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SPECTRAL AND ANGULAR DISTRIBUTIONS
OF SYNCHROTRON RADIATION

R. A. Mack

February 3, 1966

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
and HARVARD UNIVERSITY

CAMBRIDGE ELECTRON ACCELERATOR

CAMBRIDGE 38, MASSACHUSETTS

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SPECTRAL AND ANGULAR DISTRIBUTIONS

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ABSTRACT

Spectral and angular distributions of synchrotron radiation are described, tabulated, and plotted in a more general form than has previously been reported.

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INTRODUCTION

Charged particles moving in a circular arc in a magnetic field undergo radial acceleration, and radiate electromagnetic energy. This paper tabulates some of the characteristics of this radiation for electrons having total energy much greater than rest energy. In this case, the "synchrotron" radiation has a broad spectral distribution peaking at frequencies far greater than the angular rotation frequency of the electron in its orbit.

The basic equations describing synchrotron radiation are derived by Schwinger, Phys. Rev. 75 1912 (1949). The development of Schwinger's equations into the form used here for computation is presented in appendices to the present paper.

A. The total synchrotron radiation power, summed over all frequencies (or wavelengths) is given by*

$$\frac{dE}{dt} = \frac{4\pi}{3} e^2 \frac{i\gamma^4}{R} = 88.46 \frac{iE^4}{R} \frac{m}{ma \text{ gev}^4} \text{ watt}$$

for a beam of current i and energy E travelling in a full circle of radius R . For shorter arcs, the above expression may be divided by $2\pi R$ to yield the power per meter of arc.

*For derivations of equations presented in section A, see Appendix A, etc.

Magnetic accelerators excited with a fully biased sine wave have a $1-\cos$ energy variation during acceleration. If the beam is cut off just as it reaches peak energy, it is shown in Appendix A that the synchrotron radiation power averaged from the start of one acceleration cycle to the start of the next is .1367 times the synchrotron radiation power at the peak beam energy.

B. The spectral energy distribution is described here in two ways 1) the spectral power density, being the synchrotron radiation power per unit of synchrotron radiation quantum energy, and 2) the integrated spectral power, being that part of the synchrotron radiation power which is contained in radiation having quantum energy less than some given quantum energy.

The equations for spectral distribution are simplified and generalized if not expressed directly as functions of synchrotron radiation quantum energy, ϵ , but instead as functions of the ratio $r = \epsilon/\epsilon_c$, where ϵ_c is the "critical" quantum energy, defined by

$$\begin{aligned}\epsilon_c &= \frac{3}{4\pi} hc \frac{\gamma^3}{R} \\ &= 2218. \frac{E^3}{R} \frac{m}{\text{gev}^3} \text{ ev}\end{aligned}$$

The spectral power density is given by

$$\begin{aligned} \frac{d^2 E}{d\epsilon dt} &= 2\pi 3^{\frac{1}{2}} \frac{e}{h\bar{c}} i\gamma r \int_r^{\infty} K_{5/3}(s) ds \\ &= 5.316 \cdot 10^{-2} \frac{i E}{ma \text{ gev}} d_E(r) \frac{\text{watt}}{\text{ev}} \end{aligned}$$

The function $d_E(r)$ is tabulated and plotted; as indicated, it is a function of r only, and is approximately equal to $r^{1/3}$ for small r . The integrated spectral power is given by

$$\begin{aligned} \int_0^{\epsilon} \frac{d^2 E}{d\epsilon dt} d\epsilon &= \frac{dE}{dt} f_E(r) \\ &= 88.46 \frac{iE^4}{R} \frac{m}{ma \text{ gev}^4} f_E \text{ watt} \end{aligned}$$

The function $f_E(r)$ is tabulated and plotted; as indicated, it is a function of r only; it is the fraction of total synchrotron radiation power contained in radiation having r less than the given value, and approaches 1 as r becomes large. Note that half of the power is in radiation with $\epsilon > \epsilon_c$.

C. The spectral photon distribution is also described in two ways 1) the spectral photon density, being the number of synchrotron radiation photons per unit of synchrotron radiation quantum energy, and 2) the integrated photon flux, being the number of synchrotron radiation photons having quantum energies less than some given quantum energy.

The spectral photon density is given by

$$\begin{aligned} \frac{d^2N}{d\epsilon dt} &= 2\pi 3^{\frac{1}{2}} \frac{e}{hc} i\gamma \left[I(15) - I(r) \right] \\ &= 1.496 \cdot 10^{14} \frac{iR}{E^2} \frac{\text{gev}^2}{m \text{ ma}} d_N(r) \frac{\text{photon}}{\text{ev sec}} \end{aligned}$$

The function $d_N(r)$ is tabulated and plotted; it is a function of r only, and is proportional to $r^{-2/3}$ for small r .

The integrated photon flux is given by

$$\begin{aligned} \int_0^\epsilon \frac{d^2N}{d\epsilon dt} d\epsilon &= 2\pi 3^{\frac{1}{2}} \frac{e}{hc} i\gamma \left[rI(15) - P(r) \right] \\ &= 8.084 \cdot 10^{17} iE \frac{1}{m \text{ a}} \frac{\text{gev}}{\text{sec}} f_N(r) \frac{\text{photon}}{\text{sec}} \end{aligned}$$

The function $f_N(r)$ is tabulated and plotted; as indicated, it is a function of r only; it is the fraction of total synchrotron radiation quanta contained in radiation having r less than the given value, and approaches 1 as r becomes large.

D. Spectral-angular Intensity

Synchrotron radiation is concentrated in the direction of motion of the electrons. In most practical cases, the observed beam arc is sufficiently long that the distribution in the plane of the beam is not observed, the angular size of the arc being greater than the angular size of the planar distribution. The distribution perpendicular to the beam plane, but integrated over all horizontal angles, is often experimentally significant.

The presentation and interpretation of the off-plane distribution is simplified by the introduction of a new variable $\alpha = r^{1/3} \gamma \psi$ where ψ is the angle between the radiation and the beam plane.

The off-plane distribution is then given by

$$\begin{aligned} \frac{d^3 E}{d\psi d\epsilon dt} &= 3 \frac{e^2}{hc} i \gamma^2 r^2 (1 + \gamma^2 \psi^2)^2 \left[K_{2/3}^2(x) + \frac{\gamma^2 \psi^2}{1 + \gamma^2 \psi^2} K_{1/3}^2(x) \right] \\ &= 38.84 \frac{i E^2}{\text{ma gev}^2} v_0(r) \frac{v}{v_0}(\alpha, r) \frac{\text{watt}}{\text{ev radian}} \end{aligned}$$

for a full circle in the beam plane. The functions $v_0(r)$ and $\frac{v}{v_0}(\alpha, r)$ are tabulated and plotted. Qualitatively, the distribution extends to about $\alpha=1$, $\psi_{\max} = \frac{1}{r^{1/3} \gamma} = \left(\frac{3hc}{4\pi\epsilon R} \right)^{1/3}$. Then, as r increases from very small values, the spectral-angular intensity (v_0) increases as $r^{2/3}$ while the angular size ($\psi_{\max}$) decreases as $r^{-1/3}$ so that the spectral intensity, as previously noted, increases as $r^{1/3}$. As r exceeds 1, v_0 falls rapidly, while the significant values of ψ continue to decrease.

r	$d_E(r)$	f_E	$d_N(r)$	f_N	v_O
1.0E-09	0.0010	1.00E-12	1.00E 06	0.001232	1.000E-06
1.0E-06	0.0100	1.00E-08	1.00E 04	0.012316	1.000E-04
1.0E-05	0.0215	2.15E-07	2.15E 03	0.026530	4.642E-04
1.0E-04	0.0463	4.64E-06	4.63E 02	0.057131	2.154E-03
0.001	0.0992	0.0001	99.15621	0.122813	0.0100
0.002	0.1243	0.0002	62.15229	0.154478	0.0159
0.004	0.1554	0.0006	38.84133	0.194117	0.0252
0.006	0.1767	0.0011	29.44173	0.221716	0.0330
0.008	0.1933	0.0016	24.15649	0.243547	0.0399
0.010	0.2070	0.0021	20.70094	0.261874	0.0463
0.020	0.2546	0.0052	12.72929	0.327380	0.0733
0.030	0.2855	0.0088	9.51537	0.372300	0.0957
0.040	0.3083	0.0128	7.70862	0.407359	0.1155
0.050	0.3264	0.0170	6.52768	0.436431	0.1335
0.060	0.3411	0.0215	5.68534	0.461412	0.1501
0.070	0.3534	0.0261	5.04910	0.483391	0.1656
0.080	0.3639	0.0309	4.54866	0.503053	0.1802
0.090	0.3729	0.0358	4.14299	0.520867	0.1939
0.100	0.3806	0.0408	3.80635	0.537164	0.2070
0.150	0.4069	0.0672	2.71296	0.602806	0.2640
0.200	0.4203	0.0943	2.10136	0.651710	0.3104
0.250	0.4260	0.1230	1.70419	0.690509	0.3491
0.300	0.4269	0.1515	1.42311	0.722455	0.3816
0.350	0.4245	0.1799	1.21275	0.749412	0.4088
0.400	0.4196	0.2080	1.04899	0.772559	0.4315
0.450	0.4130	0.2358	0.91777	0.792697	0.4504
0.500	0.4051	0.2631	0.81024	0.810398	0.4658
0.550	0.3963	0.2898	0.72056	0.826082	0.4783
0.600	0.3868	0.3159	0.64470	0.840073	0.4881
0.650	0.3768	0.3413	0.57976	0.852623	0.4956
0.700	0.3665	0.3661	0.52362	0.863934	0.5009
0.750	0.3560	0.3902	0.47469	0.874169	0.5043
0.800	0.3454	0.4136	0.43173	0.883462	0.5061
0.850	0.3347	0.4362	0.39378	0.891927	0.5063
0.900	0.3241	0.4582	0.36008	0.899657	0.5053
0.950	0.3135	0.4795	0.33001	0.906734	0.5030
1.000	0.3031	0.5000	0.30305	0.913226	0.4996
1.500	0.2096	0.6696	0.13976	0.956107	0.4296
2.000	0.1403	0.7850	0.07016	0.976737	0.3360
2.500	0.0922	0.8614	0.03687	0.987327	0.2491
3.000	0.0598	0.9114	0.01994	0.992970	0.1785
3.500	0.0385	0.9436	0.01099	0.996049	0.1248
4.000	0.0246	0.9643	0.00614	0.997758	0.0857
4.500	0.0156	0.9775	0.00347	0.998718	0.0580
5.000	0.0099	0.9858	0.00198	0.999263	0.0389
5.500	0.0062	0.9911	0.00113	0.999574	0.0258
6.000	0.0039	0.9944	0.00065	0.999753	0.0170
6.500	0.0025	0.9965	0.00038	0.999856	0.0111
7.000	0.0015	0.9978	0.00022	0.999916	0.0072
7.500	0.0010	0.9986	0.00013	0.999951	0.0047
8.000	0.0006	0.9992	0.00007	0.999971	0.0030
8.500	0.0004	0.9995	0.00004	0.999983	0.0019
9.000	0.0002	0.9997	0.00003	0.999990	0.0012
9.500	0.0001	0.9998	0.00002	0.999994	0.0008
10.000	0.0001	0.9999	0.00001	0.999997	0.0005

v/v_0

$r =$	.0001	.01	.1	.2	.4	.6	.8	1.0	1.5	2.0	3.0	4.0
0.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.1	1.0152	1.0136	1.0085	1.0053	1.0012	0.9984	0.9962	0.9943	0.9908	0.9882	0.9843	0.9814
0.2	1.0574	1.0512	1.0311	1.0189	1.0029	0.9919	0.9833	0.9763	0.9629	0.9529	0.9380	0.9270
0.3	1.1171	1.1036	1.0602	1.0338	0.9997	0.9761	0.9581	0.9434	0.9152	0.8944	0.8640	0.8417
0.4	1.1797	1.1570	1.0842	1.0404	0.9840	0.9455	0.9161	0.8923	0.8473	0.8144	0.7669	0.7327
0.5	1.2276	1.1948	1.0901	1.0278	0.9481	0.8944	0.8538	0.8211	0.7601	0.7160	0.6537	0.6097
0.6	1.2433	1.2008	1.0662	0.9869	0.8867	0.8200	0.7700	0.7302	0.6569	0.6048	0.5327	0.4832
0.7	1.2131	1.1628	1.0050	0.9131	0.7984	0.7231	0.6673	0.6234	0.5436	0.4882	0.4132	0.3631
0.8	1.1308	1.0760	0.9055	0.8075	0.6869	0.6089	0.5519	0.5075	0.4283	0.3744	0.3035	0.2576
0.9	0.9998	0.9446	0.7743	0.6778	0.5608	0.4864	0.4328	0.3915	0.3193	0.2713	0.2101	0.1718
1.0	0.8330	0.7815	0.6243	0.5365	0.4317	0.3662	0.3198	0.2845	0.2239	0.1848	0.1363	0.1071
1.1	0.6502	0.6058	0.4717	0.3980	0.3114	0.2583	0.2213	0.1936	0.1469	0.1176	0.0825	0.0622
1.2	0.4727	0.4374	0.3320	0.2751	0.2093	0.1698	0.1426	0.1226	0.0897	0.0695	0.0463	0.0334
1.3	0.3183	0.2925	0.2165	0.1761	0.1303	0.1033	0.0851	0.0719	0.0506	0.0380	0.0240	0.0165
1.4	0.1974	0.1802	0.1300	0.1038	0.0747	0.0579	0.0467	0.0388	0.0263	0.0191	0.0114	0.0075
1.5	0.1121	0.1016	0.0715	0.0561	0.0392	0.0297	0.0235	0.0192	0.0125	0.0088	0.0049	0.0031
1.6	0.0579	0.0522	0.0358	0.0276	0.0187	0.0139	0.0108	0.0086	0.0054	0.0037	0.0019	0.0012
1.7	0.0271	0.0243	0.0162	0.0123	0.0081	0.0059	0.0045	0.0035	0.0021	0.0014	0.0007	0.0004
1.8	0.0114	0.0102	0.0066	0.0049	0.0032	0.0022	0.0017	0.0013	0.0007	0.0005	0.0002	0.0001
1.9	0.0043	0.0038	0.0024	0.0018	0.0011	0.0008	0.0006	0.0004	0.0002	0.0001	0.0001	0.0000
2.0	0.0014	0.0013	0.0008	0.0006	0.0003	0.0002	0.0002	0.0001	0.0001	0.0000	0.0000	0.0000
2.1	0.0004	0.0004	0.0002	0.0002	0.0001	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
2.2	0.0001	0.0001	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
2.3	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
2.4	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Appendix A1. Total synchrotron radiation power.

For the total power radiated by one electron travelling in a circular orbit, Schwinger, PR 75 1912 (1949) Eqn. I.9, gives

$$P = \frac{2}{3} \omega_0 \frac{e^2}{R} \beta^3 \gamma^4$$

For all equations given in this paper, we assume highly relativistic electrons; then

$$\beta = \frac{v}{c} = 1.000$$

$$\omega_0 = \frac{c}{R} = \text{orbital angular frequency}$$

$$c = 2.998 \times 10^{10} \text{ cm/sec} = \text{velocity of light}$$

$$e = 1.602 \times 10^{-19} \text{ coul} = \text{electronic charge}$$

$$(\text{note: } 1 \text{ coul}^2 = 10^{-2} \frac{\text{erg sec}^2}{\text{cm}})$$

$$\text{ev} = \text{electron volt} = 1.602 \times 10^{-19} \text{ watt sec}$$

$$R = \text{orbit radius} (= 26.26 \text{ m at CEA})$$

$$\gamma = \frac{E}{\mu} = \frac{\text{electron orbital energy}}{\text{electron rest mass energy}}$$

$$\mu = .511 \text{ mev}$$

Now let n = number of circulating electrons

i = current of circulating electrons

$$\text{Then } i = n e \frac{c}{2\pi R} = \frac{nec}{2\pi R}$$

$$\frac{dE}{dt} = \frac{2}{3} \frac{c}{R} \frac{e^2}{R} \gamma^4 n = \frac{4\pi}{3} e \gamma^4 i / R$$

$$= 88.46 \frac{i E^4}{R} \frac{\text{m}}{\text{ma gev}^4} \text{ watt}$$

Appendix A2. Average power over acceleration cycle.

If we let θ be the phase angle of the accelerator magnetic field, E_m be the energy at peak field, P_{avg} be the average synchrotron radiation power over a full magnet cycle, and P_{pk} be the peak synchrotron radiation power, then

$$\begin{aligned}
 E &= E_m (1 - \cos \theta)^4 \\
 \frac{P_{avg}}{P_{pk}} &= \frac{1}{2} \frac{\frac{1}{\pi} \int_0^\pi (1 - \cos \theta)^4 d\theta}{2^4} \\
 &= \frac{1}{32\pi} \int_0^\pi (1 - 4\cos \theta + 6\cos^2 \theta - 4\cos^3 \theta + \cos^4 \theta) d\theta \\
 &= 4.375/32 \\
 &= .1367
 \end{aligned}$$

Appendix B1. Spectral Power Density.

For the power radiated per unit angular frequency interval by one electron in a circular orbit, Schwinger, loc. cit., Eqn. II-16 gives

$$P(\omega, t) = 3^{3/2}/4\pi e^2/R (E/\mu)^4 \omega_o \omega/\omega_c^2 \int_{\omega/\omega_c}^{\infty} K_{5/3}(s) ds$$

where ω_o = orbital angular frequency = $\frac{c}{R}$

ω = synchrotron radiation angular frequency = $2\pi\nu$

$$\omega_c = \text{"critical"} \quad \omega = \frac{3}{2} \frac{c}{R} \frac{E^3}{\mu^3}$$

$K_{5/3}$ = modified Bessel function of second kind and order 5/3

Now let $\epsilon = h\nu = \frac{h\omega}{2\pi} = \text{photon energy}$

$$\epsilon_c = \frac{h\omega_c}{2\pi} = \frac{3}{4\pi} \frac{hc}{\mu^3} \frac{E^3}{R} = \text{"critical" photon energy}$$

$$r = \frac{\epsilon}{\epsilon_c}$$

$$h = 6.625 \times 10^{-27} \text{ erg sec} = \text{Planck's constant}$$

$$\begin{aligned} \text{Then } \frac{d^2 E}{d\epsilon dt} &= 2\pi/h \frac{d^2 E}{d\omega dt} = 2\pi/h n P(\omega, t) \\ &= 2\pi 3^{1/2} \frac{e}{2\mu hc} iE r \int_r^\infty K_{5/3}(s) ds \end{aligned}$$

$$\text{Now } K_n(s) = \frac{n!(-n)!}{2n} \sum_{k=0}^{\infty} \frac{(s/2)^{2k-n}}{k!(k-n)!} - \frac{(s/2)^{2k+n}}{k!(k+n)!}$$

$$\text{So } K_{5/3}(s) = .3(5/3)!(-5/3)! \sum_{k=0}^{\infty} \frac{(s/2)^{2k-5/3}}{k!(k-5/3)!} - \frac{(s/2)^{2k+5/3}}{k!(k+5/3)!}$$

The above series can be integrated term by term for any finite limits of s , but not to the infinite limit required. This difficulty can be overcome by observing that integration from some large value, say $r = 15$, to infinity yields a negligibly small result. Thus $\int_r^\infty$ can be substituted for $\int_r^\infty$ with negligible error, and then the series can be integrated term by term. A minor computational difficulty remains, in that the individual terms of the integrated series become large for large values of

the argument, a first group of terms being positive and a later group being negative to yield a small net sum. Because of this behavior, the series sum is roughly $s/2$ decimal digits less accurate than the original values of s and the factorials. Integrating term by term and regrouping the first few terms for ease of computation, we have

$$I(s) = .3(5/3)!(-5/3)! \left[\frac{(s/2)^{-2/3}}{(-5/3)!(-1/3)} + \frac{(s/2)^{4/3}}{(-2/3)! 2/3} \right. \\ \left. + \sum_{k=0}^{\infty} \frac{(s/2)^{2k+10/3}}{(k+2)!(k+1/3)!(k+5/3)} - \frac{(s/2)^{2k+8/3}}{k!(k+5/3)!(k+4/3)} \right]$$

with $\int_r^{\infty} K_{5/3}(s) ds \doteq I(15) - I(r)$ to 8 decimal digits

Note: $I(15) = -1.8137994$.

For $s \ll 1$, only the first term in the series is significant, and

$$r \int_r^{\infty} K_{5/3}(s) ds = .9(5/3)! 2^{2/3} r^{1/3} \text{ for } r \ll 1$$

A new function $d_E(r)$ is now defined as

$$d_E(r) = \frac{r \int_r^{\infty} K_{5/3}(s) ds}{.9 2^{2/3} (5/3)!} \doteq \frac{r [I(15) - I(r)]}{.9 2^{2/3} (5/3)!}$$

It is evident from the above that $d_E(r) \approx r^{1/3}$ for $r \ll 1$. Then we have

$$\begin{aligned} \frac{d^2 E}{d\epsilon dt} &= .9 \cdot 2^{5/3} 3^{1/2} \pi (5/3)! \frac{e}{\mu hc} iE d_E(r) \\ &= 5.316 \cdot 10^{-2} \frac{iE}{\text{ma gev}} d_E(r) \frac{\text{watt}}{\text{ev}} \end{aligned}$$

Appendix B2. Integrated Spectral Power.

Let f_E = fraction of total synchrotron radiation power contained in photons with energy less than ϵ . Then

$$f_E = \frac{\int_0^\epsilon \frac{d^2 E}{d\epsilon dt} d\epsilon}{\frac{dE}{dt}}$$

$$\text{Now } \int_0^\epsilon \frac{d^2 E}{d\epsilon dt} d\epsilon = \epsilon_c \int_0^r \frac{d^2 E}{d\epsilon dt} dr$$

and using previously stated expressions for ϵ_c , $\frac{\partial^2 E}{\partial \epsilon \partial t}$, and $\frac{dE}{dt}$,

we have

$$f_E = \frac{3^{5/2}}{8\pi} \int_0^r r [I(15) - I(r)] dr$$

Integrating the previous series term by term is permissible, yielding

$$f_E = \frac{3^{5/2}}{8\pi} \left\{ \frac{r^2}{2} I(15) - .6 (5/3)! (-5/3)! \left[\frac{(r/2)^{4/3}}{(-5/3)! (-1/3)! 2/3} + \frac{(r/2)^{10/3}}{-2/3 (-5/3)! 10/9} \right] + \sum_{k=0}^{\infty} \frac{(r/2)^{2k+16/3}}{(k+2)! (k+1/3)! (k+5/3) (k+8/3)} - \frac{(r/2)^{2k+14/3}}{k! (k+5/3)! (k+4/3) (k+7/3)} \right\}$$

and finally,

$$\int_0^E \frac{d^2 E}{d\epsilon dt} d\epsilon = \frac{dE}{dt} f_E$$

$$= 88.46 \frac{iE^4}{R} \frac{m}{ma \text{ gev}^4} f_E \text{ watt}$$

Appendix C1. Spectral Photon Density.

The number of synchrotron radiation photons per unit time interval and unit photon energy interval is given by

$$\frac{d^2 N}{d\epsilon dt} = \frac{1}{\epsilon} \frac{d^2 E}{d\epsilon dt} = \frac{1}{r\epsilon_c} \frac{d^2 E}{d\epsilon dt}, \text{ and defining } d_N(r) = \frac{dE}{r\epsilon_c}$$

$$= .9 \cdot 2^{5/3} 3^{1/2} \pi (5/3)! \frac{e}{\hbar c} \frac{iE}{\epsilon_c} d_N(r)$$

$$= 1.496 \cdot 10^{14} \frac{Ri}{E^2} \frac{\text{gev}^2}{m \text{ ma}} \frac{d_N(r)}{\text{ev/sec}} \text{ photon}$$

Appendix C2. Integrated Photon Flux

Let f_N = fraction of total synchrotron radiation photons having energies less than ϵ . Then

$$f_N = \frac{\int_0^\epsilon \frac{d^2 N}{d\epsilon dt} d\epsilon}{\int_0^\infty \frac{d^2 N}{d\epsilon dt} d\epsilon} = \frac{\int_0^r \frac{1}{r} \frac{d^2 E}{d\epsilon dt} dr}{\int_0^\infty \frac{1}{r} \frac{d^2 E}{d\epsilon dt} dr}$$

$$= \frac{\int_0^r [I(15) - I(r)] dr}{\int_0^\infty [I(15) - I(r)] dr}$$

Integrating term by term the previously given series for $I(s)$ is permissible, and yields

$$\int_0^r I(s) ds = P(r) = .3(5/3)!(-5/3)! \left[\frac{(r/2)^{1/3}}{(-5/3)!(-1/3)(1/6)} + \frac{(r/2)^{7/3}}{-2/3(-5/3)!(14/18)} + \sum_{k=0}^{\infty} \frac{(r/2)^{2k+13/3}}{(k+2)!(k+1/3)!(k+5/3)(k+13/6)} - \frac{(r/2)^{2k+11/3}}{k!(k+5/3)!(k+4/3)(k+11/6)} \right]$$

$$\text{Then } \int_0^\epsilon \frac{d^2 N}{d\epsilon dt} d\epsilon = \int_0^\epsilon \frac{1}{r\epsilon_c} \frac{2\pi^3}{\mu} \frac{e}{hc} e^{-iEr} [I(15) - I(r)] d\epsilon$$

$$= 2\pi^3 \frac{1}{\mu} \frac{e}{hc} iE [rI(15) - P(r)]$$

$$= 2\pi^3 \frac{1}{\mu} \frac{e}{hc} iE [15I(15) - P(15)] f_N$$

$$= 8.084 \cdot 10^{17} \frac{iE}{\text{ma gev}} f_N \frac{\text{photon}}{\text{sec}}$$

Note: $15 I(15) - P(15) = 5.235987....$

Appendix D1. Spectral-angular Intensity.

For one electron in a circular orbit, Schwinger (loc. cit., Eqn. II - 34) gives

$$P(\psi, \omega, t) = \frac{3}{4\pi^2} e^2/R (\omega/\omega_c)^2 \gamma^2 (1+\gamma^2\psi^2)^2 \left[K_{2/3}^2(x) + \frac{\gamma^2\psi^2}{1+\gamma^2\psi^2} K_{1/3}^2(x) \right]$$

where ψ = angle between synchrotron radiation and orbital plane
and $x = \frac{\omega}{2\omega_c} (1 + \gamma^2\psi^2)^{3/2}$

Introducing r and i as done previously in Appendix B1, and introducing a new variable $\alpha = r^{1/3} \gamma \psi$, we have

$$\frac{d^3E}{d\psi d\epsilon dt} = 3 \frac{e^2}{hc} \gamma^2 i r^{2/3} (\alpha^2 + r^{2/3})^2 \left[K_{2/3}^2(x) + \frac{\alpha^2}{\alpha^2 + r^{2/3}} K_{1/3}^2(x) \right]$$

$$\text{with } x = \frac{(\alpha^2 + r^{2/3})^{3/2}}{2}$$

Introducing the series approximations for the K functions yields

$$\frac{d^3E}{d\psi d\epsilon dt} = 3 \frac{e^2}{hc} i \gamma^2 r^{2/3} \left[\frac{3}{2} {}_2F_2(2/3) \right]^2$$

for $\alpha \ll r \ll 1$

A new function $v(\alpha, r)$ is now defined as

$$v(\alpha, r) = \frac{r^{2/3} (\alpha^2 + r^{2/3})^2}{\left[\frac{3}{2} {}_2F_2(2/3) \right]^2} \left[K_{2/3}^2(x) + \frac{\alpha^2}{\alpha^2 + r^{2/3}} K_{1/3}^2(x) \right]$$

for all α, r

where it is evident from the preceding equation that

$$v(\alpha, r) \doteq r^{2/3} \quad \text{for } \alpha \ll r \ll 1$$

Then, letting v_0 be the value of v when $\alpha = 0$,

$$\begin{aligned} \frac{d^3 E}{d\psi d\epsilon dt} &= 27 \left[(2/3)! / 2^{2/3} \right]^2 \frac{e^2}{\mu hc} i E^2 v_0(r) \frac{v}{v_0}(\alpha, r) \\ &= 38.84 \frac{i E^2}{\text{ma gev}^2} v_0(r) \frac{v}{v_0}(\alpha, r) \frac{\text{watt}}{\text{ev. radian}} \end{aligned}$$

For reference, the $K_{\frac{2}{3}}$ functions are listed below, with the first few terms regrouped as usual.

$$K_{\frac{2}{3}}(x) = 3/4 (2/3)! \left[(x/2)^{-2/3} + (-2/3)! \right]$$

$$\sum_{k=0}^{\infty} \frac{(x/2)^{2k+2/3}}{k! (k+2/3)!} \left\{ \frac{(x/2)^{2/3} (k+2/3)! k!}{(k+1)! (k+1/3)!} - 1 \right\}$$

$$K_{\frac{1}{3}}(x) = 3/2 (1/3)! (-1/3)!$$

$$\sum_{k=0}^{\infty} \frac{(x/2)^{2k+1/3}}{k! (k+1/3)!} \left\{ \frac{(x/2)^{-2/3} (k+1/3)!}{(k-1/3)!} - 1 \right\}$$

$$\left(-\frac{1}{3} \right)! = 1.3541179394264$$

$$\left(-\frac{2}{3} \right)! = 2.6789385347077$$

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