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MANIPULATOR CABLE WITH CONSTANT STRESS

by

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ABSTRACT

A manipulator or mechanical arm involves an upper and a lower arm, as it were, with a variable angle between them. Cables used to transmit motion and force from the upper to the lower arms pass over a pulley at the joint or elbow. A pulley, axially fixed with respect to the joint, imposes a change in length of the cable as the angle between the arms varies. Manipulation design requires a cable of constant length during this variation; this constant length may be achieved by guiding the center of the pulley along the proper path. Two acceptable solutions were obtained in terms of four variables, viz., the lengths of each arm, the radius of the pulley, and the angle between the arms. One solution moved the pulley center along a straight line with respect to the lower arm. The other solution moved the pulley center along a circular arc with respect to the upper arm. Practical and economical mechanisms based on these solutions were investigated for use in manipulator design.

INTRODUCTION

With man's exploration of the atom, methods of handling radioactive elements remotely became a necessity. This need produced a mechanical arm called a master-slave manipulator. The earliest manipulators, and subsequent mechanical manipulators, consisted of two vertical arms connected directly by tapes or cables through a horizontal member. The vertical arm within the radioactive area was called the slave, while the arm which the operator actually works was called the master. Manipulator design must acknowledge the basic requirements of minimum friction, inertia and backlash throughout the complete system.

These mechanical manipulators were sufficient for the earlier handling problems involving gamma radiation only. Containment of gamma radioactivity has been achieved through the construction of shielded facilities, usually of a dense concrete. Escape of particulate matter was controlled by negative air pressure within the facility, which produced inward air velocities through cracks or openings in excess of 100 fpm. Alpha radioactivity has been handled in tightly sealed glove boxes. Combining these two types of activity, new problems have become evident in the remote-handling field.

Gamma facilities must be made both gas- and particulate-tight as a first requirement for containment of alpha activity. Sealing of movable cables in a horizontal tube becomes a difficult problem. In addition, facilities containing both types of radiation are growing in size. These problems have led to the development of "electronically controlled" manipulators. These manipulators have the master and slave arms connected by only a multiple-conductor electrical cable, which may be sealed more easily than moving mechanical cables. An electrically connected manipulator also allows greater coverage of the slave area when coupled with some type of vehicle.

In spite of the advantages noted, certain difficulties appear. One of these is the problem of a mechanical cable of constant length. Fig. 1 shows how the completely mechanical manipulator does this inherently. Fig. 1(a) shows the normal position, whereas Fig. 1(b) shows a decrease in the angle α. The inherent compensation during this decrease is due to the fact that as the cable unwraps from pulley A, an equal amount wraps onto pulley B.

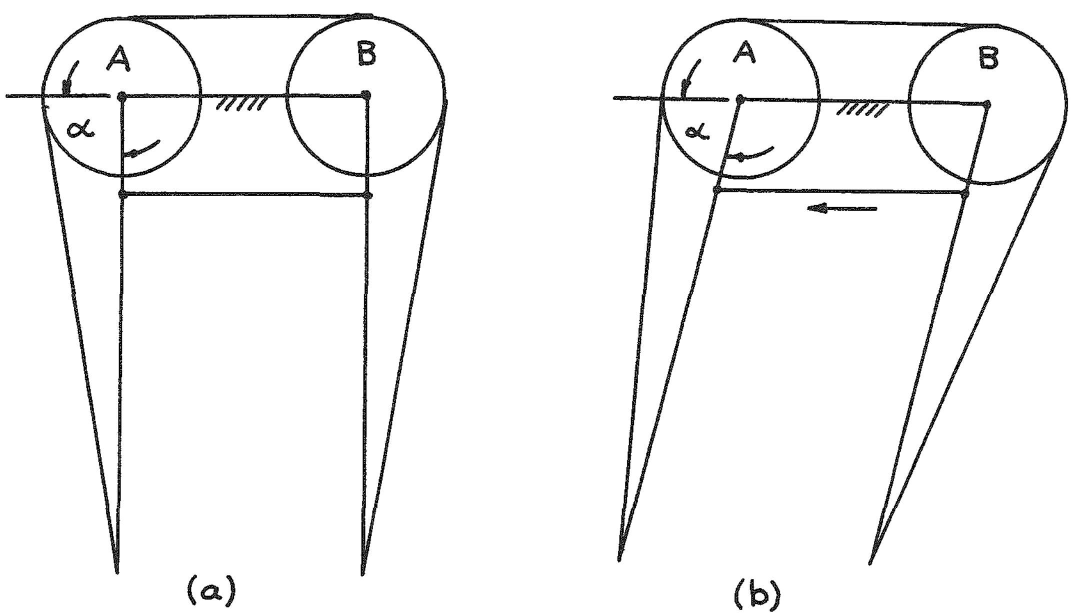


FIG. 1 MECHANICAL MANIPULATOR

Fig. 2 shows how an electronic manipulator differs. Fig. 2(a) shows the normal position, whereas Fig. 2(b) shows a decrease in the angle α. In this case, as α decreases the amount of wrap on pulley B increases, calling for an increase in cable length to maintain a constant force. If the cable length is to remain constant, the pulley center must be moved. Determining the proper path to be followed by the center of the pulley to give a constant cable length has proved to be a very laborious job on the present electronic manipulators.

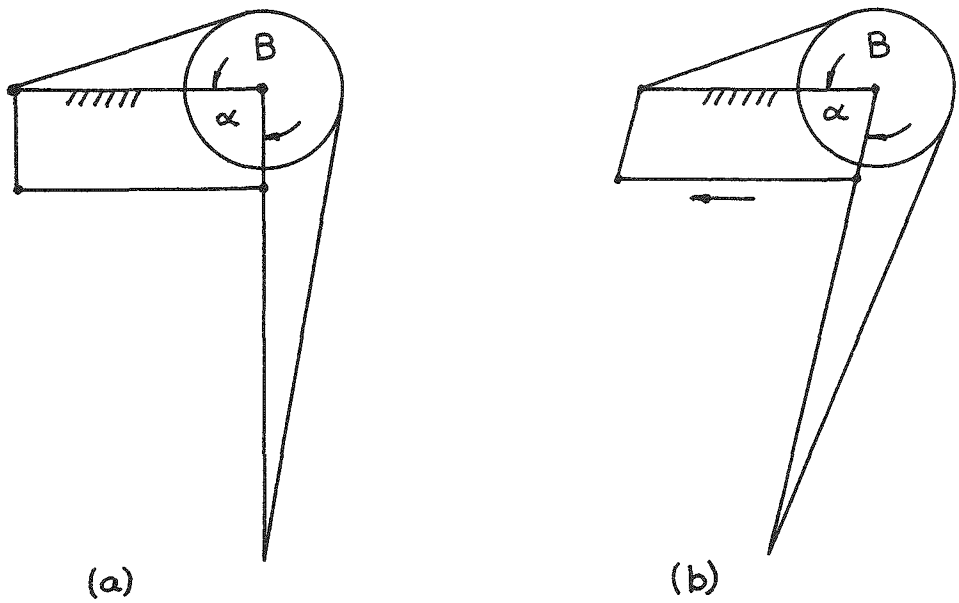


FIG. 2 ELECTRONIC MANIPULATOR

The purpose of this paper was to find a general solution for the pulley path for any manipulator, thus eliminating the need for a new investigation for each new model with different parameters.

DESCRIPTION OF PROBLEM

The problem can best be described by means of a schematic diagram (see Fig. 3). The symbols are defined as follows:

L_1 = length of upper or fixed manipulator arm, in.

L_2 = length of lower or moving manipulator arm, in.

R = pitch radius of the pulley, in.

C = length of cable, in.

α = angle between the two manipulator arms, degrees

O = center of pulley

Y = perpendicular distance of the point O from line L_1, in.

P = pivot point between L_1 and L_2

X = distance of the point O from the point P parallaled to L_1

J = point of cable origin

K = point of tangency of cable arrival at the pulley from J

M = point of tangency of cable as it leaves pulley

N	=	point of cable termination
A, B, D	=	various points of reference
γ	=	angle that the originating cable makes with L_1, degrees
δ	=	angle that the terminating cable makes with L_2, degrees
$\overline{JK}$	=	length of the originating cable, in.
$\overline{MN}$	=	length of the terminating cable, in.
$\widehat{KM}$	=	amount of wrap of the cable on the pulley, in.
ϕ, ψ	=	reference angles used to obtain γ, degrees
ϵ, λ	=	reference angles used to obtain δ, degrees
θ	=	angle of wrap of the cable on the pulley, degrees
σ	=	reference angle used to obtain θ, degrees

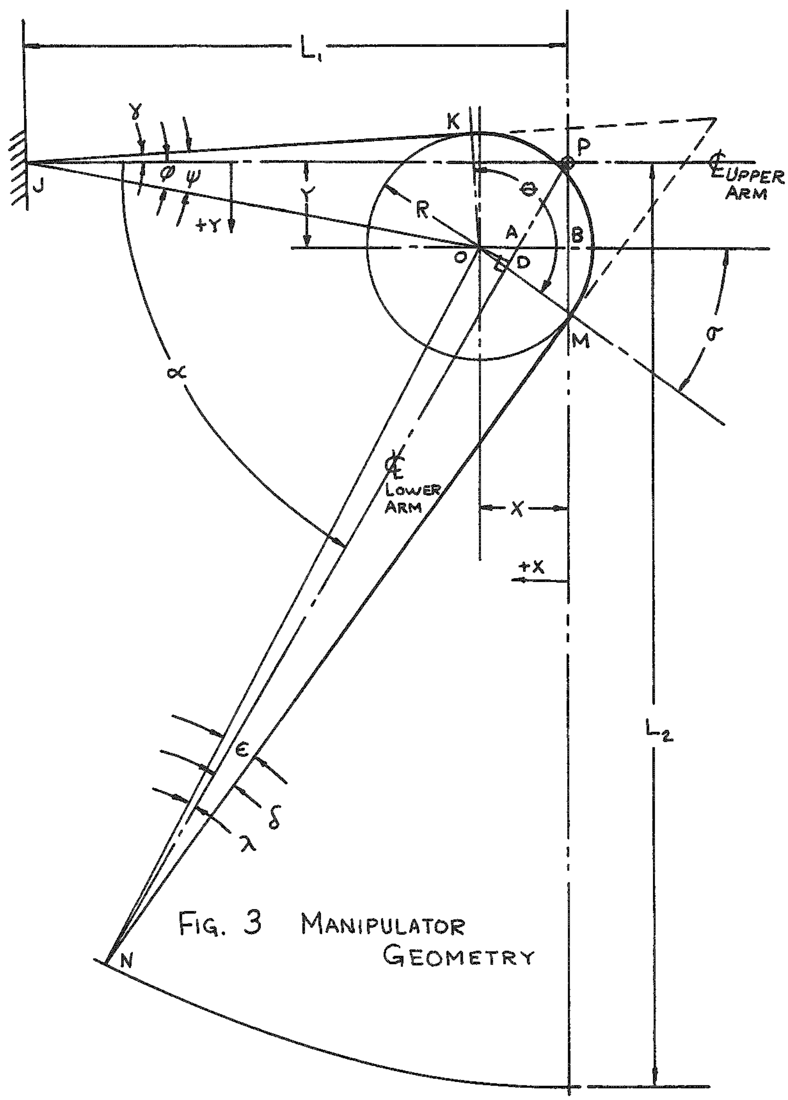


FIG. 3 MANIPULATOR GEOMETRY

Based on Fig. 3 and its symbols, the problem can be stated as follows:

Given a manipulator with upper and lower arms L_1 and L_2 and a pulley of pitch radius R, it is desired to obtain a cable of constant length C by moving the pulley center O along some path while the angle between the arms, α, varies.

The problem is to obtain the proper path for constant length as a general solution. This solution is then applied to finding a practical and economical mechanism which will move the pulley center along the prescribed path.

GENERAL SOLUTION

Using Fig. 3 and its symbols, an equation for the length of the cable or tape can be obtained. In its simplest form, the length may be expressed as

$$C = \overline{JK} + \widehat{KM} + \overline{MN} \quad . \tag{1}$$

A solution to Eq. 1 in terms of the parameters L_1, L_2 and R, the independent variable α, and the dependent variables X and Y will be obtained as follows:

$$JK = \sqrt{(JO)^2 - R^2} \tag{2}$$

$$JO = \sqrt{(L_1 - X)^2 + Y^2} \tag{3}$$

$$JP = L_1 \tag{4}$$

$$JK = \sqrt{(L_1 - X)^2 + Y^2 - R^2} \tag{5}$$

$$MN = \sqrt{(NO)^2 - R^2} \tag{6}$$

$$NO = \sqrt{(AO)^2 + (AN)^2 - 2(AO)(AN)\cos\alpha} \tag{7}$$

$$AO = X - (AB) = X - \frac{Y}{\tan\alpha} = X - Y\cot\alpha \tag{8}$$

$$AP = \sqrt{(AB)^2 + Y^2} \tag{9}$$

$$AN = L_2 - (AP) = [L_2 - \sqrt{(Y\cot\alpha)^2 + Y^2}] \tag{10}$$

$$NO = \sqrt{(X - Y\cot\alpha)^2 + [L_2 - \sqrt{(Y\cot\alpha)^2 + Y^2}]^2 - 2(X - Y\cot\alpha)[L_2 - \sqrt{(Y\cot\alpha)^2 + Y^2}]\cos\alpha} \tag{11}$$

$$MN = \sqrt{(X - Y\cot\alpha)^2 + [L_2 - \sqrt{(Y\cot\alpha)^2 + Y^2}]^2 - 2(X - Y\cot\alpha)[L_2 - \sqrt{(Y\cot\alpha)^2 + Y^2}]\cos\alpha - R^2} \tag{12}$$

$$\widehat{KM} = R\theta \tag{13}$$

$$\theta = (90° + \gamma + \sigma) \tag{14}$$

$$\sigma = (90° - \alpha) + \delta \tag{15}$$

$$\delta = \epsilon - \lambda \tag{16}$$

$$\epsilon = \sin^{-1}(R/NO) \tag{17}$$

$$\lambda = \sin^{-1}(DO/NO) \tag{18}$$

$$DO = AO \sin\alpha \tag{19}$$

$$DO = (X - Y\cot\alpha)\sin\alpha \tag{20}$$

$$\gamma = \psi - \phi \tag{21}$$

$$\psi = \sin^{-1}(R/JO) \tag{22}$$

$$\phi = \sin^{-1}(Y/JO) \tag{23}$$

$$\gamma = \sin^{-1}\frac{R}{\sqrt{(L_1-X)^2 + Y^2}} - \sin^{-1}\frac{Y}{\sqrt{(L_1-X)^2 + Y^2}} \tag{24}$$

$$\widehat{KM} = R\left\{\frac{\pi}{2} + \sin^{-1}\frac{R}{\sqrt{(L_1-X)^2 + Y^2}} - \sin^{-1}\frac{Y}{\sqrt{(L_1-X)^2 + Y^2}}\right.$$
$$+\frac{\pi}{2} - \alpha + \sin^{-1}\frac{R}{\sqrt{(X-Y\cot\alpha)^2 + \left[L_2 - \sqrt{(Y\cot\alpha)^2 + Y^2}\right]^2 - 2(X-Y\cot\alpha)\left[L_2 - \sqrt{(Y\cot\alpha)^2 + Y^2}\right]\cos\alpha}}$$
$$\left. - \sin^{-1}\frac{(X-Y\cot\alpha)\sin\alpha}{\sqrt{(X-Y\cot\alpha)^2 + \left[L_2 - \sqrt{(Y\cot\alpha)^2 + Y^2}\right]^2 - 2(X-Y\cot\alpha)\left[L_2 - \sqrt{(Y\cot\alpha)^2 + Y^2}\right]\cos\alpha}}\right\} \tag{25}$$

If L_1/R or $L_2/R > 10$, then the arc sin may be taken as equal to the angle. Then $\widehat{KM}$ can be simplified to the following:

$$\widehat{KM} = R\left\{(\pi-\alpha) + \frac{(R - Y)}{\sqrt{(L_1 - X)^2 + Y^2}}\right.$$
$$\left. + \frac{R - (X-Y\cot\alpha)(\sin\alpha)}{\sqrt{(X-Y\cot\alpha)^2 + \left[L_2 - \sqrt{(Y\cot\alpha)^2 + Y^2}\right]^2 - 2(X-Y\cot\alpha)\left[L_2 - \sqrt{(Y\cot\alpha)^2 + Y^2}\right]\cos\alpha}}\right\} \tag{26}$$

The sum of Eqs. 5, 12 and 26 yields C, Eq. 1, in terms of the variables X, Y and α. In order to find a particular path of point O, a physical relationship between X and Y must be obtained.

Particular Solution A (Pulley Pitch Line Tangent to Lower Arm Center Line)

To obtain this physical relationship, further information with respect to size and construction of both arms and pulley is needed. The upper arm is of tubular cast aluminum, while the lower arm consists of a cast aluminum upper section and an extruded aluminum lower section. At least seven cables are used; all of them pass through the upper arm, while five or more pass through the lower arm, the cable forming a "ribbon" through each tube. The width of the ribbon must be held to a minimum as dictated by the width of bearing needed for each of the several pulleys in the vicinity of the pivot P. The lower arm has four times the inertial effect of the upper arm per unit of weight; therefore, the ribbon width and the tube diameter should be held to minima in the lower arm. The ribbon must pass through the tube on a diameter to obtain the minimum size tube for a given ribbon width.

A desirable relationship between X and Y may be obtained if the cable leaves the pulley at the point M lying on the center line NP of the lower arm or tube (see Fig. 4). Using similar triangles, the following Eq. 27 for X as function of Y and α, can be obtained:

$$X = \frac{R}{\sin\alpha} + \frac{Y}{\tan\alpha} \quad . \tag{27}$$

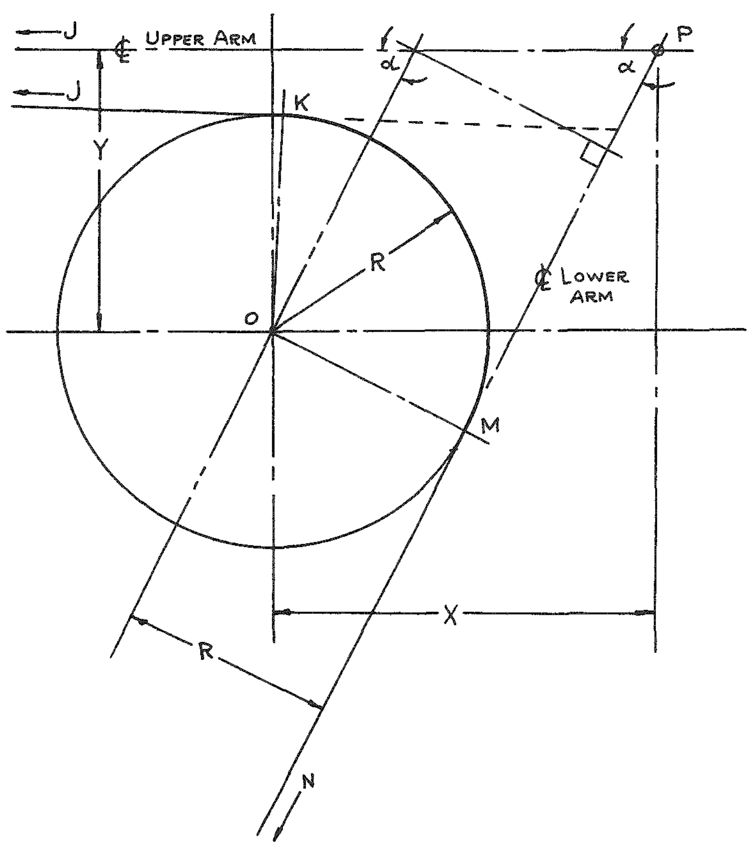


Fig. 4 Pulley Pitch Line Tangent To Arm Center Line

Eliminating the denominators,

$$X = R \csc \alpha + Y \cot \alpha \quad . \tag{28}$$

Substituting Eq. 28 into Eqs. 5, 12 and 26 results in the following:

$$JK = \sqrt{(L_1 - X)^2 + Y^2 - R^2} \tag{29}$$

or

$$JK = \sqrt{(L_1 - R \csc \alpha - Y \cot \alpha)^2 + Y^2 - R^2} \quad . \tag{30}$$

When Eq. 28 is substituted into Eq. 26, the last two terms drop out, since

$$(R \csc \alpha + Y \cot \alpha - Y \cot \alpha) \sin \alpha = R \quad . \tag{31}$$

Therefore,

$$\widehat{KM} = R \left\{ (\pi - \alpha) + \sin^{-1} \frac{R}{\sqrt{(L_1 - X)^2 + Y^2}} - \sin^{-1} \frac{Y}{\sqrt{(L_1 - X)^2 + Y^2}} \right\} . \tag{32}$$

If $\frac{R}{L_1} < 0.1$, and assuming arc $\sin \gamma = \gamma$, the last two terms can be combined and simplified to yield

$$\widehat{KM} = R \left\{ (\pi - \alpha) + \frac{(R - Y)}{\sqrt{(L_1 - R \csc \alpha - Y \cot \alpha)^2 + Y^2}} \right\} \tag{33}$$

$$MN = \sqrt{(X - Y \cot \alpha)^2 + [L_2 - \sqrt{(X + Y \cot \alpha)^2 + Y^2}]^2 - 2(X - Y \cot \alpha)[L_2 - \sqrt{(X + Y \cot \alpha)^2 + Y^2}] \cos \alpha - R^2} \tag{34}$$

$$MN = \sqrt{(R \csc \alpha)^2 + [L_2 - \sqrt{(Y \cot \alpha)^2 + Y^2}]^2 - 2(R \csc \alpha)[L_2 - \sqrt{(Y \cot \alpha)^2 + Y^2}] \cos \alpha - R^2} \tag{35}$$

$$MN = \sqrt{R^2 \csc^2 \alpha + (L_2 - Y \csc \alpha)^2 - 2R \cot \alpha (L_2 - Y \csc \alpha) - R^2} \tag{36}$$

$$\begin{aligned} C &= \sqrt{(L_1 - R \csc \alpha - Y \cot \alpha)^2 + Y^2 - R^2} \\ &\quad + R \left\{ (\pi - \alpha) + \frac{(R - Y)}{\sqrt{(L_1 - R \csc \alpha - Y \cot \alpha)^2 + Y^2}} \right\} \\ &\quad + \sqrt{R^2 \csc^2 \alpha + (L_2 - Y \csc \alpha)^2 - 2R \cot \alpha (L_2 - Y \csc \alpha) - R^2} \quad . \end{aligned} \tag{37}$$

As a first approach to a solution, it was thought to be desirable that the difference in values for Y should be kept to a minimum. In addition, Y should have values such that the following is true:

$$R - Y_{min} = Y_{max} - R \tag{38}$$

or

$$Y_{max} = 2R - Y_{min} \tag{39}$$

The reason for applying these additional conditions was again the desire to keep the size, and therefore the weight, of the upper arm to a minimum. The ideal condition is

$$Y_{max} = Y_{min} = R \tag{40}$$

To obtain a value for C for which Eq. 39 is true, the following values for parameters were used: L_1 = 30 in., L_2 = 40 in., and R = 1.0 in. Using these values, a series of curves of C vs. α for different values of Y were obtained, from which C = 69.619 in. This value for C was based on 15 increments from 45 to 135 degrees. Both the range of α and the value of the parameters chosen are those of the Argonne National Laboratory's Electronically Controlled Master-Slave Manipulator, Model 3.

With C = 69.619 in., a solution for Y vs. α from Eq. 37 was needed. The implicit form of this equation, which does not reduce to a simple algebraic equation, is awkward. Several values of Y were obtained by laborious trial and error. The success of the trial and error solution suggested the use of a digital computer, such as the IBM 650.

The method for programming selected was that outlined in IBM Technical Newsletter No. 11. This method is one developed by Dr. V. M. Wolontis of the Bell Telephone Laboratories; it is commonly known as the Bell L_1 system. Its actual name is a Complete Floating-Decimal Interpretive System for the IBM 650 Magnetic Drum Calculator. The actual program developed along with an explanation of the actual notation used may be found in the Appendices.

The program used solved for values of X and Y at 15-degree increments of α over the range from 15 to 165 degrees, a range exceeding present demands. This is a greater variation in α than that needed for the previously mentioned Model 3 manipulator, in Model 3, α varies from 45 to 135 degrees. Checking Eq. 39, using the value of Y at 45 degrees for Y_{min} gave a value of 1.8611 inches for Y_{max}. The actual value obtained at 135 degrees was 1.8422 inches. This difference was due to an inaccuracy in the value of C resulting from a graphical solution and slide rule calculation.

Further investigation, using a desk calculator for checking values of C, showed that Eq. 39 is very sensitive to small changes in C. Also, even when using a calculated value of C, the program fails to check the value at angles outside the 45- to 135- degree range of the Model 3.

Future manipulators may not be limited to the range from 15 to 165 degrees. The ultimate limit might be from 0 to 360 degrees. However, a range of 0 to 180 degrees would allow a solution which, because of symmetry, could result in 360 degrees of movement. If α were 180 degrees, there is a unique solution to C:

$$C = L_1 + L_2 \quad . \tag{41}$$

Also, the following must be true when α = 180 degrees:

$$Y = R \quad . \tag{42}$$

Although Eq. 42 must be satisfied, X can be any value and still satisfy the general equation for C at 180 degrees. However, a value of zero for X is unique, since it represents a point at which O is nearest P.

Substituting C from Eq. 41 into Eq. 37, the parameters of the Model 3 were then used to determine a new path. In addition to plotting the path of point O in coordinates of X and Y, the path of O was also plotted using coordinates of α and $L_2 - \overline{MN}$. The curve obtained from X-Y coordinates resembled a parabola. The curve of α vs. $L_2 - \overline{MN}$ was nearly a straight line. Also, the parabolic curve approached a circular arc between 45 and 180 degrees.

A single problem was solved with the use of the parameters of Model 3. In addition to this solution, it was desired to know the effect of different parameters. Table I shows the range of parameters for which solutions have also been obtained.

TABLE I

Parameters for $X = R \csc \alpha + Y \cot \alpha$

Parameter Set	L_1	L_2	R	L_1/R
1	10.0	10.0	0.5	20.0
2	10.0	10.0	0.75	13.3
3	10.0	10.0	1.0	10.0
4	20.0	20.0	0.5	40.0
5	20.0	20.0	0.75	26.6
6	20.0	20.0	2.00	10.0
7	10.0	40.0	1.0	10.0
8	30.0	40.0	1.0	30.0
9	40.0	40.0	2.00	20.0
10	36.0	48.0	2.0	18.0

The tabular and graphical solutions to these different parameter sets follow. In addition, the solutions to parameter set 12 are given as a special problem having C = 69.619 in.

TABLE II

Parameter Set 1 L_1 = 10 0, L_2 = 10 0, R = 0 5 and L_1/R = 20 0

[illegible]	Y	X	MN	L_2 - MN
15	-0.306	0 790	9 316	0 684
30	-0 143	0 752	9 421	0 579
45	-0 007	0 700	9 509	0 491
60	0 109	0 640	9 586	0 414
75	0 206	0 573	9 653	0 347
90	0.287	0.500	9 713	0 287
105	0.353	0.423	9.768	0 232
120	0.406	0 343	9 820	0 180
135	0 446	0 261	9.868	0 132
150	*	*	*	*
165	*	*	*	*
180	0 500	0 000	10 000	0.000

TABLE III

Parameter Set 2 L_1 10 0, L_2 = 10 0, R = 0 75 and L_1/R = 13 3

	Y	X	MN	L_2 - MN
15**	-0 455	1 200	8 959	1 041
30**	-0 210	1 136	9 121	0 879
45	-0 005	1 056	9 257	0 743
60	0 167	0 963	9 373	0 627
75	0 312	0 860	9 476	0 524
90	0 433	0 750	9 567	0 433
105	0 552	0 634	9 650	0 350
120	0 610	0 514	9 728	0 272
135	0 671	0 390	9 802	0 198
150	0 712	0 266	9 875	0 125
165	0 735	0 156	9 960	0 040
180	0 750	0 000	10 000	0 000

* Data missing

** Data do not satisfy assumptions because of small angle approximation for [illegible] .

TABLE IV

Parameter Set 3 L_1 = 10 0, L_2 = 10 0, R = 1 0 and L_1/R = 10 0

[illegible]	Y	X	MN	L_2 - MN
15	-0 601	1 622	8 589	1 411
30	-0 273	1 527	8 814	1 186
45	0 000	1 414	9 005	0 995
60	0 228	1 287	9 159	0 841
75	0 420	1 148	9 297	0 703
90	0 580	1.000	9 420	0 580
105	0 710	0.845	9 532	0 468
120	0 815	0 684	9 636	0 364
135	0 895	0 519	9 734	0 266
150	0 951	0 353	9.830	0 170
165	0.981	0 202	9.941	0.059
180	1 000	0 000	10 000	0.000

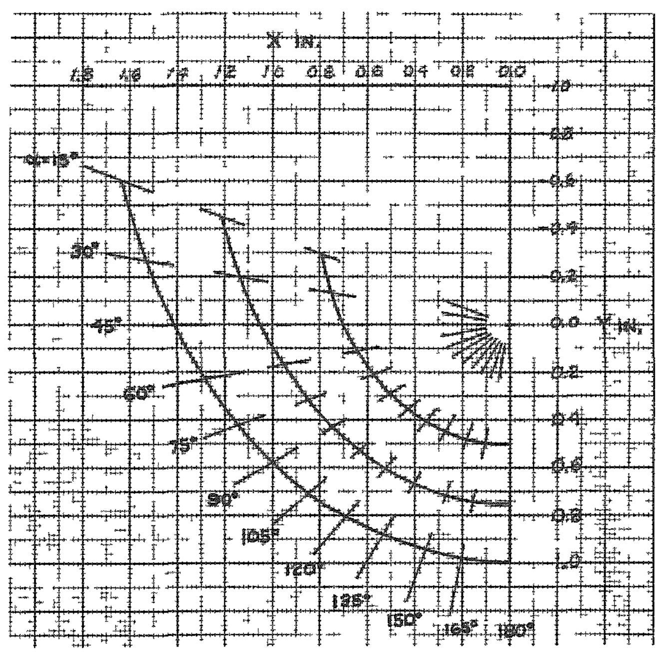


FIG 5 PARAMETER SETS 1,2 & 3
X = R csc α + Y cot α
L_1 = 10.0, L_2 = 10.0 WITH
R = 0.5, 0.75 & 1.0

TABLE V

Parameter Set 4 L_1 = 20 0, L_2 = 20 0, R = 0.5 and L_1/R = 40

	Y	X	MN	L_2 - MN
15	-0.308	0 782	19 324	0.676
30	-0.147	0 746	19 427	0 573
45	-0.010	0 697	19.514	0 486
60	0 106	0 639	19 588	0 412
75	0 204	0 572	19.655	0 345
90	0 286	0 500	19 714	0 286
105	0.352	0 423	19 769	0 231
120	0 406	0 343	19 820	0 180
135	0.446	0 261	19 869	0 131
150	0 473	0.180	19 919	0 081
165	0 487	0 115	19 985	0 015
180	0.500	0 000	20.000	0.000

TABLE VI

Parameter Set 5 L_1 = 20 0, L_2 = 20 0, R = 0 75 and L_1/R = 26 6

	Y	X	MN	L_2 - MN
15	-0 461	1 179	18 981	1 019
30	-0 217	1.124	19 136	0 864
45	-0 012	1 049	19 267	0.733
60	0.162	0.959	19 380	0 620
75	0 308	0 859	19 480	0 520
90	0 430	0 750	19 570	0.430
105	0.530	0.635	19 653	0 347
120	0.610	0 514	19 729	0 271
135	0.670	0 391	19 802	0.198
150	0.712	0 267	19 875	0 125
165	0.734	0 159	19.964	0.036
180	0.750	0.000	20 000	0.000

TABLE VII

Parameter Set 6: L_1 = 20.0, L_2 = 20.0, R = 2.00 and L_1/R = 10.0

α	Y	X	MN	L_2 - MN
15*	-1.202	3.243	17.178	2.822
30*	-0.546	3.054	17 628	2.372
45*	0.000	2.828	18.000	2.000
60	0.457	2.573	18 317	1.683
75	0.841	2.296	18 594	1 406
90	1.160	2.000	18.840	1.160
105	1.422	1.690	19.064	0.936
120	1 632	1.367	19.270	0.730
135	1.792	1.036	19.465	0.535
150	1.905	0 700	19.654	0.346
165	1.974	0.360	19.837	0.163
180	2 000	0.000	20.000	0.000

* Data do not satisfy assumptions because of small angle approximation for γ.

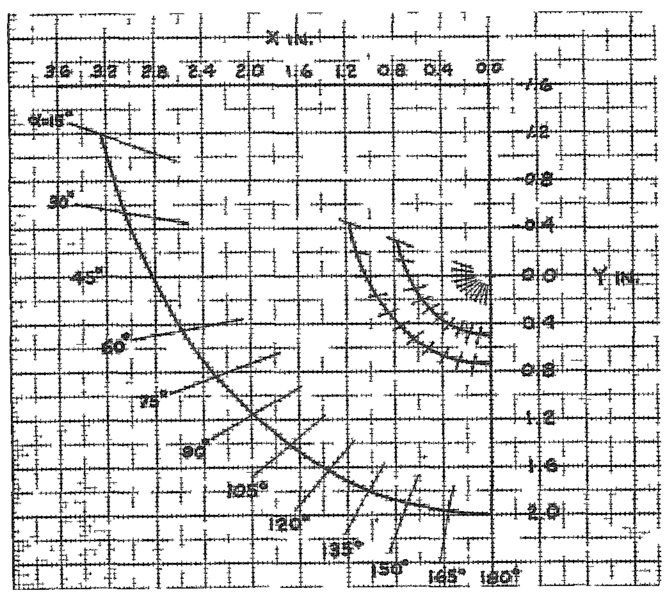


FIG 8 PARAMETER SETS 4,5 & 6
X = R csc α + Y cot α
L_1 = 20.0, L_2 = 20.0 WITH
R = 0.5, 0.75 & 2.0

TABLE VIII

Parameter Set 7: L_1 = 10.0, L_2 = 40.0, R = 1.0 and L_1/R = 10.0

α	Y	X	MN	L_2 - MN
15	-0.601	1.622	38.589	1.411
30	-0.273	1.527	38.814	1.186
45	0.000	1.414	39.000	1.000
60	0.228	1.287	39.159	0.841
75	0.419	1.148	39.297	0.702
90	0.580	1.000	39.420	0.580
105	*	*	*	*
120	*	*	*	*
135	*	*	*	*
150	*	*	*	*
165	*	*	*	*
180	*	*	*	*

* Data missing

TABLE IX

Parameter Set 8: L_1 = 30.0, L_2 = 40.0, R = 1.0 and L_1/R = 30.0

α	Y	X	MN	L_2 - MN
15	-0.615	1.569	38.643	1.357
30	-0.291	1.496	38.850	1.150
45	-0.017	1.397	39.024	0.976
60	0.215	1.279	39.175	0.825
75	0.410	1.145	39.307	0.693
90	0.573	1.000	39.426	0.574
105	0.707	0.846	39.536	0.464
120	0.813	0.685	39.638	0.362
135	0.894	0.520	39.735	0.265
150	0.951	0.354	39.830	0.170
165	0.981	0.202	39.941	0.059
180	1.000	0.000	40.000	0.000

TABLE X

Parameter Set 9: L_1 = 40.0, L_2 = 40.0, R = 2.0 and L_1/R = 20.0

α	Y	X	MN	L_2 - MN
15	-1.223	3.162	37.262	2.738
30	-0.573	3.007	37.682	2.318
45	-0.026	2.803	38.036	1.964
60	0.437	2.562	38.341	1.659
75	0.826	2.292	38.609	1.391
90	1.150	2.000	38.850	1.150
105	1.417	1.691	39.069	0.931
120	1.629	1.369	39.273	0.727
135	1.793	1.035	39.464	0.536
150	1.905	0.700	39.654	0.346
165	1.970	0.376	39.854	0.146
180	2.000	0.000	40.000	0.000

TABLE XI

Parameter Set 10: L_1 = 36.0, L_2 = 48.0, R = 2.00 and L_1/R = 18.0

α	Y	X	MN	L_2 - MN
15	-1.221	3.171	45.253	2.747
30	-0.570	3.012	45.677	1.968
45	-0.023	2.805	46.032	1.968
60	0.439	2.563	46.339	1.661
75	0.827	2.292	46.607	1.393
90	1.151	2.000	46.849	1.151
105	1.417	1.691	47.069	0.931
120	1.629	1.369	47.273	0.727
135	1.791	1.037	47.467	0.533
150	1.905	0.700	47.654	0.346
165	*	*	*	*
180	2.000	0.000	48.000	0.000

*Data missing

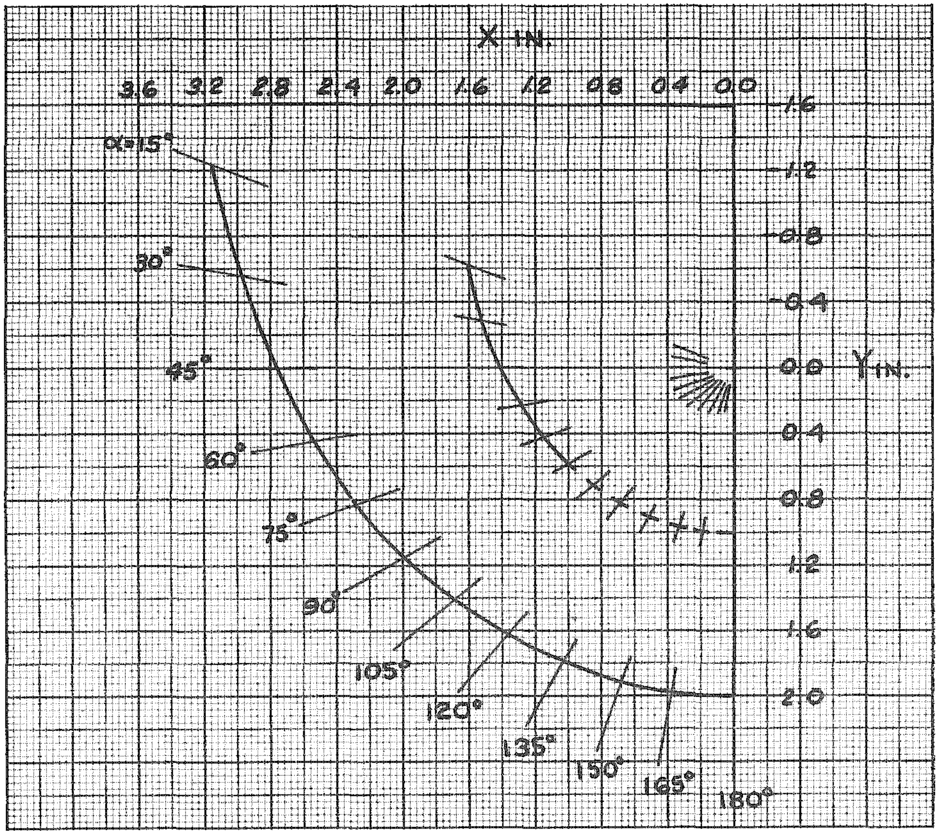


FIG. 7 PARAMETER SETS 7 & 9

$X = R\csc\alpha + Y\cot\alpha$

$L_1 = 10.0$, $L_2 = 40.0$ WITH $R = 1.0$

$L_1 = 40.0$, $L_2 = 40.0$ WITH $R = 2.0$

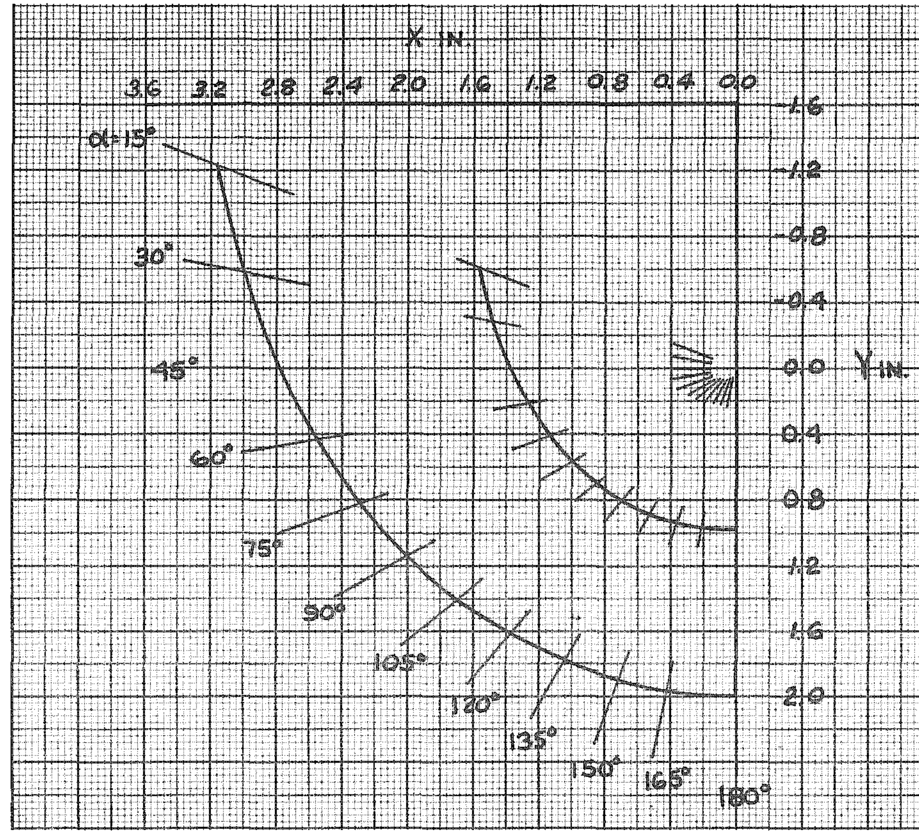


FIG. 8(a) PARAMETER SETS 8 & 10

$X = R\csc\alpha + Y\cot\alpha$

$L_1 = 30.0$, $L_2 = 40.0$ WITH $R = 1.0$

$L_1 = 36.0$, $L_2 = 48.0$ WITH $R = 2.0$

TABLE XII

Parameter Set 11 $L_1 = 30.0$, $L_2 = 40.0$,
$R = 1\ 0$ and $L_1/R = 30\ 0$

SPECIAL PROBLEM
X = R csc + Y cot
Also
$JK + \widehat{KM} + MN = 69\ 619$

	Y	X	MN	L_2 - MN
15**	-0.563	1.764	38.442	1.558
30**	-0.188	1.675	38 643	1 357
45	0.139	1 553	38.804	1 196
60	0.434	1.405	38 921	1 079
75	0.701	1.223	39.006	0.994
90	0.952	1.000	39 048	0.952
105	1.203	0.713	39.022	0.978
120	1.479	0.301	38.869	1.131
135	1.842	-0.428	38 395	1.605
150**	2.503	-2.336	36.725	3.275
165**	*	*	*	*
180**	*	*	*	*

* Data missing
** Data do not satisfy assumptions because of small angle approximation for γ or $C \neq 69.619$

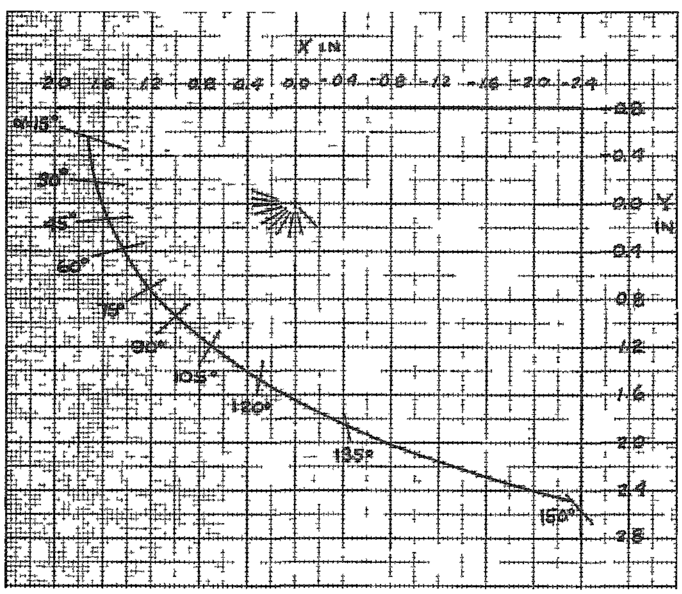


Fig. 8(b) Parameter Set 11
X = R csc α + Y cot α
C = 69.619 = JK + $\widehat{KM}$ + MN
$L_1 = 30.0$, $L_2 = 40.0$ with R = 1.0

Summary

A review of the results can best be had from the graphical presentations in Figs. 5 through 8. The magnitudes of X and Y vary almost exclusively as a function of R. The tabular data, however, do indicate that the length of L_1 does influence the result; L_2 does not affect the result since the point M remains on the Line NP. It will be noticed that the path, although parabolic in shape, does approach the arc of a circle for a large portion of the curve.

Mechanisms to cause the pulley center to follow the analytically determined paths more or less accurately were next investigated.

Mechanism 1 used a cam, the precedent being Model 3, with the future possible requirement of a large α and the maintenance of cam weight. The cam proposal appears impractical. In addition, accurate cams are very expensive.

Mechanism 2 considered the use of a gear and rack to drive a slider on the lower arm. This mechanism is feasible since the movement of the pulley center with respect to the lower arm is nearly linear. As in the case of the cam, weight and cost are high.

Other mechanisms, such as a four-bar linkage or a slidecrank, were impracticable for one reason or another.

Particular Solution B (Pulley Center Moving on the Arc of a Circle)

As a possible method for reducing complexity, other physical relations between X and Y were investigated. For angles of α greater than 45 degrees, the parabolic curves obtained as previously mentioned appear as an arc of a circle. Using graphical means, the center of the circle arc for each of the Parameter Sets of Table I was found. Using these centers an approximate general solution was obtained when the center of the circular arc lay at the coordinates, $x = 0$, and $y = -0.5R$, the radius of the circular arc being 1.5R. This relation is accurate at $\alpha = 180$ degrees. For other angles the relation is a very close approximation to the solutions obtained. Figure 9 summarizes the new relation between X and Y.

This relationship allows a simpler solution to the four-bar linkage, since the point O lies on the follower rather than on the coupler. The length of the fixed link is 0.5R for all parameters. This allows easier graphical solution for the remaining links.

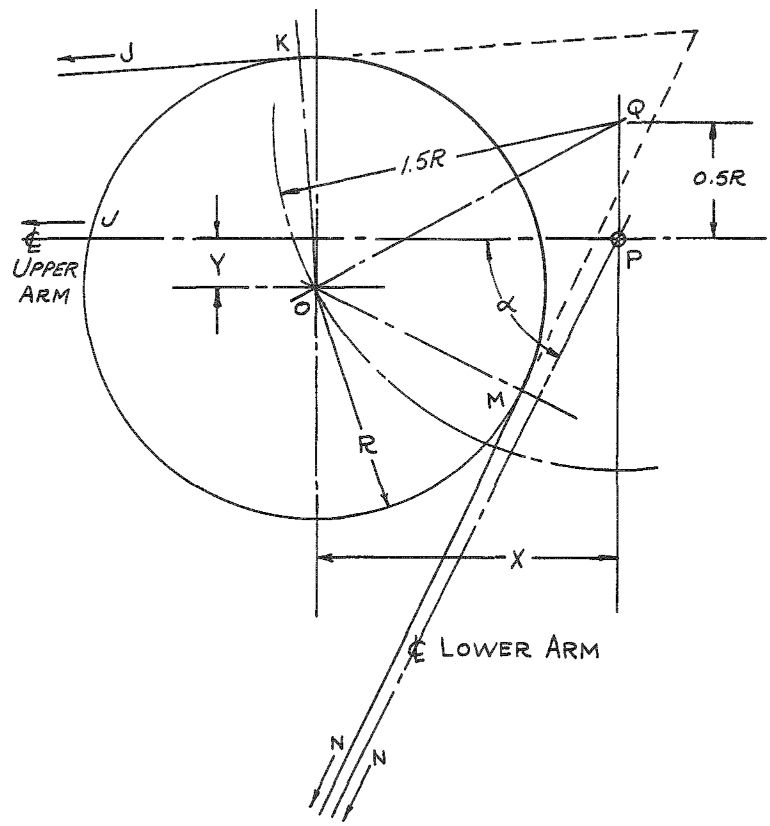


FIG 9 PULLEY CENTER MOVING
ON THE
ARC OF A CIRCLE

A second simplification may be the use of a cam whose working profile is a straight line cam. A third solution involving a single link is applicable to cases in which a line parallel to the link bisects the included angle between the lines JK and MN.

The geometry shown in Fig. 9 gives the following equations needed for computing X and Y. Since Q, the center of the circular arc, has the coordinates x = 0 and y = 0.5R, and the radius of the circle is equal to 1.5R, an equation for this particular circle can be written as

$$X^2 + (Y + 0.5R)^2 = 2.25R^2 \quad . \tag{43}$$

From this,

$$X = \sqrt{2.25R^2 - (Y + 0.5R)^2} \quad . \tag{44}$$

Substituting this equation for X into Equations 5, 12 and 26,

$$JK = \sqrt{\left[L_1 - \sqrt{2.25R^2 - (Y + 0.5R)^2}\right]^2 + Y^2 - R^2} \tag{45}$$

$$\widehat{KM} = R\left\{(\pi-\alpha) + \frac{(R-Y)}{\sqrt{\left[L_1 - \sqrt{2.25R^2 - (Y+0.5R)^2}\right]^2 + Y^2}}\right. \tag{46}$$

$$+ \frac{R - (\sin\alpha)\sqrt{2.25R^2 - (Y+0.5R)^2} + Y\cos\alpha}{\sqrt{\left[\sqrt{2.25R^2-(Y+0.5R)^2}-Y\cot\alpha\right]^2 + \left[L_2-\sqrt{(Y\cot\alpha)^2+Y^2}\right]^2 - 2\cos\alpha\left[\sqrt{2.25R^2-(Y+0.5R)^2}-Y\cot\alpha\right]\left[L_2-\sqrt{(Y\cot\alpha)^2+Y^2}\right]}} \tag{47}$$

$$MN = \sqrt{\left[\sqrt{2.25R^2-(Y+0.5R)^2}-Y\cot\alpha\right]^2 + \left[L_2-\sqrt{(Y\cot\alpha)^2+Y^2}\right]^2 - 2\cos\alpha\left[\sqrt{2.25R^2-(Y+0.5R)^2}-Y\cot\alpha\right]\left[L_2-\sqrt{(Y\cot\alpha)^2+Y^2}\right] - R^2}$$

$$C = L_1 + L_2 = \sqrt{\left[L_1 - \sqrt{2.25R^2 - (Y+0.5R)^2}\right]^2 + Y^2 - R^2}$$

$$+R\left\{(\pi-\alpha) + \frac{(R-Y)}{\sqrt{\left[L_1 - \sqrt{2.25R^2 - (Y+0.5R)^2}\right]^2 + Y^2}}\right.$$

$$\left. + \frac{\left[R - (\sin\alpha)\sqrt{2.25R^2 - (Y+0.5R)^2} + Y\cos\alpha\right]}{\sqrt{\left[\sqrt{2.25R^2-(Y+0.5R)^2}-Y\cot\alpha\right]^2 + \left[L_2-\sqrt{(Y\cot\alpha)^2+Y^2}\right]^2 - 2\cos\alpha\left[\sqrt{2.25R^2-(Y+0.5R)^2}-Y\cot\alpha\right]\left[L_2-\sqrt{(Y\cot\alpha)^2+Y^2}\right]}}\right\}$$

$$+ \sqrt{\left[\sqrt{2.25R^2-(Y+0.5R)^2}-Y\cot\alpha\right]^2 + \left[L_2-\sqrt{(Y\cot\alpha)^2+Y^2}\right]^2 - 2\cos\alpha\left[\sqrt{2.25R^2-(Y+0.5R)^2}-Y\cot\alpha\right]\left[L_2-\sqrt{(Y\cot\alpha)^2+Y^2}\right] - R^2} \tag{48}$$

Using Eq. 48, a new IBM program, similar to the one used for Eq. 37, was developed. This program can be found in the appendices.

Table XIII shows the variation of parameters for which solutions to Eq. 48 have been obtained. The program was arranged to produce γ and δ (Fig. 3) during the course of the computation for Y.

TABLE XIII

Parameters for $X = \sqrt{2.25R^2 - (Y + 0.5R)^2}$

Parameter Set	L_1	L_2	R	L_1/R	L_2/R
12	10.0	10.0	0.5	20.0	20.0
13	10.0	10.0	0.75	13.3	13.3
14	10.0	10.0	1.0	10.0	10.0
15	20.0	20.0	0.5	40.0	40.0
16	20.0	20.0	1.0	20.0	20.0
17	20.0	20.0	2.0	10.0	10.0
18	10.0	40.0	0.5	20.0	80.0
19	10.0	40.0	1.0	10.0	40.0
20	40.0	40.0	2.0	20.0	20.0
21	10.0	20.0	0.5	20.0	40.0
22	10.0	20.0	1.0	10.0	20.0
23	30.0	30.0	1.0	30.0	30.0
24	30.0	40.0	1.0	30.0	40.0

These angles become important when a possible solution utilizing a single link is to be investigated. Figure 10 shows the relationship of a single link. Several symbols, not shown on Fig. 3, are necessary for further clarification:

β = the included angle between the lines JK and MN extended, degrees
μ = the angle between the lines OQ and JP, degrees
ρ = the angle between the bisector of β and the line JP, degrees

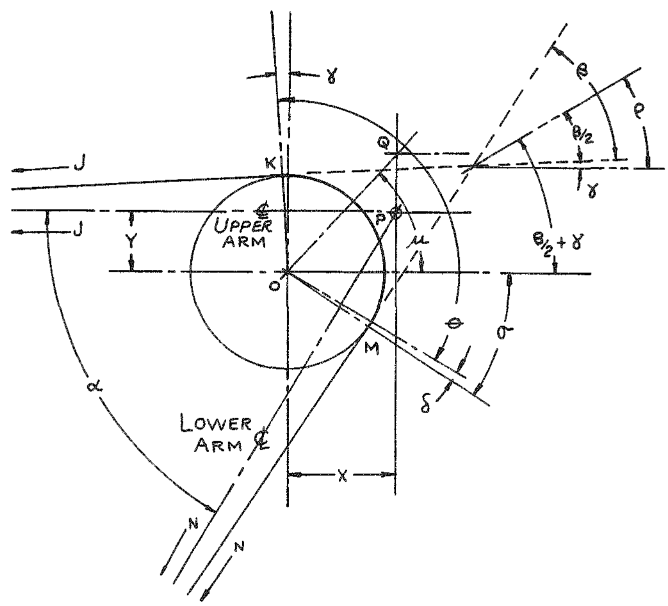


FIG. 10 SINGLE LINK RELATIONSHIPS

In order to have an exact solution with a single link, certain conditions among these angles must be satisfied. In addition to C being constant, ρ must be equal to μ. The following equations simplify the angles ρ and μ into terms which may be obtained during computations for X and Y:

$$\rho = \tfrac{1}{2}\beta + \gamma \qquad (49) \qquad\qquad \tfrac{1}{2}\beta = 90° - \tfrac{1}{2}\theta \quad . \qquad (50)$$

Using Equations 14 and 15,

$$\theta = (180° - \alpha) + \gamma + \delta \qquad (51)$$

$$\tfrac{1}{2}\theta = 90° - \tfrac{1}{2}(\alpha - \gamma - \delta) \qquad (52)$$

$$\tfrac{1}{2}\beta = \tfrac{1}{2}(\alpha - \gamma - \delta) \qquad (53)$$

$$\rho = \tfrac{1}{2}(\alpha - \gamma - \delta) + \gamma \qquad (54)$$

$$\rho = \tfrac{1}{2}(\alpha + \gamma - \delta) \qquad (55)$$

$$\mu = \tan^{-1}\left(\frac{Y + 0.5R}{X}\right) = \sin^{-1}\left(\frac{Y + 0.5R}{1.5R}\right) \qquad (56)$$

TABLE XIV

Parameter Set 12:
L_1 = 10.0, L_2 = 10.0, R = 0.50,
L_1/R = 20.0, and L_2/R = 20.0

α	Y	X	ρ	μ
45	*	*	*	*
60	0	0.707	31°53'	19°30'
75	0.081	0.673	39°10'	26°15'
90	0.169	0.622	46°22'	33°57'
105	0.255	0.555	53°32'	42°20'
120	0.333	0.471	60°43'	51°5 '
135	0.400	0.375	67°56'	59°59'
150	0.449	0.271	75°23'	68°49'
165	0.479	0.177	82°36'	76°20'
180	0.500	0.000	90°0 '	90°0 '

*Data missing

TABLE XV

Parameter Set 13:
L_1 = 10.0, L_2 = 10.0, R = 0.75,
L_1/R = 13.3, and L_2/R = 13.3

α	Y	X	ρ	μ
45	*	*	*	*
60	0.029	1.050	32°45'	21°25'
75	0.144	0.998	39°57'	27°26'
90	0.269	0.922	47°2 '	34°27'
105	0.393	0.822	54°4 '	43°3 '
120	0.507	0.699	61°4 '	51°41'
135	0.604	0.554	68°8 '	60°30'
150	0.678	0.396	75°19'	69°24'
165	0.722	0.247	82°34'	77°24'
180	0.750	0.000	90°0 '	90°0 '

*Data missing

TABLE XVI

Parameter Set 14:
L_1 = 10.0, L_2 = 10.0, R = 1.0,
L_1/R = 10.0, and L_2/R = 10.0

α	Y	X	ρ	μ
45	*	*	*	*
60	0.079	1.384	31°54'	22°42'
75	0.220	1.316	39°11'	28°41'
90	0.378	1.216	46°22'	35°50'
105	0.536	1.084	53°33'	43°41'
120	0.683	0.922	60°34'	52°6 '
135	0.810	0.731	67°57'	60°50'
150	0.907	0.520	75°13'	69°41'
165	0.966	0.316	82°36'	77°50'
180	1.000	0.000	90°0 '	90°0 '

*Data missing

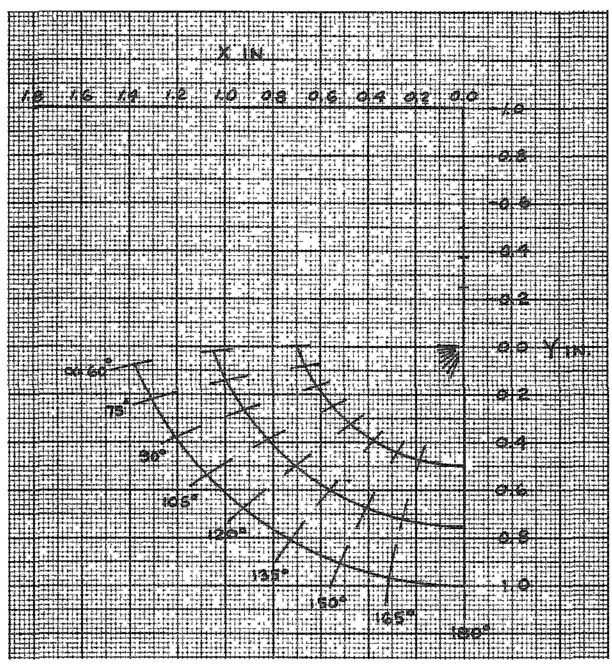


FIG. 11 PARAMETER SETS 12, 13 & 14
$X = \sqrt{2.25R^2 - (Y+0.5R)^2}$
L_1 = 10.0, L_2 = 10.0 WITH
R = 0.5, 0.75 & 1.0

TABLE XVII

Parameter Set 15·
L_1 = 20.0, L_2 = 20.0, R = 0.5,
L_1/R = 40.0, and L_2/R = 40.0

α	Y	X	ρ	μ
45	*	*	*	*
60	*	*	*	*
75	0.068	0.679	38°22'	25°6 '
90	0.160	0 628	45°41'	33°4 '
105	0.249	0 560	53°1 '	41°42'
120	0 330	0 476	60°22'	50°37'
135	0.398	0 377	67°43'	59°48'
150	0.449	0 272	75°7 '	68°44'
165	0.478	0.179	82°33'	76°11'
180	0.500	0.000	90°0 '	90°0 '

*Data missing

TABLE XVIII

Parameter Set 16
L_1 = 20.0, L_2 = 20 0, R = 1.0,
L_1/R = 20.0, and L_2/R = 20.0

	Y	X		μ
45	*	*	*	*
60	0 001	1 414	31°53'	19°31'
75	0.165	1 344	39°10'	26°20'
90	0 341	1 242	46°22'	34°4 '
105	0.512	1 107	53°32'	42°26'
120	0 670	0 938	60°42'	51°17'
135	0 804	0 741	67°56'	60°23'
150	0.905	0.525	75°13'	69°31'
165	0 966	0 318	82°35'	77°46'
180	1 000	0 000	90°0 '	90°0 '

*Data missing

TABLE XIX

Parameter Set 17
L_1 = 20.0, L_2 = 20.0, R = 2 0,
L_1/R = 10 0, and L_2/R = 10.0

γ	Y	X	ρ	μ
45	*	*	*	*
60	0 160	2.767	33°33'	22°45'
75	0.443	2.630	40°43'	28°45'
90	0.759	2.430	47°40.	35°54'
105	1.076	2.166	54°32'	43°46'
120	1.370	1.839	61°24'	52°11'
135	1.624	1.454	68°20'	61°1 '
150	1.821	1.022	75°24'	70°5 '
165	1.943	0.583	82°38'	78°47'
180	2.000	0.000	90°0 '	90°0 '

*Data missing

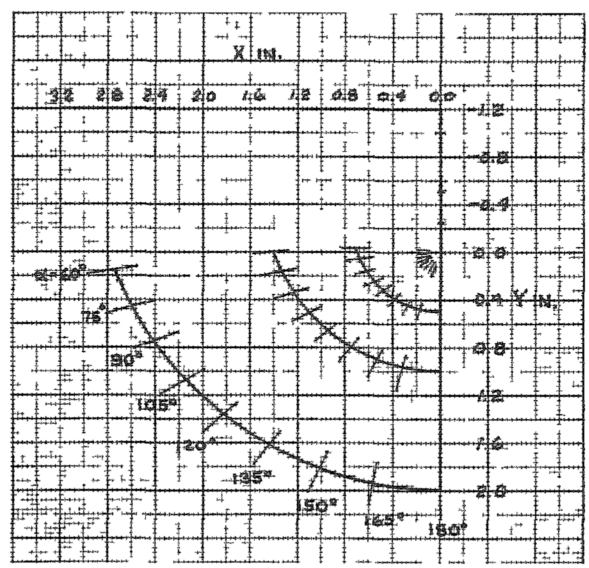


FIG 12 PARAMETER SETS 15, 16 & 17
$X = \sqrt{2.25R^2 - (Y + 0.5R)^2}$
L_1 = 20.0, L_2 = 20.0 WITH
R = 0.5, 1.0 & 2.0

TABLE XX

Parameter Set 18:
$L_1 = 10.0$, $L_2 = 40.0$, $R = 0.5$,
$L_1/R = 20$, and $L_2/R = 80$

α	Y	X	ρ	μ
45	*	*	*	*
60	0.000	0.707	31°38'	19°28'
75	0.080	0.674	38°53'	26°5 '
90	0.167	0.623	46°7 '	33°48'
105	0.253	0.556	53°20'	42°8 '
120	0.332	0.472	60°34'	50°58'
135	0.399	0.376	67°50'	59°55'
150	0.449	0.271	75°10'	68°48'
165	0.479	0.178	82°35'	76°15'
180	0.500	0.000	90°0 '	90°0 '

*Data missing

TABLE XXI

Parameter Set 19
$L_1 = 10.0$, $L_2 = 40.0$, $R = 1.0$,
$L_1/R = 10$, and $L_2/R = 40$

	Y	X	ρ	μ
45	*	*	*	*
60	0.075	1.385	33°12'	22°33'
75	0.215	1.319	40°15'	28°28'
90	0.373	1.220	47°12'	35°35'
105	0.532	1.089	54°9 '	43°28'
120	0.680	0.926	62°7 '	51°53'
135	0.808	0.734	68°10'	60°42'
150	0.906	0.522	75°19'	69°38'
165	0.966	0.317	82°37'	77°48'
180	1.000	0.000	90°0 '	90°0 '

*Data missing

TABLE XXII

Parameter Set 20:
$L_1 = 40.0$, $L_2 = 40.0$, $R = 2.0$,
$L_1/R = 20.0$, and $L_2/R = 20.0$

	Y	X	ρ	μ
45	*	*	*	*
60	0.004	2.827	31°53'	19°27'
75	0.333	2.688	39°10'	26°22'
90	0.684	2.483	46°22'	34°9 '
105	1.028	2.211	53°32'	42°32'
120	1.344	1.872	60°42'	51°25'
135	1.613	1.475	67°55'	60°33'
150	1.817	1.032	75°12'	69°53'
165	1.942	0.586	82°34'	78°44'
180	2.000	0.000	90°0 '	90°0 '

*Data missing

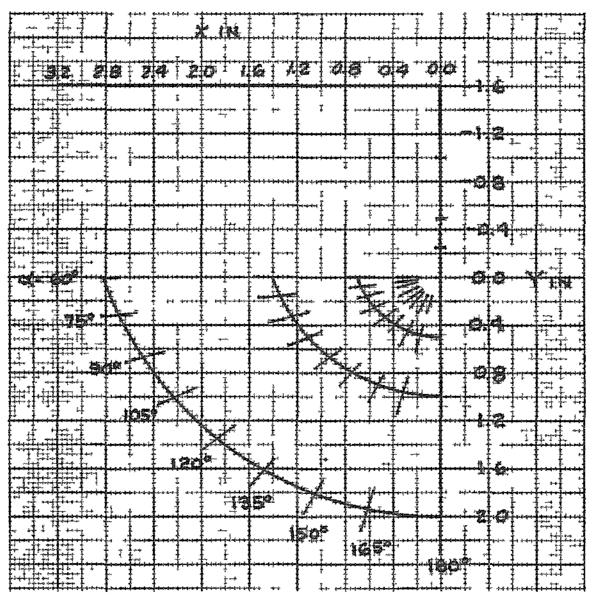


FIG. 13 PARAMETER SETS 18, 19 & 20
$X = \sqrt{2.25R^2 - (Y + 0.5R)^2}$
$L_1 = 10.0$, $L_2 = 40.0$ WITH $R = 0.5$ & 1.0
$L_1 = 40.0$, $L_2 = 40.0$ WITH $R = 2.0$

TABLE XXIII

Parameter Set 21
$L_1 = 10.0$, $L_2 = 20\ 0$, $R = 0.5$,
$L_1/R = 20\ 0$, and $L_2/R = 40\ 0$

θ	Y	X	ρ	μ
45	*	*	*	*
60	0.000	0.707	31°43'	19°28'
75	0.080	0.673	38°59'	26°20'
90	0.168	0.623	46°12'	33°51'
105	0.254	0.556	53°24'	42°12'
120	0.333	0.472	60°37'	51°1 '
135	0.399	0.375	67°52'	59°59'
150	0.449	0.271	75°11'	68°49'
165	0.479	0.178	82°34'	76°16'
180	0.500	0.000	90°0 '	90°0 '

*Data missing

TABLE XXIV

Parameter Set 22
$L_1 = 10\ 0$, $L_2 = 20\ 0$, $R = 1\ 0$,
$L_1/R = 10\ 0$, and $L_2/R = 20\ 0$

	Y	X	ρ	μ
45	*	*	*	*
60	0 076	1 385	33°19'	22°35'
75	0.216	1 318	40°24'	28°31'
90	0.374	1 219	47°22'	35°39'
105	0.533	1 088	54°13'	43°31'
120	0.681	0.924	61°12'	51°58'
135	0.809	0.733	68°13'	60°45'
150	0.906	0.522	75°21'	69°40'
165	0 966	0 317	82°38'	77°48'
180	1.000	0.000	90°0 '	90°0 '

*Data missing

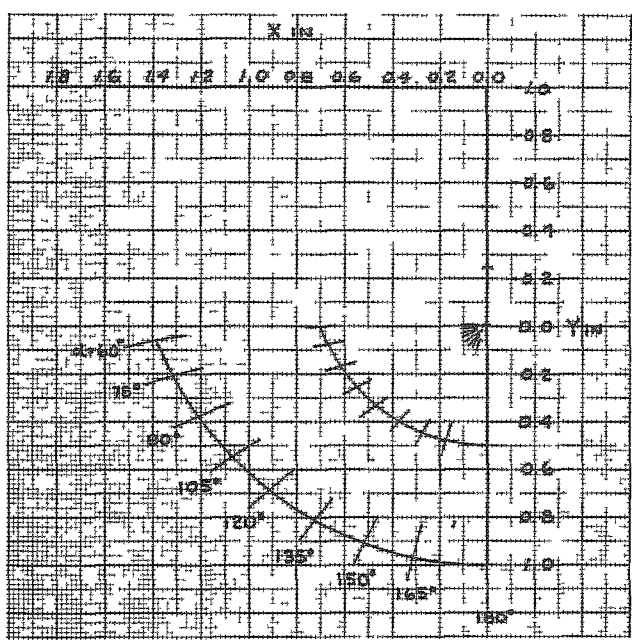


FIG. 14 PARAMETER SETS 21 & 22
$X = \sqrt{2.25R^2 - (Y + 0.5R)^2}$
$L_1 = 10.0$, $L_2 = 20.0$ WITH
$R = 0.5$ & 1.0

TABLE XXV

Parameter Set 23:
$L_1 = 30.0$, $L_2 = 30.0$, $R = 1.0$,
$L_1/R = 30.0$, and $L_2/R = 30.0$

α	Y	X	ρ	μ
45	*	*	*	*
60	*	*	*	*
75	0.147	1.353	38°37'	25°33'
90	0.328	1.251	45°55'	33°30'
105	0.505	1.114	53°12'	42°4 '
120	0.666	0.944	60°29'	51°1 '
135	0.802	0.745	67°47'	60°13'
150	0.905	0.527	75°9 '	69°26'
165	0.966	0.318	82°33'	77°46'
180	1.000	0.000	90°0 '	90°0 '

*Data missing

TABLE XXVI

Parameter Set 24:
$L_1 = 30$, $L_2 = 40$, $R = 1.0$,
$L_1/R = 30$, and $L_2/R = 40$

	Y	X	ρ	μ
45	*	*	*	*
60	*	*	*	*
75	0.146	1.354	38°33'	25°31'
90	0.327	1.251	45°51'	33°28'
105	0.504	1.114	53°9 '	42°2 '
120	0.665	0.944	60°27'	50°59'
135	0.802	0.745	67°46'	60°13'
150	0.904	0.527	75°8 '	69°25'
165	0.966	0.318	82°33'	77°48'
180	1.000	0.000	90°0 '	90°0 '

*Data missing

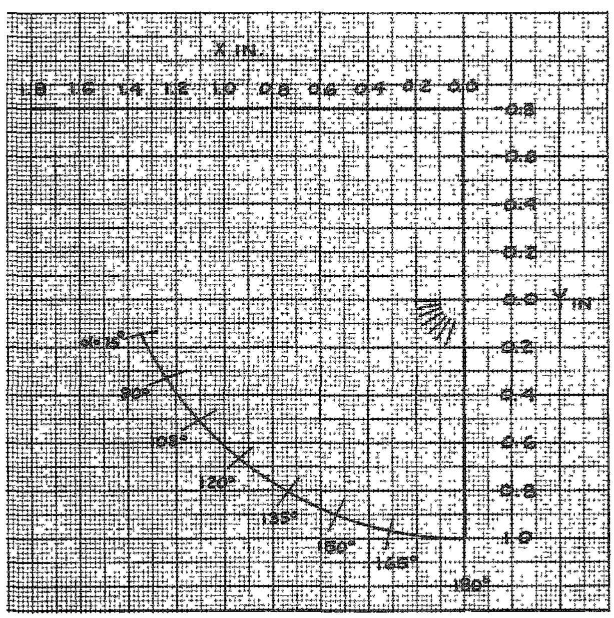


FIG 15 PARAMETER SETS 23 & 24
$X = \sqrt{2.25R^2 - (Y + 0.5R)^2}$
$L_1 = 30.0$, $L_2 = 30.0$ WITH $R = 1.0$
$L_1 = 30.0$, $L_2 = 40.0$ WITH $R = 1.0$

Summary

A review of the results obtained for the Parameter Sets of Table XIII can best be obtained from the tabular data. This differs from results of the Particular Solution A, which were mainly evident on the graphical presentation. The reason for this difference is that with the Particular Solution A the path of the point O was laid out. In Particular Solution B the path is already known, but the points on the path are undetermined.

Of the ten Parameter Sets of Table I, eight were duplicated in Table XIII. Five additional parameter sets were also used as inputs for Eq. 48. Duplication allowed comparison between Particular Solutions A and B. The comparison resulted in a comparative path but a difference in position on the path.

Although the program for Particular Solution B was similar to that for Particular Solution A, difficulty in obtaining a solution was encountered. This was found to be caused by terms under radical signs becoming negative. Physically meaningless answers were obtained, since the program used the absolute value of all terms whose roots were taken. A comparison of the radical terms with the graphical solution obtained with the first (or tangential) relationship showed that, if the point for a given angle fell outside of the circular arc, certain radicals became negative. The points that fell outside of the circular arc were for values of α equal to 15 and 30 degrees. When α was equal to 45 degrees, conditions were marginal.

Due to these marginal conditions, the program was revised to start with α = 60 degrees. This seemed to solve the difficulties, since solutions were obtained for eleven sets of parameters; however, the other three gave erratic results again. The sets of parameters which failed to give a solution with α starting at 60 degrees are:

Parameter Set	L_1	L_2	R	L_1/R	L_2/R
15	20.0	20.0	0.5	40.0	40.0
23	30.0	30.0	1.0	30.0	30.0
24	30.0	40.0	1.0	30.0	40.0

A comparison of these parameters with those that gave solutions showed that somewhere between L_1/R = 20.0 and L_1/R = 30.0 there is a change to the imaginary zone.

The first attack was to start at α = 75 degrees. This resulted in a solution for all three sets. This, as well as starting the other sets at 60 degrees, does not satisfy the needs of a manipulator to obtain values of α = 45 degrees. Further investigation into the error in cable length has shown that, for α = 45 degrees, L_1/R = 10.0 and Y = 0.0, Eq. 56 is satisfied.

For $\alpha = 45$ degrees, the error varies from 0.000 in. for $L_1/R = 10.0$ to 0.023 in. at $L_1/R = 20.0$ and 0.47 in. at $L_1/R = 30.0$. The degree of error was obtained by a slight revision to the program, which eliminated the trial-and-error portion.

The program for calculating the length of cable is very valuable, since it can be used to determine deviations from the true path which can be tolerated. This tolerance had been assumed to be 0.001 in. for use in computing the solutions to Tables I and XIII. The method used for solving a four-bar linkage involves the use of four or five precision points; the variation between points may cause a change in length which may or may not be acceptable. The program for computing length easily determines this variation. Also, in the case of a single link, this variation can be found when a line parallel to the link bisects the included angle between the lines JK and MN.

The amount of allowable variation is dependent on the length and size of cable: the longer the cable, the greater the tolerance, and the larger the cable, the greater the tolerance.

DESIGN OF MANIPULATOR MECHANISM

In addition to the information obtained during this investigation, data with regard to the physical properties of the cables or tapes are also necessary in order to design a working mechanism. At this point, the word tape is introduced in addition to cables. Either may be used in an actual mechanism, and both satisfy the conditions used during the solution of the pulley center path. Cables have the advantage of changing directions in more than one plane, along with a lower bending stress in the cables for a given pulley size. The tapes are restricted to changing direction in a single plane where short arm lengths are involved. The tapes have the advantages of less friction and of less deflection.

The type of cable used for most manipulator designs is of the 7 x 19 construction. There is one exception, since 3 x 7 is used for $\frac{1}{32}$-in. diameter cable. The cables are usually stainless steel. The tapes are of two sizes: $\frac{3}{16}$ in. wide by 0.005 in. thick, or $\frac{1}{4}$ in. wide by 0.010 in. thick. The tapes are made from Elgiloy, the trade name for a cobalt, chromium, nickel and molybdenum alloy manufactured by the Elgin National Watch Company of Elgin, Illinois. An experimental lot of $\frac{3}{64}$-in. diameter cable has also been made using Elgiloy.

There is a desirable relationship between cable or tape size and pulley size, since the bending stress in the cable is directly proportional to the pulley diameter. If the diameter of the pulley is 32 times the cable diameter, the bending stress is approximately 10 to 15 per cent of the ultimate stress for 7 x 19 cables. For 3 x 7 cables, the bending stress becomes 20 per cent. With tapes, the pulley diameter is 400 times the tape thickness;

this gives a bending stress of approximately 25 to 30 per cent of the ultimate stress. The relationship for tapes was based on the equal pulley size for equal deflection when compared to a $\frac{1}{16}$-in. diameter cable. The relations for pulley size are minimum recommendations. A large pulley is preferable since in addition to a lower stress, a large pulley reduces friction. Friction becomes additive for each pulley, while the maximum bending stress is obtained only at the smallest pulley.

One approximate load on the cables due to loading the manipulator is four times the specified load. For example, a 50-pound manipulator will usually have cables loaded to 200 pounds. A comparison of the load capacity of the various cables and tapes used in manipulator design, and their relative deflections, are given in Table XXVII. The recommended pulley size and load due to bending are included.

TABLE XXVII

Load Capacities and Relative Deflections of Cables and Tapes

Cable Diameter, in.	Ultimate Strength, lb	Deflection in. (100 lb, 6 ft lg)	Pulley Diameter, in.	Bending Load, lb
0.032 ($\frac{1}{32}$)	120	1	1.000	24
0.045($\frac{3}{64}$)	215	$\frac{3}{8}$	1.500	22
0.070 ($\frac{1}{16}$)	480	$\frac{1}{4}$	2.000	66
0.100 ($\frac{3}{32}$)	1000	$\frac{1}{8}$	3.000	120
0.135 ($\frac{1}{8}$)	1900	$\frac{1}{16}$	4.000	220
Tape Size				
0.188 x 0.005	250	$\frac{1}{4}$	2.000	69
0.250 x 0.010	625	$\frac{1}{8}$	4.000	188

The ultimate strength and deflections were found from tests. These check with calculated and published data within test accuracy. The bending load was calculated from the following formulas:

$$S_b = E_c d_w / D_p \qquad (57)$$

where

S_b = bending stress in outer wire, psi

E_c = modulus of elasticity of cable or tape, psi

d_w = diameter of wire, in.

D_p = pulley diameter, in.

The equivalent bending load in pounds is

$$F_b = A_c S_b = \frac{A_c E_c d_w}{D_p} \quad , \tag{58}$$

where

A_c = Area of cable, in^2.

The diameter of wire can be found in two ways. One is to simply measure a sample; the other is based on geometrical construction which gives the following relationships between the diameter of the cable, d_c, and the diameter of the wire. The area of a cable can also be obtained geometrically.

$$7 \times 19 \text{ cable - } d_w = 0.066\, d_c \tag{59}$$

$$7 \times 19 \text{ cable - } A_c = 0.45\, d_c^2 \tag{60}$$

$$3 \times 7 \text{ cable - } d_w = 0.156\, d_c \tag{61}$$

$$3 \times 7 \text{ cable - } A_c = 0.40\, d_c^2 \quad . \tag{62}$$

Based on Table XXVII and the relationship between manipulator capacity and cable load, the computation of permissible load due to items other than load and bending stress, such as the forces due to stopping or starting, the force equivalent for wear, pre-tension in the cable, and most particularly the amount permissible for change in length due to possible variation from the prescribed paths, is possible. In addition to the foregoing, there should be a safety allowance to account for possible variations in physical properties of the cables or tapes. In manipulator design, there are also changes of length due to temperature or rotation out of a flat plane. The temperature change may be neglected, since usually the temperature difference is limited to 40 degrees. The change in length due to rotation out of a flat plane must be subtracted from any permissible change in length for a prescribed pulley center path. This change in length could range from 0.049 in. to 0.198 in. for L_2 = 10.0 in., depending on pulley size. If L_2 increases to 40.0 in., the change in length is from 0.012 in. to 0.050 in. for the same variation in pulley size.

The per cent of ultimate load for each of the above items approaches constant values for cables or tapes. The size of the cable or tape has little effect on the magnitude of this per cent of ultimate for each item. Table XXVIII shows these various percentages.

The loads due to the manipulator are based on present manipulator designs. The loads due to bending are based on the pulley diameters of Table XXVII. Load due to acceleration can only be estimated, but it must be accounted for, since all power to the cables and tapes of electronically

TABLE XXVIII

Percentages of Ultimate Load for Cables and Tapes

Type of Load	% of Ultimate Load for Cables	% of Ultimate Load for Tapes
Manipulator	37	28
Bending	13	27
Acceleration	5	5
Wear	10	5
Change in Length	25	25
Safety Allowance	10	10
TOTAL	100	100

controlled manipulators is provided by high-accelerating, quick-reversing servo-motors. The difference between cable and tape wear is because the internal and pulley friction are much greater for cables, thus resulting in more wear. Overlooking the change in length, the allowance is needed for variations in physical properties of materials. Although the change in length has the same percentage for cables and tapes, the amount of deflection is more for cables. A deflection of 0.0050 in. per inch of length can be tolerated for cables; a deflection of 0.0021 in. per inch of length can be tolerated for tapes. The difference is due to the apparent modulus of elasticity of the cable.

The change in length consists of initial tension, rotation out of plane and deviation from a prescribed path. This change in length ranges between 0.100 in. and 0.400 in. for cables, and between 0.042 in. and 0.168 in. for tapes. The amount of initial tension is dependent on physical take-up and anticipated temperature variations. For the purpose of this investigation, temperature effects were neglected, since the temperature is approximately constant during use. An initial tension of approximately 3 per cent of ultimate has proved satisfactory to date. This reduced the allowable deflection by 0.0006 in. per inch for cables, and by 0.00025 in. per inch for tapes.

In order to get some feel for the problems involved in design of the actual mechanism, assume that the change in length due to rotation out of plane is an average value of 0.155 in. With this value, only very long tapes could be used, or the amount of rotation out of plane would have to be restricted. This value would also eliminate most of the very short-arm manipulators. With tapes, rotation out of plane is not as common as with cables; hence, this problem is not as serious as it looks. With cables, however, it means that for very short-arm manipulators, small size pulleys or restriction of rotation must be incorporated. This portion of the design is straightforward, and the best case for each manipulator can be obtained easily. Neglecting this part of the design, but making allowance for the magnitude of its change, a figure of ten per cent of ultimate for deviation from a prescribed path becomes feasible.

Ten per cent means a tolerable limit of 0.002 in. per inch of cable length or 0.0008 in. per inch of tape length. This means a manipulator having two 10-inch arms may have a deviation of 0.040 in., while one with two 40-inch arms may have a deviation of 0.160 in. For tapes, these values would be 0.016 in. and 0.064 in. respectively. This means that, for shorter arms, the use of a single link solution becomes impossible, since the deviation needed approaches 0.055 in. This also eliminates the use of a single link for a solution with tapes. Figure 16 shows a possible design of a single link system.

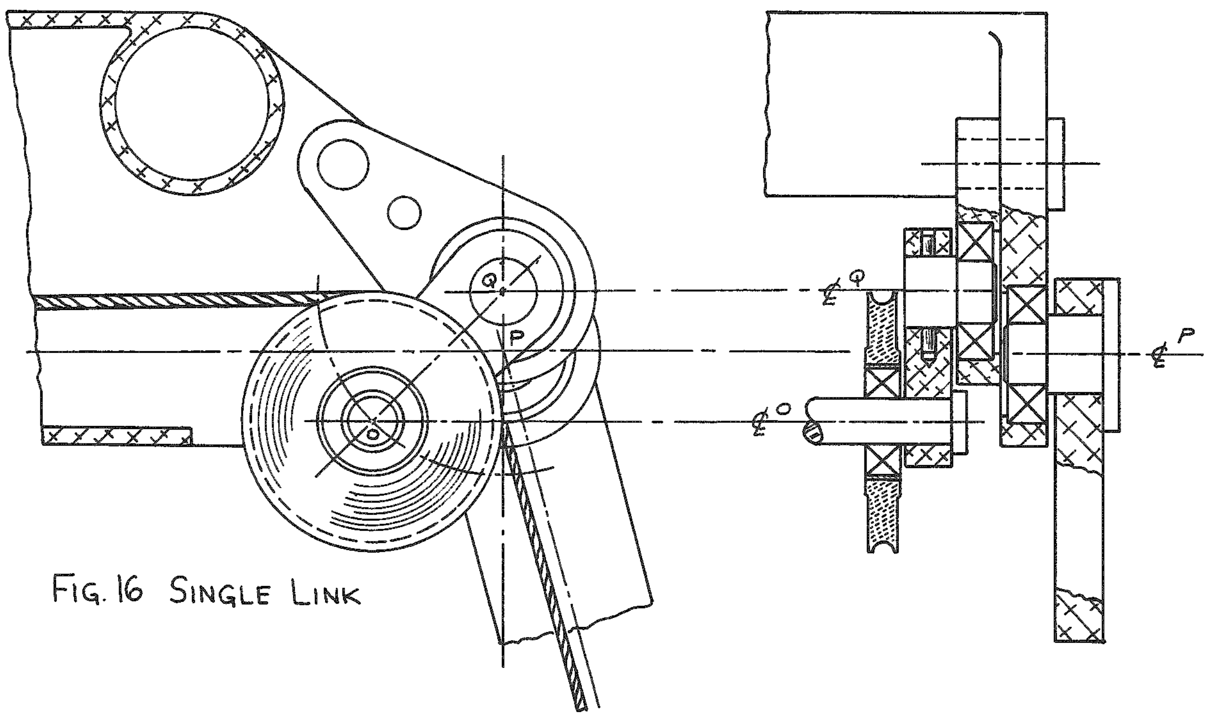


FIG. 16 SINGLE LINK

The single link as shown in Fig. 16 would work for a pair of 20-inch or longer arms. This solution would, however, be limited to 45 to 180 degrees of motion, since, when using a single link, its length would have to be equal to 2R in order to have one pulley clear the other. This is needed for symmetrical operation to 315 degrees. The additional link to which the point Q is mounted is a feature needed for remote replacement of broken cables. During operation of the manipulator, this link acts as part of the upper arm, L_1. During cable replacement, this link pivots, giving slack to the cable.

Figure 17 shows a possible "straight-cam" solution. This solution was suggested by Professor Denavit from preliminary data obtained. This could be a very good solution for the short-arm or tape manipulators which need exact solutions. A link to slacken the cable is also present. This system would need additional design consideration to the problems of remote repair. Remote repair is an important item for all electronically controlled manipulators since, once in a shielded facility, they are never removed. All servicing must be by other manipulators or through neoprene gloves.

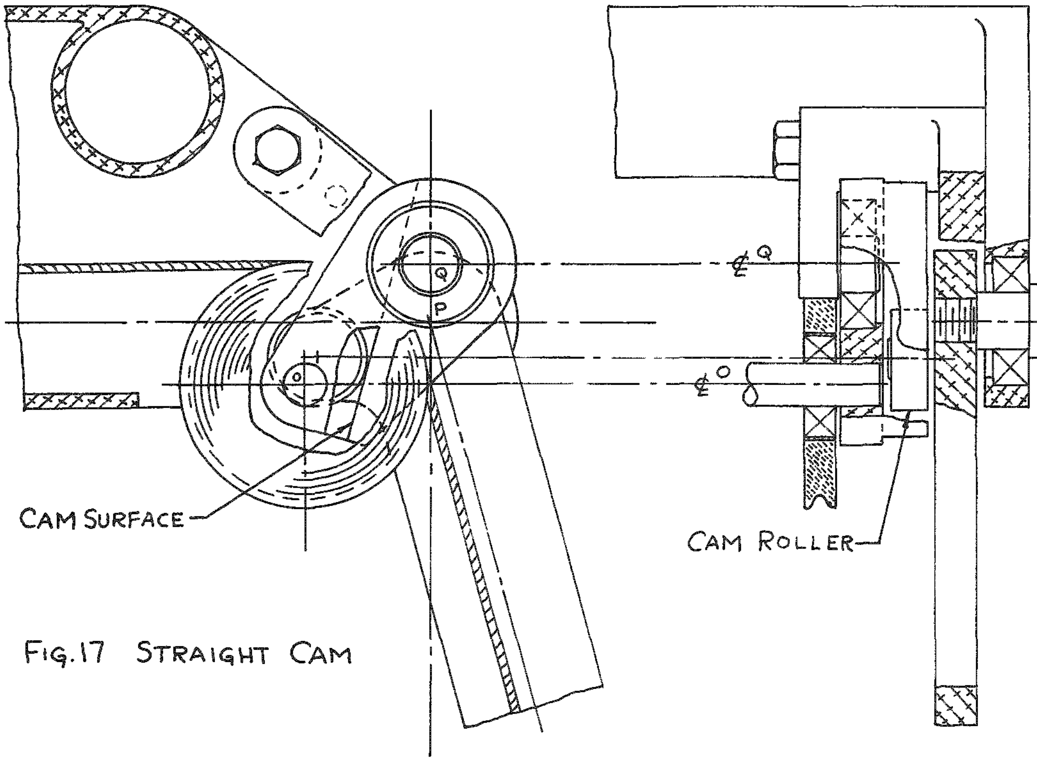


FIG. 17 STRAIGHT CAM

Figure 18 shows a schematic of four-bar linkage which gives a very close approximation to the exact solution. This scheme and that involving the straight cam are limited to a maximum of 180 degrees, like the single link. Figure 18(a) shows an intermediate position, whereas Fig. 18(b) shows the 180-degree position.

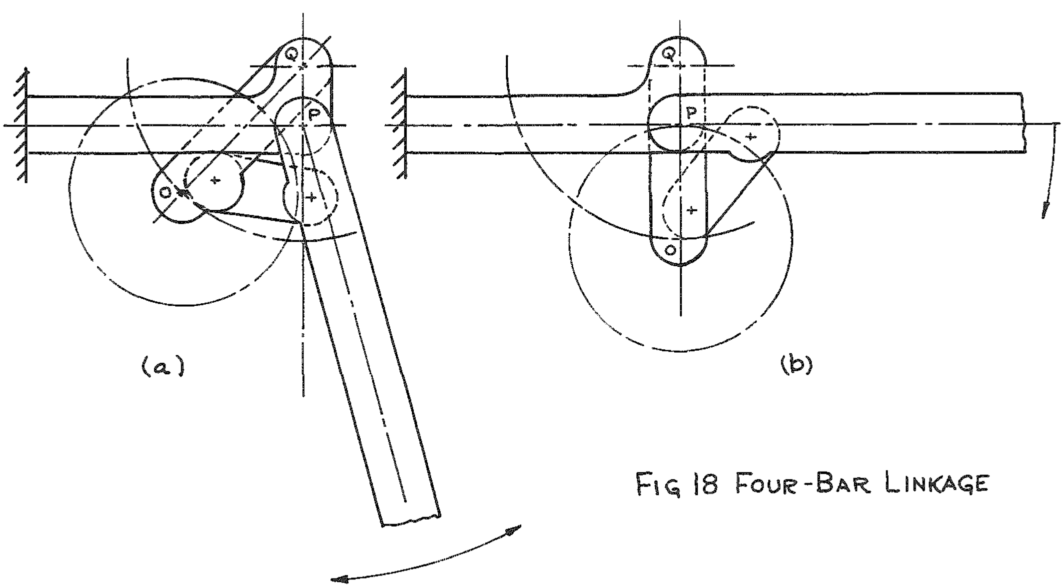


FIG 18 FOUR-BAR LINKAGE

In addition to the solutions presented, there are a multitude of cam (exact) solutions to either of the two particular solutions obtained here. Heavy and expensive to build, they will not be discussed.

CONCLUSIONS

The purpose of this investigation was to find a general solution for the path of a pulley center giving a constant cable length while the cable changes direction over the pulley. The involved relations were solved by an IBM 650 digital computer. A solution for the path for any manipulator, given the lengths of the arms and pulley size, can be obtained in forty minutes from the programs developed. The solutions allow a variation of α between 45 and 315 degrees; practical design limitations hold α between 45 and 180 degrees. Further investigation to different relationships between X and Y may result in a design which would allow the complete travel.

Based on weight and cost, the single link is the best solution. Limitation to manipulators having long arms and cables eliminates the single link as a general solution. The relationship between X and Y could be changed by increasing the length of the single link and moving its pivot center from the point directly above the arm. The amount of movement depends on the length of the single link, which in turn is dependent on the angle μ approaching ρ. The movement of the single-link pivot would be away from the arm pivot in the quadrant for which X and Y are negative. A longer link would allow greater angles of α. As μ approaches ρ, manipulators having shorter arms or those using tapes may use the single link, thus approaching a general solution.

The use of a digital computer for manipulator kinematics has shown the way to other problems encountered. For example, a linkage system which could be used to balance a manipulator for greater angles of α may become practical. To date, the only approach to such a system has been an electronic one, and this is expensive.

APPENDIX A

Bell L_1 Computer System

Use of the IBM 650 digital computer necessitates the need for a programming system since machine language, although faster, is more difficult to program for engineering problems. The system commonly known as the Bell L_1 system was used.

The Bell L_1 system is a floating point system; this indicates the decimal point of a number by a characteristic number similar to the characteristic number used for logarithms. The two-digit characteristic follows the eight-digits of number. Since the base characteristic is 50, this allows a range of 0 to 99. A characteristic of 50 indicates a number A_1 for which the following is true: $1.0 \leq A_1 < 10$.

If the characteristic is 51, then $10 \leq A_1 < 100$; if the characteristic is 49, then $0.1 \leq A_1 < 1.0$, etc.

In addition to numbers as such, the system's instructions are signed ten-digit numbers. The instructions have the following form:

±	O_1	A or O_2	B	C

O_1 is a one-digit operation code and B and C are three-digit addresses. A or O_2 is also interpreted as a three-digit address if $O_1 \neq 0$. If $O_1 = 0$, then O_2 is used as an operational code of three digits. A summary of these operational codes may be found in Table XXIX.

Using this system, the part of the program which covers the equation is straightforward. The remaining part of the program requires the setting up of repeating loop systems; they, upon reading in a value for Y, will decide if the value for Y satisfies the equation. If it does, the answers are punched out; if not, a new value for Y is selected and tried. This process is repeated for each value of Y until it satisfies the equations. In addition to the trial-and-error loop for Y, loops were needed to add increments to obtain 15-degree increments automatically. With the solution of more than one set of parameters, a loop to call for a new problem read-in was also incorporated.

The programs and loop-flow diagrams for each of the two particular solutions follow. In addition to the program shown, the program for Particular Solution B was used by changing one card and eliminating the trial-and-error loop in order to calculate the length of cable for a given value of Y. This allowed a comparison with the solutions of constant length to determine amounts of variation in length which may be tolerated. This is very important during the design of the mechanism, since it allows certain deviations which might permit the use of simpler mechanisms.

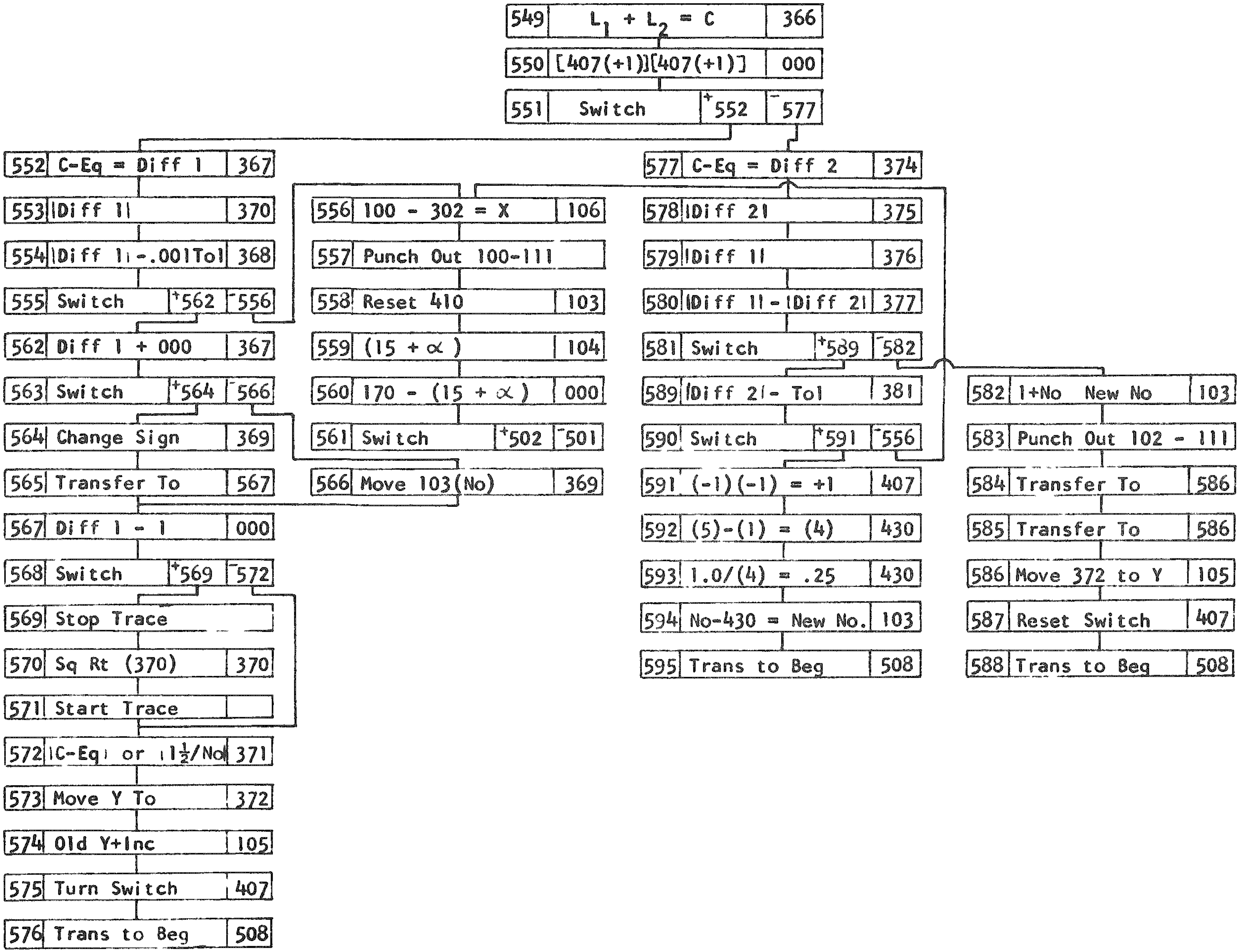
TABLE XXIX

Summary of Operation Codes

O_1 Operations		O_2 Operations			
Num.	Alpha	Num.	Alpha	Num.	Alpha
0	GO to O_2	000	UNC STOP	350	ABS
1	ADD	200	COND STOP	351	EXP 10
2	SUB	201	TR SGN	352	LOG 10
3	MPY	202	TR EXP	353	SIN Degrees
4	DIV	203	TR	354	COS Degrees
5	NGMPY	204	TR SUBR	355	ART Degrees
6	TR A	205	TR OUT	400	READ
7	TR B	300	SQRT	401	CONS
8	TR C	301	EXP E	410	PCH
9	MOVE	302	LOG E	450	START TR
		303	SIN Radians	451	STOP TR
		304	COS Radians	452	ST TR ERAS
		305	ART Radians	454	NOOP

APPENDIX B
Program For Particular Solution A

500	0400400411	0400100111	0451000000	0354104060	0353104061	0450000000
506	4060061200	4401061201	5105200300	5102201301	1000100302	1000300302
512	3302302303	3105105305	3102102306	1303305307	2307306308	0451000000
518	0300308107	0450000000	0454000000	0451000000	0200307311	0450000000
524	2102105312	4000311313	2000402314	9001313110	9001314111	2403104315
530	4000403315	3000408315	1000313316	3000102108	3301301350	3105201353
536	2101353354	3000000355	5102200356	1000000357	3000354358	2000306359
542	1000355360	1000350362	0451000000	0300000109	0450000000	1107108364
548	1000109365	1100101366	1407407000	0201552577	2366365367	0350000370
554	2000409368	0201562556	2100302106	0410100111	9001410103	1404104104
560	2405104000	0201502501	1367406367	0201564566	5103401369	0203000567
566	9001103369	2370401000	0201569572	0451000000	0300370370	0450000000
572	4370369371	9001105372	1372371105	5401401407	0203000508	2366365374
578	0350374375	0350367376	2376375377	0201589582	1401103103	0410102111
584	0203000586	0203000586	9001372105	3407407407	0203000508	2375409831
590	0201591556	3407407407	2410401430	4401430000	2103000103	0203000508
500	*	*	*	*	*	*
400	2000000050	1000000050	1000000049	1800000052	1500000051	1700000052
406	0000000000	1000000050	3141592650	1000000047	5000000050	0000000000

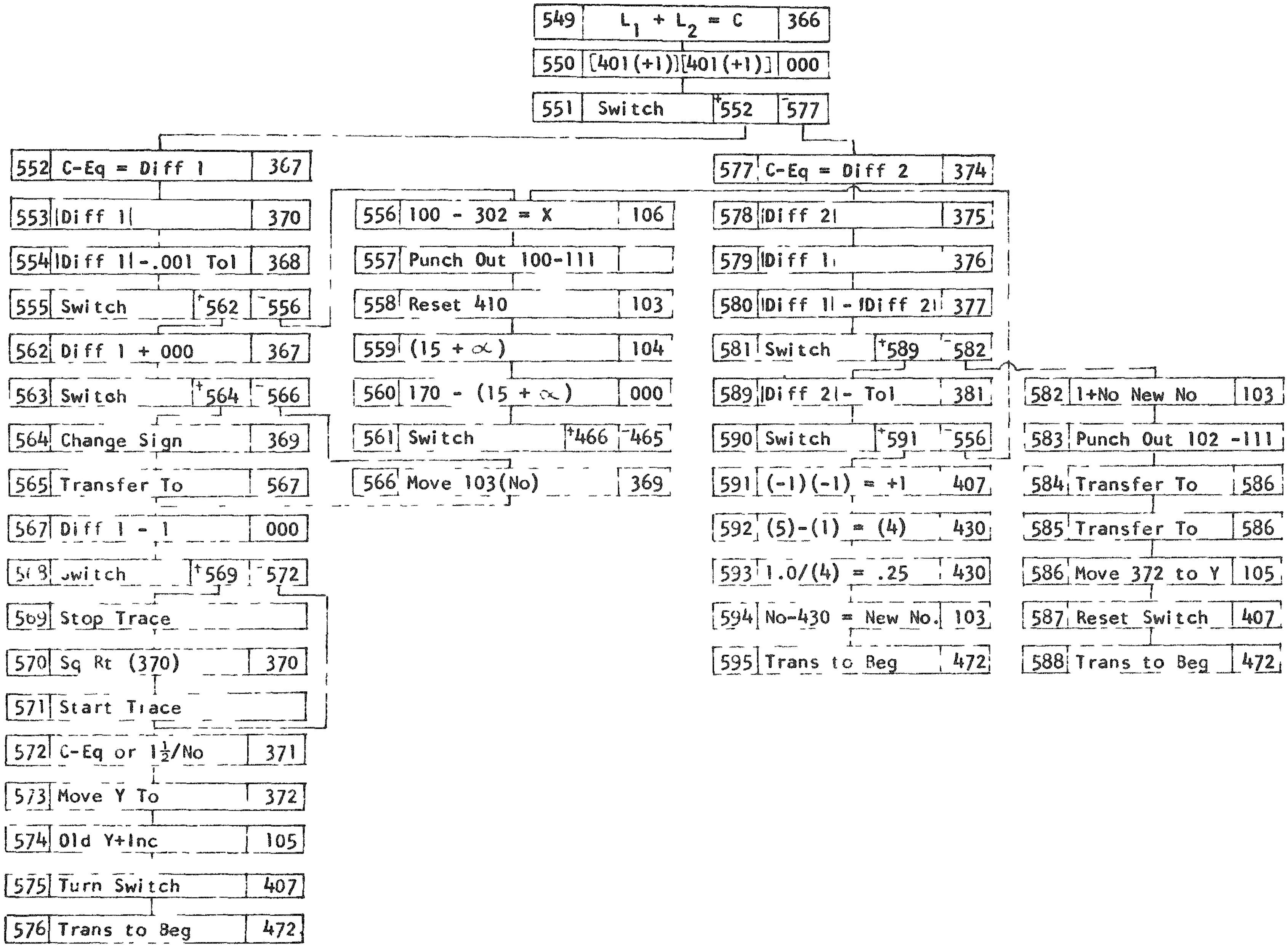


PARTICULAR SOLUTION A
LOOP-FLOW DIAGRAM

APPENDIX C
Program For Particular Solution B

464	0400400411	0400100111	0451000000	0354104200	0353104201	0454000000
470	0450000000	4200201202	3102102300	3000402301	3400102302	0454000000
476	1105302303	3000000304	2301304305	0350305306	0451000000	0454000000
482	0300306307	0450000000	2100307308	3000000309	3105105310	0454000000
488	1310309311	2000300312	0350312313	0451000000	0300313107	0454000000
494	0450000000	3105202314	3000000315	2307314316	3000000317	0454000000
500	1315310318	0451000000	0300318319	0450000000	2101319320	0454000000
506	3320320321	1000317322	3316320323	1200200324	3000323325	0454000000
512	2322325326	2000300327	0350327328	0451000000	0300328109	0454000000
518	0450000000	2403104329	4000403329	3000408329	2102105330	0454000000
524	0451000000	0300311331	0450000000	4330331332	3307201333	0454000000
530	3105200334	0350326335	0451000000	0300335336	0450000000	0454000000
536	2102333337	1000334338	4000336339	1000332340	1000329341	0454000000
542	3341102108	1000107342	1000109365	1100101366	9001332110	9001339111
548	0454000000	0454000000	3401401407	0201552577	2366365367	0350000370
554	2000409368	0201562556	9001307106	0410100111	9001410103	1404104104
560	2405104000	0201466465	1367406367	0201564566	5103401369	0203000567
566	9001103369	2370401000	0201569572	0451000000	0300370370	0450000000
572	4370369371	9001105372	1372371105	5401401407	0203000472	2366365374
578	0350374375	0350367376	2376375377	0201589532	1401103103	0410102111
584	0203000586	0203000536	9001372105	3407407407	0203000472	2375409381
590	0201591556	3407407407	2410401430	4401430000	2103000103	0203000472
464	*	*	*	*	*	*
400	5000000049	1000000050	2250000050	1800000052	1500000051	1700000052
406	0000000000	1000000050	3141592650	1000000047	5000000050	0000000000

PARTICULAR SOLUTION B
LOOP-FLOW DIAGRAM



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