EFFECTS OF RETARDATION ON ELECTROMAGNETIC SELF-ENERGY
OF ATOMIC STATES *

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A B S T R A C T

The significance of retardation effects in photon emissions and absorption is emphasized in the calculation of self-energy. It is explicitly demonstrated that inclusion of such effects leads to a finite answer for the shifts of atomic energy levels in a non-relativistic theory without cutoff. Ambiguities that exist in the mass renormalization in the non-relativistic approach are pointed out. Such ambiguities vanish from the relativistic theory. Explicit calculation is carried out in the case of hydrogen by utilizing the Coulomb Green's function. The advantage of the present approach in calculating self-energy shift in high Z hydrogenic ions is suggested.

I. INTRODUCTION

It is well-known that Bethe's original calculation of the Lamb shift,^{1/} which introduces a photon energy cutoff at the electron mass, gives surprisingly close agreement with the experimental value. Without the cutoff, Bethe's result would be logarithmically divergent. In addition to being non-relativistic, Bethe's calculation neglects retardation in the photon emission and absorption, an approximation which has been argued to be accurate.^{2/} While a correct treatment of electromagnetic radiative correction must be based on a relativistic formalism, it is interesting to note that finite value for the Lamb shift can be obtained from a non-relativistic calculation without an ultraviolet photon cutoff, but including all multipole interactions.^{3/} In such a calculation, there exist ambiguities in the correct mass renormalization counter term, which can only be properly treated by considering the non-relativistic model as a suitable limit of a relativistic theory, where such ambiguities are absent. However, the ambiguities do not affect the convergence of the result.

In this paper, we carry out a non-relativistic calculation including retardation, and show how and why it leads to a finite Lamb shift. Our technique involves the relation between energy shift and a part of the forward Compton scattering amplitude, a relation known in particle physics.^{4/} In the present case, the crossed forward scattering amplitude can be calculated exactly, and the necessary integral over photon momentum can easily be carried out numerically. In Section II of the paper, we present and derive the relevant formula for energy shift and the Compton amplitude. In Section III, we discuss the reason for the convergence of the Lamb shift, the renormalization prescription, and the ambiguities

thereof. In Section IV, we give some numerical results for hydrogen. These results are not especially useful for hydrogen, where very accurate results from a relativistic theory including retardation ~~are~~ available. However, our technique may be more useful for hydrogenic ions of high Z , where retardation corrections may be substantial, and there are not equally accurate calculations available.

II. FORMULATION

In this section, we derive the relation between the self-energy shift and the on-shell Compton amplitude. The unrenormalized energy shift is well-known to be given by^{4/}

$$\Delta E^{\text{unr}} = \text{Re} \Delta M \quad (\text{II-1})$$

$$\Delta M = \frac{i^2}{(2\pi)^4} \frac{1}{2!} \int_{-\infty}^{\infty} \frac{T_{\mu\nu}(k, k_0) g_{\mu\nu}}{k^2 - k_0^2 - i\epsilon} d^4 k \quad (\text{II-2})$$

where

$$\begin{aligned} T_{\mu\nu}(k, k_0) = & \frac{1}{2} \int_{-\infty}^{+\infty} e^{ik(\tilde{y}-\tilde{x})} d^3 x d^3 y \int_0^{\infty} e^{ik_0 x_0} dx_0 \langle n | j_{\mu}(\tilde{x}, x_0) \sum_m |m\rangle \langle m | j_{\nu}(\tilde{y}, 0) | n \rangle \\ & + \frac{1}{2} \int_{-\infty}^{+\infty} e^{ik(\tilde{y}-\tilde{x})} d^3 x d^3 y \int_{-\infty}^0 e^{ik_0 x_0} dx_0 \langle n | j_{\nu}(\tilde{y}, 0) \sum_m |m\rangle \langle m | j_{\mu}(\tilde{x}, x_0) | n \rangle \\ & + (\mu \leftrightarrow \nu) \end{aligned} \quad (\text{II-3})$$

and $\epsilon_{\mu} \epsilon_{\nu} T_{\mu\nu}$ is the forward Compton scattering amplitude for the scattering of a virtual photon with three-momentum k and energy k_0 , and polarization vector ϵ_{μ} from an atom in state n . Using $j_{\mu}(t) = e^{iH_a t} j_{\mu}(0) e^{-iH_a t}$, where H_a is the atomic Hamiltonian, $\int_0^{\infty} dt = \int_{-\infty}^{+\infty} dt \theta(t)$, $\int_{-\infty}^0 dt = \int_{-\infty}^{\infty} dt \theta(-t)$ and

$$\theta(t) = \frac{1}{2\pi i} \int \frac{e^{i\omega t}}{\omega - i\epsilon}, \quad \text{Eq. (II-3) can be reduced to}$$

$$\begin{aligned} T_{\mu\nu}(k, k_0) = & \frac{-i}{2} \sum_m \int_{-\infty}^{+\infty} e^{ik(\tilde{y}-\tilde{x})} d^3 x d^3 y \langle n | j_{\mu}(\tilde{x}) | m \rangle \langle m | j_{\nu}(\tilde{y}) | n \rangle \\ & \times \left[\frac{1}{E_m - E_n - k_0 - i\epsilon} + \frac{1}{E_m - E_n + k_0 - i\epsilon} \right] + (\mu \leftrightarrow \nu) \end{aligned} \quad (\text{II-4})$$

Let us now examine the poles in k_0 in the integrand in (II-2). The poles coming from the photon propagator are determined by the Feynman

prescription (Fig. 1). The poles in the first term in (II-4) are all below the real axis and those in the second term are all above. Hence in each case, we close the k_0 contour accordingly, as in Fig. 2 and Fig. 3, and pick up only the contribution from the photon pole. Therefore only Compton amplitude of on mass-shell photons enters. These two terms give equal contribution, and correspond to the crossed Born term in the on-shell forward Compton amplitude. The $(\mu \leftrightarrow \nu)$ term also gives a factor of 2. Hence, after doing the k_0 integration, we obtain

$$\Delta M = \frac{-1}{(2\pi)^3} \int d^3k \frac{g_{\mu\nu}}{2|k|} \sum_m \int d^3x d^3y e^{ik(\chi-x)} \frac{\langle n | j_\mu(x) | m \rangle \langle m | j_\nu(y) | n \rangle}{E_m - E_n + |k| - i\epsilon} \quad (\text{II-5})$$

which is exactly what one gets from second order perturbation theory, according to the interaction $\vec{j} \cdot \vec{A}$.

However, in the present work, we only evaluate the scattering amplitude in (II-5) non-relativistically where a closed form exists and show how it leads to a finite Lamb shift with mass renormalization. Ambiguities that exist in the mass renormalization counter-term in non-relativistic calculations are discussed in Section III.

In the non-relativistic limit, $g_{\mu\nu} j_\mu j_\nu$ in (II-5) is replaced by $\sum_\lambda \epsilon_i^\lambda \epsilon_j^\lambda \left(\frac{e}{m}\right)^2 p_i p_j$, which just comes from the interaction $-\frac{e}{m} \vec{p} \cdot \vec{A}$ taken to second order. However, it is also well-known that in the non-relativistic interaction Hamiltonian, there is also the $\frac{e^2 A^2}{2m}$ term. This term gives rise to the process in Fig. 4, whose matrix element $\langle n | \frac{e^2 A^2}{2m} | n \rangle$ is independent of the wave function of the particle. This graph is present in both cases of bound and free electrons. The net result is that it does not contribute to the energy shift after mass renormalization. Thus in subsequent discussions, we shall ignore this term. Also,

to bring out the dependence on the state in question, we shall write

W_n in place of M in Eq.(II-5), and we have

$$W_n = \frac{1}{4\pi} \int k dk \sum_{\lambda} d\Omega_{\vec{k}} \mathcal{M}_n(\lambda, \vec{k}, \omega) \quad (\text{II-6})$$

where λ is the polarization mode, $\omega = |\vec{k}|$ is the on-shell photon energy,

and $\mathcal{M}_n$ is the forward Compton scattering amplitude in the given state n .

$\mathcal{M}_n$ can be written as

$$\mathcal{M}_n(\lambda, \vec{k}, \omega) = \frac{e^2}{m} \langle n | e^{\lambda \cdot \vec{p}} e^{i\vec{k} \cdot \vec{x}} G e^{\lambda \cdot \vec{p}} e^{-i\vec{k} \cdot \vec{x}} | n \rangle \quad (\text{II-7})$$

where the dipole approximation is not used. In (II-7), $G = \frac{1}{\vec{E} - H + i\epsilon}$ is the coulomb Green's function; it can be written, in the momentum representation, according to Schwinger,^{5/} as

$$G(\vec{p}_2, \vec{p}_1, \Omega_n) = -\frac{m}{2\pi^3} X^2 \left(\frac{1}{2\sin\pi\tau} e^{i\pi\tau} \right) \int_1^0 + \rho^{-\tau} \frac{d}{d\rho} \frac{1-\rho^2}{\rho} \times \left(\frac{1-\rho^2}{\rho} \frac{1}{[X^2(\vec{p}_1 - \vec{p}_2)^2 + (\vec{p}_1^2 + X^2)(\vec{p}_2^2 + X^2)(1-\rho)^2/4\rho^2]} \right) d\rho \quad (\text{II-8})$$

$$\text{where } \tau = \lambda/X \quad (\text{II-9A})$$

$$X^2 = -2m\Omega_n \quad (\text{II-9B})$$

$$\lambda = \alpha Z m \quad (\text{II-9C})$$

$$\alpha = \frac{e^2}{\hbar c} = e^2 = \frac{1}{137}, \quad \hbar = c = 1 \quad (\text{II-9D})$$

$$\Omega_n = E_n + \omega + i\epsilon \quad \text{for the direct Born graph} \quad (\text{II-9E})$$

$$= E_n - \omega + i\epsilon \quad \text{for the crossed Born graph} \quad (\text{II-9F})$$

Note that the Ω_k in (II-6) is the solid angle. The $i\epsilon$ is inserted to prevent divergence at resonances. A similar calculation of the Lamb shift using similar techniques but without the inclusion of retardation has been carried out by Lieber.^{6/}

Since the photon momentum vector is to be averaged over angles in the calculation of the Lamb shift, we are allowed to replace tensor terms such as $\epsilon_i \epsilon_j$ and $k_i k_j$ in the scattering amplitude by $\frac{\epsilon \cdot \epsilon}{3} \delta_{ij}$ and $\frac{k^2 \delta_{ij}}{3}$ and for forward scattering $\epsilon \cdot \epsilon = 1$. Thus the summation over polarization modes reduces to a factor of two after the above modifications are made; W_n in (II-6) then becomes

$$W_n = \frac{2}{\pi} \int k dk \tilde{m}_n(k) \quad (\text{II-10})$$

where $\tilde{m}_n$ is the scattering amplitude m_n with the above modifications. The problem thus reduces in part to the calculation of $\tilde{m}_n$. Calculations utilizing Coulomb Green's functions for Compton scattering amplitudes have been carried out by Gavrilă^{7/} in the dipole approximation for the 1s hydrogenic state, by Gavrilă and Costescu^{8/}, with retardation, on the same state, and also by Klarsfeld^{9/} in the dipole approximation for general s states in hydrogenic atoms. We have used techniques similar to those used by Gavrilă and Costescu to obtain $\tilde{m}_{2s}$ and $\tilde{m}_{2p}$. We find

$$\tilde{m}_{2s} = \frac{-\lambda^2 \alpha^2 a}{[(X+\beta)^2 + k^2]} \left\{ \frac{4[(X^2 - \beta^2) + k^2]^2}{[(X+\beta)^2 + k^2]^2} \frac{F(3; -1-\tau, 5-\tau; u)}{(4-\tau)(3-\tau)(2-\tau)} + \frac{1}{2} \frac{F(1; -2-\tau, 4-\tau; u)}{3-\tau} \right\} \quad (\text{II-11})$$

and

$$\begin{aligned} \tilde{m}_{2p} = & \frac{-\lambda^2 \alpha^2 a}{[(X+\beta)^2 + k^2]} \left\{ \frac{u}{3} \frac{F(1; -1-\tau, 3-\tau; u)}{(2-\tau)} - \frac{2}{3} \frac{[\beta^2 + X^2 + k^2]}{[(X+\beta)^2 + k^2]} \frac{F(1; -1-\tau, 4-\tau; u)}{(3-\tau)} \right. \\ & + \frac{5}{6} \frac{F(1; -2-\tau, 4-\tau; u)}{(3-\tau)} + \frac{16k^2 \beta^2}{3[(X+\beta)^2 + k^2]^2} \frac{F(3; -1-\tau, 5-\tau; u)}{(4-\tau)(3-\tau)(2-\tau)} + \frac{1}{12} \frac{F(1; -\tau, 4-\tau; u)}{(3-\tau)} \\ & \left. + \frac{1}{12} \frac{F(1; -2-\tau, 2-\tau; u)}{(1-\tau)} \right\} \end{aligned} \quad (II-12)$$

where $a =$ Bohr radius (II-13)

$\beta = \lambda/2$ (II-14)

$u = \frac{[(X-\beta)^2 + k^2]}{[(X+\beta)^2 + k^2]}$ (II-15)

and $F(\alpha; \beta, \gamma; z)$ is the Gauss type of hypergeometric function.^{10/}

The energy variable is contained exclusively in the factors τ and X . The k^2 terms come strictly from the higher multipoles. We therefore recover the dipole form of the scattering amplitude by putting k^2 equal to zero. However, by retaining the k^2 term, we see that the integrand in (II-~~12~~⁶) is now less divergent, in fact, only logarithmically divergent. More interesting is that the 2s and 2p state have the same logarithmic divergence. We shall see in the next section that by including retardation, the leading divergence in the mass renormalization counter-term has identical logarithmic behavior.

III. CONVERGENCE AND RENORMALIZATION

It is both interesting and instructive to see why the non-relativistic Lamb shift including retardation is convergent. While this fact follows from the asymptotic behavior of the integrands $\mathcal{M}_{2s}$ and $\mathcal{M}_{2p}$, it can be seen more readily from Eq.(II-7). We can write the Coulomb Green's function as an operator

$$G = \frac{1}{E - \frac{p^2}{2m} - V - k} \quad (III-1)$$

By a well-known identity

$$e^{ik \cdot x} G e^{-ik \cdot x} = \frac{1}{E - V - k - \frac{(p-k)^2}{2m}} \quad (III-2)$$

The physical context of this is just that the inclusion of retardation in the photon emission and absorption just imposes momentum conservation between the photon and the electron, changing the intermediate electron momentum to $p-k$. From the identity, we can see that for large k , $\mathcal{M}_n$ behaves as $2m/k^2$, compared to the $1/k$ behavior of the non-retarded integral. This is sufficient to make the unrenormalized energy shift of each state, such as 2s, only logarithmically divergent. Furthermore, for degenerate states like 2s and 2p, the coefficient of the logarithmic divergence are equal, as shown below, so that the Lamb shift is convergent without renormalization. However, the energy shift of an individual state does require mass renormalization to be finite. We consider how to carry out that renormalization below.

It is worth noting that the extra convergence does not appear to occur in a relativistic calculation in which

$$G \sim \frac{1}{E - \alpha \cdot p - \beta m - V - k}$$

$$\text{and } e^{ik \cdot x} G e^{-ik \cdot x} \sim \frac{1}{E - \alpha \cdot p - \beta m - V - \alpha \cdot k - k}$$

which still goes as $1/k$ for large k . The finiteness of the relativistic Lamb shift occurs for other reasons which we do not consider here.

In the relativistic theory of bound state energy shift, mass renormalization is accomplished by introducing a counter-term $\delta m \bar{\psi} \psi$ into the Hamiltonian, and evaluating this term, together with the usual second order shift, between the bound states in question. The effect of the mass counter-term is to cause the propagator of a dressed particle near the mass shell to differ by only a multiplicative constant from that of a bare free particle.^{11/} This can be accomplished in the relativistic theory because the electron propagator depends only on the Lorentz invariant combination $\not{p}$, and therefore once the electron is on the mass shell ($\not{p} = m$), the propagator has no further dependence on the momentum.

On the other hand, in the model we have been using, in which the electron is treated non-relativistically and the photon relativistically, the dressed propagator depends both on the energy E and 3-momentum squared p^2 of the particle. Therefore, even when the particle is put on the mass shell ($E = p^2/2m$), the propagator retains a complicated dependence on p . As a result, it is impossible by a simple mass renormalization, to make the propagator of a dressed particle agree with that of a bare particle, for all momenta on the mass-shell.

It is possible, by a simple mass renormalization as first carried out by Bethe,^{1/} to make the dressed and bare propagator agree at

small p^2 . When this is done in the model neglecting retardation, the dressed propagator still has a pole at $E = p^2/2m$, for all value of p^2 . On the other hand when retardation is included, the pole of the dressed propagator occurs at $E = f(p^2)$ where $f(p^2)$ is a complicated function which reduces to $p^2/2m$ only for small p^2 . In order to make the dressed propagator have a simple pole at $E = p^2/2m$, when retardation is included, it would be necessary to add a counter-term with complicated dependence on p^2 . This is equivalent to requiring that the self-energy of a free particle should vanish identically, and is not obviously justified. A more convincing way to solve this problem would be to treat the non-relativistic model as a suitable limit of the relativistic theory where the normalization prescription is unambiguous. This we have not yet done. What we do here is to adopt the simple mass renormalization prescription of Bethe, with the mass renormalization parameter δm evaluated so as to make the propagator, including the effect of retardation, agree at small p^2 . It is not difficult to see that the result of this is to give δm by

$$\delta m = \frac{4\alpha}{3\pi} \int_0^\infty \frac{k dk}{k + \frac{k^2}{2m}}$$

which is logarithmically rather than linearly divergent. The counter term that must be added to the Hamiltonian to give the proper mass renormalization is

$$\frac{\delta m}{m} \frac{p^2}{2m}$$

so that the correct energy shift is given by adding this to W_n given by Eq.(II-10). Now the asymptotic behavior of the integral $\mathcal{M}_n(k)$ in

Eq. (II-~~6~~⁶) can be seen to be

$$\tilde{m}_{2s} \xrightarrow{k \rightarrow \infty} -\frac{Z^2 \alpha^3 m}{6k^2}$$

and
$$\tilde{m}_{2p} \xrightarrow{k \rightarrow \infty} -\frac{Z^2 \alpha^3 m}{6k^2}$$

from the properties of the Gaussian hypergeometric functions.^{10/}

$$\therefore W_n \sim \int \frac{-\alpha^3 m Z^2}{3\pi} \frac{dk}{k}$$

For 2s and 2p, where only the asymptotic contribution is written, i.e., the divergent contributions are equal. On the other hand, the quantities $\frac{\delta m}{m} \frac{p^2}{2m}$ for these states are also equal by the virial theorem

$$\langle \frac{p^2}{2m} \rangle_n = -E_n = \frac{Z^2 \alpha^2 m}{8} \text{ for each state, and } \frac{\delta m}{m} \text{ is a number.}$$

The renormalized energy shift of either state, given by

$$W_n^{\text{ren}} = \frac{2}{\pi} \int_0^\infty k dk \tilde{m}_n + \frac{\alpha^2}{6\pi} Z^2 \int_0^\infty \frac{k dk}{k + \frac{k}{2m}}$$

is now clearly finite because the asymptotic form of the terms with $\tilde{m}$ is equal but with opposite sign to the δm term.

Another consequence of these considerations is that if we are interested only in the energy difference of the 2s and 2p states, it is unnecessary to perform any mass renormalization, because this will cancel between the two states anyway. This is also true in the absence of retardation, but then the energy difference is still logarithmically divergent, while here it is finite.

If we instead adopt the prescription of adding a momentum dependent contribution to the Hamiltonian in order to make $\Delta E = 0$ for a free

particle, we still obtain a finite result for W_n^{ren} , but this result differs somewhat from the result with the simple mass renormalization.

The details are given in Appendix A.

Numerical results are given in the next section.

IV. NUMERICAL RESULTS

In this section, we present numerical results for W_{2s}^{ren} and $[W_{2s}^{\text{ren}} - W_{2p}^{\text{ren}}]$, where in the latter case, the mass renormalization terms cancel because of the virial theorem as shown in Section III. In order to illustrate the effect of the inclusion of retardation, we give in Table I the value of the integrand in $W_{2s}^{\text{ren}} - W_{2p}^{\text{ren}}$ (apart from constant factors) as a function of k as well as the would be value of $W_{2s}^{\text{ren}} - W_{2p}^{\text{ren}}$ in MHz, had there been an ultraviolet photon cutoff at k , under the column "Lamb shift," for both cases with and without retardation. The numerical integration is done by trapezoidal rule up to a cutoff about ten times the electron mass. The contribution from frequencies higher than this is estimated by assuming that the integrand scales simply as $1/k^2$. The table illustrates where retardation becomes significant and where contribution to the energy shift is maximum, facts which help explain why Bethe's answer is so surprisingly close to the actual value, as will be discussed in the conclusion of this paper.

We note that with a cutoff at the electron mass, we do obtain in our non-retarded calculation $W_{2s}^{\text{ren}} = 1056.7$ MHz, and $W_{2s}^{\text{ren}} - W_{2p}^{\text{ren}} = 1053.9$ MHz, which is in close agreement with Bethe's. In the retarded calculation, we obtain

$$W_{2s}^{\text{ren}} = 931.1 \text{ MHz}$$

$$W_{2s}^{\text{ren}} - W_{2p}^{\text{ren}} = 950.3 \text{ MHz} .$$

In the Appendix, we discuss an alternative approach to treat the mass renormalization term. There we obtain

$$\tilde{w}_{2s}^{\text{ren}} = 1330 \text{ MHz}$$

$$\tilde{w}_{2s}^{\text{ren}} - \tilde{w}_{2p}^{\text{ren}} = 996.6 \text{ MHz} .$$

V. CONCLUSION

We have shown that the inclusion of higher multipole effects in the calculation of the non-relativistic Lamb shift produces a finite answer for the energy shift of the hydrogen bound states, without the introduction of an ultraviolet cutoff as in the calculation of Bethe. The modification that is introduced in the integrand of the integral over photon energy by higher multipole effects become significant in the range $Z\alpha m < k < m$, whereas the dominant contribution of the non-relativistic calculation to the Lamb shift (the $\ln Z\alpha$ term) comes from $k \sim (Z\alpha)^2 m$. Therefore, it may be argued that the surprisingly accurate agreement of Bethe's answer with experiment is a consequence of two facts. In the region of maximum contribution, the integrand with and without cutoff do not sensibly differ, and in the region where Bethe introduces a cutoff, the retardation effects produces an effective cutoff anyway.

Apart from whatever light our calculation may shed on this point, we wish to emphasize that retardation effects evidently play a sizable role in the numerical value of the Lamb shift even in hydrogen, and would contribute much more significantly in heavy hydrogenic ions, where $Z\alpha \sim (Z\alpha)^2 \sim 1$. In view of experimental interest in such ions,^{12/} it would appear worthwhile to examine the contribution of retardation effects there in details. Of course, this must be done within a relativistic theory, since $(v/c)^2 \sim (Z\alpha)^2$ also. It may, however, be feasible to use the technique of the Coulomb Green's function in that case also,^{13/} or at least to carry out the low frequency part of the calculation in that way. We hope to return to that question elsewhere.

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Appendix A. An Alternative Approach for Mass Renormalization

As discussed in Section III, a second alternative for the mass renormalization counter term is momentum dependent and makes $\Delta E = 0$ for a free particle. In such cases, the counter-term would be given by

$$\Delta E' = \frac{2\alpha}{3\pi} \frac{p^2}{2m^2} \int \frac{kdk d \cos\theta}{\frac{k^2}{2m} + k - \frac{pk \cos\theta}{m}} \quad (A1)$$

as compared to $\frac{p^2}{2m} \frac{\delta m}{m}$ where

$$\delta m = \frac{4\alpha}{3\pi} \int_0^\infty \frac{kdk}{k + \frac{k^2}{2m}}$$

as given in Section III. The difference between these two forms of counter-terms is equal to $\frac{p^2}{2m^2} F(p)$, where

$$\begin{aligned} F(p) &= \frac{2\alpha}{3\pi} \int \frac{kdk d \cos\theta}{\frac{k^2}{2m} + k - \frac{pk \cos\theta}{m}} - \frac{2\alpha}{3\pi} \int_0^\infty \frac{kdk}{\frac{k^2}{2m} + k} \\ &= \frac{4\alpha m^2}{3\pi p} \left\{ \left(1 - \frac{p}{m}\right) \ln\left(1 - \frac{p}{m}\right) - \left(1 + \frac{p}{m}\right) \ln\left(1 + \frac{p}{m}\right) + \frac{2p}{m} \right\} \\ &\equiv \frac{4\alpha m^2}{3\pi p} G(p) \end{aligned} \quad (A2)$$

This would give an additional contribution to the energy shift

$$\begin{aligned} \Delta E'_n &= \langle n | \frac{F(p)}{m} \frac{p^2}{2m} | n \rangle \\ &= \langle n | \frac{2\alpha}{3\pi} p G(p) | n \rangle \end{aligned} \quad (A3)$$

In particular, for the 2s and 2p states in hydrogen, we have

$$\Delta E'_{2s} = 399 \text{ MHz} \quad (A4)$$

$$\Delta E'_{2p} = 352 \text{ MHz} \quad (A5)$$

Thus
$$\Delta E'_{2s} - \Delta E'_{2p} = 47 \text{ MHz} ; \quad (A6)$$

so
$$\tilde{W}_{2s}^{\text{ren}} = W_{2s}^{\text{ren}} + \Delta E_{2s} = 1330 \text{ MHz} , \quad (A7)$$

and
$$\tilde{W}_{2s}^{\text{ren}} - \tilde{W}_{2p}^{\text{ren}} = 996.6 \text{ MHz} \quad (A8)$$

in such an approach.

Table I. Numerical Integration for $W_{2s}^{\text{ren}} - W_{2p}^{\text{ren}}$ ^{a)}

k	Retarded Calculation		Non-retarded Calculation	
	Integrand	Lamb shift	Integrand	Lamb shift
0.1	-.92302(-2)	-.58286(0)	-.92302(-2)	-.58286(0)
0.5	.97261(-1)	.12149(1)	.97263(-1)	.12152(1)
1.0	.30352(-1)	.40699(2)	.30354(-1)	.40700(2)
5.0	.81463(-2)	.13092(3)	.81487(-2)	.13094(3)
1.0(1)	.47409(-2)	.18079(3)	.47440(-2)	.18083(3)
5.0(1)	.12293(-2)	.32470(3)	.12339(-2)	.32500(3)
1.0(2)	.66066(-3)	.39699(3)	.66568(-3)	.39768(3)
b) 1.4(2)	.48492(-3)	.43382(3)	.49013(-3)	.43485(3)
2.0(2)	.34746(-3)	.47385(3)	.35286(-3)	.47539(3)
5.0(2)	.14353(-3)	.57995(3)	.14926(-3)	.58424(3)
1.0(3)	.71092(-4)	.66106(3)	.76907(-4)	.67008(3)
5.0(3)	.10979(-4)	.83238(3)	.16051(-4)	.87719(3)
1.0(4)	.39447(-5)	.88624(3)	.81116(-5)	.96857(3)
c) 1.9(4)	.12241(-5)	.91910(3)	.43002(-5)	.10539(4)
5.0(4)	.11915(-6)	.94050(3)	.16463(-5)	.11836(4)
1.0(5)	.13920(-7)	.94419(3)	.82608(-6)	.12770(4)
2.0(5)	.10684(-8)	.94493(3)	.41411(-6)	.13707(4)
d) ∞	---	.9503 (3)	---	---

a) k is given in units of $\frac{\alpha}{a}$.

Integrand is $k(\tilde{M}_{2s} - \tilde{M}_{2p})$ in units such that $\frac{\alpha}{a} = 1$.

Lamb shift is given in MHz.

Numbers in parentheses indicate powers of ten to be multiplied.

b) Where $k \sim 1$, i.e., where retardation starts to be significant.

c) Where Bethe introduces his cutoff. Note that we recover the 1053.9 MHz in our non-retarded calculation.

d) Estimated by assuming integrand scales as $\frac{1}{k^2}$.

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FIGURE CAPTIONS

Fig. 1: Equivalent pole distribution in the photon propagator according to the Feynman contour.

Fig. 2: The k_0 contour for first integrand in (II-4).

Fig. 3: The k_0 contour for second integrand in (II-4).

Fig. 4: Feynman diagram for process arising from the $\frac{e^2 A^2}{2m}$ term in the non-relativistic interaction Hamiltonian. The dashed line represents the photon.

Fig. 1

Fig. 2

Fig. 3

Fig. 4