in the tank, i.e.,

at steady state

$$\left| \frac{\Delta M}{\Delta t} \right|_{o, ss} = C_3 x_o + C_4 p_o$$

at $t = \Delta t$, The same mass rate for an increase in the opening x and the pressure p'

$$\left| \frac{\Delta M}{\Delta t} \right|_{o, \Delta T} = C_3 (x_o + x) + C_4 (p_o + p')$$

For a fixed inlet, the net mass leaving at time Δt is

$$\begin{aligned} \left| \frac{\Delta M}{\Delta t} \right|_{o, \Delta T} - \left| \frac{\Delta M}{\Delta t} \right|_{o, ss} &= C_3 (x_o + x) + C_4 (p_o + p') - C_3 x_o - C_4 p_o \\ &= C_3 x + C_4 p' \end{aligned}$$

The net mass increase per unit time in the tank when both inlet and outlet are made variable will then be:

$$\begin{aligned} \frac{\Delta M}{\Delta t} &= \left| \frac{\Delta M}{\Delta t} \right|_{in, \Delta T} - \left[\left| \frac{\Delta M}{\Delta t} \right|_{o, \Delta t} - \left| \frac{\Delta M}{\Delta t} \right|_{o, ss} \right] \\ &= C_1 x - C_2 p' - C_3 x - C_4 p' = (C_1 - C_3)x - (C_2 + C_4)p' \end{aligned}$$

$$\text{or} \quad \frac{\Delta M}{\Delta t} = (C_1 - C_3)x - (C_2 + C_4)p' \text{ ----- (5)}$$

Following the same procedure as in System I, we can obtain with the indicated simplification, i.e.,

$$\beta A \dot{x} = \frac{\Delta M}{\Delta t}$$

Substituting (5) in this, we get

$$\beta A \dot{x} = (C_1 - C_3)x - (C_2 + C_4)p'$$

which in terms of p' is

$$p' = \frac{(C_1 - C_3)x - \beta A \dot{x}}{(C_2 + C_4)} \text{ ----- (6)}$$

Inserting this is the dynamic equilibrium equation previously derived, i.e.,

$$\ddot{x} + \frac{r}{m} \dot{x} + \frac{K}{m} x - p' \frac{A}{m} = 0$$

We obtain

$$\ddot{x} + \frac{r}{m} \dot{x} + \frac{K}{m} x - \frac{A}{m} \cdot \frac{(C_1 - C_3)x - \beta A \dot{x}}{(C_2 + C_4)} = 0$$

which rearranged provide the stability equation

$$\ddot{x} + \frac{\dot{x}}{m} \left[r + \frac{A^2 \beta}{(C_2 + C_4)} \right] + \frac{x}{m} \left[K - \frac{A (C_1 - C_3)}{(C_2 + C_4)} \right] = 0$$

Following the same reasoning as in System I, the condition for stability is:

$$K - \frac{A (C_1 - C_3)}{(C_2 + C_4)} > 0$$

or

$$K > \frac{A (C_1 - C_3)}{(C_2 + C_4)}$$

which is the relation given by Den Hartog and used in the analysis.

System III.

This system is the same as System I with the flow reversed. The same assumptions of the other two systems are applicable.

Referring to System II, holding fixed the inlet, the net mass leaving at time Δt is:

$$C_3 x + C_4 p'$$

Referring to System I, the increase in mass entering above the steady state value is zero since the inlet opening remains constant ($x = 0$). Therefore, the net decrease in mass in the tank is:

$$\frac{\Delta M}{\Delta t} = 0 - C_3 x - C_4 p'$$

Substituting this expression in equation (2).

$$\beta A \dot{x} = -C_3 x - C_4 p'$$

we obtain

$$p' = - \frac{(\beta A \dot{x} + C_3 x)}{C_4}$$

Upon substitution of this p' in the dynamic equilibrium equation (4), we obtain:

$$\ddot{x} + \frac{\dot{x}}{m} \left[r + \frac{\beta A^2}{C_4} \right] + \frac{x}{m} \left[K + \frac{AC_3}{C_4} \right] = 0$$

The terms of this equation including the term in $\frac{x}{m}$

$$K + \frac{AC_3}{C_4}$$

are all positive. This indicates a stable system as discussed for System I.

FIG. 1 - SYSTEM I & III
(SYSTEM III SAME AS I WITH REVERSE FLOW)

FIG. 2 - SYSTEM II

FIG. 3 STATIC STABILITY OF CORE. LATERAL SUPPORT SPRING VS. LEAKAGE RADII