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A NUMERICAL STUDY OF PROTOSTELLAR FORMATION

Charles Knowlton Westbrook
Ph.D. Thesis

March 8, 1974

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Abstract

The earliest phase in the evolution of a star is the formation of a gravitationally bound gas cloud from the interstellar medium. This cloud mass is not, in general, in hydrostatic equilibrium, and will evolve hydrodynamically until equilibrium is attained. This thesis examines the protostellar evolution of these gas clouds.

A mathematical model is presented treating all the appropriate physical processes which are important during protostar formation including hydrodynamics, thermodynamics and equation of state, radiation transport, and convective energy transport. Important features of the mathematical model are: (1) use of temperature or specific energy density as an independent parameter rather than energy flux or luminosity, and (2) inclusion of all time derivatives from the differential equations. These provisions cast the stellar structure equations into the form of an initial value problem that can be evolved in time.

A numerical model is then developed. An important feature of the difference equations is their ability to evolve configurations which are far from radiative, convective, or hydrodynamic equilibrium. This feature is a direct result of the form presented for the differential equations. This numerical model is applied to a wide range of initial conditions. The evolution of cloud masses from 0.01 to 50 solar masses is investigated.

Results of evolutionary calculations are presented which are consistent with the existence of a lower mass limit for formation of main sequence stars. Other results for large mass clouds demonstrate mechanisms which severely limit the maximum size for stable stars. Calculations for protostars of one solar mass are compared with calculations in the astrophysical literature and indicate that such a protostar first appears as a stellar object with an effective temperature of about 3100°K and a luminosity of about 500 times that of the sun.

All the calculations included assume spherical symmetry. Convective energy transport is treated with mixing length theory which has been extended to deal with conditions far from convective equilibrium by the introduction of a convective flux limiter. A numerical model is developed for material in hydrostatic equilibrium, with radiative and convective energy transport, coupling implicitly all the stellar structure equations into one system. This system of numerical equations is then joined when necessary to the decoupled hydrodynamics and energy balance equations at the boundary of the region which is in hydrostatic equilibrium.

All of these numerical models allow the evolution to be followed far beyond the stage where previous calculations in the literature had to be terminated. These calculations are therefore able to include the entire evolution of a protostar from its formation until well into the quasistatic contraction phase in a single consistent numerical treatment.

I. General Features of Protostar Formation

INTRODUCTION

Most of the phases of stellar evolution are well understood in principle. Observational and theoretical knowledge have combined to describe the great majority of the lifetimes of most of the known stars. Only in dealing with the very first and very last portions of stellar evolution does current knowledge lack a significant degree of certainty. This paper is an attempt to examine some of the details involved in the first of these areas, the formation of protostars out of gas clouds in the interstellar medium.

OBSERVATIONAL BACKGROUND

Observationally, this phase of stellar evolution is very poorly documented. The principal reason is that this phase is of extremely short duration, even by human standards, probably of the order of years or even less. The two best known observations, of FU Orionis in 1936 and V1057 Cygni in 1969, showed increases of six orders of magnitude in less than a year. When this time scale is compared with the average lifetime of a star, the frequency of these events becomes quite small. Another factor which further reduces the probability of observing the formation of protostars is that most protostars are associated with dense clouds of interstellar dust and gas which greatly obscure observational efforts. Also, most theoretical work, including results below, predict that the majority of the radiation emitted by forming protostars will be at infrared frequencies, making them more difficult to observe.

Once protostars are formed, observational evidence is somewhat more available, although still not copious. The general class of T Tauri stars are believed to be very young stars which have not reached the Hertzsprung-Russell (HR) main sequence. These stars provide a rough idea of the end result of early stellar evolution; of course, different theories of protostar formation which all arrive at similar models of T Tauri stars are equally valid, at least as far as observational evidence is concerned.

GENERAL DESCRIPTION OF THE MODEL

In this project we wish to describe a theoretical model for calculating the formation of protostars. An attempt has been made to include as many as possible of the physical processes which are significant. If all the appropriate physics can be included, the question of the validity of the results of the calculations can be reduced to a question of the validity of the initial conditions. This statement points out the most important feature of the theoretical model used—its solution of the relevant equations in a deterministic manner.

Most calculations of stellar evolution have been based on the Henyey method¹⁻³ which consists of calculating a sequence of static stellar models, coupled by some prescription for variation of internal parameters in time from one model to the next. This type of approach is ideally suited to calculations of ordinary stellar structure in which all the time constants are extremely long. The driving force between successive models usually becomes a change in material composition which affects thermonuclear energy production rates. There is no appreciable hydrodynamic motion, and only very gradual changes in luminosity. An important feature of the solution of the equations of stellar structure in this method is the use of luminosity as an independent variable, with temperature as a dependent variable. Physically, a temperature (or energy density) gradient determines an energy flux; the Henyey method instead uses the luminosity or energy flux to determine the temperature gradient which is consistent with that energy flux. The stellar structure equations are thus reduced to a set of boundary value problems in space⁴ for a given model. The key to the use of this type of calculation is that no time derivative in the stellar structure equations is appreciable. The Henyey method has been extended to cases in which there is hydrodynamical motion over the time duration of one stellar model. This can be done, as by Bodenheimer,⁵ by solving the hydrodynamic equations implicitly in time while retaining the use of luminosity as an independent variable.

In this work we wish to examine processes which have shorter time constants than those which occur in static models. We include all time derivatives in the equations of stellar structure. We find that the relative magnitudes of the time scales for the relevant physical processes can vary considerably during the evolution of a protostar, and it is this variation that is responsible for a number of interesting results.⁶ These time scales can in fact differ significantly at different regions in a single protostellar model, with the result that those regions may be evolving independently.

Therefore, for physically deterministic reasons we have chosen energy density as an independent variable and energy flux or luminosity as the dependent variable, after Colgate and White,⁷ Christy,⁸ and Hayashi *et al.*⁹ Inclusion of the time derivatives changes the nature of the equations to a set of initial value problems, with each successive model determined through the time derivatives and the energy flux term. As a result, we are strongly dependent upon assumptions concerning the state of the interstellar medium out of which the protostar is formed. Conversely, we might wish to infer information about possible conditions in the interstellar medium by comparing our results with observations. This requires that we be able to distinguish observationally between final models which differ only quantitatively, not qualitatively, and is probably not possible in practice.

The previous discussion is, to a varying degree, valid for nearly all stellar evolution calculations. In this work we wish to focus specifically on the period in the evolution of a protostar, beginning with its creation as a gravitationally bound entity and ending with its attainment of overall hydrostatic equilibrium, with or without mass

ejection into space. We wish to examine the detailed structure of this collapsing protostar in time. Given the structure, we may calculate the effects of the energy transport processes and deduce an observational history of the protostar in the form of a track on the HR diagram. We may also be able to assess the relative and absolute importances of different transport processes, particularly radiation and convection, and their dependence upon physical data and theoretical models for those physical processes. We have attempted to determine the differences in protostellar evolution as a function of several parameters. The most obvious of these is total mass, varying from $0.01 M_{\odot}$ to $50 M_{\odot}$. Other parameters that can and have been varied include boundary conditions, low-temperature opacity, convective energy transport efficiency, and the degree of initial gravitational binding.

In our calculations we shall assume that spherical symmetry is valid. The basic motivation for this assumption is the great simplification in the equations of motion and energy conservation that result. We are therefore unable to treat quantitatively any effects due to rotation, magnetic fields, or any other nonspherical effects. It is possible to provide qualitative descriptions for influences these effects might have. For example, both uniform and differential rotation are almost certainly significant in a large fraction of protostars, and it is likely that rotation will retard the gravitational collapse of the protostar, but quantitative results cannot be produced by the methods to be used here. Moss¹⁰ points out how rotation can change many features in the structure of a protostar, including the effective temperature, contraction time, luminosity, and the relative extent of the radiative core as it forms. Our results will therefore be strictly valid for a restricted set of cases, those in which nonspherical effects are unimportant. Attempting to include rotational and other nonspherical effects would also have expanded this effort far beyond the present scope of this thesis.

INITIAL CONDITIONS AND BOUNDARY CONDITIONS

We wish to be as general as possible about the initial conditions for our calculations. If we try to begin evolutionary calculations at the earliest possible time, we are led to initial conditions which are specified by a modified Jeans criterion. The question of how this density perturbation is produced, satisfying the requirement of being gravitationally bound, is a difficult problem which will not be discussed further here. Nevertheless, we assume that a spherical gas cloud is somehow formed. The initial structure of this cloud is somewhat arbitrary. We shall find that the succeeding evolution is extremely nonhomologous, so that the initial density and temperature distributions are not critically important. Because in every case the initial opacities are so low,¹¹ we can be safe in assuming the initial protostar is isothermal. For simplicity, we assume that the initial model is constant in density also. To be gravitationally bound, the density must exceed a critical value given by¹²

$$\rho_c \propto \frac{T^3}{m^2}. \quad (1)$$

For $T = 10^4 \text{K}$ and $M = M_{\odot}$, $\rho_c \approx 10^{-19} \text{g/cm}^3$. The question of how this density is achieved is not the question here. However it is achieved, whether by Hoyle's fragmentation model,¹³ by gravitational pressure waves or some other mechanism, we assume that the critical density is reached.

A number of calculations of protostellar evolution, notably those of Hayashi and collaborators^{9,14,15} assume polytropic¹⁶ initial conditions with a high degree of central condensation and appreciable initial velocities calculated from a free-fall approximation. Using these initial conditions amounts to assuming that the prior collapse has been homologous, maintaining a polytropic configuration in density. These assumptions are not confirmed by comparison with calculations with initially uniform density; the most important advantage of using those initial conditions turns out to be the fact that they enable the entire subsequent evolution to be done using explicit hydrodynamics, a noteworthy simplification. We have repeated some of Hayashi's calculations, largely for pedagogical reasons, to ensure proper calculational methods, but polytropic initial conditions are probably not physically realistic.

The question of external boundary conditions on a protostar is related to the physical model assumed for the initial formation of the protostar. If we assume that the bound mass was imbedded in a much larger cloud which is otherwise in pressure equilibrium, it would be reasonable to assume that we should apply a constant external boundary pressure to the protostar. If the protostar is assumed to be an isolated entity, we should instead choose vacuum boundary conditions. In that case, however, some fraction of the outer portion of the cloud would be expelled into the vacuum. We would then wish to calculate the evolution of only the mass which remained bound. Both of these procedures were tried. Since internal pressures eventually become many orders of magnitude higher than the externally applied pressure, both cases eventually became very similar.

II. Physical Principles

EQUATION OF MOTION

The equations of motion and energy conservation for a gas in local thermodynamic equilibrium are well known. With the assumption of spherical symmetry, the equation of motion is

$$\rho \frac{d^2 r}{dt^2} = - \frac{dP}{dr} - \frac{GM(r)}{r^2} \rho \quad (2)$$

where

$$M(r) = \int_0^r \rho(r') 4\pi r'^2 dr' \quad (3)$$

Therefore,

$$dm = 4\pi r^2 \rho dr, \quad (4)$$

and we rewrite (2)

$$\frac{d^2 r}{dt^2} = -4\pi r^2 \frac{dP}{dm} - \frac{GM(r)}{r^2}. \quad (5)$$

The great majority of the theoretical investigations of stellar structure and evolution treat Eq. (5) with $\ddot{r} = 0$. The reasoning behind this assumption can best be seen by solving Eq. (2) under the idealized conditions where $dP/dr = 0$ and $\rho(r) = \text{constant}$. The resulting equation, integrated twice, yields the approximate result

$$t_{ff} \approx \frac{1}{\sqrt{4\pi G\rho}}. \quad (6)$$

Since Eq. (2) with $dP/dr = 0$ is the equation for free fall, t_{ff} is interpreted as the time required for a gas sphere of uniform initial density to collapse to zero volume. The actual solution to the free-fall equation is given by Colgate and White.⁷ We will use their analytic solution to verify the results of calculations, but the approximate solution, Eq. (6), provides us with a very useful time scale to use in comparisons. In most cases of interest, calculations of stellar structure can ignore the time derivative in (2) because t_{ff} is extremely small compared to the known evolutionary time scales for the stellar model. For example, using the average density for the sun, $t_{ff} \approx 10^3$ sec or about 1/4 hr. As a result, deviations from hydrostatic equilibrium have a lifetime of the order of t_{ff} which is negligible compared with the lifetime of the sun. In this paper, however, we are dealing specifically with situations that are initially far from hydrostatic equilibrium, so we will include the acceleration term in the equation of motion and t_{ff} will be the characteristic time scale for subsequent calculations.

ENERGY CONSERVATION

The equation for energy conservation is the first law of thermodynamics.

$$\frac{dQ}{dt} = \frac{dE}{dt} + P \frac{dV}{dt} \quad (7)$$

where $E = E(\rho, T)$, the specific internal energy. For quasistatic changes, the rate of heat addition is

$$dQ/dt = T(dS/dt). \quad (8)$$

The rate of heat addition to a differential spherical shell is equal to the heat generated in that shell by nuclear reactions, plus the heat excess added to the shell by transport processes,

$$dQ/dt = \epsilon(r) - \nabla \cdot \vec{F} \quad (9)$$

where $\vec{F}$ is the total heat flux, including radiative, convective, and conductive processes, and $\epsilon(r)$ is the local heat generation rate. In practice, neutrino emission, although strictly a transport process, is usually included as a negative contribution to $\epsilon(r)$. Expanding dE/dt and rearranging, we have

$$\frac{dT}{dt} = \frac{1}{\left(\frac{\partial E}{\partial T}\right)_V} \left\{ \epsilon(r) - \left[P + \left(\frac{\partial E}{\partial V}\right)_T \right] \frac{dV}{dt} - \nabla \cdot \vec{F} \right\}. \quad (10)$$

Flux contributions that will be considered here will be radiative, conductive, and convective. The radiative flux is calculated using the assumption that asymptotic diffusion theory¹⁷ is appropriate. This approximation breaks down in one important area which is discussed in Appendix B. Further, we assume that electron conductivity, where significant, can be included with the radiative flux by defining a conductive opacity.⁴ This requires the validity of Fick's law for the radiative flux. The flux is

$$F_{r+c} = F_{rad} + F_{cond} = - \left[4ac\lambda_d T^3 \frac{dT}{dr} + \lambda_c \frac{dT}{dr} \right] \quad (11)$$

$$F_{r+c} = - \frac{4ac}{3} \lambda T^3 \frac{dT}{dr}$$

where

$$\lambda \equiv 3\lambda_d + \frac{3\lambda_c}{4acT^3}.$$

Of course, densities are low enough in pre-main-sequence conditions such as these that electron conductivity is negligible. It is included formally only for completeness. Noting that for spherical symmetry the divergence operator can be written $\nabla \rightarrow \frac{\partial}{\partial V} A$, we can rewrite Eq. (10):

$$\frac{dT}{dt} = \frac{1}{\left(\frac{\partial E}{\partial T}\right)_V} \left\{ \epsilon(r) - \left[P + \left(\frac{\partial E}{\partial V}\right)_T \right] \frac{dV}{dt} + v \frac{\partial}{\partial V} \left[A \frac{4ac}{3} \lambda T^3 \frac{dT}{dr} \right] - \nabla \cdot \vec{F}_{conv} \right\}. \quad (12)$$

For calculations in which convection is not included, $\vec{F}_{conv} = 0$ and the above equation is the desired one for energy conservation.

For many cases of interest, convective energy transport is not important. Even for stellar models in which convection does contribute to the energy transport, the convection is usually confined to some subregion of the star. Often either the central core or the outermost atmosphere is convective while the remainder of the star is totally radiative. As a result, Eq. (12), without the convective flux term, is used for the majority of our calculations. We now will develop an expression for the remaining term in (12), the convective flux.

CONVECTIVE ENERGY TRANSPORT

Perhaps the most serious theoretical deficiency in present knowledge of stellar physics is the lack of a definitive theory of convection. The enormous complexity of the problem, involving macroscopic turbulent fluid motion coupled with continuous energy exchange between regions of the fluid, is probably intractable in general. The customary visualization of the physical process is that in a fluid that is unstable to convection, eddies or convective fluid elements are formed which move through the fluid, often covering distances that are large compared to the dimension of the fluid element itself, retaining a collective identity for some period of time. These eddies exchange energy and momentum between themselves and with the bulk of the fluid by way of viscous interactions, hydromagnetic waves, radiation, and other processes. Each fluid element has a size, shape, chemical composition, velocity, and lifetime, and all of these parameters can change with time. Thus a general theory of convection becomes a description of the creation, evolution, and dissipation of these fluid elements. A great deal of work is currently being done in this area (for example, Spiegel¹⁸ and Ledoux,¹⁹ but a general theory that can be applied to astrophysical calculations does not yet exist.

The simplest approximation to convective energy transport that is commonly used, the mixing length theory, is due originally to Bohm-Vitense.^{20,21} This model substitutes an "average" fluid element, with an average size, velocity, and lifetime, for the entire distribution of elements. These averages must be carefully defined; usually this involves some kind of *a priori* knowledge or assumptions, often based on laboratory experiments, the results of which are then extrapolated to solar dimensions. The resulting expressions are therefore suspect in several ways. The simplifications or assumptions may themselves be erroneous. For example, collapsing an entire distribution function of eddy properties into an average value may seriously distort resulting averages. Further, the parameters which are evaluated on the basis of laboratory experiments or even dimensional analysis may be appropriate only over a limited range of conditions.

The resulting expressions must then be applied with the realization that those solutions which are sensitive to details of the convective model are not very reliable. We can expect, however, to be able to supply rough corrections to our evolutionary calculations because of the effects of convection. We will consider only the energy transport effects of convection, with none of the material transport effects. This limitation is quite acceptable in these calculations since most of the models calculated do not evolve far enough to begin burning nuclear fuel; therefore, the chemical composition is very nearly uniform. In regions where there is a sharp gradient in fractional ionization, for example, convection will tend to reduce that gradient by the turbulent mixing of material from both sides of the ionization front. We neglect this effect, but since such a front is usually accompanied in these calculations by a strong hydrodynamic

shock wave which rapidly passes through the medium, the effects of turbulent mixing due to convection should be relatively unimportant.

The general framework of mixing length theory has been accepted here with the usual reservations. We will not derive most of the relations, which are derived by Cox and Giuli.²² We will, however, discuss how one important feature has been redefined for use in time-dependent calculations.

The expression for convective energy flux is

$$F_c(\Delta r) = \frac{1}{2} \rho v c_p(\Delta r) (\Delta \nabla T) \quad (13)$$

where $\Delta \nabla T$ is the difference between the temperature change which the fluid element would experience if it were displaced by dr , $\left(\frac{dT'}{dr}\right)$, and the actual temperature gradient in the material, $\left(\frac{dT}{dr}\right)$.

$$\Delta \nabla T = \left[\left(-\frac{dT}{dr} \right) - \left(-\frac{dT'}{dr} \right) \right]. \quad (14)$$

In these expressions ρ is the density in the medium, v is the radial velocity of the moving fluid element, c_p is the heat capacity of the fluid element, and Δr is the radial displacement of the element. We average $F_c(\Delta r)$ over all displacements Δr , defining the average displacement Λ ,

$$\Lambda \equiv (\Delta r)_{\text{average}}$$

such that the energy flux is correctly given. That is,

$$\int_0^\infty F_c(\Delta r) d(\Delta r) = F_c(\Lambda) \Lambda. \quad (15)$$

Therefore, (13) becomes

$$F_c(\Lambda) = \frac{1}{2} \rho \bar{v} c_p \Lambda (\Delta \nabla T). \quad (16)$$

The average velocity $\bar{v}$ is approximated by assuming that the eddy has been accelerated in the nonuniform density medium that is convectively unstable,

$$v^2 = -\frac{1}{8} g \left[\Delta \nabla \rho(\Lambda) \right] \frac{\Lambda}{\rho}. \quad (17)$$

Thus we have

$$F_c(\Lambda) = \frac{c_p}{4\sqrt{2}} (\rho g)^{1/2} \Lambda^{3/2} \left[-\Delta \nabla \rho(\Lambda) \right]^{1/2} (\Delta \nabla T). \quad (18)$$

Finally, we relate the unstable superadiabatic density gradient to the excess temperature gradient,^{23,24}

$$- \frac{\Delta \nabla p}{\rho} = \frac{1}{T} (\Delta \nabla T) \Lambda \quad (19)$$

giving as a final expression for the flux

$$F_c(\Lambda) = \frac{c_p}{4\sqrt{2}} g^{1/2} \Lambda^2 \frac{\rho}{T^{1/2}} (\Delta \nabla T)^{3/2}. \quad (20)$$

Let us attempt to summarize as many as possible of the major approximations included in the derivation of (20).

- All numerical constants are uncertain, often by as much as an order of magnitude. For example, the factor of 8 in (17) is obtained by assuming that half of the kinetic energy the fluid element acquires is dissipated by turbulent viscosity.²² If it is assumed that viscous effects reduce the amount of kinetic energy that is retained by the eddies, this constant would be greater; the effects on static models of varying this value are examined by Henyey *et al.*²⁵ In principle, of course, the "correct" value is some function of density and temperature.

- The medium under consideration is assumed to be in hydrostatic equilibrium. This assumption was needed to derive (17), in which the acceleration is a result of a small perturbation from hydrostatic equilibrium.

- Expression (19) assumes an ideal gas law. This approximation is made only in this expression and can be made more general to account for changes in fractional ionization.²³ This latter effect has not been included in these calculations.

- The term dT'/dr covers a great deal of physics. Most authors define an efficiency of convection γ ,

$$\gamma \equiv \frac{\nabla - \nabla'}{\nabla' - \nabla_{ad}} \quad (21)$$

where

$$\nabla \equiv \frac{dT}{dr}, \quad \nabla' \equiv \frac{dT'}{dr}, \quad \nabla_{ad} \equiv \left(\frac{dT}{dr} \right)_{ad}$$

where the subscript "ad" refers to the temperature change for a fluid element that is displaced adiabatically. The efficiency is intended to account for the fact that as an eddy moves through the fluid it loses energy, primarily by radiation, reducing the net convective energy flux. This parameter γ can vary over a great range, depending upon the optical thickness of the fluid elements. Most calculations substitute an expression for γ which includes some information about the suspected size or other properties of the fluid elements to be calculated. Stothers and Chin²⁶ further develop the principle of convective efficiency by examining the possibility of reducing the amount of convection in stellar cores as a result of magnetic pressure. Another approach that has been tried is to use a slightly different expression for $\Delta \nabla T$, requiring that the excess gradient be greater than some minimum value.²⁷

- The velocity of the moving fluid elements, calculated from (17), is usually not permitted to exceed the local sound speed, arguing that convection is considerably subsonic. This limitation is discussed further by Cox and Giuli.²² In protostellar environments sound speeds are much greater than convective fluid velocities, so this limitation is not involved in our model.

- By far the best known approximation is the attempt to define Λ , the mixing length. Clayton⁴ states that fluid cells will not dissipate until they have moved through a pressure or density scale height; this conclusion is obtained by analogy with laboratory fluid convection. Chiu²³ derives Eq. (20) by a different method and points out that the derivation is valid only if Λ is much smaller than the scale height. He disregards that limitation because experience in laboratory experiments indicates larger mixing lengths. Most authors assume $\Lambda = \alpha H_p$ or $\Lambda = \alpha H_\rho$, where H_p and H_ρ are the pressure and density scale heights, and $\alpha \approx 1.5$. Since scale heights can change rapidly, some authors (e.g. Kippenhahn *et al.*²⁸) use a non-local mixing length, but the principle (and α) are usually the same as in local mixing length models.

In some work the mixing length is further defined as being related to the physical size of the fluid cells; for example, in the sun convective elements are identified with solar granules which have diameters of the order of magnitude of local pressure scale heights. Zhilyaev and Kolesnik²⁹ extend this argument to T Tauri stars and conclude that mixing lengths must be many times larger in T Tauri stars than in the sun and that the spatial scale for convection in T Tauri stars is much the larger.

Perhaps the most important general assumption in the conventional use of mixing length theory is that the medium involved is assumed to be close to convective equilibrium. Under those conditions, the temperature gradients are closely related to the density and pressure gradients, so that it is very reasonable to believe that all the spatial scales are interrelated. However, we shall find that conditions in protostars that are rapidly evolving in time can be very far from convective equilibrium, creating a great deal of doubt in the choice of a way to relate the mixing length to other physical variables.

With the above limitations in mind, we should point out that in conventional calculations of stellar models, mixing length theory is surprisingly reliable. There are several reasons why this is true. Solar observations can supply appropriate values for most necessary parameters to within an order of magnitude. For models which are in convective equilibrium (most stellar models must be in such equilibrium if they are not evolving rapidly) we would expect ΔT to be small. Finally, and perhaps most important, most calculations treat the energy flux as the independent variable. If F_c is specified, Eq. (20) may be integrated for $T(r)$ without ambiguity. Since the flux is known, all of the other parameters are effectively restricted to "reasonable" values. Some generality is lost in solving the equation in this manner. For example, the existence of density inversions is not allowed when the equations are solved in this manner.³⁰

As discussed earlier, we treat temperature as the independent variable, determining the flux as a function of temperature. As a result, no artificial constraints have been placed upon the structure of the model. If nonequilibrium regions appear during the evolution of a model, fluxes are produced which will act to restore the equilibrium. A problem which arises, however, for regions which are far from equilibrium is that there is now no limit on the convective flux in (20). The calculation of an absurdly high flux indicates a need for modifying the mathematical model; this has been done by the inclusion of a term which could be called a convective flux limiter, suggested by the use of a similar device for diffusion calculations.

We consider Eq. (20) to be true only for $\Delta\nabla T \ll \nabla T$, that is, for regions where the deviations from convective equilibrium are small. We then include a correction term on the right-hand side of (20) to gradually limit the convective flux as $\Delta\nabla T$ increases. The actual limiting value adopted and the analytic form of the correction term are not completely specified; as in diffusion calculations, the selection of the "best" flux limiter approaches an art form. Clearly, we have

$$F'_c = f F_c. \quad (22)$$

We begin by assuming a flux limiter of the form

$$f = \left[\frac{1}{1 + \left(\frac{\Delta\nabla T}{\nabla T} \right)^{f_1}} \right]^2. \quad (23)$$

Thus for $\Delta\nabla T \ll \nabla T$, $f \approx 1$ and $F'_c = F_c$. We can choose f_1 by a simple physical argument. If $\Delta\nabla T = \nabla T$, then $\Delta\nabla\rho$ is approximately $\nabla\rho$. We can thus evaluate the velocity of the fluid elements from (17), and if necessary limit $\bar{v}$ to v_{sound} . Then

$$F'_c = \frac{\frac{1}{2} \rho \bar{v} c_p \Delta\nabla T}{(1 + f_1)^2}. \quad (24)$$

We can then require that the flux through a spherical surface at radius r cannot exceed some fraction of the total thermal energy contained within r . In the calculations to be described f_1 was generally chosen to be 1000, limiting the convective flux to roughly 0.1% of the thermal content in a spherical zone.

It is interesting to examine the implications of incorporating the limiter term into some of the factors in (20). For example, if we redefine the excess temperature gradient,

$$(\Delta\nabla T)_1^{3/2} = (\Delta\nabla T)_0^{3/2} f, \quad (25)$$

we are implying that the criterion for convective instability changes as we depart from convective equilibrium. Our physical model for convection, however, indicates that

the criterion should be a simple function of ΔT . Equation (25) further suggests that the fluid elements respond in a nonlinear way to the excess temperature gradient. Alternatively, if we incorporate f into the mixing length, we have

$$\Lambda_1^2 = \Lambda_0^2 f. \quad (26)$$

Physically, this relation suggests that under conditions far from convective equilibrium, fluid elements do not travel as far as they would when closer to equilibrium. A possible mechanism for this effect might be an increase in turbulent friction as conditions depart from equilibrium; increased turbulence should be expected under such conditions. Another very similar interpretation can be made if we include f in the fluid element velocity calculation (17). This would suggest that, due perhaps to turbulent effects, the velocity of the moving elements does not increase as rapidly as (17) would indicate as the medium departs from convective equilibrium.

Of course, we must not lose sight of the fact that the reason we had to introduce a convective flux limiter was that our mixing length model breaks down as we depart from equilibrium. By including such a term we hope to extend the range of applicability of the model. By attempting to include the flux limiter in one of the factors in (20) we are really trying to identify that part of the model which breaks down first. In the calculations performed with this model, the limiter was included in the mixing length as in (26), reasoning that besides having the advantage of simplicity, the physical implications of including it in that term seemed most reasonable.

EQUATION OF STATE

All equation-of-state information is available in either of two forms, tabular data for pressure and specific internal energy as functions of density and temperature, or data calculated internally by the program. These internal calculations involve solving a coupled set of Saha-like equations, including all of the relevant molecular and atomic species present. It became evident that in most cases tabular data were not accurate enough, and most of the calculations were carried out with the pressure and energy expressions which are described in Appendix F. A further advantage of using an internal computation is that the mass fractions of hydrogen and helium are included as input parameters to the program, allowing a much easier method of examining differences in results due to changes in composition. Figure 1 shows solutions for pressure as a function of temperature and density for the relative composition used in these model calculations, and Fig. 2 gives the specific internal energy. To better show the structure of the energy relations, the contribution to the specific internal energy due to the radiation field is not included. The pressure plots include the scalar radiation pressure.

Fig. 1. Pressure as a function of temperature. Densities included are 10^{-19} (lowest curve) to 10^{-1} (highest curve), increasing in powers of 10.

Fig. 2. Specific internal energy, not including radiation energy. Lowest curve is for density of 10^{-19} , increasing in powers of 100 to 10^{-1} (highest curve).

OPACITY

Rosseland mean opacities were evaluated by using an analytic fit to the data of Cox and Stewart,³¹ with extensions to lower temperatures from Auman and Bodenheimer.³² At very low temperatures a floor was placed on the opacities; this floor value is an input parameter, and for most of the models this floor was $0.01 \text{ cm}^2/\text{g}$. A plot of the fit to the given data, again as a function of temperature and density, is given by Fig. 3.

The low-temperature opacity is at the same time one of the most troublesome and most interesting points in the specification of input physical parameters. It is troublesome for several reasons. Published data are available for only a few compositional mixtures, and none of these cover as wide a range of density and temperature as we need. Suitable interpolations and extrapolations are therefore necessary. Extrapolations to low temperatures are a subject of some debate. Auman and Bodenheimer³² use a constant of $1.5 \times 10^{-2} \text{ cm}^2/\text{g}$ below 1000°K and note that an increase of opacity by a factor of 40 in this range has a marked effect on the amount of emitted radiation. Nakano *et al.*¹⁴ use much higher values for $T < 1200^\circ\text{K}$, assuming that ice crystals and mineral grains are the dominant contributors to the opacity below 150°K and between 200°K and 1200°K , respectively. Larson³⁴ uses a constant below 1400°K , where the constant value is specified by the postulated grain abundance in the gas cloud. Above this temperature, Larson uses a rough power law for the opacity, proportional to $\rho^{1/2}T^7$. Use of data of any kind in these temperature ranges requires an a priori knowledge or assumption of just what are the most important species that account for the opacity; not enough is known of the state and composition of protostellar clouds to be very sure of the exact values to be used for the opacity. The amounts of heavier materials, including complex molecules, are very poorly known, and such impurities may make important contributions to the low-temperature opacity. Forms of the opacity below 1000°K can be varied in our program, and the effects of varying these values are a subject we wish to examine further below. In general, increasing the low-temperature opacity has the effect of decreasing the density at which the infalling portions of the protostar become opaque to the radiation from the core. This also means that the collapsing material becomes opaque at an earlier time in its evolution, and hydrostatic equilibrium and convective energy transport become important at an earlier time. Each of these points will be discussed further, but it can be seen that the actual numerical values for the low-temperature opacity can indeed influence later results to some degree. It should be remembered that, in accordance with earlier warnings, those results which depend strongly on what values are used for these parameters are most suspect. We shall try only to see the qualitative dependence of our results on these parameters.

Fig. 3. Rosseland mean opacity for densities from 10^{-12} g/cm^3 (lowest curve) to 10^{-1} g/cm^3 (highest curve), increasing in powers of 10.

III. Numerical Model

INTRODUCTION

Solving sets of coupled differential equations by solving corresponding difference equations is both a science and an art form. Choice of an optimum set of difference equations usually involves a great deal of trial and error, as well as experience and intuition. In most cases, including this one, the difference equations finally adopted are usually justified by the fact that they work while other difference equations did not. It is essentially impossible to perform the type of stability and convergence analyses for these equations that are done for simple systems by Richtmyer and Morton.³⁵ Therefore, if the equations produce "well-behaved" solutions, one assumes that most stability and convergence criteria have been satisfied. Certain general criteria remain including a Courant stability condition for explicit differencing, and other sensible limitations, particularly the choice of time steps and mesh spacing, must be observed. In most cases, the first indication of instabilities in the difference scheme comes from some failure in an actual calculation. Then the numerical model must be modified to remove the error.

We wish to express the above differential equations in difference form. This is done, assuming spherical symmetry, by dividing the sphere into a number, J , of concentric spherical shells. Using these shells to define a Lagrangian grid and using zone (shell) mass as the Lagrangian variable, we can write the difference equations in terms of zonal variables. In all difference equations below, a subscript j will refer to the value of a variable at the outer boundary of the j th shell, while a subscript $j-\frac{1}{2}$ will refer to a variable that is evaluated at the center of the j th zone. Most of the variables can change as a function of time; a superscript will indicate the time at which the variable is evaluated. Thus $P^n = P(t^n)$. In some cases this superscript will be omitted and a superscript $n+\frac{1}{2}$ is assumed in those cases, indicating that the variable is evaluated at the midpoint of the time interval (t^n, t^{n+1}) . When the subscript is omitted, $j+\frac{1}{2}$ is to be understood. Most of the variables and their definitions are given in the following List of Symbols.

List of Symbols

t^n	Time, iteration n
$\Delta t^{n+\frac{1}{2}}$	Time step, $t^{n+1} - t^n$
r_j	Outer radius of zone j
$dm_{j-\frac{1}{2}}$	Mass of zone j
M_j	$\sum_{i=0}^j dm_{i-\frac{1}{2}}$ total mass interior to r_j

u_j	Velocity of outer boundary of zone j
$\rho_{j-\frac{1}{2}}$	Density of zone j
$V_{j-\frac{1}{2}}$	Specific volume of zone j
$T_{j-\frac{1}{2}}$	Temperature of zone j
$E_{j-\frac{1}{2}}$	Specific internal energy density of zone j
$\kappa_{j-\frac{1}{2}}$	Rosseland mean opacity of zone j
$P_{j-\frac{1}{2}}$	Pressure of zone j
$Q_{j-\frac{1}{2}}$	Richtmyer-von Neumann artificial viscosity in zone j

HYDRODYNAMICS

The definition of the Lagrangian variable becomes the difference equation for conservation of mass:

$$dm_{j+\frac{1}{2}} = \frac{4\pi}{3} \rho_{j+\frac{1}{2}} (r_{j+1}^3 - r_j^3). \quad (27)$$

The equation of motion (5) is differenced explicitly:

$$\frac{u_{j+\frac{1}{2}}^{n+\frac{1}{2}} - u_j^{n-\frac{1}{2}}}{\Delta t^{n+\frac{1}{2}}} = -4\pi (r_j^n)^2 \frac{(P_{j+\frac{1}{2}}^n + Q_{j+\frac{1}{2}}^n - P_{j-\frac{1}{2}}^n - Q_{j-\frac{1}{2}}^n)}{dm_j} - \frac{GM_j}{(r_j^n)^2} \quad (28)$$

where $dm_j = \frac{1}{2} (dm_{j+\frac{1}{2}} + dm_{j-\frac{1}{2}})$ and $Q_{j-\frac{1}{2}}$ is the artificial viscosity, enabling us to treat shock waves in the conventional manner. In addition to the Richtmyer-von Neumann form of artificial viscosity,³⁵ in some cases we used a form of artificial viscosity derived by White.³⁶ Comparison of results from different forms of artificial viscosity will be presented later. From $u^{n+\frac{1}{2}}$ we can calculate the new interface radius,

$$r_j^{n+1} = r_j^n + u_j^{n+\frac{1}{2}} \Delta t^{n+\frac{1}{2}}. \quad (29)$$

We can then evaluate the change in specific volume over the time interval,

$$\left(\frac{dV}{dt}\right)_{j+\frac{1}{2}}^{n+\frac{1}{2}} = \frac{V_{j+\frac{1}{2}}^{n+1} - V_{j+\frac{1}{2}}^n}{\Delta t} = \frac{4\pi}{3 dm_{j+\frac{1}{2}} \Delta t} \left[(r_{j+1}^{n+1})^3 - (r_j^{n+1})^3 - (r_{j+1}^n)^3 + (r_j^n)^3 \right]. \quad (30)$$

In principle, all that is still needed to complete the time step is a prescription for advancing the pressure to P^{n+1} . Since pressure is a function of temperature and density, this prescription involves the energy conservation equation, but if P^{n+1} can be calculated, the entire model has been advanced by one time step.

The time step used in these integrations can depend on several parameters, but perhaps the most fundamental time step limit is the Courant condition which limits t to

Table 1. Summary of density in free-fall calculation, with 100 equal-mass zones. $\Delta\rho_{\max} \equiv \max_j |\bar{\rho} - \rho_j|$, and (Dev.) indicates the difference between $\bar{\rho}$ and the analytic solution to the free-fall equation.⁷

Time (yr)	$\bar{\rho}$ (g/cm ³)	$\frac{\Delta\rho_{\max}}{\bar{\rho}}$	Dev. (%)
0	1×10^{-11}	0	0
13.00	2.444×10^{-11}	0	0
19.45	3.755×10^{-10}	0	0.1
20.22	1.263×10^{-9}	0	0.1
21.02	3.820×10^{-7}	0	0.2
21.06	1.199×10^{-4}	0	0.2
21.066	8.161×10^{-3}	0.001	0.2
21.067	3.387	0.01	0.4
21.068	1.1×10^6	0.15	1

with $P_{j-\frac{1}{2}} = 0$ for all j . If the sphere starts with a constant density, then the analytic solution⁷ predicts that the density at any later time will also be a constant over the entire volume; that constant of course increases in time. The results of the numerical integration are summarized in Table 1.

The second problem where analytic solutions are available is for the polytropic equation of state. Temperatures and densities for the polytrope with polytropic index

Table 2. Summary for problem with polytropic equation of state, with no energy transport. $\Delta r_{\max} \equiv \max_n |r^n - \bar{r}|$.

$\frac{M_r}{M_\odot}$	$\frac{\bar{r}}{R_\odot}$	$\left(\frac{F_g - F_p}{F_g}\right)$	$\frac{r_{\max}}{\bar{r}}$
0.00012	0.025	0.007	0.09
0.00754	0.109	0.005	0.09
0.1040	0.271	0.003	0.08
0.2483	0.380	0.003	0.08
0.4399	0.488	0.002	0.08
0.6417	0.596	0.003	0.08
0.8815	0.754	0.004	0.08
1.000	1.000	0.004	0.09

the time it takes for a sound wave to traverse a zone radially. A result of differencing the force equation explicitly, the Courant time step is required to maintain the stability of the difference solutions,³⁵ and it will cause problems as will be shown subsequently.

There are two interesting problems in which the applicability of Eq. (28) can be tested by comparing analytic solutions to numerical results. The first of these is the free-fall problem, where the pressure gradient is set identically to zero in the difference equation. For the same initial conditions, we compare the solution of the differential equation (2) with the results of a computer integration of (28)

$n = 1.5$, an ideal gas with $\gamma = 5/3$, were used as input to the program. Initial interface velocities are zero, and Eq. (28) is integrated over a long period of time. After each time step, the pressure was updated with the polytropic equation of state. Results given in Table 2 show that the velocities remain negligible, the zone interfaces remain close to their equilibrium values, and the differential equation solution is accurately retained by the difference equation.

ENERGY BALANCE—RADIATION FLOW

For energy conservation, Eq. (12) is differenced implicitly, where all variables are evaluated at $t^{n+\frac{1}{2}}$ unless otherwise noted:

$$\left(\frac{\partial E}{\partial T}\right)_V \frac{T_{j+\frac{1}{2}}^{n+\frac{1}{2}} - T_{j+\frac{1}{2}}^n}{\Delta t/2} = \dot{S}_{j+\frac{1}{2}} - \left[P_{j+\frac{1}{2}} + Q_{j+\frac{1}{2}} + \left(\frac{\partial E}{\partial V}\right)_T \right] \frac{dV}{dt} \\ - \nabla \cdot \vec{F}_{\text{rad}} - \nabla \cdot \vec{F}_{\text{cond}} - \nabla \cdot \vec{F}_{\text{conv}} \quad (31)$$

If we neglect convection for the moment and use Fick's law to evaluate the radiative flux, we have

$$\left(\frac{\partial E}{\partial T}\right)_V \frac{T_{j+\frac{1}{2}}^{n+\frac{1}{2}} - T_{j+\frac{1}{2}}^n}{\Delta t/2} = \dot{S}_{j+\frac{1}{2}} - \left[F + Q + \left(\frac{\partial E}{\partial V}\right)_T \right] \frac{dV}{dt} \\ + \frac{V}{V} \left[A_{j+1} \tilde{T}_{j+1}^3 \frac{(T_{j+\frac{1}{2}}^{n+\frac{1}{2}} - T_{j+\frac{1}{2}}^n)}{\left(\frac{\Delta r}{\lambda}\right)_{j+1}} - A_j \tilde{T}_j^3 \frac{(T_{j+\frac{1}{2}}^{n+\frac{1}{2}} - T_{j-\frac{1}{2}}^n)}{\left(\frac{\Delta r}{\lambda}\right)_j} \right] \quad (32)$$

To assure continuous energy flow across zone boundaries, we evaluate the optical depths at zone interfaces in the following way:

$$\left(\frac{\Delta r}{\lambda}\right)_j = \frac{1}{2} \left[\frac{r_{j+1} - r_j}{\lambda_{j+\frac{1}{2}}(\tilde{T}_j)} + \frac{r_j - r_{j-1}}{\lambda_{j-\frac{1}{2}}(\tilde{T}_j)} \right] \quad (33)$$

This uses the "average" temperature at the zone interface to calculate optical depths in each adjacent zone, and these resulting depths are then linearly averaged. In both the above equations, the temperature used at the interface between two zones is an appropriate average of the zone temperatures. In the calculations done here,

$$\tilde{T}_j^4 = \frac{1}{2} (T_{j+\frac{1}{2}}^4 + T_{j-\frac{1}{2}}^4) \quad (34)$$

This averaging is equivalent to a linear averaging of the radiation energy density, and was chosen because radiation transport dominates all other energy transport processes for many cases of interest here. A different average will be used when convective fluxes are evaluated, but the above scheme (34) will still be used for calculating the radiative fluxes. A flux limiter¹⁷ is used in the radiation diffusion. This is very important for problems that will be calculated in which one has a hot, optically thick region surrounded by a cold, optically thin region. Without a flux limiter, diffusion could overestimate the amount of energy radiated into the cold region. The flux limiter

enables the diffusion to limit the flux to the blackbody result when the temperature gradients are very large. Numerically, inclusion of a flux limiter is equivalent to a redefinition of $(\Delta r/\lambda)_j$ in Eq. (33), and we shall assume it is included.

We solve the equation by back-substitution, a form of Gaussian elimination, finding (see Appendix A for details)

$$T_{j-\frac{1}{2}}^{n+\frac{1}{2}} = e_j T_{j+\frac{1}{2}}^{n+\frac{1}{2}} + f_j. \quad (35)$$

We require boundary conditions at the inner and outer boundaries, using:

$$\left. \frac{dT}{dr} \right|_{r=0} = 0 \quad T_{J+\frac{1}{2}} = \text{constant}.$$

Time-step controls limit the temperatures to a change of 1% in any one time step. Temperatures are then advanced to t^{n+1} by

$$T^{n+1} = T^n + 2 (T^{n+\frac{1}{2}} - T^n). \quad (36)$$

NUMERICAL MODEL FOR CONVECTION

If we now consider difference equations for the convective flux (20) where the flux limiter term for convection has been included, either in the mixing length or in the excess temperature gradient, we note that the flux must be evaluated at zone interfaces to ensure continuity of the flux, just as above in the calculation of the radiative flux. We then define

$$\begin{aligned} (c_p)_j &\equiv \frac{1}{2} \left[(c_p)_{j+\frac{1}{2}} + (c_p)_{j-\frac{1}{2}} \right] \\ \frac{\rho_j}{T_j^{\frac{1}{2}}} &\equiv \frac{1}{2} \left[\frac{\rho_{j+\frac{1}{2}}}{T_{j+\frac{1}{2}}^{\frac{1}{2}}} + \frac{\rho_{j-\frac{1}{2}}}{T_{j-\frac{1}{2}}^{\frac{1}{2}}} \right] \\ \Lambda_j &= \frac{1}{2} \left[\Lambda_{j+\frac{1}{2}} + \Lambda_{j-\frac{1}{2}} \right]. \end{aligned} \quad (37)$$

The gravitational acceleration is evaluated at the zone interface. Different averages for the above variables can be used, but for simplicity the above averages were used in all of these calculations. The remainder of the model is uncertain enough to make refinement of these quantities somewhat pedantic. For $\Delta \nabla T$ we use the approximation

$$\frac{dT'}{dr} \approx \left(\frac{dT}{dr} \right)_{\text{adiabatic}}.$$

This is strictly appropriate only near convective equilibrium. Due to radiative losses,

$$\left| \frac{dT}{dr} \right| \geq \left| \left(\frac{dT}{dr} \right)_{\text{adiabatic}} \right|.$$

and use of the adiabatic gradient will tend to underestimate the flux. For simplicity we will use this approximation for all cases. As a result, we can write

$$\Delta \nabla T_j = - \frac{(T_{j+\frac{1}{2}} - T_{j-\frac{1}{2}})}{(r_{j+\frac{1}{2}} - r_{j-\frac{1}{2}})} - \left(\frac{\Gamma_2 - 1}{\Gamma_2} \right) \frac{T_j}{\lambda_{P_j}} \quad (38)$$

where Γ_2 is the second adiabatic exponent and λ_P is the pressure scale height. Since Γ_2 and λ_P are zone-centered variables, we average them at the zone interfaces as for the other zone-centered variables in (37). In principle, we should also average $T_j = \frac{1}{2}(T_{j+\frac{1}{2}} + T_{j-\frac{1}{2}})$, but in practice calculation of fluxes this choice tends to result in numerical instabilities.³⁷ The solution here is to use a technique called "upstream differencing" to ensure stability, using $T_j = T_{j-\frac{1}{2}}$ or $T_j = T_{j+\frac{1}{2}}$, whichever is larger. In effect, this means that one uses that temperature where the flux originates; since the sign of the second term in (38) is negative, this choice will somewhat reduce the flux. In our calculations convection will be radially outward, so $T_j = T_{j-\frac{1}{2}}$. Thus we rewrite (38)

$$(\Delta \nabla T)_j = - \frac{(T_{j+\frac{1}{2}} - T_{j-\frac{1}{2}})}{(r_{j+\frac{1}{2}} - r_{j-\frac{1}{2}})} - \left(\frac{\Gamma_2 - 1}{\Gamma_2} \right) \frac{T_{j-\frac{1}{2}}}{\lambda_{P_j}}. \quad (39)$$

Since the flux term involves $(\Delta \nabla T)^{3/2}$, we linearize by setting

$$(\Delta \nabla T)^{3/2} = (\Delta \nabla T)^{1/2} (\Delta \nabla T).$$

We include this term in the implicit solution of the total energy balance, so we evaluate $(\Delta \nabla T)^{1/2}$ at t^n , and $(\Delta \nabla T)$ at t^{n+1} , producing a flux that is linearized in the solution variable $T^{n+\frac{1}{2}}$. We then include the convective flux in (31) in the following form:

$$-\nabla \cdot \vec{F}_{\text{conv}} = \frac{v}{V} \left[\delta_{j+1} A_{j+1} b_{j+1} (\Delta \nabla T)_{j+1}^{n+\frac{1}{2}} - \delta_j A_j b_j (\Delta \nabla T)_j^{n+\frac{1}{2}} \right] \quad (40)$$

where

$$b_j \equiv \left[\frac{(\Delta \nabla T)^{1/2}}{4\sqrt{2}} c_{P_j} g_j^{1/2} \Lambda_j^2 \frac{\rho_j}{T_j^{1/2}} \right]^n \quad (41)$$

and $\delta_j = 1$ if $(\Delta \nabla T)_j \geq 0$, and $\delta_j = 0$ if $(\Delta \nabla T)_j < 0$. Expanding $\Delta \nabla T$, we have

$$\begin{aligned}
-\nabla \cdot \vec{F}_{\text{conv}} = \frac{v}{V} \left\{ A_{j+1} b_{j+1} \delta_{j+1} \left[\frac{(T_{j+1}^{n+\frac{1}{2}} - T_{j+2}^{n+\frac{1}{2}})}{\Delta r_{j+1}} + \left(\frac{\Gamma_2 - 1}{\Gamma_2} \right)_{j+1} \frac{T_{j+1}^{n+\frac{1}{2}}}{\lambda_{P_{j+1}}} \right] \right. \\
\left. - A_j b_j \delta_j \left[\frac{(T_{j+\frac{1}{2}}^{n+\frac{1}{2}} - T_{j-\frac{1}{2}}^{n+\frac{1}{2}})}{\Delta r_j} + \left(\frac{\Gamma_2 - 1}{\Gamma_2} \right)_j \frac{T_{j-\frac{1}{2}}^{n+\frac{1}{2}}}{\lambda_{P_j}} \right] \right\}. \quad (42)
\end{aligned}$$

This term is added to (32) to provide the difference equation form of Eq. (12).

SOLUTION OF THE DIFFERENCE EQUATIONS

The structure of the protostar at time t^n can be described by specifying $\{r_j^n, T_{j+\frac{1}{2}}^n, dm_{j+\frac{1}{2}}^n, u_j^{n-\frac{1}{2}}, t^n\}$. The remaining variables can be calculated as functions of these, including P, Q, V, E, λ, S , and any others (see List of Symbols). Equation (28) is used to advance all of the velocities to time $t^{n+\frac{1}{2}}$. Then Eqs. (29) and (30) are used to advance r and V to time t^{n+1} . The assumption is that $u^{n+\frac{1}{2}}$ is the average velocity over the time interval (t^n, t^{n+1}) . v^{n+1} and V^n are then used to calculate

$$\frac{dV}{dt} = \frac{V^{n+1} - V^n}{\Delta t^{n+\frac{1}{2}}}.$$

Finally, using temperatures extrapolated to $t^{n+\frac{1}{2}}$ (see Appendix A for details) to calculate $P^{n+\frac{1}{2}}$, energy derivatives, and optical depths, the temperatures are advanced to t^{n+1} using equations (35) and (36). This completes the integration procedure from t^n to t^{n+1} . At each time step an energy "check" is performed, a verification that energy has been conserved for that time step. The time step for the next integration step is then chosen on the basis of the previous time step. These include limitations that temperatures, volumes, and internal energies in each zone not change by more than some fraction during any time step. There are a number of criteria which must be satisfied and terms which must be defined in a particular manner in order to exactly conserve energy for the difference equations being used. These questions and an examination of the truncation errors are dealt with in detail in Appendices C and D. For our purposes here we shall assume that, subject to time-step limitations, Eqs. (28) and (31) give reliable solutions.

IV. Implicit Hydrostatics

INTRODUCTION

As calculations followed the time evolution of each model, several features appeared in almost every case. A core formed at the center of the collapsing protostar. The properties of the core varied from model to model, depending on a number of factors; these will be discussed in detail in a subsequent section of this paper. However, general features of all of the cores were similar. All were much more dense and had higher temperatures than the infalling outer shells, and in each case the material in the core was in hydrostatic equilibrium. Due to the numerical nature of the solutions, zone interface velocities are never identically zero, but the velocities of zones in the core approach zero as the zone boundaries oscillate about their equilibrium values. To see the sharp distinction between zones in the core and those outside it, Table 3 gives a number of characteristic parameters for a protostar model of $1 M_{\odot}$ at a time soon after the core had been formed. Figure 4 shows the density and temperature as functions of radius for that model. The quantities tabulated as being illustrative are density, temperature, pressure, Rosseland mean free path for radiation, $\Delta t_{\text{Courant}}$, t_{ff} , zone interface velocity, kinetic energy per zone, specific internal energy, and the ratio of gravitational to pressure force. It is easy in this case to determine the extent of the core and to make several observations. The hydrostatic equilibrium condition in the

Table 3. Comparison of representative quantities inside and outside the protostellar core.

	Inside core	Outside core
ρ (g/cm ³)	10^{-5}	10^{-18}
T(°K)	10^4	10
λ_{Ross} (cm)	10^2	10^{17}
Δt_C (sec)	10^5	10^{11}
t_{ff} (sec)	10^{10}	10^{16}
v (cm/sec)	10^{-6}	10^{-3}
Kinetic energy (ergs)	10^3	10^9
Internal energy (ergs)	10^{-3}	10^{-7}
$ \frac{G}{P} $ forces	1	3

core means that at least one of the necessary conditions for inclusion of our convective energy transport model is certainly satisfied; whether the core is convective or not will depend on other parameters as well. There is a strong density and pressure gradient at the boundary of the core, equivalent to the conditions in a spherical shock wave. This shock wave moves outward through the Lagrangian grid, but actually remains almost stationary in radius. Infalling material passes through the shock and is heated and compressed. The material behind the shock wave, in the core, very quickly achieves hydrostatic equilibrium. This can be seen by calculating the free-fall time t_{ff} for zones in the core. The factor $\rho^{-1/2}$ means that t_{ff} is smaller by many orders of magnitude for the core, and as explained earlier, t_{ff} is roughly the time scale for restoration of hydrostatic equilibrium. Another approach to the same conclusion is to evaluate the

Fig. 4. Structure of one $M_{\odot}$ model at the time of core formation.

Courant time-step limit, roughly the time required for a pressure wave to traverse a zone. The smaller zone thicknesses and the higher densities in the core combine to create the great difference in Δt_C between the core and the outer material. As a result, the core has some of the properties of a resonating sphere. Spherical sound or pressure waves traverse the core and these motions persist for many periods before the artificial viscosity and the energy fluxes can damp them out. If time steps larger than the Courant time step were used, hydrodynamic instabilities would arise from these spherical pressure waves and errors would destroy the calculations. Comparison of t_{ff} for the low-density outer portions of the protostar with the Δt_C of the core show that we have arrived at a computational crisis. The exact numbers vary from model to model, but a rough estimate might be that it would require 10^6 or $10^7 \times \Delta t_C$ (core) to equal t_{ff} for the outer material, which might amount to more than 1000 hours of computer time, clearly an unreasonable amount of time. It should be emphasized that this problem arises from the use of explicit hydrodynamics. The work of Hayashi^{9,14,15} on the collapse of protostars uses explicit hydrodynamics, but the initial conditions in those calculations, including the use of more mass in zones near the center than farther from the center and the high degree of central condensation, have been chosen in such a way that the disparity in time scales is much less severe. It is important to see whether those initial conditions are realistic in a physical sense; they have the important advantage of allowing the evolution to be calculated using a very simple set of difference equations. For different, more realistic initial conditions, we find the conditions of Fig. 4. To further evolve that model we must deal with the time-step question.

In a physical sense, the equilibrium in the core is a product of a balance between several processes. There is release of gravitational potential energy by contraction; from the virial theorem we can relate the rate of gravitational energy release to the rate of heating of the core and its radiation of energy. Also, the core is transporting energy by radiation and convection out to the core boundary and, depending on values used for opacities, radiating from the surface of the core into or completely through the infalling cloud. There is also compressional work being exchanged by zones in the core. A balance of all of these processes determines the hydrostatic equilibrium. If any one of these processes is not important or has a much longer time scale than the others, it could either be ignored or its effect could be calculated separately. In practice, none of these processes is negligible in all cases; therefore, it was decided to include all processes all the time.

Numerically, the time-step problems, caused by the explicit differencing of Eq. (5), suggest that an implicit solution might be preferable. Since the pressures in (5) are functions of temperature and density, the energy equation, already differenced implicitly, must be included and solved simultaneously with the force equation. The following development was suggested by the treatment of Cox *et al.*³⁸ The principal differences in the following treatment are the way radiation pressure is included and in the inclusion of convective energy transport. In those cases where convection is

important, the time scale for convection can be comparable to the hydrodynamic time scale. Therefore, if the convective transport were solved separately, it would be difficult to avoid a "push-pull" type of instability where two or more operators are competing. Solved together, there is very little change in the equilibrium positions, but each operator taken separately would cause a large change in zone position and temperature.

EQUATIONS FOR IMPLICIT HYDROSTATICS

We begin by assuming that during the interval (t^n, t^{n+1}) the force balance has been slightly perturbed. Then

$$\begin{aligned} r_j^{n+1} = r_j^n + \frac{\partial r_j^n}{\partial r_{j-1}^n} (\Delta r)_{j-1}^{n+\frac{1}{2}} + \frac{\partial r_j^n}{\partial r_j^n} (\Delta r)_j^{n+\frac{1}{2}} + \frac{\partial r_j^n}{\partial r_{j+1}^n} (\Delta r)_{j+1}^{n+\frac{1}{2}} \\ + \frac{\partial r_j^n}{\partial T_{j-\frac{1}{2}}^n} (\Delta T)_{j-\frac{1}{2}}^{n+\frac{1}{2}} + \frac{\partial r_j^n}{\partial T_{j+\frac{1}{2}}^n} (\Delta T)_{j+\frac{1}{2}}^{n+\frac{1}{2}}. \end{aligned} \quad (43)$$

We require

$$r_j^{n+1} = 0.$$

In the energy equation, the only change from the earlier solution is that dV/dt is now evaluated implicitly rather than from the results of the explicit force equation. We difference dV/dt :

$$\frac{dV}{dt} \rightarrow \frac{(\Delta V)_{j+\frac{1}{2}}^{n+\frac{1}{2}}}{\Delta t} = \frac{1}{\Delta t} \frac{4\pi}{\Delta m_{j+\frac{1}{2}}} \left[(r_{j+1}^n)^2 (\Delta r)_{j+1}^{n+\frac{1}{2}} - (r_j^n)^2 (\Delta r)_j^{n+\frac{1}{2}} \right] \quad (44)$$

where

$$\begin{aligned} (\Delta r)^{n+\frac{1}{2}} &= r^{n+1} - r^n \\ (\Delta T)^{n+\frac{1}{2}} &= T^{n+1} - T^n \end{aligned}$$

and using a more general form of back substitution,

$$\begin{aligned} r_j^{n+1} &= H_j T_{j+\frac{1}{2}}^{n+1} + I_j r_{j+1}^{n+1} + J_j \\ T_{j-\frac{1}{2}}^{n+1} &= E_j T_{j+\frac{1}{2}}^{n+1} + F_j r_j^{n+1} + G_j. \end{aligned} \quad (45)$$

We solve for $\{\rho, T\}^{n+1}$ which simultaneously satisfies the requirements of hydrostatic equilibrium and implicit energy balance, including radiation flow and convection. These equations are solved iteratively since the coefficients in the coupled equations are nonlinear functions of density and temperature. Further, since the zones in the core coupled so strongly hydrodynamically, an under-relaxation technique is used for updating ρ and T . A complete derivation of the difference equations as well as a description of the iteration and under-relaxation method are included in Appendix E.

DISCUSSION

By solving both equations implicitly, we have gained an important advantage in that we are no longer limited by a Courant time step. What has been sacrificed is the ability to follow the detailed motion of pressure waves in the higher density core. Over a large number of periods, the oscillation of the core due to these waves was found to have no effect upon the time-averaged configuration of the core. These shock waves are the means by which the remaining kinetic energy in the core material is converted to thermal energy through the dissipative action of the artificial viscosity. In the program, the kinetic energy lost by neglecting zone boundary motion is calculated and included in the $\dot{S}$ term as an energy source in the energy equation.

A limitation we have introduced into the calculation is a result of setting the accelerations to zero. If, during the subsequent evolution, some process should destroy the hydrostatic equilibrium, the assumptions of this treatment would be invalidated. There are two ways in which this could happen. A sudden change in boundary conditions could so sharply perturb the core that it could be significantly displaced from equilibrium. This problem is greatly reduced by a careful choice of time steps during periods when material is being accreted onto the core. It should be noted that this problem is strictly a numerical one; the hydrodynamic time scales in the core are so small compared with even the reduced time steps during mass accretion that the core immediately resumes an equilibrium configuration. We must choose smaller time steps so that the iteration scheme will converge to the correct solution. The second process that could affect the equilibrium would be the sudden introduction of a new $\dot{S}$ energy source into Eq. (34). Physically, this might arise due to initiation of nuclear reactions. Most of the possible reactions are so sensitive to temperature that if the temperature in some portion of the core reached a value where reactions were possible, a strong energy source might result, in which case it would be necessary to examine the solutions very carefully. Temperatures in the calculations to follow remain less than 10^6 °K in all but the last model calculated, far below that required for most important reactions, so the only possible energy source would be from the burning of deuterium or lithium, normally of such low abundances that they would not provide a significant amount of energy in comparison with that provided by gravitational contraction. Therefore, we shall assume that the approximation of zero acceleration is a reasonable one and that

our set of coupled difference equations should be applicable. The additional problem encountered when the protostellar core is still surrounded by an infalling gas cloud is that some portions of the protostar are subject to implicit hydrostatic equations while explicit hydrodynamics should still be used in other portions. The general strategy for matching solutions in the two regions is also discussed in Appendix E.

V. Applications to Protostellar Models

INTRODUCTION

A wide variety of protostellar models has been included in this study. The general strategy was to use the simplest possible numerical models as long as they were reliable. In fact, most of the more sophisticated numerical schemes described above were devised after it became obvious that the simpler methods were no longer adequate. For example, the need for using an implicit form of differencing for the hydrodynamics arose because the Courant time step eventually became prohibitively short in some portions of the protostar. An interim difference scheme, not described here, was used for some time to solve only the hydrodynamics of the protostellar core. The energy balance was still done implicitly, but was not coupled to the hydrodynamics. This scheme was suitable until the opacity of the core became low enough for radiation transport to have an appreciable effect on the equilibrium in the core. Instabilities then arose due to the competing effects of the hydrodynamics and energy flow, requiring further work that led to the numerical method described above. As a further example, the flux limiter in the treatment of convection became necessary when the expressions for the convective flux became absurdly and unphysically high.

In order to examine as many independent parameters as possible, a large number of problems were run, most of them on a CDC 7600 computer. A typical problem consumed between 40 and 60 min of computer time, so many of the steps taken in developing improvements to the numerical methods were made in the interests of saving computation time.

The most important independent parameter considered was total mass. The range of masses considered was from $0.01 M_{\odot}$ to $50 M_{\odot}$, with some emphasis on $1 M_{\odot}$. This range, while perhaps somewhat too broad to hope for really full understanding of the entire range, has a number of very interesting features. First, most of the range of observed stellar masses is included. The $0.01 M_{\odot}$ case is almost certainly lower than the lower limit for main-sequence stars.³⁹ Also, with the greatly oversimplified expression used [Eq. (1)] for initial density as a function of total mass, it is highly questionable whether so dense an object as would be required for our initial conditions could actually be formed. However, arguing that dense objects might be formed in

some manner, whether by fragmentation of larger clouds, by dynamical effects, or some other means, it was decided to include such a problem if only for pedagogical reasons. Other problems of $0.1 M_{\odot}$ and $0.2 M_{\odot}$ were also run. All the protostellar models for masses less than or equal to $1 M_{\odot}$ turn out to be convectively unstable throughout a varying fraction of the protostar's volume. The effects of convection are an interesting problem by themselves. A number of models with masses greater than $1 M_{\odot}$ were also run, including $5 M_{\odot}$, $10 M_{\odot}$, and $50 M_{\odot}$. For these cases convection never becomes very important, so uncertainties of the convection model could be avoided. It became evident that a new effect appeared in the $50 M_{\odot}$ problem that was not present in problems below about $5 M_{\odot}$. Subsequent runs showed that in the $50 M_{\odot}$ model and, to a lesser degree in the $5 M_{\odot}$ and $10 M_{\odot}$ cases as well, radiation pressure dominates the material pressure in the outer portions of very massive protostars, creating new situations of interest. Also, in the cores of very hot, very massive protostars, including only the $50 M_{\odot}$ case here, radiation pressure also dominates the pressure of the protostellar core. Since one of the purposes of this entire project was to develop a versatile computer program that could handle widely disparate configurations, the choice of this large range of initial masses was very desirable. As was mentioned earlier, the question of how the "blob" of gravitationally bound gas arrived at our chosen initial conditions is not answered here. The only way in which that question enters into our calculations is in the choice of appropriate boundary conditions. One model was run in which the required density-mass relationship was not satisfied; fortunately no protostar was formed in that case, as will be described below.

Two types of initial conditions were considered. These may be categorized as polytropic initial conditions and uniform initial conditions. The motivations for choosing one set of initial conditions are quite different from those for choosing the other. The real question that must be dealt with involves some sort of a priori knowledge of the state of the interstellar medium out of which the protostars form. If single, isolated, gravitationally bound globules are formed by thermal instabilities, a Jeans-type analysis would suggest that the cloud might possess a moderate radial density gradient. If fragmentation in an already collapsing gas cloud is instead responsible for protostar initiation, it is unclear what types of initial conditions would be most common. It is clear, however, from a consideration of the expression for the free-fall time that the initial density distribution must be nearly constant. If this were not true, the portions of the cloud which were of higher density would collapse independently into separate entities. If the cloud had a centrally-peaked density distribution, it will be shown that under many conditions it could have evolved from a uniform cloud and can be described by our model. And if the density distribution is sufficiently non-uniform, the free-fall times would differ by so large an extent that the evolution of the denser portions would be complete before the less dense portions had moved at all.

There is a considerable body of theoretical work based on polytropic configurations. Most of this is due to historical reasons, since before the advent of large computers polytropes were among the few models for which stellar structure equations

could be solved analytically. The work of Hayashi and his collaborators^{6,9,14,15} assumes that the initial density distribution for calculating the evolution of a protostar is essentially polytropic. That is a very fortunate assumption since with their use of explicit hydrodynamics formally coupled to the energy balance calculation only through the dV/dt and Q terms, those are among the few problems which can feasibly be completed on a computer. Nearly all other initial conditions run into time-step difficulties through the Courant time step. The central density of about 10^{-11} g/cm³ used by Hayashi in those papers is far too high to have resulted from any reasonable physical mechanism, so it must be due to prior evolution. Similarly, the initial velocities in their calculations are free-fall velocities. It is unlikely that such velocities could be maintained after the material becomes opaque to radiation at a density of about 10^{-13} g/cm³, when material pressure gradients begin to appear. We shall see from the results of calculations with uniform initial conditions that the subsequent evolution is highly non-homologous, suggesting that the polytropic configuration of Hayashi could not have resulted from an initially uniform density distribution.

Polytropic initial conditions have been included for several reasons, in spite of such questions concerning their validity. First, the work of Hayashi provided a very useful standard against which to compare our results to be sure any possible errors in the program had been eliminated. Secondly, the calculation of problems with these initial conditions could be done quickly in terms of computer time. Also, several of the numerical schemes had free parameters included which required "calibration," and these problems were convenient test vehicles. Included among such parameters were those in the implicit hydrostatics and in the convective energy transport.

The other class of initial conditions that were included were those with initially uniform conditions in temperature and density. For a uniform spherical mass of constant temperature, there is a critical density (or equivalently a critical radius) for a given mass to be gravitationally bound. For densities smaller than that critical value, the sphere will not remain bound. Larson³⁴ points out that, since the pressure is a function of density and temperature only, any uniform sphere will begin to collapse at the free-fall rate since $dP/dr = 0$, except possibly at the outer boundary. This free fall continues until a pressure gradient is established that is sufficient to retard or reverse the collapse. For densities that are less than this critical density, the pressures rise fast enough to retard the collapse at very early stages. The value of the critical density as a function of mass can be estimated by a number of arguments, similar to the Jeans analysis⁴⁰ or dimensional analysis.⁴¹ For further discussions of this point, see Bonner.¹² For our initial conditions, the initial density is chosen from Eq. (1),

$$\rho_c = \rho_0 \left(\frac{M_\odot}{M} \right)^2. \quad (46)$$

Following Larson, a value of 10^{-19} g/cm^3 was used for ρ_0 , with an initial temperature of 10^4 K . Larson also includes models with higher initial temperatures, not included here, and for those models ρ_0 would be somewhat changed; the evolution does not appear to depend much if at all on the initial temperature of the cloud, so that dependence was not included in our study. At these temperatures and densities the opacity is at its floor value in every case, so the Rosseland mean free path is $(\kappa_0 \rho_c)^{-1}$, and in every case this meant that the sphere was initially transparent to radiation.

EARLY STAGES OF THE COLLAPSE

Until the pressure gradient in the protostellar cloud becomes appreciable, free-fall conditions will prevail and the material in the sphere will accelerate radially inward. A number of things will happen during this collapse, depending somewhat on the initial density and which set of initial conditions was used. As the material in the protostellar mass moves inward, it releases gravitational potential energy. This energy is radiated away as long as the surrounding material remains optically thin.

In the polytropic case, the centrally peaked density distribution will cause the free-fall time at the center to be shorter than at the outside of the sphere. Therefore, the central region will collapse at a much faster rate than will the outer portion. As the center continues its collapse, the increasing density eventually causes a rise in the opacity until the central portion is radiatively opaque. After this stage is reached, the gravitational potential energy that is being released by the collapse cannot be effectively radiated away from the core. The temperature begins to rise quickly and some portions of the protostar come, at least temporarily, into equilibrium. However, first the dissociation of hydrogen molecules and later the ionization of hydrogen and helium reduce the effective γ of the gas below the critical value of $4/3$, and the free fall then continues until most of the material is ionized. Finally, the pressure increases enough to establish a stable hydrodynamic equilibrium in the central portion of the sphere. All this has occurred on a time scale which is short compared with the free-fall time of the material farther from the center of the sphere. At this point, usually the core that has been formed is optically very thick while the surrounding material has an optical depth much less than unity. As a result this core, which is considerably hotter than the surrounding gas, radiates from its outer surface through the exterior portion of the cloud into space. The total luminosity of the still collapsing protostar is being supplied by this hotter core, but the radiation is considerably reddened spectrally by absorption and re-emission in the cooler portions of the cloud. It is this core, with its effective temperature somewhat modified, which first appears in the HR diagram. Energy transport in the core may be important, depending on how high the opacity rises and what the temperature distribution becomes. The free-fall time in the core of the protostar is now many orders of magnitude shorter than in the outer portion. Therefore deviations from equilibrium in the core, as well as shock waves due to infalling

material, cause a multitude of transient perturbations in the core. Since these perturbations are incidental to the further evolution of the remainder of the protostar, they are damped out numerically by using the implicit model described in Chapter IV. The remaining material in the surrounding cloud falls into the core, passing through the same transitions as the core material.

Conditions at the core boundary are equivalent to a spherical shock wave (see Table 3). As material falls into the core, this shock wave heats the material as it compresses. The shock wave moves radially outward in a Lagrangian sense; because of the density disparity the shock wave is nearly stationary in space and the process is much the same as a rapid accretion phase.

One feature of the evolution of the polytropic problem that does not occur for uniform problems is a result of the high initial density in the former case. This means that the outer portions of the protostar become optically thick some time before they fall into the core. They then absorb a substantial amount of radiation and heat up before they encounter the shock wave. The shock wave then heats up this material further to its final value. This radiation front, preceding the shock front and causing the initial flareup of the protostar, is due simply to the high initial density of the protostellar cloud which in turn reduces the radiation mean free path in the infalling cloud.

The uniform initial conditions produce a number of qualitative differences from polytropic conditions. Larson³⁴ has described some of the general features of this evolution. The earliest feature is that, since the density is uniform, the free-fall time is the same for the entire spherical mass. The only means of disturbing the free fall is the propagation of the effects of the outer boundary condition through the collapsing sphere. For all of the boundary conditions considered, this produced a spherical rarefaction wave that traveled radially inward at the local sound speed. Material inside this wave continues to follow the analytic free-fall curve⁷ until disturbed by the rarefaction. The early structure of the protostar is determined by how far the free fall has progressed by the time the rarefaction wave reaches the center of the sphere. If the rarefaction wave reaches the center quickly enough and a stable pressure gradient is established before the free fall has progressed far enough, then the collapse will stop. The sphere will oscillate about that equilibrium but will not evolve further. That density at which the center of the sphere is marginally unstable to gravitational collapse when the rarefaction reaches the center is another way of defining the critical density for a given mass. One model was calculated with a density less than this critical density to verify the above reasoning and will be further discussed below. With this one exception, all the problems included here equalled or exceeded this critical density.

Until the density of the infalling material reaches about 10^{-13} g/cm³, the collapse is very nearly isothermal due to the low opacity of the gas. If the low-temperature opacity of the gas is increasing from 0.01 cm²/g to 1.0 cm²/g this density at which the material begins to be opaque increases to 10^{-15} g/cm³, but this has little effect on the further evolution of the protostar. From this stage on, the collapse becomes increasingly adiabatic. The free-fall time becomes far shorter than the characteristic

time scale for radiative transport. There is a thin spherical shell at the outer limit of this optically thick center which radiates into the surrounding gas cloud. This shell is the source of the protostar's luminosity. The core thus formed has an effective temperature and can now be regarded as an essentially stellar object. The source of its luminosity is the release of gravitational potential energy and its effective temperature observationally can be modified by the surrounding cloud, but with these limitations it is still possible to locate the protostar in the HR diagram.

If the optical depth of the surrounding cloud is high enough, it is possible for the radiation pressure from the protostellar core to reverse the infall of some of the surrounding material. For several reasons this phenomenon occurs principally for protostars with masses greater than $1 M_{\odot}$, as will be discussed below. This effect could contribute to the low number of very massive stars which are observed.

In all of the calculations performed, the numerical scheme described in Chapter IV was employed from the time of formation of the core until the completion of the evolution.

GROWTH OF THE PROTOSTELLAR CORE

Two major processes contribute to the further evolution of the core, which is in hydrostatic equilibrium. First, the material in the core radiates from its outer surface and must therefore contract to maintain this hydrostatic equilibrium. The time scale for this contraction is the Kelvin contraction time; in the absence of other influences this contraction would eventually result in a normal star appearing on the main sequence. The second process which takes place is the gradual addition of mass from the remainder of the protostellar cloud to the core. An important feature of these calculations is the manner in which this addition is done. While the core evolution is done implicitly, the mass infall is calculated explicitly, using the complete set of difference equations of Chapter III. This is different from the procedure used by Larson and most other stellar evolution calculations which add mass by means of an empirical, analytic mass accretion rate formula. If the infall of material could be retarded or reversed by radiation or material pressure, it is important to actually perform hydrodynamic calculations for the motion of that material instead of adding that material analytically.

An algorithm is needed to decide when a given mass shell has become part of the core and to include its motion in the implicit core calculation. This is necessary because as that zone increases in density and decreases in radial thickness, the Courant time step for that zone becomes very short, slowing down the calculation of the rest of the problem. In these calculations the algorithm included a test on the density and velocity of the zone. These additional tests were required because in many cases a temporary force balance is achieved, even though the zone is moving rapidly inwards and is still far from its equilibrium radius and density. This occurs just before the zone temperature and density reach a dissociation or ionization region. The gamma of the material in the zone is greater than $4/3$ and the pressure is increasing rapidly. If

the zone were not moving inward at an appreciable velocity, the material would come to rest at this point. However, the density and opacity continue to increase, the temperature rises, and the material passes into a dissociation or ionization region where the gamma of the gas falls below $4/3$ and the gas is again gravitationally unstable. It is this temporary pseudo-equilibrium which is the source of what Larson calls a first protostellar core. As can be seen from the radial velocities it is not a core from the standpoint of being a static configuration, and the small shock that is produced occurs because material immediately outside this region is still being accelerated inward while material in this pseudo-equilibrium region is moving at constant velocity. The piling-up of material between these two regions is what Larson refers to as the shock wave from the first protostellar core. This "shock" or compression region does travel outward through the infalling mass, but this is superimposed on a uniform inward velocity of the gas. The shock moves outward through the gas, but remains nearly fixed in radius. This is an extremely weak and short-lived phenomenon, and is described here only to show how it derives from the equation of state behavior near dissociation and ionization regions. This next zone to be added passes through the true shock region (see Table 3) and by the time it approaches the conditions in the core it is almost completely ionized. The core contracts slightly as this zone imparts most of its kinetic energy to the core. Shock waves in the core convert that energy into thermal energy. The new zone is allowed to oscillate several times about its equilibrium position in an explicit mode to be certain that it will not be ejected or blown off from the core. At that point it is formally added to the core, and its remaining kinetic energy is added uniformly to the core zones. Since the core exhibits a strong collective behavior hydrodynamically, any kinetic energy would quickly be spread throughout the core in a complicated way; depositing excess kinetic energy uniformly through the $\dot{S}$ term in Eq. (31) is probably adequate.

The above sequence is followed as zone after zone is added to the core. This replacement of what is physically a continuous process by a numerical and necessarily discrete process causes a slight oscillatory behavior in some of the macroscopic parameters of the core, such as the effective temperature, core radius, and luminosity. For convenience, these variables have been smoothed before plotting or tabulation.

The addition of mass to the core ceases when the mass of the surrounding cloud has all been added to the core or has been blown off. The remaining calculations are fully implicit, and involve the further evolution of the core that has been formed. In some cases, the core has evolved considerably between the time of its initial formation and the time the last zone has been added to the core. The extent of this further evolution during the accretion phase is important because it has been suggested⁴² that under some circumstances the core can have even evolved into the nuclear burning period before all the mass has been accreted. This might be a possible explanation for the existence of many stars with extended cold envelopes and encircling gas clouds. This point will be examined in the discussion of the $50 M_{\odot}$ model.

EVOLUTION OF THE STABLE PROTOSTAR

The further evolution of the protostar has been calculated in some of the cases below. This phase of the evolution is not the principal interest in this project and is done in a more complete fashion by other authors. However, it was possible to examine the general features of the subsequent evolution and the way they were affected by several factors. The most significant result of this study was that the track of the protostar in the HR diagram was very sensitive to the amount of convection in the envelope of the protostar. This was studied by varying the constant in the convective flux limiter in Eq. (23). If the convective flux was reduced below some minimum value, the effective temperature and luminosity of the protostar fell rapidly below the values indicated by other authors for time-independent models; for the same model when the convective flux is higher than some other value, the effective temperature and luminosity begin to rise immediately after the core has been formed and rise much too quickly. Part of the uncertainty in these calculations is due to the inadequacy of the present model in the protostellar atmosphere. This question is discussed in Appendix B. However, some of the behavior must be due to a real dependence of the luminosity on convection in the envelope. We can therefore use these evolutionary tracks, together with published evolutionary calculations in which the envelopes are done more carefully but in which the time evolution is more crudely done, to determine a "best" value for the input parameter in the convective flux. This problem does not arise until the infall of material onto the core is complete, since the temperature change from compressional heating in the shock region is larger than even a large convective flux could provide. The constant that is determined in this manner is different for different mass protostars, generally increasing monotonically with mass. This can be understood by noting that in small (i.e., less than $1 M_{\odot}$) stars convection is relatively more important than in larger stars. The superadiabatic temperature gradients in the envelopes of these low mass stars are proportionately larger than in heavier stars, so the overestimate of the convective flux is greater in the smaller stars. In protostars larger than about $3 M_{\odot}$, the effects of convection are nearly negligible.

PROTOSTELLAR MODELS CALCULATED

The numbers assigned to each evolutionary calculation are for expository purposes and are not a reflection of the order in which the problems were run. In fact, the descriptions are generally in order of increasing mass, but the polytropic cases are described first.

Polytropic Models $1 M_{\odot}$

Case 1

This problem has an initial polytropic density distribution, and is identical to the initial conditions for Hayashi's $1 M_{\odot}$ calculations for polytropic index $n = 1.5$. Of course, since a real equation of state is used and not a polytropic equation of state, the polytropic index indicates only the degree of initial central condensation. This problem was actually done twice, the first time with an initially uniform temperature and later with a centrally peaked temperature distribution. The two were very similar, and only the latter is described here. The initial density and temperature distributions are given in Fig. 5. The free-fall time is not a constant because of the non-uniform density, but for the central region it is approximately 5 years. The evolution to approximately 5 years is calculated using explicit hydrodynamics and implicit energy balance. At that

Fig. 5. Initial structure for polytropic models.

time the central core has formed, comprising about 15% of the total mass. Most of the material in the core is ionized, with temperatures in the range of 10^4 to 10^5 K. From this point on, the core was evolved by means of the implicit hydrostatics method while the infalling material was still done explicitly. In the core, the constant f_1 in the convection, Eq. (23), was chosen to be 1000.

The evolution closely followed the path calculated by Narita *et al.*¹⁵ with a large flareup at approximately 6.9 years due to the diffusion of photons from the hot central core through the infalling cloud. It should be emphasized that the entire phenomenon of the heating of the infalling cloud is in large measure due to the relatively high densities in that material, of the order of 10^{-11} g/cm³. If these densities were much lower, the radiation from the core would pass directly through the infalling gas with essentially no absorption.

In these calculations the maximum luminosity was reached at approximately 8.2 years. This precedes the arrival of the shock front at the surface which occurs at approximately 9.0 years. All of these times are somewhat later than those found by Hayashi, but the deviations are probably not important to the general features of the evolution. The structure of the protostar at the onset of hydrostatic equilibrium agrees very closely with that calculated by Hayashi. The time variation of the luminosity is given in Fig. 6. Because the explicit calculation had gradually given way to an implicit one as the core came to include all of the mass, it was possible to follow the evolution of this model far beyond the point at which Hayashi was forced to abandon it. We find that following the addition of all of the mass to the protostellar core, the protostar contracts and cools. This stage was followed for about 3×10^4 years, at which point the structure was that given by Table 4. The convective region, which at the onset of

Table 4. Final configurations for polytropic cases. Column 1 is for Case 1 and Column 2 is for Case 2, with enhanced convection.

$\frac{M_r}{M_\odot}$	$\frac{r}{R_\odot}$		$-\log \rho$ (g/cm ³)		$\log T$ (°K)		$\frac{L_{\text{conv}}}{L_{\text{rad}}}$	
	1	2	1	2	1	2	1	2
0.0081	0.147	0.146	0.53	0.53	5.14	5.14	0	0
0.0149	0.948	0.944	1.87	1.86	5.17	5.17	0	0
0.0603	3.00	2.98	2.83	2.82	5.23	5.24	0	0
0.1462	6.22	6.09	3.35	3.34	5.16	5.16	220	225
0.2706	10.9	10.9	4.00	4.00	5.17	5.17	0	0
0.4228	18.2	18.3	4.52	4.52	5.07	5.07	0	0
0.5836	27.8	27.8	5.00	5.00	4.97	4.97	0	0
0.7336	39.1	38.9	5.40	5.39	4.82	4.83	0	0
0.8550	51.9	51.4	5.83	5.80	4.69	4.70	0	0.0005
0.9387	65.8	63.9	6.24	6.16	4.51	4.50	0.002	1.44
0.9864	75.6	71.9	5.75	6.14	3.64	4.05	0.086	49.3
0.9983	75.9	72.9	5.77	5.90	3.08	3.22	0	0.56
1.000	76.2	73.3	6.58	6.70	2.81	3.01	0	0

Fig. 6. Luminosity for polytropic models.

hydrostatic equilibrium included only the outer 10% of the mass, finally has grown to include the outer 20% of the mass. Presumably this region will grow inwards as the slow contraction continues, as outlined by Hayashi.⁹ Figure 7 shows the evolutionary track of the protostar. The general features of this path agree substantially with Hayashi, with the final phase of this calculation showing the predicted fall in luminosity at a nearly constant effective temperature. From the onset of equilibrium until the end of the calculations, the radius has decreased from $96 R_{\odot}$ to $76 R_{\odot}$. Central temperatures at this final stage remain about $2 \times 10^5 \text{K}$.

Fig. 7. Evolutionary track for Case 1.

Case 2—1 M_☉

This problem is initially the same as Case 1. In fact, Case 1 was evolved until the formation of the core at 5 years when the physical conditions were saved on tape. This same configuration was subsequently further evolved as described above for Case 1, then again for Case 2, with the only change being a variation of the convective efficiency. In Case 2 the value adopted for f_1 was 100. As a result, for the same physical conditions as in Case 1 with $f_1 = 1000$, Case 2 would calculate a convective flux 100 times larger since the flux depends on the square of the mixing length. The two models compared almost exactly until all the material had been added to the core. This indicates that the details of the transport in the interior of the protostar are relatively unimportant to its evolution through this infall period. This is because the central 75% of the mass at this time is stable to convection. In the envelope, however, the amount of convection is very important in determining the observational history of the protostar in terms of its effective temperature and luminosity. The final structure in temperature and density for Case 2 is shown in Fig. 9. Comparison in Table 4 of the two polytropic cases shows that the two are very similar except near the outer boundary where the Case 2 temperatures are somewhat higher than for Case 1. The higher convective flux in the outer 10-20% of the mass in Case 2 has resulted in a slightly greater compression near the interior of that convective region than for Case 1, and it is that compression and release of potential energy that is the source of the increased temperature. The evolutionary track for Case 2 in Fig. 8 shows the effect of this increased surface temperature. Both cases will slowly evolve along roughly parallel tracks, compressing and increasing the temperature in the central region. At this stage the centers of both models are roughly isothermal. As the temperature gradient becomes more centrally peaked the convection zone will grow inwards, since the opacity of the material is too high for radiation to provide enough energy transport in the presence of a large temperature gradient.

Uniform Models

Case 3—0.01 M_☉

Two problems were run with masses of 0.01 M_☉, much less than the lower limit of about 0.085 M_☉ for main-sequence stars derived on theoretical grounds by Graboske and Grossman.³⁹ Using the Jean's criterion of Eq. (46), we find that it is necessary to begin at a uniform density of 10^{-15} g/cm³. This density is probably achievable only through fragmentation of a much larger cloud that had collapsed from an initially lower density. In this run, the cloud is unstable to gravitational collapse and the free-fall phase follows the general pattern described above. However, the core reaches an equilibrium before the dissociation of H₂ has occurred to any appreciable degree. Subsequent contraction of the core increases the central density and temperature further, but the core temperature remains about 1.6×10^3 °K and the density less than 10^{-7} g/cm³.

Fig. 8. Evolutionary track for Case 2.

Fig. 9. Final structure for Case 2 (one $M_{\odot}$).

Table 5. Structure of $0.01 M_{\odot}$ gas cloud at end of evolutionary calculation.

$\frac{M_r}{M_{\odot}}$	$\frac{r}{R_{\odot}}$	$-\log \rho$ (g/cm^3)	$\log T$ ($^{\circ}K$)
0.001	28.5	7.30	3.21
0.002	40.0	7.50	3.21
0.003	50.5	7.69	3.18
0.004	60.9	7.91	3.12
0.005	71.2	8.01	3.08
0.006	82.3	8.19	3.01
0.007	95.0	8.39	2.93
0.008	112	8.68	2.83
0.009	114	9.28	2.63
0.010	2 (4)	16.2	1.32

until the calculation of this model was ended at about 3×10^3 years, about three times the free-fall time. The final structure in our calculations is shown in Fig. 10 and Table 5.

Case 4— $0.01 M_{\odot}$ (unbound)

The second problem with $0.01 M_{\odot}$ was designed to test the necessity of having a minimum density determined by the Jean's criterion. With an initial density of $10^{-17} g/cm^3$, only 1% of the minimum density required by Eq. (46), the free fall was in fact halted very quickly by the pressure gradient that was

a result of the rarefaction described earlier. The cloud oscillates about its initial configuration without any significant central condensation.

Case 5— $0.1 M_{\odot}$

This model, $0.1 M_{\odot}$, is just above the lower mass limit, and is the smallest mass included which collapsed through the dissociation and ionization regions in the ρ - T plane. The initial density for this model was $10^{-17} g/cm^3$ with a free-fall time scale of approximately 1.1×10^4 years.

The core is formed at a time of 2.15×10^4 years and includes approximately 38% of the total mass, or about $0.038 M_{\odot}$. The structure of the protostar at this time is shown in Table 6. At this point, we are particularly interested in the interaction

Table 6. Structure for $0.1 M_{\odot}$ model. Column 1 is for time just after formation of protostellar core, containing $0.038 M_{\odot}$. Column 2 is for final structure.

$\frac{M_r}{M_{\odot}}$	$\frac{r}{R_{\odot}}$		$-\log \rho$ (g/cm^3)		$\log T$ ($^{\circ}K$)		$\log \lambda_{Ross}$	
	1	2	1	2	1	2	1	2
0.01	1.01	0.60	1.66	1.20	4.82	5.26	-4.26	-3.59
0.02	1.36	0.82	2.19	1.45	4.83	5.21	-3.18	-3.26
0.03	2.15	1.12	2.97	1.91	4.75	5.32	-2.00	-2.03
0.04	24.3	1.45	6.68	2.11	4.00	5.29	5.58	-1.78
0.05	6.7 (3)	1.75	13.9	2.23	1.95	5.21	15.9	-1.82
0.06	2 (4)	2.03	15.0	2.35	1.70	5.13	17.0	-1.88
0.07	3 (4)	2.31	15.5	2.48	1.58	5.04	17.5	-1.94
0.08	5 (4)	2.61	15.9	2.63	1.50	4.93	17.9	-2.03
0.09	7 (4)	2.99	16.3	2.87	1.44	4.77	18.3	-2.09
0.10	1 (5)	4.09	16.9	3.90	1.36	4.45	18.9	-1.51

Fig. 10. Final structure for Case 3 ($0.01 M_{\odot}$).

between the hot core and the remainder of the infalling cloud. This question is one of the subjects in dispute in the theory of protostar formation. If the energy that is radiating from the outer surface of the protostellar core at a fairly high effective temperature passes through the infalling gas with negligible absorption, the core will contract fairly rapidly. Also, the luminosity of the protostar will be the luminosity of the core. This is the position adopted by Larson,⁴³ although he does point out that uncertainties in the opacity in the temperature range of 2000-2500°K might reduce the amount of energy that is radiated through the cloud. As a result of the loss of most of this energy, Larson predicts that when the protostar has reached complete hydrostatic equilibrium, the radius is quite small. Narita *et al.*¹⁵ predict that most of the energy emitted by the core is instead absorbed in the region just outside the core, the material in the shock region. As a result, the luminosity of the protostar is much smaller than the core luminosity. From examining the Rosseland mean free paths λ_R in Table 6, it is clear that the shock region is optically very thick; the outer portion of the shock region has a temperature less than 1000°K while the outer part of the core is about 3×10^3 °K. The shock region, characteristically containing two or three zones, remains optically thick throughout the rest of the infall phase of the evolution. In Fig. 11 we plot the luminosity of the protostellar core and the flareup at the surface of the protostar as the shock arrives at the surface. The luminosity plotted is not the "observed" luminosity due to the absorption in the shock, but it gives some idea of the amount of energy being radiated by the hot protostellar core.

The end of the infall phase of the evolution occurs when the last zone is added to the core at 2.94×10^4 years. The subsequent evolution of the quasistatic protostar is dominated by the convective region at the surface which grows inward toward the center. There is also a small convective region at the very center of the protostar, and it is probable that once the inversion region in temperature (covering the remaining non-convective portion of the mass) has disappeared, the entire protostar will be convective. In this calculation, this temperature inversion region has decreased in size from the time of its formation until the time the model was terminated. This process was due to the combined efforts of contraction of the core, transport by radiation and convection in the core, and radiation at the outer edge of the inversion region. Also, the inversion region is contracting slowly and heating itself. We can conclude that this inversion region is a temporary phenomenon, but it does occur and our model is able to deal with it and follow the evolution of the protostar as the inversion disappears. The final structure, at 2.46×10^7 years, is given in Table 6 and Fig. 12. The extremely small values for λ_R at all points are the reason the energy transport is dominated by convection. The time required for a $0.1 M_\odot$ protostar to contract to the main sequence has been estimated^{6,44} at about 5×10^8 years, so that our final model covers the end result of only 5% of that evolution time.

Fig. 11. Luminosity for Case 5 ($0.1 M_{\odot}$).

Fig. 12. Final structure for Case 5 ($0.1 M_{\odot}$).

Case 6—0.2 M_⊙

Three related problems with 0.2 M_⊙ were included. All were initially identical, with an initial uniform density of 2.5×10^{-18} g/cm³. One problem was evolved using explicit hydrodynamics until the core formed at approximately 4.3×10^3 years. At this time, approximately 40% of the original mass is in the protostellar core in which the density ranges from 2×10^{-4} to 10^{-6} g/cm³, and the temperature is nearly constant at about 2×10^4 °K. This configuration was then independently evolved in three models, each one with a varying amount of convection specified by a different value for f_1 in Eq. (23). The values used were 10000, 2000 and 1000. In Case 6, with $f_1 = 10000$, the entire core was unstable to convection at the time of its formation. The ratio of convective flux to radiative flux in the core was approximately 100; this ratio increased as more material was added to the core. This is because the material in the core is close to the conditions where the opacity reaches a maximum value (see Fig. 3). Slow contraction of the core tends to first establish and eventually accentuate a radial temperature gradient by releasing gravitational potential energy faster close to the core center than near the edge of the core. The combination of a rather steep temperature gradient and very high opacity results in convection dominating the energy transport. As the core contracts its temperature increases, reducing its opacity by more than a factor of 10 by the time the shock front has reached the surface of the protostar.

By the time all the material has been added to the core, at about 5.6×10^3 years, there are a number of concentric regions in the protostar. The central 15-20% of the mass is extremely unstable to convection; in this model the convective flux is 10^4 times the radiative flux and the ratio of the temperature gradient to the adiabatic temperature gradient is about 10. The next 20% of the mass is also convective, but convection carries just over half the energy and the two temperature gradients are nearly equal. This region is much closer to a conventional convective equilibrium region than is the innermost region. The next 40% of the mass is a fully radiative region. The opacities are lower by a factor of 10 than in the deep interior and the temperature gradients are much smaller. The outer 10% of the mass is close to convective equilibrium, with only a small fraction of the energy flux being convective.

It is important to note that at the time the shock front arrives at the surface, the γ of that material in the last zone is about 1.2, suggesting that it might be somewhat unstable hydrodynamically. With the zoning used in these calculations it would be impossible to determine if there is any mass ejected by the shock wave. We can say from these results that it appears possible that some small amount of mass might be blown off, but the amount of mass ejected is a small fraction of the zone mass, or less than $0.001 M_{\odot}$.

The flareup of the protostar as the shock wave reaches the surface reaches a maximum at about $67.5 L_{\odot}$ with the effective temperature at about 3750°K, with a radius of $19 R_{\odot}$. As explained in Appendix B, these values for L and T_e are only estimates. However, as the now quasistatic phase of the evolution continues, this model contracts

and cools at its outer surface. The five distinct portions of the protostar continue to exist as relatively separate regions even as long as 1.6×10^6 years when this model was terminated. The luminosity of the core is plotted as a function of time in Fig. 13. The convective region at the surface remains very small, including less than 10% of the mass. There is not a sufficient convective flux to keep the surface temperature at its shocked value. Therefore, the evolutionary path of this protostar in the HR diagram falls sharply towards lower luminosities and lower effective temperatures as shown in Fig. 14.

In this model temperature gradients in the central portion of the protostar remain large for a much longer time than in models with larger amounts of convection. Even at 1.6×10^6 years, the temperature gradient in the core exceeded the adiabatic gradient by factors of about 4. Later models with more convection establish an equilibrium gradient much more quickly. This, plus the rather extreme cooling at the outer surface, tends to suggest that a larger convective flux is needed to establish the structure that is predicted by other calculations.^{6,44}

Case 7—0.2 $M_{\odot}$

This model was started from Case 6 at the time the core formed, but with $f_1 = 1000$. This gives 100 times the convective flux as in Case 6 for the same conditions. The early evolution is generally the same as for Case 6. Convective fluxes in the deep core exceed radiative fluxes by factors from 10^4 to 10^6 as opposed to factors from 10 to 10^4 in the prior case, but the same qualitative regions still exist.

The evolution after the onset of hydrostatic equilibrium is quite different from that of Case 6, however. In the central core, higher convective fluxes have smoothed out the temperature gradients that persisted with smaller fluxes. Also, the material temperature in this central core has increased past the peak in the opacity. As a result, the core, including 25% of the mass, is in convective equilibrium at about 1.5×10^5 K. A second region, containing another 25% of the mass, is close to equilibrium, with $L_{\text{conv}}/L_{\text{rad}}$ about equal to 10. The non-convective region now contains only about 20% of the mass, and the outer convective region has grown inward and includes about 30% of the protostar's mass. This extended convective region contracts much more rapidly than in Case 6. The energy released by this gravitational contraction is transported, largely by convection, to the surface region. As a result, the surface temperature of this model is much higher than Case 6 as the protostar contracts. The track of this quasistatic evolution is plotted roughly in Fig. 14. From that track we see that the effective temperature increases rapidly. The luminosity falls initially but soon begins to increase again, having reached a minimum of $57 L_{\odot}$ at 7.6×10^6 years. The model was terminated at 1.1×10^7 years, at which time the luminosity and T_e were still increasing. Without further calculation it is difficult to guess what track the protostar would follow, but we should expect the track to be nearly vertical in the HR diagram for some time.

Fig. 13. Luminosity of the core for Case 6 ($0.2 M_{\odot}$).

Fig. 14. Evolutionary tracks for $0.2 M_{\odot}$ indicating dependence on convection.

Case 8—0.2 M_☉

To partially answer the question raised by the evolutionary track of Case 7, another model based on Case 6 was tried with an intermediate value of 2000 for f_1 . Results for this model combine features of Cases 6 and 7. The inner 50% of the final structure is very close to convective equilibrium, with the temperature gradient very nearly equal to the adiabatic gradient. The outer convective region extends inward through 35% of the mass, where the temperature gradient exceeds the adiabatic gradient by about one-third. The path of this model in the HR diagram, also plotted in Fig. 14, falls vertically for a much longer period than in Case 7, reaching a minimum of about 35 L_☉ at 4.7×10^7 years.

None of these evolutionary tracks should be considered as definitive, but rather an illustration of the sensitivity of the observed luminosity to the convective model in the protostellar atmosphere. It is unrealistic to expect more than general results for the stellar atmosphere using a model such as the one used here. However, the three models included do give a reasonable demonstration of the response of the atmosphere to variations in convection. As stated earlier, when we find a feature in the evolution which is very sensitive to variations in some parameter, we must be careful about any conclusions that may be drawn. We cannot claim any quantitative accuracy for the evolutionary paths because a complete description of the physics of the stellar atmosphere has not been included, but we can be fairly sure of the qualitative features. The structure of the protostar is summarized in Table 7; the first values, at the onset of quasistatic equilibrium, are very similar for Cases 6, 7, and 8, and the second columns are for Case 8, the model which was followed for the longest time, shown at 6.3×10^7 years.

Table 7. Structure for 0.2 M_☉, Case 8. Column 1 is for the structure just after the addition of the last mass to the core. Column 2 is for the final structure.

$\frac{M_r}{M_\odot}$	$\frac{r}{R_\odot}$		$-\log \rho$ (g/cm ³)		$\log T$ (°K)		$\frac{L_{\text{conv}}}{L_{\text{rad}}}$	
	1	2	1	2	1	2	1	2
0.004	2.07	1.04	3.17	2.27	5.66	5.27	9.81	6.38
0.008	2.35	1.29	2.83	2.25	5.31	5.21	1.9 (3)	10
0.024	2.61	1.91	2.21	2.34	4.75	5.13	2 (6)	2.29
0.060	3.64	3.04	2.92	2.74	5.02	5.15	2 (3)	0.76
0.080	4.32	3.62	3.02	2.86	4.92	5.08	5 (3)	1.80
0.100	4.88	4.19	3.20	3.00	4.86	5.02	0	0
0.120	5.82	4.88	3.57	3.26	4.91	5.05	0	0
0.140	7.23	5.75	3.91	3.45	4.88	5.00	0	4.41
0.160	9.14	6.62	4.23	3.57	4.78	4.85	0	38.1
0.180	11.8	7.51	4.64	3.71	4.61	4.64	0.21	748
0.196	19.0	9.20	5.61	4.43	4.18	4.28	0.05	3.14 (4)

One Solar Mass

A number of models were included with $1 M_{\odot}$, with each independent parameter varied separately. The low-temperature opacity floor was raised to $1.0 \text{ cm}^2/\text{g}$ for some of the runs, and the amount of convection was varied by changing f_1 in Eq. (23). The problems included are summarized in Table 8.

Runs in Cases 9, 10, 11, and 12 were all based on a single explicit evolution through the formation of the protostellar core at 2.14×10^5 years. That core contained $0.38 M_{\odot}$; the density and temperature of the protostar at this time are plotted in Fig. 4. Energy transport in the core is dominated by convection, but neither convection nor

Table 8. Summary of the $1 M_{\odot}$ problems included.

Case	Opacity floor (cm^2/g)	f_1
9	0.01	1000
10	0.01	200
11	0.01	100
12	0.01	10
13	1.0	1000
14	1.0	10

radiation provides any appreciable fraction of the heating at the outer boundary of the core that is produced by the shock wave. The shock region remains optically thick throughout the infall phase of the evolution in all the cases included. The time scale for the propagation of the shock wave is much shorter than the energy transport time scales in the core. Therefore, even for $f_1 = 0$ in the convective model, transport in the core does not

significantly affect the shock region or the core luminosity. As a result, Cases 9, 10, 11, and 12 are nearly identical throughout the infall phase of the evolution until the shock wave reaches the outer surface of the protostar. In all cases, the final mass shell passes through the shock and becomes part of the core at about 2.66×10^5 years. The structure of the protostar at this time is summarized in Fig. 15. It is important to note that the radius of the protostar when it begins the quasistatic contraction phase is $90 R_{\odot}$. As discussed earlier, this large value of the radius does not agree with Larson's calculation of approximately $2 R_{\odot}$ for the protostar at this stage. Again, as suggested by Hayashi, the discrepancy is possibly due to excessive energy losses through the shock region during the mass infall phase in Larson's calculation. Less than $0.01 M_{\odot}$ in our model is within a radius of $1 R_{\odot}$ at this time.

All of the models have a convective core at the onset of hydrostatic equilibrium. Case 12, with the highest amount of convective flux, has established a convective equilibrium in the core, where $\frac{dT}{dr} = \left(\frac{dT}{dr}\right)_{\text{ad}}$ to within 1%. The other models show a monotonic departure from convective equilibrium, and in Case 9 the ratio of temperature gradients is about 10 to 1.

There is very little convection at the surface of the protostar at this time. During the contraction phase that follows the onset of quasistatic equilibrium, each model follows a slightly different path. The central 75% of the protostellar mass contracts and heats up; the key to the behavior of the protostellar atmosphere is the rate at which this energy, generated by the contraction, can be transported to the surface and

radiated into space. As t_1 decreases, the efficiency of convection increases; in Cases 9, 10, 11, and 12 convection becomes progressively more important. In each model the outermost 10% of the mass heats up, but the rate of increase in temperature is roughly proportional to the amount of convection. The four tracks during this phase are plotted in Fig. 16.

Fig. 15. Structure of $1 M_{\odot}$ model at the time the final mass shell is added to the core.

Fig. 16. Evolutionary tracks for $1 M_{\odot}$ models.

The convective surface region gradually grows inward until it joins the central convection zone, whereupon the evolution proceeds through a completely convective phase. None of these models was continued into the totally convective phase. It was found that the surface convective region evolves fastest in Case 12 and slowest in Case 9, with the other models in between. The final structures are summarized in Table 9. There are several general features that may be seen from Table 9. First, the elapsed time is different for each model, yet Case 9, the longest run, is less evolved than any of the others. Its mean density is lower and there is only a very small surface convection region. The temperature gradient in the outer half of the star is much smaller than in the other models, consistent with the lack of a convective region. Case 10 has contracted to nearly half the radius of Case 12, yet the surface regions in the two are at nearly the same temperature. This difference in the structure is due largely to the greater efficiency of Case 12 in transporting energy from the inner half of the protostar to the outer portion; Case 10 is distinctly hotter than Case 12 in the inner half of its mass. The convective region in Case 12 has progressed inward in mass to a slightly greater degree than in Case 10. Case 11 has evolved quasistatically to about the same density as Case 12, but the enhanced convection in Case 12 has raised the temperature somewhat more. Also, model 12 has evolved more quickly than model 11, contracting to nearly the same radius in less than one-fourth the time.

Table 9. Summary of the structure of the final model in Cases 9, 10, 11, and 12. Problem times are (9: 1.66×10^7 years), (10: 4.51×10^6 years), (11: 1.33×10^6 years), (12: 2.84×10^5 years).

$\frac{M_r}{M_\odot}$	$\frac{r}{R_\odot}$				$-\log \rho$ (g/cm ³)				$\log T$ (°K)				$\frac{L_{\text{conv}}}{L_{\text{rad}}}$			
	9	10	11	12	9	10	11	12	9	10	11	12	9	10	11	12
0.1	5.90	5.44	6.66	6.14	3.19	3.04	3.31	3.20	5.42	5.51	5.31	5.34	0	0.35	6.57	318
0.2	7.80	6.98	8.57	7.92	3.30	3.11	3.39	3.29	5.37	5.45	5.22	5.24	0	0.12	16.1	801
0.3	9.36	8.22	10.2	9.44	3.41	3.21	3.51	3.42	5.33	5.42	5.15	5.17	0	0	5.36	1250
0.4	10.8	9.55	12.2	11.5	3.52	3.39	3.86	3.84	5.29	5.46	5.26	5.34	0	0	0	0
0.5	12.3	10.9	15.0	14.4	3.65	3.48	4.11	4.06	5.24	5.42	5.27	5.33	0	0.94	0	23.9
0.6	13.9	12.1	18.1	17.1	3.81	3.52	4.30	4.16	5.19	5.32	5.23	5.22	0	7.67	0	145
0.7	15.8	13.1	21.1	19.5	3.99	3.54	4.40	4.25	5.13	5.18	5.11	5.10	0	69.7	4.02	705
0.8	18.1	14.1	23.8	21.9	4.23	3.56	4.47	4.36	5.06	5.01	4.94	4.96	0	944	57.4	4 (3)
0.9	21.7	15.0	26.5	24.5	4.60	3.61	4.58	4.54	4.94	4.77	4.71	4.78	0.21	3 (4)	1 (3)	2 (4)
1.0	37.6	16.4	31.0	29.0	6.01	4.13	5.21	5.27	4.61	4.34	4.27	4.34	0	1 (7)	4 (5)	6 (5)

Clearly we have determined that increased convection, particularly in the outer part of the protostar, increases the contraction rate. The surface temperatures increase slowly in all cases, also increasing more rapidly with more convection. The contraction time scale is shorter than the heating time scale, so the luminosity falls rapidly. Case 10 follows the $1 M_\odot$ HR diagram evolutionary track of Ezer and Cameron⁴⁴ closest of all the models calculated, indicating that a value of $f_1 = 200$ give a fairly good estimate of convective energy transport in the outer portion of the protostar. As noted in Appendix B, it is not possible to obtain quantitative results for

photospheric physics using our present numerical techniques. The rather high sensitivity of the approximate evolutionary paths to changes in f_1 indicates a basic deficiency in the treatment of this part of the problem, so the evolutionary tracks must be accepted as rough estimates only. Further work in producing an improved photospheric model would be extremely valuable in making it possible to predict the output from the proto-star. Ideally, a multifrequency radiation transport calculation in the optically thin outer part of the photosphere would be most useful, where the diffusion treatment breaks down. However, the frequency-dependent photon cross-section data at these temperatures and densities, with largely unknown trace contributions from other elements and molecules, are extremely uncertain; a more sophisticated model might still be quite uncertain. The simplest improvement in our model would be to make f_1 a function of temperature, density, and radius. Of course, further improvement in the form of mixing length theory used here, to include the reduced effectiveness of convection in the stellar atmosphere, would make the surface region calculations more reliable.

It is possible that the use of mixing length theory, even with the use of complicated flux limiters or other devices to extend its validity, is completely unreliable in stellar envelopes except for rough estimates of convective fluxes. The low densities and the nearly free boundary at the surface of the star, combined with large temperature gradients, mean that the stellar envelope will probably be characterized by large-scale turbulent effects which we are trying to represent by a model which includes a requirement or assumption of hydrodynamic equilibrium. Although there is not any net mass motion in a stellar envelope such as those being calculated in these models, still there is a great deal of macroscopic mass motion which can carry a great deal of energy. It would require a complete treatment of the nonlinear hydrodynamical problem, coupled with a treatment of the nonlinear radiation problem, to produce a good model for the stellar atmosphere. Even with those tools, the physical data are unreliable enough to still limit the model. The production of such a physical model was not the purpose of this project, and the model used is a simple model to try to get usable results from the structures calculated.

Two further models, Cases 13 and 14, were included to determine whether an increase in the low-temperature opacity of the infalling gas cloud would make any difference in the infall dynamics. If most of the core radiation were not absorbed in the shock region, increasing the opacity outside the shock might substantially retard the collapse of the cloud. These models were nearly identical to Cases 9 and 12, a further indication that the absorption in the shock region is nearly total for $1 M_{\odot}$.

Case 15— $5 M_{\odot}$

A model of $5 M_{\odot}$ was calculated from an initial density of $4 \times 10^{-21} \text{ g/cm}^3$. At this density and at a temperature of about 10^4 K , radiation pressure provides about 90% of the total pressure. The free-fall phase of the gas cloud evolution proceeds just as

for smaller mass clouds. The core forms at approximately 1.1×10^6 years, containing 44% of the protostellar cloud mass, or $2.2 M_{\odot}$. The structure at this time is shown in Table 10. The temperatures and densities in the core are considerably higher than they are for $1 M_{\odot}$. In the surrounding cloud, the density is roughly proportional to $r^{-2.5}$; this density distribution has been established by the propagation of the rarefaction wave from the outer surface and agrees well with Larson's density which is proportional to $r^{-2.4}$. This cloud is nearly isothermal; the values of the temperatures are uncertain to a factor of about 2 due to the use of the diffusion equation in this region where it is not really valid. Therefore, the radiation pressure here is also uncertain to within an order of magnitude. Still, it is clear that in this outer gas cloud radiation contributes a large fraction of the total pressure. The material pressures are low because of the low density. If $\rho \propto r^{-2.5}$, then $P_{\text{mat}} \propto r^{-2.5\gamma}$ where γ is about 1.4. With the material pressure falling rapidly with increasing radius and the radiation pressure roughly constant in the nearly isothermal outer cloud, the ratio $P_{\text{rad}}/P_{\text{total}}$ increases rapidly as can be seen from Table 10. This ratio was always small for smaller mass protostars due to their larger initial densities.

Table 10. Structure for $5.0 M_{\odot}$. Column 1 is for the structure just after the formation of the core, including $2.2 M_{\odot}$. Column 2 corresponds to structure at end of calculation.

$\frac{M_r}{M_{\odot}}$	$\frac{r}{R_{\odot}}$		$-\log \rho$ (g/cm^3)		$\log T$ ($^{\circ}\text{K}$)		$\frac{P_{\text{rad}}}{P_{\text{tot}}}$	
	1	2	1	2	1	2	1	2
0.5	13.8	11.1	1.95	3.32	4.81	5.83	6 (-7)	0.012
1.0	13.84	14.8	1.81	3.44	4.68	5.78	2 (-7)	0.011
1.5	13.9	17.8	1.89	3.55	4.60	5.74	2 (-7)	0.011
2.0	14.1	20.6	2.15	3.68	4.51	5.69	2.(-7)	0.011
2.5	4.2 (5)	23.5	17.6	3.81	1.25	5.64	0.13	0.010
3.0	1 (6)	26.7	18.4	3.97	1.14	5.58	0.35	0.010
3.5	2 (6)	30.4	18.9	4.17	1.11	5.51	0.57	0.009
4.0	3 (6)	35.4	19.3	4.45	1.09	5.42	0.74	0.009
4.5	4 (6)	44.3	19.7	4.97	1.03	5.26	0.87	0.010
5.0	5 (6)	2 (8)	20.3	20.5	0.96	0.37	0.91	0.220

As the shock wave heats up more of the protostellar cloud, the temperature in the infalling cloud very slowly increases. The opacity of the infalling cloud is equal to its minimum possible value, so it is effectively constant in the cloud. Therefore, radiation energy is absorbed by the cloud at a rate proportional to the local density. This tends to increase the radiation pressure gradient until, in the outer 10-15% of the mass, the pressure gradient force exceeds the gravitational force and the infalling material there is decelerated. This process is illustrative of the problem of competing time scales. Since the opacities and densities in the cloud are very small, a long period of time is

required for enough radiation energy to be absorbed to significantly retard the infall. If the material falls into the core before that occurs, then the effects of the radiation pressure will be negligible, but if the shock region has not yet reached a given radius before the infall has been stopped, then the material will be blown off. In this $5 M_{\odot}$ case, $4.7 M_{\odot}$ forms the final quasistatic core while $0.3 M_{\odot}$ is blown off at velocities of about 10^4 cm/sec, although this velocity is uncertain by a factor of about 10.

Several other features of the core evolution should also be noted. First, convection is almost nonexistent in the core. Transient irregularities in the temperature gradients produce convective fluxes, but the convection quickly smooths out those regions. At the end of our evolution, there was a very small convective zone at the outer edge of the quasistatic core, providing less than 1% of the total energy flux.

A second major feature is that the core temperatures are significantly higher at a similar stage in the evolution than for $1 M_{\odot}$ and smaller clouds. Temperatures in the core are higher than those corresponding to peak opacity values. For this reason, the higher temperatures are responsible for the relative insignificance of convection, since radiation is a relatively efficient transport mechanism in these temperature and density ranges. The subsequent evolution of the protostar will be essentially non-convective, as described elsewhere.^{9,34,44} However, the protostellar radius as determined in these calculations is larger by more than a factor of 2 than found for $5 M_{\odot}$ by Larson.³⁴ Our results tend to confirm his statement that no convective Hayashi phase of pre-main sequence evolution occurs. The final structure calculated is summarized in Table 10.

Case 16 — $10 M_{\odot}$

The physics of the collapse of a $10 M_{\odot}$ cloud is very similar to that for $5 M_{\odot}$. The initial density is lower, 10^{-21} g/cm³, and the free-fall time scale is correspondingly longer, about 1.1×10^6 years. The core in this model forms at 2.15×10^6 years and contains $3.6 M_{\odot}$. The structure at this time is summarized in Table 11. Radiation pressure is an even larger fraction of the total pressure for this model than for $5 M_{\odot}$. The increased contribution of radiation pressure, plus the longer time scale for gravitational collapse, result in a larger fraction of the infalling cloud being blown off for $10 M_{\odot}$ than for $5 M_{\odot}$. In this case $2.6 M_{\odot}$ is blown off, leaving $7.4 M_{\odot}$ in the remaining quasistatic core. The last material is added to the core at 2.56×10^6 years, at which time the structure is that given in Table 11. During the time that mass has been accreted onto the core, the central temperature has increased from just over 10^4 °K to almost 2×10^6 °K. Densities in the core have also increased by almost four orders of magnitude. All of the energy is transported by radiation, and there appears to be no convective pre-main-sequence phase for this model. The protostar that has been formed is really one of $7.4 M_{\odot}$, with an initial radius of $46 R_{\odot}$. We note that this radius is smaller than the initial radius for a $1 M_{\odot}$ protostar as it begins its quasistatic phase. The $7.6 M_{\odot}$ model is much more condensed and much hotter at the beginning of this phase than the smaller models.

Table 11. Structure for $10 M_{\odot}$ protostar. Column 1 is for structure just after formation of the core, Column 2 is for structure at end of calculation.

$\frac{M_r}{M_{\odot}}$	$\frac{r}{R_{\odot}}$		$-\log \rho$ (g/cm^3)		$\log T$ ($^{\circ}\text{K}$)		$\frac{P_{\text{rad}}}{P_{\text{tot}}}$	
	1	2	1	2	1	2	1	2
1.0	460	6.61	6.58	2.36	3.80	6.24	3 (-5)	0.222
2.0	462	8.98	6.23	2.53	3.50	6.19	2 (-6)	0.023
3.0	467	11.1	6.45	2.71	3.48	6.15	4 (-6)	0.023
4.0	9 (5)	13.3	18.3	2.91	1.20	6.07	0.357	0.024
5.0	3 (6)	16.0	19.4	3.16	0.74	5.99	0.249	0.025
6.0	5 (6)	19.6	20.0	3.50	0.74	5.89	0.553	0.027
7.0	8 (6)	27.0	20.5	4.19	0.74	5.70	0.780	0.036
8.0	1 (7)	6 (6)	20.9	20.7	0.74	1.33	0.905	0.997
9.0	2 (7)	1 (7)	21.4	21.5	0.73	1.02	0.964	0.996
10.0	3 (7)	3 (7)	27.2	22.2	0.61	0.81	0.988	0.996

Case 17 — $50 M_{\odot}$

A very large protostellar mass, initially of $50 M_{\odot}$, was included as the largest model to be evolved. This model is especially interesting because the initial density of $4 \times 10^{-23} \text{ g/cm}^3$ is much closer to commonly occurring interstellar gas cloud densities than is the case for the smaller models included. At this low density, pressures are dominated by radiation pressure to an even greater extent than for $5 M_{\odot}$ and $10 M_{\odot}$.

The free fall proceeds as in earlier models; a central core forms at approximately 1.08×10^7 years. The free-fall time scale is 5.5×10^6 years. This general feature of the formation time for the core being very close to twice the free-fall time appears to be true for all the models included, independent of cloud mass. The collective oscillations of the core are much larger in amplitude than in previous models, with the radius of the core varying from $31.7 R_{\odot}$ to $820.1 R_{\odot}$. The period of one such oscillation is approximately 2.53 years. These oscillations nearly damp themselves out after about 50 periods, reaching an equilibrium state summarized in Table 12. The core contains $18 M_{\odot}$ with a radius of $244 R_{\odot}$. Temperatures in the core are slightly above $2 \times 10^4 \text{ K}$ at this time.

In this calculation, only 3% of the remaining cloud is accreted onto the core. The remaining material, amounting to $31 M_{\odot}$, is blown off by radiation pressure. As pointed out by Larson and Starrfield,⁴⁵ the calculation of mass limits of protostars due to the effects of radiation pressure is extremely difficult. This is largely due to the sensitivity of these calculations to opacity data; slight variations in assumed composition can change the amount of mass in the core by at least a factor of 2. The size of the core in this model, $19 M_{\odot}$, is entirely consistent with the predictions of Larson and Starrfield.⁴⁵

Table 12. Structure for 30 $M_{\odot}$ protostar. Column 1 is for structure just after formation of core, Column 2 is for structure at end of calculation.

$\frac{M_r}{M_c}$	$\frac{r}{R_0}$		$-\log \rho$ ($\mu \text{ cm}^3$)		$\log T$ (°K)		$\frac{P_{\text{rad}}}{P_{\text{tot}}}$	
	1	2	1	2	1	2	1	2
5	241	29.4	4.61	3.65	4.33	6.04	1 (-5)	0.100
10	242	43.4	4.59	4.02	4.25	5.93	7 (-6)	0.104
15	243	63.0	4.28	4.58	3.79	5.75	1 (-7)	0.114
20	4 (6)	4 (6)	19.7	19.6	0.68	1.39	0.268	0.974
25	1 (7)	1 (7)	20.8	20.6	0.68	1.18	0.837	0.994
30	3 (7)	3 (7)	21.4	21.4	0.68	0.97	0.951	0.993
35	4 (7)	4 (7)	21.9	21.9	0.68	0.97	0.082	0.998
40	5 (7)	5 (7)	22.3	22.3	0.68	0.83	0.993	0.998
45	8 (7)	8 (7)	22.8	22.8	0.68	0.77	0.998	0.999
50	1 (8)	1 (8)	23.6	23.6	0.65	0.69	0.999	0.999

Numerical difficulties halted the evolution of this model soon after the cloud material began to move outward. The extremely high contribution of radiation pressure and energy density in the entire protostar, including the core, created problems in the difference equations that could not easily be resolved. Since the primary interest here was in determining the fraction of the initial cloud that became part of the core, this model was considered essentially complete at this point. Further refinement of the numerical model will be necessary to further evolve this protostar.

Core temperatures rise even more rapidly in this core than for the 10 $M_{\odot}$ model, reaching 10^6 °K before the mass ejection begins. This behavior is a verification of the fact that the rate of core evolution toward the main sequence increases rapidly with increasing mass.⁴² Since the evolution of this model was ended at a fairly early stage, it was not possible to follow the evolution of the core any closer to the main sequence, but it is clear that the evolution of this model, with a core of 19 $M_{\odot}$, is much faster than any of the other models calculated.

VI. Conclusion

The intent of this study has been to perform a rather broad survey of the physical processes involved in the collapse of gravitationally bound gas clouds. The sizes of the clouds considered span nearly the entire range of observed stellar masses, including one model with a mass below the theoretical lower mass limit for reaching the main sequence. While it has not been one of the major subjects for investigation, our results are consistent with the lower mass limit of Graboske and Grossman.³⁹ At the other

end of the mass spectrum, we find that radiation pressure and rapid core evolution place severe limits on the maximum size attainable under these assumed formation processes. It would be necessary to include many more high-mass calculations to further investigate this regime, but our results do tend to confirm the predictions of Larson and Starrfield⁴⁵ on the existence and rough numerical values of this upper limit to stellar masses.

The model for convective energy transport, while simple and fairly crude, still predicts quite well all the qualitative features that are predicted by other calculational models. The important contribution from this model is that it is deterministic in a sense not found elsewhere in the astrophysical literature in that it calculates convective fluxes directly from instantaneous values for temperature gradients. A benefit of this treatment is that there is no need for any kind of equilibrium assumption to calculate the proper flux; it is quite possible and even common to find configurations in the calculations where the temperature distributions in no way resemble an equilibrium distribution. This should not be at all surprising since gravitational collapse is a violent process and energy transport time scales are often not short enough to immediately restore equilibrium in a protostellar core. Even density and temperature inversions can be easily evolved over long periods of time with this model. The convective model is best in the interior of protostars where the necessary assumptions of pressure equilibrium, radiative diffusion, and low mass motion are accurate. In those regimes this model accurately represents the evolution of convective cores over all mass ranges in which they exist. This model is less reliable in stellar atmospheres where the above assumptions are often quite poor. It is possible, with some amount of "tuning," to get fairly reliable energy fluxes in the envelopes of lower mass protostars, but the procedures involved are often subject to valid criticism, the most severe being that the evolution in those cases is not as self-determining as possible.

The protostellar atmosphere is troublesome in another case as well. The assumptions of radiative diffusion are not valid through the last few mean free paths at the outer edge of the protostar. As mentioned earlier, even a vastly improved model might not improve the situation here. However, it may be possible to include a calculation to improve the prediction of the evolutionary paths in the HR diagram, joining that solution to a diffusion solution at a depth of several mean free paths.

A number of parameters were included which had little or no effect on our calculations. The opacity floor at low temperatures was not important for low-mass stars, largely because of the great amount of absorption in the shock region which reduced the amount of radiation passing through the low-density clouds. The effect of this type of variation on the large protostellar clouds has not been included in this study. It is likely that such variation in the floor opacity might change the amount of mass that is blown off by radiation pressure; this change will certainly be small, provided that the variations in the low-temperature opacity are within physically realistic limits.

Treating the shock region with an improved form of artificial viscosity³⁶ had little effect on the subsequent progress of the shock front. The first one or two zones at the center of the core in nearly every model were found to have lower densities and higher temperatures than was thought reasonable. This effect was traced to an anomalous shock heating at the center of the collapsing sphere due to a special case in the calculation of artificial viscosity. Simple spherical convergence in the absence of a shock still resulted in a positive artificial viscosity. Neither the standard method nor the improved artificial viscosity in the form used could remove this error. Since this anomaly was small and did not appear to influence the subsequent evolution and since energy transport eventually erased this problem, no further effort was spent on dealing with it.

The principal areas in which future development is planned are mainly involved with the radiation transport calculation. The atmosphere, including better effective temperature and luminosity calculations, will be improved. Perhaps most interesting, the energy and momentum deposition of radiation in the infalling cloud should be improved. With that tool it will be possible to be much more precise in finding an upper mass limit, which would be a substantial contribution.

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Appendix A

Implicit Energy Balance

We begin with Eq. 32:

$$\left(\frac{\partial E}{\partial T}\right)_V \frac{(T_{j+\frac{1}{2}} - T_{j+\frac{1}{2}}^n)}{\frac{\Delta t}{2}} = \left\{ \dot{S} - \left[P + Q + \left(\frac{\partial E}{\partial V}\right)_T \right] \frac{dV}{dt} \right. \\ \left. + \frac{v}{V} \left[A_{j+1} \tilde{T}_{j+1}^3 \frac{(T_{j+\frac{3}{2}} - T_{j+\frac{1}{2}})}{\left(\frac{\Delta r}{\lambda}\right)_{j+1}} - A_j \tilde{T}_j^3 \frac{(T_{j+\frac{1}{2}} - T_{j-\frac{1}{2}})}{\left(\frac{\Delta r}{\lambda}\right)_j} \right] \right\} \quad (A-1)$$

where

$$\left(\frac{\Delta r}{\lambda}\right)_j = \frac{1}{2} \left[\frac{r_{j+1} - r_j}{\lambda_{j+\frac{1}{2}}(\tilde{T}_j)} + \frac{r_j - r_{j-1}}{\lambda_{j-\frac{1}{2}}(\tilde{T}_j)} \right]. \quad (A-2)$$

Defining

$$a_j \equiv \frac{A_j \tilde{T}_j^3}{\left(\frac{\Delta r}{\lambda}\right)_j} \\ \beta_{j+\frac{1}{2}} \equiv dm_{j+\frac{1}{2}} \left\{ \dot{S} - \left[P + Q + \left(\frac{\partial E}{\partial V}\right)_T \right] \frac{dV}{dt} \right\}_{j+\frac{1}{2}} \\ \alpha_{j+\frac{1}{2}} \equiv \frac{2}{\Delta t} dm_{j+\frac{1}{2}} \left(\frac{\partial E}{\partial T}\right)_V \quad (A-3)$$

and noting that $v/V = 1/dm$, we multiply (A-1) by $dm_{j+\frac{1}{2}}$ and, using the above definitions, we arrive at

$$a_{j+1} T_{j+\frac{3}{2}} + (-\alpha_j - a_{j+1} - a_j) T_{j+\frac{1}{2}} + a_j T_{j-\frac{1}{2}} + \beta_{j+\frac{1}{2}} + \alpha_{j+\frac{1}{2}} T_{j+\frac{1}{2}}^n = 0. \quad (A-4)$$

Then using back substitution, we have

$$T_{j-\frac{1}{2}} = e_j T_{j+\frac{1}{2}} + f_j. \quad (A-5)$$

Substituting this into (A-4) and solving for $T_{j+\frac{1}{2}}$,

$$T_{j+\frac{1}{2}} = \left[\frac{a_{j+1}}{(\alpha_{j+\frac{1}{2}} + a_{j+1} + a_j - a_j e_j)} \right] T_{j+\frac{3}{2}} + \frac{(a_j f_j + \beta_{j+\frac{1}{2}} + \alpha_{j+\frac{1}{2}} T_{j+\frac{1}{2}}^n)}{(\alpha_{j+\frac{1}{2}} + a_{j+1} + a_j - a_j e_j)}. \quad (A-6)$$

Therefore,

$$e_{j+1} = \frac{a_{j+1}}{(a_{j+1} + a_{j+1} + a_j - a_j e_j)}$$

$$f_{j+1} = \frac{(a_j f_j + \beta_{j+1} + a_{j+1} T_{j+1}^n)}{(a_{j+1} + a_{j+1} + a_j - a_j e_j)}. \quad (\text{A-7})$$

We must know e_0 and f_0 from central boundary conditions. For these calculations, we assume that at $r = 0$,

$$\left. \frac{dT}{dr} \right|_{r=0} = 0.$$

This boundary condition determines e_0 and f_0 , from which we use (A-7) to calculate the $[e_j]$ and $[f_j]$. For a boundary condition at the outside boundary, we use $T_{J+1/2} = \text{constant}$. This constant is known from consideration of the medium from which the protostar was formed, and is an input parameter to the program. Once $T_{J+1/2}$ is given Eq. (A-5) is used, with $[e_j]$ and $[f_j]$, to solve for $[T_{j-1/2}]$.

ITERATION SCHEME

Equation (A-1) is fully implicit, and some terms in the coefficients are functions of $T^{n+1/2}$. The way this is dealt with is to use $T^{n-1/2}$ and T^n to extrapolate an estimate for $T^{n+1/2}$, say $\tilde{T}^{n+1/2}$. Then $\left(\frac{\partial E}{\partial T}\right)_V$, $\left(\frac{\partial E}{\partial V}\right)_T$, $P^{n+1/2}$, and S are evaluated as functions of $\tilde{T}^{n+1/2}$, and these values are used in (A-1). Solutions from the back substitution are compared with $\tilde{T}^{n+1/2}$. If these differ by more than a minimum amount, we iterate, redefining $\tilde{T}^{n+1/2}$. This procedure is repeated until the two temperatures agree. There are two methods which are used for redefining $\tilde{T}^{n+1/2}$ in terms of the previous estimate of $\tilde{T}^{n+1/2}$ and the solution of the back substitution, $T^{n+1/2}$. The first method is to use $T^{n+1/2}$ directly as the next estimate. This is best suited for cases in which temperatures are changing rapidly in time; even exponential growth can be adequately approximated by only a few iterations. However, near equilibrium, where temperatures are nearly constant, this method can produce serious oscillatory errors. To avoid this problem, for cases close to equilibrium, the next iteration estimate is calculated as the average of the solution $T^{n+1/2}$ and previous estimate $\tilde{T}^{n+1/2}$ of the previous iteration.

Appendix B

Effective Temperature and Luminosity

When a zone is optically very thick, ordinary diffusion theory will usually be adequate for calculations of energy transport. Use of an extrapolated boundary condition, requiring that the pseudoflux go to zero at an optical depth of $2/3$ outside a vacuum boundary, gives a surprisingly accurate proper flux at the vacuum boundary for a wide variety of problems. However, in these calculations, zoning mass fractions had to be large enough that the outermost zone often contained as many as 10^{10} mean free paths. Use of the diffusion equation here, where only the zone-centered temperature is known, leads to exceedingly small surface temperatures. It is well established that a more sophisticated treatment is needed in this outer region, the protostellar envelope, where most of the observed radiation is emitted.

It is not the purpose of this project to solve the stellar atmosphere problem, a substantial effort in itself. It would be useful and interesting, however, to be able to arrive at some reasonable estimate of the effective temperatures and luminosities of each stellar model calculated.

As indicated above, the complete solution to this problem would require a complete solution of the energy-dependent transport equation for radiation and matter since absorption and re-emission processes are dominant in the regimes studied. Even if that were done here, there would be a great amount of uncertainty in the solutions for the spectral output due to a lack of experimental or theoretical data on the frequency-dependent photon cross sections. Much theoretical work in this area assumes some sort of analytic form for the opacities rather than tabular physical data, so even in those calculations there is an inadequacy in the model. Therefore, we recognize a deficiency in our model and seek a simple yet reasonable way of estimating effective surface temperatures.

To motivate and then verify our choice, we will calibrate our model with solutions from the astrophysical literature of the more sophisticated and more correct physical models. If we were to assume that the temperature of the outermost zone had a constant value and dropped to zero immediately at the outer surface, the effective temperature would equal the last zone temperature. On physical grounds, we can immediately rule this out as being too high. In fact, from several sources including Hayashi,⁶ Ezer and Cameron⁴⁴ and others, we know that the effective temperature soon after hydrostatic equilibrium has been reached for a $1 M_{\odot}$ protostar is approximately 3400°K . In our own $1 M_{\odot}$ models, we have a temperature in the last zone of about $1.4 \times 10^4^{\circ}\text{K}$. Since the luminosity is proportional to T_e^4 , an overestimate by this much of T_e is very serious. If we should accept the estimate of T_e from simple diffusion theory, we would find that T_e would be about 10°K for the $1 M_{\odot}$ model. This is due to the extremely high optical depth of the zones once they become part of the core. The problem here is due in part to the zoning; finer zoning near the outside would help. But most of the error is because

the very outermost region is not in fact optically thick and is thus not adequately represented by diffusion theory.

These two estimates of T_e must be considered upper and lower bounds. From the sources quoted above, we can see that the effective temperature should be roughly one-third of the temperature in the center of the last zone for the $1 M_\odot$ case. For the purpose of plotting evolutionary tracks only we shall follow the simple device of multiplying the zone-centered temperatures by a constant of $1/3$. While there is no theoretical justification for doing this, it will enable us to view, very roughly, these evolutionary tracks. Since we seek only to examine qualitatively behavior of observational properties, we will use this very crude model. It should be noted, however, that this model in no way affects the calculated structure of the protostar, since the fluxes are calculated independently of this crude model and the luminosity and effective temperature are not physical variables in our calculations.

One further observation can be made with respect to calculation of luminosities in most of the literature. In most models the luminosity is one of the independent physical variables, with the luminosity at the outer radius used to define an effective radiation temperature, $L = 4\pi R^2 \sigma T_e^4$, where R is the outer radius of the protostar, L is its luminosity, and σ is the Stefan-Boltzmann constant. The partial differential equation involving dL/dm is then integrated inward, with the temperature at each point being a dependent variable that is a result of that integration for $L(r)$. In an optically thick medium, the contribution to the local energy density from the integral of dL/dm is negligible compared with the other contributing terms. This means that a huge error in the surface luminosity or temperature would make almost no difference in the structure of the protostar except in the outermost several mean free paths. This same effect is why solution-matching and implicit techniques at a number of radii are necessary in most stellar structure codes. Solutions of the stellar structure equations are very insensitive to radial gradients. Thus, an estimate of T_e that was high or low by even thousands of degrees would not affect the structure of the rest of the star. It is not particularly surprising then, using temperature as an independent variable which is subject to more extreme and rapid fluctuation than is luminosity, to give an inadequate estimate of the surface temperature from a diffusion model. The radiative flux is proportional to the gradient of T^4 ; near the outer boundary the temperature falls rapidly, and thus the flux is proportional to $(T_1^4 - T_0^4)$, where T_0 is the effective temperature. As long as $T_1 \gg T_0$, as is true in these problems, T_0 can vary by a large amount while the flux is nearly constant.

Appendix C

Energy Conservation in a Sphere

Let us separate our examination of the energy conservation of the difference equations into two parts, dealing first with the pressure-derivative term. We begin with the force equation:

$$\frac{u_j^{n+\frac{1}{2}} - u_j^{n-\frac{1}{2}}}{\Delta t} = - \frac{4\pi (r_j^n)^2}{\Delta m_j} (P_{j+\frac{1}{2}}^n - P_{j-\frac{1}{2}}^n). \quad (C-1)$$

We will assume for the present that all Δm are equal and that all time steps Δt are equal. Then defining the specific volume,

$$v_{j+\frac{1}{2}}^n = \frac{4\pi}{3\Delta m} [(r_{j+1}^n)^3 - (r_j^n)^3]. \quad (C-2)$$

Thus we can write, after rearranging,

$$\begin{aligned} \frac{v_{j+\frac{1}{2}}^{n+1} - v_{j+\frac{1}{2}}^n}{\Delta t} &= \frac{4\pi}{3\Delta m \Delta t} [(r_{j+1}^{n+1})^3 - (r_{j+1}^n)^3 - (r_j^{n+1})^3 + (r_j^n)^3] \\ &= \frac{4\pi}{\Delta m \Delta t} [(r_{j+1}^{n+\frac{1}{2}})^2 (r_{j+1}^{n+1} - r_{j+1}^n) - (r_j^{n+\frac{1}{2}})^2 (r_j^{n+1} - r_j^n)] \end{aligned} \quad (C-3)$$

where

$$(r_j^{n+\frac{1}{2}})^2 = \frac{1}{3} [(r_j^{n+1})^2 + r_j^{n+1} r_j^n + (r_j^n)^2].$$

Then

$$\frac{v_{j+\frac{1}{2}}^{n+1} - v_{j+\frac{1}{2}}^n}{\Delta t} = \frac{4\pi}{\Delta m} [(r_{j+1}^{n+\frac{1}{2}})^2 u_{j+1}^{n+\frac{1}{2}} - (r_j^{n+\frac{1}{2}})^2 u_j^{n+\frac{1}{2}}]. \quad (C-4)$$

We can write (C-1) for the time step $(t^{n+\frac{1}{2}}, t^{n+\frac{3}{2}})$ by replacing n by $n+1$. If we multiply (C-1) and the later time-step version of the same equation by $u_j^{n+\frac{1}{2}}$ and add these two equations together, we find

$$\frac{u_j^{n+\frac{3}{2}} u_j^{n+\frac{1}{2}} - u_j^{n+\frac{1}{2}} u_j^{n-\frac{1}{2}}}{\Delta t} = - \frac{4\pi}{\Delta m} u_j^{n+\frac{1}{2}} [(r_j^{n+1})^2 (P_{j+\frac{1}{2}}^{n+1} - P_{j-\frac{1}{2}}^{n+1}) + (r_j^n)^2 (P_{j+\frac{1}{2}}^n - P_{j-\frac{1}{2}}^n)]. \quad (C-5)$$

Therefore, if we define the kinetic energy associated with interface j by

$$K_j^{n+1} \equiv \frac{1}{2} u_j^{n+\frac{3}{2}} u_j^{n+\frac{1}{2}}, \quad (C-6)$$

the change in kinetic energy can be written

$$\frac{(\Delta K)_j^{n+1}}{\Delta t} = - \frac{4\pi}{\Delta m} \frac{u_j^{n+1}}{2} \left[(r_j^{n+1})^2 (P_{j+\frac{1}{2}}^{n+1} - P_{j-\frac{1}{2}}^{n+1}) + (r_j^n)^2 (P_{j+\frac{1}{2}}^n - P_{j-\frac{1}{2}}^n) \right]. \quad (C-7)$$

We write the energy equation

$$\frac{E_{j+\frac{1}{2}}^{n+1} - E_{j+\frac{1}{2}}^n}{\Delta t} = S_{j+\frac{1}{2}}^{n+1} - P_{j+\frac{1}{2}}^{n+1} \frac{(V_{j+\frac{1}{2}}^{n+1} - V_{j+\frac{1}{2}}^n)}{\Delta t} - (\nabla \cdot \vec{F})_{j+\frac{1}{2}}^{n+1}. \quad (C-8)$$

If we write $(\nabla \cdot \vec{F})_{j+\frac{1}{2}} = f(j+1) - f(j)$, where the flux across each zone boundary is a function only of variables that are averaged (the averaging procedure used is arbitrary for present purposes) at the interfaces, then summing $(\nabla \cdot \vec{F})_{j+\frac{1}{2}}$ over all zones gives strict cancellation of terms at every interface except at $j=J$, the outer boundary of the problem. Note that due to the averaging of optical depths and temperatures in Eq. (32), the positive flux term for zone $j+\frac{1}{2}$ is exactly cancelled by the negative flux term for zone $j+\frac{3}{2}$. Thus, the transport terms strictly conserve energy over every time step except for energy lost or added through the outer problem boundary. Therefore, neglecting flux terms and assuming no sources, we have

$$\frac{E_{j+\frac{1}{2}}^{n+1} - E_{j+\frac{1}{2}}^n}{\Delta t} = - P_{j+\frac{1}{2}}^{n+1} \frac{(V_{j+\frac{1}{2}}^{n+1} - V_{j+\frac{1}{2}}^n)}{\Delta t}. \quad (C-9)$$

Inserting (C-3), we have

$$\frac{E_{j+\frac{1}{2}}^{n+1} - E_{j+\frac{1}{2}}^n}{\Delta t} = - \frac{4\pi}{\Delta m} P_{j+\frac{1}{2}}^{n+1} \left[(r_{j+\frac{1}{2}}^{n+1})^2 u_{j+\frac{1}{2}}^{n+1} - (r_j^{n+1})^2 u_j^{n+1} \right]. \quad (C-10)$$

Centering the energies at zone interfaces,

$$E_j^n \equiv \frac{1}{2} (E_{j+\frac{1}{2}}^n + E_{j-\frac{1}{2}}^n). \quad (C-11)$$

Adding (C-10) to the same equation for $j-\frac{1}{2}$ and using (C-11),

$$\begin{aligned} \frac{E_j^{n+1} - E_j^n}{\Delta t} = & - \frac{4\pi}{2\Delta m} \left[P_{j+\frac{1}{2}}^{n+1} (r_{j+\frac{1}{2}}^{n+1})^2 u_{j+\frac{1}{2}}^{n+1} + P_{j-\frac{1}{2}}^{n+1} (r_j^{n+1})^2 u_j^{n+1} \right. \\ & \left. - P_{j+\frac{1}{2}}^{n+1} (r_j^{n+1})^2 u_j^{n+1} - P_{j-\frac{1}{2}}^{n+1} (r_{j-\frac{1}{2}}^{n+1})^2 u_{j-\frac{1}{2}}^{n+1} \right]. \end{aligned} \quad (C-12)$$

Now if we add (C-12) and (C-7) and define

$$\frac{\Delta H_j^{n+1}}{\Delta t} = \frac{K_j^{n+1} + E_j^{n+1} - K_j^n - E_j^n}{\Delta t}, \quad (C-13)$$

we have, after rearranging,

$$\begin{aligned} \frac{\Delta H_j^{n+1}}{\Delta t} = & -\frac{4\pi}{\Delta m} \left\{ \frac{1}{2} P_{j+\frac{1}{2}}^{n+\frac{1}{2}} \left[(r_{j+1}^{n+\frac{1}{2}})^2 u_{j+1}^{n+\frac{1}{2}} + (r_j^{n+\frac{1}{2}})^2 u_j^{n+\frac{1}{2}} \right] \right. \\ & - \frac{1}{2} P_{j-\frac{1}{2}}^{n+\frac{1}{2}} \left[(r_j^{n+\frac{1}{2}})^2 u_j^{n+\frac{1}{2}} + (r_{j-1}^{n+\frac{1}{2}})^2 u_{j-1}^{n+\frac{1}{2}} \right] \\ & + u_j^{n+\frac{1}{2}} \left[\frac{P_{j+\frac{1}{2}}^{n+1} (r_j^{n+1})^2 + P_{j+\frac{1}{2}}^n (r_j^n)^2}{2} - P_{j+\frac{1}{2}}^{n+\frac{1}{2}} (r_j^{n+\frac{1}{2}})^2 \right] \\ & \left. - u_j^{n+\frac{1}{2}} \left[\frac{P_{j-\frac{1}{2}}^{n+1} (r_j^{n+1})^2 + P_{j-\frac{1}{2}}^n (r_j^n)^2}{2} - P_{j-\frac{1}{2}}^{n+\frac{1}{2}} (r_j^{n+\frac{1}{2}})^2 \right] \right\}. \quad (C-14) \end{aligned}$$

Defining the rate at which work is being done as

$$W_{j+\frac{1}{2}}^{n+\frac{1}{2}} = 4\pi P_{j+\frac{1}{2}}^{n+\frac{1}{2}} \left[(r_{j+1}^{n+\frac{1}{2}})^2 u_{j+1}^{n+\frac{1}{2}} + (r_j^{n+\frac{1}{2}})^2 u_j^{n+\frac{1}{2}} \right], \quad (C-15)$$

then

$$\begin{aligned} \frac{W_{j+\frac{1}{2}}^{n+\frac{1}{2}} - W_{j+\frac{1}{2}}^{n-\frac{1}{2}}}{\Delta m} = & -\frac{4\pi}{\Delta m} \left\{ \frac{1}{2} P_{j+\frac{1}{2}}^{n+\frac{1}{2}} \left[(r_{j+1}^{n+\frac{1}{2}})^2 u_{j+1}^{n+\frac{1}{2}} + (r_j^{n+\frac{1}{2}})^2 u_j^{n+\frac{1}{2}} \right] \right. \\ & \left. - \frac{1}{2} P_{j-\frac{1}{2}}^{n+\frac{1}{2}} \left[(r_j^{n+\frac{1}{2}})^2 u_j^{n+\frac{1}{2}} + (r_{j-1}^{n+\frac{1}{2}})^2 u_{j-1}^{n+\frac{1}{2}} \right] \right\}. \quad (C-16) \end{aligned}$$

Thus, if we desire exact energy conservation, namely

$$\frac{H_j^{n+1} - H_j^n}{\Delta t} + \frac{W_{j+\frac{1}{2}}^{n+\frac{1}{2}} - W_{j+\frac{1}{2}}^{n-\frac{1}{2}}}{\Delta m} = 0, \quad (C-17)$$

by comparing terms in (C-14) and (C-16) we must define $P_{j+\frac{1}{2}}^{n+\frac{1}{2}}$ by

$$P_{j+\frac{1}{2}}^{n+\frac{1}{2}} (r_j^{n+\frac{1}{2}})^2 = \frac{1}{2} \left[P_{j+\frac{1}{2}}^{n+1} (r_j^{n+1})^2 + P_{j+\frac{1}{2}}^n (r_j^n)^2 \right]. \quad (C-18)$$

This is effectively a weighting of the time-step averaged pressure due to radial changes during the time interval. For static geometry, this reduces to a simple linear average in time,

$$P_{j+\frac{1}{2}}^{n+\frac{1}{2}} = \frac{1}{2} (P^n + P^{n+1}). \quad (C-19)$$

This defines how the averaging must be done to strictly conserve energy. It is, of course, a function of a number of slightly arbitrary definitions, but the result is characteristic of many averaging schemes for intensive variables in spherical geometries. Here the pressure is weighted by a combination of geometrical and time factors. We now have a prescription for defining the properly weighted pressure to be used in iterative solutions of the energy balance equation.

Appendix D

Energy Conservation with Gravity

We return to the force equation to consider the gravitational term. We will make a similar set of definitions to those in Appendix C to strictly conserve energy; we will also obtain an expression for the truncation error resulting from that definition. For the gravitational contribution to Δu ,

$$\frac{u_j^{n+\frac{1}{2}} - u_j^{n-\frac{1}{2}}}{\Delta t} = - \frac{GM_j(r_j^n)}{(r_j^n)^2}. \quad (D-1)$$

Again, for simplicity, we assume constant Δt and Δm , and we consider only the gravitational term, not the pressure derivative term. Multiplying (D-1) and the analogous expression for $(t^{n+\frac{1}{2}}, t^{n-\frac{1}{2}})$ by $u_j^{n+\frac{1}{2}}$ and adding them together and including the definition used earlier (C-6) for the kinetic energy per unit mass, we find

$$\frac{K_j^{n+1} - K_j^n}{\Delta t} = - \frac{GM_j}{2} u_j^{n+\frac{1}{2}} \left[\frac{1}{(r_j^{n+1})^2} + \frac{1}{(r_j^n)^2} \right]. \quad (D-2)$$

But since $u_j^{n+\frac{1}{2}} = (r_j^{n+1} - r_j^n)/t$, we may write (D-2) as

$$\frac{K_j^{n+1} - K_j^n}{\Delta t} = - \frac{GM_j}{2\Delta t} \left[(r_j^{n+1} - r_j^n) \left(\frac{1}{(r_j^{n+1})^2} + \frac{1}{(r_j^n)^2} \right) \right]. \quad (D-3)$$

and after some manipulation,

$$\begin{aligned} \frac{K_j^{n+1} - K_j^n}{\Delta t} = & - \frac{GM_j}{2\Delta t} \left[\frac{1}{r_j^{n+1}} - \frac{1}{r_j^n} \right] - \frac{GM_j}{2\Delta t} \left[\frac{(r_j^{n+1})^2 + r_j^{n+1} r_j^n + (r_j^n)^2}{(r_j^n)^2 r_j^{n+1}} \right. \\ & \left. - \frac{(r_j^{n+1})^2 + r_j^{n+1} r_j^n + (r_j^n)^2}{r_j^n (r_j^{n+1})^2} \right]. \end{aligned} \quad (D-4)$$

If we use an approximate expression,

$$r_j^{n+1} r_j^n \approx \frac{1}{3} \left[(r_j^{n+1})^2 + r_j^{n+1} r_j^n + (r_j^n)^2 \right] \quad (D-5)$$

we can simplify (D-4) to give

$$\frac{K_j^{n+1} - K_j^n}{\Delta t} = \frac{GM_j}{\Delta t} \left(\frac{1}{r_j^{n+1}} - \frac{1}{r_j^n} \right). \quad (D-6)$$

If we define the rate of change of gravitational potential energy in the obvious manner,

$$\frac{G_j^{n+1} - G_j^n}{\Delta t} = - \frac{GM_j}{\Delta t} \left(\frac{1}{r_j^{n+1}} - \frac{1}{r_j^n} \right). \quad (D-7)$$

by adding (D-6) and (D-7), we have exact energy conservation.

We can examine the implications of (D-5), the only approximation used, to determine the error involved. We use

$$\begin{aligned} (r_j^{n+1})^2 + r_j^{n+1} r_j^n + (r_j^n)^2 &= 3r_j^{n+1} r_j^n + (r_j^{n+1} - r_j^n)^2 \\ &= 3r_j^{n+1} r_j^n + (\Delta r_j)^2 \end{aligned} \quad (D-8)$$

and substitute into (D-4) and find, after manipulation,

$$\begin{aligned} \frac{K_j^{n+1} - K_j^n}{\Delta t} &= - \frac{GM_j}{\Delta t} \left(\frac{1}{r_j^n} - \frac{1}{r_j^{n+1}} \right) + \frac{GM_j}{2\Delta t} \frac{(\Delta r_j)^2}{r_j^{n+1} r_j^n} \left(\frac{1}{r_j^{n+1}} - \frac{1}{r_j^n} \right) \\ &= - \left(\frac{G_j^{n+1} - G_j^n}{\Delta t} \right) - \frac{GM_j}{2\Delta t} \frac{(\Delta r_j)^3}{(r_j^{n+1})^2 (r_j^n)^2}. \end{aligned} \quad (D-9)$$

Therefore, adding the kinetic energy and gravitational potential energy, we find that the excess is given by

$$\frac{|\Delta E_j|}{\Delta t} = \frac{|\Delta K_j + \Delta G_j|}{\Delta t} = \frac{GM_j}{2\Delta t} \frac{|(\Delta r_j)^3|}{(r_j^{n+1})^2 (r_j^n)^2}. \quad (D-10)$$

This is approximately

$$\frac{|\Delta E_j|}{\Delta t} \approx \frac{GM_j}{r_j} \frac{1}{2\Delta t} \left(\frac{\Delta r_j}{r_j} \right)^3 \quad (D-11)$$

where

$$r_j^4 = (r_j^{n+1})^2 (r_j^n)^2. \quad (D-12)$$

Therefore,

$$|\Delta E_j| = \frac{GM_j}{r_j} \left\{ \frac{1}{2} \left[\left(\frac{\Delta r_j}{r_j} \right)^3 \right] \right\}. \quad (D-13)$$

Since $\Delta r_j = r_j^{n+1} - r_j^n$, the change in one zone radius over one time step, then $\Delta r_j = u_j \Delta t$, and the error in the energy conservation is roughly third-order in Δt , compared with the magnitude of G_j and K_j . Therefore, the error in energy conservation that is a result of differencing the gravitational term in the force equation is in general smaller than that due to the pressure gradient term, which is first-order or second-order in Δt , depending on the time-centering of $P_{j+1/2}$.³⁵

Appendix E

Equations for Implicit Hydrostatics

CASE I—NO CONVECTION

We begin with the basic difference equations

$$r_j^n = \frac{4\pi}{\Delta m_j} (r^2)_j^n (P_{j-\frac{1}{2}}^n - P_{j+\frac{1}{2}}^n) - \frac{GM_j}{(r^2)_j^n} \quad (E-1)$$

$$\begin{aligned} \frac{c_V}{\Delta t} (T_{j+\frac{1}{2}}^{n+1} - T_{j+\frac{1}{2}}^n) = S - [P + (\partial E)] \frac{\Delta V_{j+\frac{1}{2}}^{n+\frac{1}{2}}}{\Delta t} \\ + \frac{v}{V} \left[A_{j+1} \tilde{T}_{j+1}^3 \frac{(T_{j+\frac{1}{2}}^{n+1} - T_{j+\frac{1}{2}}^n)}{(\frac{\Delta r}{\lambda})_{j+1}} - A_j \tilde{T}_j^3 \frac{(T_{j+\frac{1}{2}}^{n+1} - T_{j+\frac{1}{2}}^n)}{(\frac{\Delta r}{\lambda})_j} \right] \end{aligned} \quad (E-2)$$

where all suppressed indices are $(n+\frac{1}{2})$ or $(j+\frac{1}{2})$. We require $r_j^{n+1} = 0$, for all j . We expand r_j^{n+1} to first order in the independent variables r and T .

$$\begin{aligned} r_j^{n+1} = r_j^n + \frac{\partial r_j^n}{\partial r_{j-1}^n} (\Delta r)_{j-1}^{n+\frac{1}{2}} + \frac{\partial r_j^n}{\partial r_j^n} (\Delta r)_j^{n+\frac{1}{2}} + \frac{\partial r_j^n}{\partial r_{j+1}^n} (\Delta r)_{j+1}^{n+\frac{1}{2}} \\ + \frac{\partial r_j^n}{\partial T_{j-\frac{1}{2}}^n} (\Delta T)_{j-\frac{1}{2}}^{n+\frac{1}{2}} + \frac{\partial r_j^n}{\partial T_{j+\frac{1}{2}}^n} (\Delta T)_{j+\frac{1}{2}}^{n+\frac{1}{2}} = 0, \end{aligned} \quad (E-3)$$

where

$$(\Delta r)_{j-1}^{n+\frac{1}{2}} \equiv r_{j-1}^{n+1} - r_{j-1}^n, \text{ etc.}$$

We define

$$L_j \equiv r_j^n - \left(\frac{\partial r_j^n}{\partial r_{j-1}^n} r_{j-1}^n + \frac{\partial r_j^n}{\partial r_j^n} r_j^n + \frac{\partial r_j^n}{\partial r_{j+1}^n} r_{j+1}^n + \frac{\partial r_j^n}{\partial T_{j-\frac{1}{2}}^n} T_{j-\frac{1}{2}}^n + \frac{\partial r_j^n}{\partial T_{j+\frac{1}{2}}^n} T_{j+\frac{1}{2}}^n \right).$$

Equation (E-3) then becomes

$$L_j + \frac{\partial r_j^n}{\partial r_{j-1}^n} r_{j-1}^{n+1} + \frac{\partial r_j^n}{\partial r_j^n} r_j^{n+1} + \frac{\partial r_j^n}{\partial r_{j+1}^n} r_{j+1}^{n+1} + \frac{\partial r_j^n}{\partial T_{j-\frac{1}{2}}^n} T_{j-\frac{1}{2}}^{n+1} + \frac{\partial r_j^n}{\partial T_{j+\frac{1}{2}}^n} T_{j+\frac{1}{2}}^{n+1} = 0. \quad (E-4)$$

We approximate $\Delta V_{j+\frac{1}{2}}^{n+\frac{1}{2}}$

$$\Delta V_{j+\frac{1}{2}}^{n+\frac{1}{2}} = \frac{4\pi}{3\Delta m_{j+\frac{1}{2}}} \left[3(r^2)_{j+1}^n (r_{j+1}^{n+1} - r_{j+1}^n) - 3(r^2)_j^n (r_j^{n+1} - r_j^n) \right].$$

We define

$$\begin{aligned} a_j &= \frac{A_j^n T_j^3}{4\pi \left(\frac{\Delta r}{\lambda} \right)_j} \\ \alpha_{j+\frac{1}{2}} &= \frac{\Delta m_{j+\frac{1}{2}}}{4\pi} \frac{c_V}{\Delta t} \\ e_{j+\frac{1}{2}} &= \left(P + \frac{\partial E}{\partial V} \right)_{j+\frac{1}{2}}^{n+\frac{1}{2}} \frac{A_{j+1}^n}{4\pi \Delta t} \\ f_{j+\frac{1}{2}} &= \left(P + \frac{\partial E}{\partial V} \right)_{j+\frac{1}{2}}^{n+\frac{1}{2}} \frac{A_j^n}{4\pi \Delta t} \\ C_{j+\frac{1}{2}} &= \frac{\Delta m_{j+\frac{1}{2}}}{4\pi} S_{j+\frac{1}{2}} + e_{j+\frac{1}{2}} r_{j+1}^n - f_{j+\frac{1}{2}} r_j^n + \alpha_{j+\frac{1}{2}} T_{j+\frac{1}{2}}^n \end{aligned}$$

and multiply (E-2) by $\Delta m_{j+\frac{1}{2}}/4\pi$ to finally write

$$a_{j+1} T_{j+\frac{3}{2}}^{n+1} + (-\alpha_{j+\frac{1}{2}} - a_{j+1} - a_j) T_{j+\frac{1}{2}}^{n+1} + \left(a_j T_{j-\frac{1}{2}}^{n+1} \right) - \left(e_{j+\frac{1}{2}} r_{j+1}^{n+1} \right) + \left(f_{j+\frac{1}{2}} r_j^{n+1} \right) + (C_{j+\frac{1}{2}}) = 0. \quad (\text{E-5})$$

Equations (E-4) and (E-5) are now the simplified forms of the two original equations. Note that the expansion of (E-3) has made the differencing of the force equation implicit in time. Also, the coupling between the two equations now includes the different treatment of the $\Delta V/\Delta t$ term in (E-2). Most of the coefficients are nonlinear in r and T , with the result that solution of the equations will probably require iteration.

We solve the equations using a form of the back-substitution method.³⁵ We write

$$\begin{aligned} T_{j-\frac{1}{2}}^{n+1} &= E_j T_{j+\frac{1}{2}}^{n+1} + F_j r_j^{n+1} + G_j \\ r_{j-1}^{n+1} &= H_{j-1} T_{j-\frac{1}{2}}^{n+1} + I_{j-1} r_j^{n+1} + J_{j-1}. \end{aligned} \quad (\text{E-6})$$

Inserting these terms into (E-4) and (E-5), we solve for $T_{j+\frac{1}{2}}^{n+1}$ and r_j^{n+1} , finding

$$\begin{aligned} r_j^{n+1} &= H_j T_{j+\frac{1}{2}}^{n+1} + I_j r_{j+1}^{n+1} + J_j \\ T_{j+\frac{1}{2}}^{n+1} &= E_{j+1} T_{j+\frac{3}{2}}^{n+1} + F_{j+1} r_{j+1}^{n+1} + G_{j+1} \end{aligned}$$

where

$$H_j = \left(\frac{\partial r_j^n}{\partial r_{j-1}^n} H_{j-1} E_j + \frac{\partial r_j^n}{\partial T_{j-\frac{1}{2}}^n} E_j + \frac{\partial r_j^n}{\partial T_{j+\frac{1}{2}}^n} \right) / D_1$$

$$I_j = \left(\frac{\partial r_j^n}{\partial r_{j+1}^n} \right) / D_1$$

$$J_j = \left(L_j + \frac{\partial r_j^n}{\partial r_{j-1}^n} J_{j-1} + \frac{\partial r_j^n}{\partial r_{j-1}^n} H_{j-1} G_j + \frac{\partial r_j^n}{\partial T_{j-\frac{1}{2}}^n} G_j \right) / D_1$$

where

$$D_1 = - \left(\frac{\partial r_j^n}{\partial r_{j-1}^n} H_{j-1} F_j + \frac{\partial r_j^n}{\partial T_{j-\frac{1}{2}}^n} F_j + \frac{\partial r_j^n}{\partial r_{j-1}^n} I_{j-1} + \frac{\partial r_j^n}{\partial r_j^n} \right) \quad (E-7)$$

$$E_{j+1} = a_{j+1} / D_2$$

$$F_{j+1} = (a_j F_j I_j + f_{j+\frac{1}{2}} I_j - e_{j+\frac{1}{2}}) / D_2$$

$$G_{j+1} = (a_j F_j J_j + f_{j+\frac{1}{2}} J_j + a_j G_j + C_{j+\frac{1}{2}}) / D_2$$

where

$$D_2 = -(a_j E_j - \alpha_{j+\frac{1}{2}} - a_{j+1} - a_j + a_j F_j H_j + f_{j+\frac{1}{2}} H_j).$$

We require two boundary conditions at the inner boundary and two more at the outer boundary. At the inner boundary,

$$r_0^{n+1} = 0$$

$$\left. \frac{dT}{dr} \right|_{r=0} = 0.$$

Beginning with these inner boundary conditions, we can use the recursion relations (E-7) to generate the full set of coefficients.

At the outside boundary, we have two types of situations. When we have a proto-stellar core imbedded in a lower density atmosphere, there are still more zones outside the region described by our implicit hydrostatic equations. For those cases, let $j=J$ indicate the outermost zone in the core. Then $T_{J+\frac{1}{2}}^{n+1}$ and r_{J+1}^{n+1} are known from separate calculations, and these provide the necessary outer boundary conditions. In Fig. E-1 the core includes all of the material within r_J . We visualize the hydrostatic problem

as a problem with time-dependent boundary conditions. The set of radii (r_j , $j=1, J$) are free to move. Moving them will exchange energy between zones, including possible changes in the energy density in the zone immediately outside the core, E_{J+1} .

The first theoretical model tried was to consider the perturbations of the zone radii separately from any energy transport mechanisms. Thus each zone was perturbed

Fig. E-1. Schematic diagram of protostar with a core imbedded inside material which is still falling inward. The relative magnitudes of the above radii are not drawn to scale; if a linear scale were used, the core would be too small to be even visible on the page. The outer radius of the core is r_J .

adiabatically, and with information from the equation of state it was possible to directly calculate changes in zone internal energy densities and temperatures. This scheme turned out to be unstable in practice in most cases. The adiabatic perturbation could be done implicitly and stably with no problems, but the subsequent transport calculations, including radiation and convection, resulted in increasingly large oscillations in radii and temperatures. The cause of this type of instability is the fact that an equilibrium exists with two opposing operators, both of which, if operating

alone, would produce large changes in configuration. A small compression results in a large temperature increase which is immediately radiated or convected away. This type of "push-pull" instability is common to many types of equilibrium calculations, and the usual cure is to couple the opposing operators together and solve the combined system in one implicit scheme. Physically, we can interpret the situation as one in which the relevant time scales for radiation or convection and gravitational contraction are roughly equal. In the coupled scheme the energy fluxes are determined implicitly; since the flux through the outer boundary of the core involves E_{J+2}^{n+1} or T_{J+2}^{n+1} , we cannot

use $\rho_{j+\frac{1}{2}}$ or $T_{j+\frac{1}{2}}$ for prescribing boundary conditions for calculation of the core, but we can use $\rho_{j+\frac{1}{2}}$. The other boundary value for this case is the known value of r_{j+1} .

When there is a core imbedded in a low-density cloud, the numerical integrations are done in a very specific order. The first calculation that is done for each time step is the explicit hydrodynamic calculation for the surrounding gas cloud that is not in hydrostatic equilibrium. In every case, the density and opacity of this material is so low that energy transport processes are not significant, leaving the free-fall time scale, Eq. (6), as the smallest relevant time scale. With the zone boundaries advanced to t^{n+1} , the core can be advanced to the same time using the numerical model above. The only required approximation is that $E_{j+\frac{1}{2}}^{n+1} \approx E_{j+\frac{1}{2}}^n$. This is quite good to about 1% because of the selection of the time step. In practice, $E_{j+\frac{1}{2}} \gg E_{j+\frac{1}{2}}$ so the core equilibrium is extremely insensitive to even large variations in $E_{j+\frac{1}{2}}$.

The other situation that arises is one where the entire protostellar mass is in hydrostatic equilibrium, in which case r_j is the outer boundary of the protostar. In that case $T_{j+\frac{1}{2}}$ is set equal to a constant, the same value as the initial temperature of the entire cloud, reasoning that conditions in the surrounding interstellar medium do not change appreciably during the evolution of the bound protostar. As in the case above, solutions are insensitive to the boundary temperature in any case. The other exterior boundary condition is $r_{j+1} \rightarrow \infty$, but in this case $I_j = 0$ in (E-7) so that r_j is coupled to $T_{j+\frac{1}{2}}$ only.

CASE II—CONVECTION INCLUDED

We have previously obtained the numerical form for the convective flux, Eq. (42). If we include the convective flux in (E-2), we add a term

$$\frac{v}{V} \left\{ A_{j+1} b_{j+1} \delta_{j+1} \left[\frac{(T_{j+\frac{1}{2}}^{n+1} - T_{j+\frac{1}{2}}^{n+1})}{\Delta r_{j+1}} + \left(\frac{\Gamma_2 - 1}{\Gamma_2} \right)_{j+1} \frac{T_{j+\frac{1}{2}}^{n+1}}{\lambda_{P_{j+\frac{1}{2}}}} \right] \right. \\ \left. - A_j b_j \delta_j \left[\frac{(T_{j+\frac{1}{2}}^{n+1} - T_{j-\frac{1}{2}}^{n+1})}{\Delta r_j} + \left(\frac{\Gamma_2 - 1}{\Gamma_2} \right)_j \frac{T_{j-\frac{1}{2}}^{n+1}}{\lambda_{P_{j-\frac{1}{2}}}} \right] \right\}$$

where

$$\delta_j = 1 \quad \text{if} \quad (\Delta \nabla T)_j \geq 0$$

$$\delta_j = 0 \quad \text{if} \quad (\Delta \nabla T)_j < 0.$$

As before, we multiply by $\frac{\Delta m_{j+1/2}}{4\pi}$ and define

$$\begin{aligned} f_{j+1/2} &= \frac{A_{j+1} b_{j+1} \delta_{j+1}}{4\pi \Delta r_{j+1}} \\ P_{j+1/2} &= \left(\frac{\Gamma_2 - 1}{\Gamma_2} \right)_{j+1} \frac{A_{j+1} b_{j+1} \delta_{j+1}}{4\pi \lambda P_{j+1}} - \frac{A_{j+1} b_{j+1} \delta_{j+1}}{4\pi \Delta r_{j+1}} - \frac{A_j b_j \delta_j}{4\pi \Delta r_j} \\ q_{j+1/2} &= \frac{A_j b_j \delta_j}{4\pi \Delta r_j} - \left(\frac{\Gamma_2 - 1}{\Gamma_2} \right)_j \frac{A_j b_j \delta_j}{4\pi \lambda P_j} \end{aligned}$$

Then (E-5) becomes

$$\begin{aligned} (a_{j+1} + f_{j+1/2}) T_{j+1/2}^{n+1} + (-\alpha_{j+1/2} - a_{j+1} - a_j + P_{j+1/2}) T_{j+1/2}^{n+1} + (a_j + q_{j+1/2}) T_{j-1/2}^{n+1} \\ - e_{j-1/2} r_{j+1}^{n+1} + f_{j+1/2} r_j^{n+1} + C_{j+1/2} = 0 \end{aligned} \quad (E-9)$$

Following the procedure above, E_{j+1} , F_{j+1} , and G_{j+1} include new terms, while H_j , I_j , and J_j are unchanged in form. Including convection, we now have

$$\begin{aligned} E_{j+1} &= (a_{j+1} + f_{j+1/2})/D_3 \\ F_{j+1} &= (a_j F_j I_j + f_{j+1/2} I_j - e_{j+1/2} + q_{j+1/2} F_j I_j)/D_3 \\ G_{j+1} &= (a_j F_j J_j + f_{j+1/2} J_j + a_j G_j + C_{j+1/2} + q_{j+1/2} F_j J_j + q_{j+1/2} G_j)/D_3 \end{aligned} \quad (E-10)$$

where

$$D_3 = -(a_j E_j - \alpha_{j+1/2} - a_{j+1} - a_j + a_j F_j H_j + f_{j+1/2} H_j + q_{j+1/2} E_j + P_{j+1/2} + q_{j+1/2} F_j H_j).$$

The boundary conditions are the same as for the nonconvective case. Since our model for convection requires that no net mass motion occur, we require $\delta_j = 0$ for the situation where the entire protostar is in equilibrium. For the case where there is a core surrounded by additional material, we also set $\delta_j = 0$, since the convective medium must be in hydrostatic equilibrium.

Iteration Scheme

As noted earlier, the nonlinearity of the coefficients in the equations of motion and energy balance means that the solutions from the back-substitution may not exactly satisfy (E-2) and (E-3). These nonlinearities arise in three major ways:

1. Zone interface conduction coefficients and opacities depend on spatial averages. For example, opacities at interfaces are evaluated using a temperature T_j defined by $\bar{T}_j^4 = \frac{1}{2} (T_{j+1}^4 + T_{j-1}^4)$.

2. Parameters must be appropriately averaged in time over $t^{n+1/2}$. For most problems it is sufficient to evaluate many of the coefficients at t^n , hoping that they vary slowly in time. This is questionable generally only in the region near the boundary of the protostellar core, the only portion of the problem where conditions are changing rapidly. In these calculations, most coefficients are evaluated at t^n . The rationale for this procedure is that the benefits resulting from the increased stability of the difference equations outweighed any inaccuracies produced. To test the validity of using parameters evaluated at t^n , several trial models were calculated fully implicitly with all coefficients evaluated at t^{n+1} . After each iteration, all coefficients were re-evaluated and another iteration was performed until the solutions for zone radii and temperatures both converged to some arbitrary tolerance. The results of this experiment were completely negative: for some problems the method became unstable, and for all cases which were stable no differences could be seen between those results and those found by evaluating parameters at t^n .

3. Spatial nonlinearities in the force equation are due to the dependence on (r, T) . The partial derivatives in (E-3) can be large. To reduce these errors an iterative procedure is employed, derived from (E-3). We re-interpret (E-3) as an expansion of r_j^{n+1} in Taylor series through first-order in each independent variable. If we replace the index n by an iteration index, say m , then we interpret (E-3) as a prescription for successive approximation of $\{r^{n+1}, T^{n+1}\}$, using a multidimensional Newton's method. The $\Delta r, \Delta T$ terms are now

$$\begin{aligned} (\Delta r^{n+1})^{m+1/2} &= (r^{n+1})^{m+1} - (r^{n+1})^m \\ (\Delta T^{n+1})^{m+1/2} &= (T^{n+1})^{m+1} - (T^{n+1})^m \end{aligned} \quad (E-11)$$

Note that we iterate with the force equation but not the energy equation (E-2), so that the $\{r^n\}$ that appear in the $(\Delta V)^{n+1/2}$ continue to be evaluated at t^n . This is because (E-2) relates the change of temperature or energy density to the amount of work done over the entire time step.

Due to sensitivity of the forces to small changes in temperature and radii and to reduce the chances of calculating oscillatory solutions (a distinct possibility since we are dealing with equilibrium configurations), an under-relaxation method is used, defined by

$$\{r^{n+1}, T^{n+1}\}^{m+1} = \{r^{n+1}, T^{n+1}\}^m + \alpha [\{r^{n+1}, T^{n+1}\}^{m+1} - \{r^{n+1}, T^{n+1}\}^m]. \quad (E-12)$$

Ordinarily $\alpha = 1$, and the new results are accepted. For most calculations here we used $\alpha = 0.2$ and allowed at most ten iterations to converge. Usually two or three iterations were sufficient. Of course α need not be a scalar. A vector under-relaxation parameter could be employed, where α_j might be proportional to $(r_j^{n+1})^{m+1}$. The simpler scalar method was used for these calculations.

The first approximation in the iteration is given by

$$\{r^{n+1}, T^{n+1}\}^{m=0} = \{r^n, T^n\}.$$

Convergence is defined by

$$\begin{aligned} \left| (r_j^{n+1})^{m+1} - (r_j^{n+1})^m \right| &< \delta (r_j^{n+1})^{m+1} \\ \left| (T_j^{n+1})^{m+1} - (T_j^{n+1})^m \right| &< \delta (T_j^{n+1})^{m+1} \end{aligned} \quad (E-13)$$

for all j , where δ is an input parameter, usually set to 0.005.

Appendix F

Equation of State Calculations

We assume that for calculations of pressure and specific internal energy we can include hydrogen and helium only; other elements or molecules, while possibly contributing significantly to the opacity, make an insignificant contribution to the pressure and energy. We include H_2 , H_1 , H_1^+ , He , He^+ , and He^{++} , represented respectively by x_m , x_0 , x_1 , y_0 , y_1 , and y_2 . The pressure (material pressure) is given by

$$P = \frac{kT}{H} \left[\left(1 - \frac{x_m}{2} + x_1\right) X + (1 + y_1 + 2y_2) \frac{Y}{4} \right] \text{ erg/cm}^3 \quad (\text{F-1})$$

and the energy is given by

$$E = \frac{3}{2} \frac{kT}{H} \left[\left(1 + x_1 - \frac{x_m}{6}\right) X + (1 + y_1 + 2y_2) \frac{Y}{4} \right] + \frac{X}{H} \left[\left(\frac{x_0 + x_1}{2}\right) I_m + x_1 I_{10} + x_m I_{\text{vib}} \right] + \frac{Y}{4H} \left[(y_1 + y_2) I_{20} + y_2 I_{21} \right] \text{ erg/g} \quad (\text{F-2})$$

where X and Y are the total mass fractions of hydrogen and helium respectively, I_m is the dissociation potential of H_2 , $I_{\text{vib}} = h\nu$ where ν is the vibrational frequency of the hydrogen molecule, and I_{10} , I_{20} , and I_{21} are the ionization potentials for H_1 , He , and He^+ respectively. The fractional concentrations satisfy the following equations:

$$x_m + x_0 + x_1 = 1$$

$$y_0 + y_1 + y_2 = 1$$

$$\frac{x_1}{x_0} = \frac{AT^{3/2}}{\epsilon\rho} e^{-I_{10}/kT} = r_{1h}$$

$$\frac{y_1}{y_0} = \frac{4AT^{3/2}}{\epsilon\rho} e^{-I_{20}/kT} = r_{1y}$$

$$\frac{y_2}{y_1} = \frac{AT^{3/2}}{\epsilon\rho} e^{-I_{21}/kT} = r_{2y}$$

$$\frac{x_0^2}{x_m} = \frac{BT^{1/2}}{\rho} \left(1 - e^{-I_{\text{vib}}/kT}\right) e^{-I_m/kT} = r_m \quad (\text{F-3})$$

where

$$\epsilon = x_1 X + (y_1 + 2y_2) \frac{Y}{4}. \quad (F-4)$$

Using a method of solution suggested by Tarter,^{*} we assume a value for ϵ . Once ϵ is known, Eq. (F-3) can be solved in order,

$$\begin{aligned} a &= r_m (1 + r_{1h})^2 \\ x_m &= 1 / \left(1 + \frac{a}{2} + \frac{1}{2} \sqrt{a^2 + 4a} \right) \\ x_0 &= \sqrt{r_m x_m} \\ x_1 &= r_{1h} x_0 \\ y_0 &= 1 / (1 + r_{1y})(1 + r_{2y}) \\ y_1 &= r_{1y} y_0 \\ y_2 &= r_{2y} y_1. \end{aligned} \quad (F-5)$$

ϵ can then be re-evaluated by (F-4) and compared with the initial estimate. The new value can then be used as a new estimate, and an iteration is carried out until ϵ converges. In practice, this convergence is very rapid; furthermore, if ϵ is saved from one time step to the next in the program for each zone the previous solution is used as the new first approximation, and convergence is usually achieved in one iteration since the fractional concentrations do not change rapidly except in fairly narrowly defined regimes of temperature and density.

To evaluate derivatives with respect to density and temperature a perturbation technique is used, simply perturbing the temperature while holding the density constant and explicitly calculating the derivative,

$$\begin{aligned} T_0^1 &= (1 + \delta) T_0 \\ \left. \frac{\partial P}{\partial T} \right|_{T=T_0} &= \frac{P(T_0^1) - P(T_0)}{T_0 \delta} \end{aligned}$$

and similarly for density derivatives and derivatives of the energy.

^{*}C. B. Tarter, Lawrence Livermore Laboratory, private communication.

In some calculations radiation pressure and energy density are important, so we add $\frac{3T^4}{3}$ to (F-1) and $\frac{2T^4}{\rho}$ to (F-2) in all cases. The values used for the required constants are:

$$I_{\text{vib}} = 0.53 \text{ eV}$$

$$I_m = 4.48 \text{ eV}$$

$$I_{10} = 13.6 \text{ eV}$$

$$I_{20} = 24.58 \text{ eV}$$

$$I_{21} = 54.4 \text{ eV}$$

$$A = 2 \left(\frac{2\pi m_e k}{h^2} \right)^{3/2}$$

$$B = \frac{H^{5/2}}{2hI} \pi^{-1/2}$$

$$I = h^2 / (8\pi^2 k T_{\text{rot}})$$

$$T_{\text{rot}} = 85.4^\circ \text{K}$$

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Appendix G

Glossary

a	radiation constant
A	area, cm^2
c	velocity of light, 3×10^{10} cm/sec
c_p	specific heat at constant pressure
ϵ	specific energy density, ergs/cm^3 or erg/g
f	convective flux limiter
$\bar{F}$	energy flux, $\text{erg/cm}^2\text{-sec}$
$\bar{F}_c$	convective energy flux, $\text{erg/cm}^2\text{-sec}$
g	acceleration due to gravity, cm/sec^2
G	gravitational constant, 6.67×10^{-8} $\text{cm}^3/\text{g-sec}^2$
H_p	pressure scale height, cm
H_ρ	density scale height, cm
K	opacity, cm^2/g
L	luminosity, ergs/sec
$L_\odot$	luminosity of sun,
m	mass, grams
dm	zone or shell mass, grams
$m_\odot$	mass of the sun
P	pressure, erg/cm^3
Q	artificial viscosity or quantity of heat energy
r	radius, cm
$R_\odot$	radius of the sun
S	entropy, $\text{erg}/^\circ\text{K}$
T	temperature, $^\circ\text{K}$
t	time, seconds or years
t_{ff}	free-fall time, seconds or years
Δt_c	Courant time step
u	material velocity, cm/sec
v	volume, cm^3
V	specific volume, cm^3/g
γ	ratio of specific heats for a gas
γ_{conv}	convective "efficiency" $\frac{\Gamma_2}{\Gamma_2 - 1}$
Γ_2	adiabatic exponent in $P \propto T^{\Gamma_2}$
Λ	convective mixing length, cm
λ_{Ross}	Rosseland radiative mean free path, cm
ρ	density, g/cm^3
$\Delta \nabla T$	excess adiabatic temperature gradient, $^\circ\text{K/cm}$
$\Delta \nabla \rho$	excess adiabatic density gradient, g/cm^4

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