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ANALYSIS OF TAPERED CIRCULAR PLATES

OAK RIDGE NATIONAL LABORATORY

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for the

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ANALYSIS OF TAPERED CIRCULAR PLATES

F. J. Stanek

DATE ISSUED

MAR 18 1959

OAK RIDGE NATIONAL LABORATORY
Oak Ridge, Tennessee
operated by
UNION CARBIDE CORPORATION
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U.S. ATOMIC ENERGY COMMISSION

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PREFACE

The basic theory for the stress analysis of circular plates with variable thickness has been understood by the fundamental analyst for a period of years, but its application had not been feasible prior to the development of the high-speed computing machines. Now, however, with the aid of the high-speed machines it is possible to present the theory in a convenient and usable form for use by the design engineer. Hence it is the purpose of this report to provide tables of numerical values of all necessary functions and instructions on their use for the stress analysis of circular plates which have a linear thickness-radius relationship and which are loaded axisymmetrically. Several illustrative examples and the derivations of the plate formulas are also given. Parts 1 and 2 are sufficient for a complete analysis of a specific plate.

The analysis presented is based on the classical small-deflection theory. Emphasis is placed primarily on plates whose thickness at the inner radius is *greater* than the thickness at the outer radius. However, a limited number of plates for which the opposite is true may also be analyzed by the procedure presented. The limitation on the dimensions of the latter type of plate is explained.

The author wishes to acknowledge the assistance of the personnel of the Numerical Analysis Department (headed by J. E. Rowe) at the Oak Ridge Gaseous Diffusion Plant for evaluating and tabulating of the numerical values included in part II. The supervision of the programing and computing was done by V. E. Anderson and G. B. Knight.

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Part I

PLATE FORMULAS AND THEIR APPLICATION

INTRODUCTION

The dimensionless coordinate x is defined by the coordinate transformation

$$x = 1 - \frac{r}{c} ,$$

where r is the radial coordinate of the plate, and c is the radius at which the thickness of the plate would be zero if the plate were extended (see Fig. 1). The plate formulas are expressed in terms of various functions of x , which are defined in part IV. However, it is not necessary to understand the derivation in order to understand the application.

The x values for plates thicker at the inner radius are in the range $0 < x < 1$ (since $c > r$), while for plates thicker at the outer radius the values are always negative (since $c < r$). Two complete sets of tables of numerical values of all functions are provided in part II. Set A contains the values of the functions for the range of x values $0 < x < 1$, while set B contains the values for the range $-1 < x < 0$. Hence almost any plate thicker at the inner radius loaded with any combination of the axisymmetrical loads shown in Fig. 1 may be analyzed by the procedure presented. Plates thicker at the outer radius may be analyzed only if their x values fall in the range $-1 < x < 0$, or more directly, only if

$$\frac{b}{2} < c < a ,$$

where a and b are the inner and outer radii of the plate, respectively. Poisson's ratio for the material of the plate should be approximately $\frac{1}{3}$, and the maximum thickness must be small relative to the radius. The lateral displacement w must be small relative to the other dimensions.

The stresses and displacements due to the bending of the plate (the bending analysis) are independent of the stresses and displacements due to the membrane forces acting upon the plate (the membrane analysis). Hence each analysis may be treated independently of the other, and the total results are then obtained by merely adding the two. Each analysis requires, initially, the evaluation of constants of integration which will satisfy the boundary conditions of the specific plate under consideration. Once the constants of integration are evaluated, the desired results are obtained by substitution in the appropriate formula.

SYMBOL AND SIGN CONVENTION FOR CIRCULAR PLATES*

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Fig. 1

r = radial coordinate of the plate (in.)

a = radius of central hole in plate (in.)

b = outer radius of plate (in.)

b_a and b_b = thickness of plate at radius a and at radius b , respectively (in.)

$c = \frac{bb_a - ab_b}{b_a - b_b}$, the radius at which thickness of plate would be zero if plate were extended (in.)

*The designated units are the most common set used in the USA; however, any consistent set may be used, since the numerical values in the tables are dimensionless.

$x = 1 - \frac{r}{c}$, the dimensionless coordinate of the plate

$m = \frac{h_a - h_b}{b - a}$, the slope in the point-slope form of the mathematical equation for the thickness of the plate

$h = m(c - r) \equiv mcx$, the thickness of the plate at radius r (in.)

w = lateral displacement of plate, positive downward (in.)

u = radial displacement of plate, positive outward (in.)

p = uniform pressure on plate, assumed to act normal to the middle surface of the plate, positive downward (psi)

M = bending moment in plate, positive when tensile stress is in top surface (lb-in./in. = lb)

Q = vertical shearing force in plate, positive as shown in Fig. 1 (lb/in.)

N = normal membrane force in plate, tension positive (lb/in.)

E = Young's modulus of elasticity (psi)

$\rho = \frac{c^2 p}{2}$, the load parameter due to the uniform pressure p (lb)

$k = a \left(Q_a - \frac{ap}{c^2} \right) \equiv b \left(Q_b - \frac{bp}{c^2} \right)$, the load parameter due to edge shear forces and the uniform pressure (lb)

C_1, C_2 , and C_3 = constants of integration in the bending analysis (lb)

A_1 and A_2 = constants of integration in the membrane analysis (psi)

Subscripts a and b = inner and outer edges of plate, respectively

Subscripts r and θ = radial and circumferential directions, respectively

Subscripts B and m = bending and membrane stresses, respectively

PLATE FORMULAS

Each of the capital letters M , W , N , and U , with a subscript, appearing on the right side of formulas 1 through 11, denotes a function of x whose numerical values are tabulated in part II.

Plate Constants

Geometrical constants

$$(i) \quad m = \frac{b_a - b_b}{b - a}$$

$$(ii) \quad c = \frac{bb_a - ab_b}{b_a - b_b}$$

Loading parameters

$$(iii) \quad \rho = \frac{c^2 p}{2}$$

$$(iv) \quad k = a \left(Q_a - \frac{a\rho}{c^2} \right) \\ \equiv b \left(Q_b - \frac{b\rho}{c^2} \right)$$

Coordinate Transformation

$$(v) \quad x = 1 - \frac{r}{c}$$

$$(vi) \quad r = c(1 - x)$$

$$(vii) \quad x_a = 1 - \frac{a}{c}$$

$$(viii) \quad x_b = 1 - \frac{b}{c}$$

Bending Analysis

Internal bending moments

$$(1) \quad M_r = x^2 (C_1 M_{r1} + C_2 M_{r2} + k M_{rk} + \rho M_{rp})$$

$$(2) \quad M_\theta = x^2 (C_1 M_{\theta1} + C_2 M_{\theta2} + k M_{\theta k} + \rho M_{\theta\rho})$$

Internal bending stresses (+ for top surface, - for bottom surface)

$$(3) \quad \sigma_{Br} = \frac{\pm 6}{(mc)^2} (C_1 M_{r1} + C_2 M_{r2} + k M_{rk} + \rho M_{r\rho})$$

$$(4) \quad \sigma_{B\theta} = \frac{\pm 6}{(mc)^2} (C_1 M_{\theta 1} + C_2 M_{\theta 2} + k M_{\theta k} + \rho M_{\theta \rho})$$

Lateral displacements

$$(5) \quad \frac{dw}{dr} = \frac{32}{3m(mc)^2 E} (C_1 W'_1 + C_2 W'_2 + k W'_k + \rho W'_\rho)$$

$$(6) \quad w = \frac{32}{3m^3 c E} (C_1 W_1 + C_2 W_2 + k W_k + \rho W_\rho + C_3)$$

Membrane Analysis

Internal membrane forces

$$(7) \quad N_r = mcx (A_1 N_{r1} + A_2 N_{r2})$$

$$(8) \quad N_\theta = mcx (A_1 N_{\theta 1} + A_2 N_{\theta 2})$$

Internal membrane stresses

$$(9) \quad \sigma_{mr} = A_1 N_{r1} + A_2 N_{r2}$$

$$(10) \quad \sigma_{m\theta} = A_1 N_{\theta 1} + A_2 N_{\theta 2}$$

Radial displacement

$$(11) \quad u = \frac{c}{E} (A_1 U_1 + A_2 U_2)$$

Total or Principal Stresses (+ for top surface, - for bottom surface)

$$(12) \quad \sigma_r = \sigma_{mr} \pm \sigma_{Br}$$

$$(13) \quad \sigma_\theta = \sigma_{m\theta} \pm \sigma_{B\theta}$$

PROCEDURE

Numerical values for each of the various functions of x appearing in formulas 1 through 11 are given in Tables 1 through 6 of each set. The formulas appear at the top of their corresponding table.

If a plate thicker at the inner radius is analyzed, the values in set A are used. If a plate thicker at the outer radius with $b/2 < c < a$ is analyzed, the values in set B are used. Otherwise, the procedures are the same.

When the geometrical constant c is obtained for the specific plate under consideration by formula ii, x_a and x_b are determined by means of formulas vii and viii, respectively. This will determine the range of x values which will be required from the tables for a complete analysis of the plate. The exact values of x_a and x_b may not be listed in the tables, in which case the nearest values are selected. This has the physical effect of changing the radii a and b of the plate slightly, provided the constant c is held constant at its initial calculated value. The adjusted values of the radii a and b should be used in the definition of the loading parameter k (formula iv). Also, if the geometrical constant m is held constant at its initial calculated value (formula i), the thicknesses b_a and b_b will change slightly. It is not necessary to determine their adjusted values in order to complete the analysis of the plate, since this is automatically compensated for through the constants m and c and the selected values for x_a and x_b . However, it is well to determine their value upon completing the analysis so that the appropriate physical interpretation may be given to the results.

Either of the definitions for the loading constant k (formula iv) may be used, depending upon which is the more convenient for the plate under consideration. It should be noted that of the three external loads, Q_a , Q_b , and p (hence ρ), only two may be arbitrarily designated. The third must maintain equilibrium; that is, the summation of forces parallel to the axis of the plate must be zero.

The constants of integration are determined by solving a set of simultaneous linear equations. Each equation of this set is a mathematical statement of a "boundary condition" or "edge condition" which must be specified for the plate under consideration. These boundary conditions are usually obtained by placing restrictions upon certain formulas at either or both edges of the plate (at x_a and/or x_b). Or more directly, in the bending analysis, the boundary conditions necessary to determine the constants of integration C_1 and C_2 are usually obtained by placing

restrictions (only two are required) upon the radial bending moment M_r (formula 1) and/or the slope dw/dr (formula 5) at either or both edges of the plate. The boundary condition for determining the constant of integration C_3 is usually obtained by specifying the displacement w (formula 6) at either edge of the plate. The two necessary boundary conditions for determining the constants of integration A_1 and A_2 are usually obtained by specifying N_r (formula 7) and/or u (formula 11) at either or both edges of the plate.

Once the constants of integration are determined, the desired results are obtained by substitution in the appropriate formula.

The theory fails at $x = 0$ and at $x = \pm 1$. The case of $x = 0$ has no practical design application because this requires that b_a or $b_b = 0$. The case of $x = -1$ is outside the inequality range stipulated previously. The case of $x = +1$ is a plate thicker at the inner radius without a central hole and may be treated by taking the slope $dw/dr = 0$ at $x_a = 0.9875$. This has the physical effect of having a rigid plug in the small central hole. The case of a plate thicker at the inner radius containing a circular plate of uniform thickness b_a is treated in part III and is referred to as "the partially tapered circular plate problem."

Some of the above points and others are illustrated in several examples in part III.

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Part II

TABLES OF NUMERICAL VALUES

INTRODUCTION

Numerical values for all functions of x are tabulated in this section. Set B contains the values for negative x and immediately follows set A. The numerical increment of x in each set is 0.0125. The value is the six-digit number (preceded by the minus sign, if negative) multiplied by ten raised to the power (sign and digit) which follows the six-digit number. For example, the values of M_{r1} and M_{r2} at $x = 0.25$ in Table 1A are

$$M_{r1} = -1.23456 -2 = -1.23456 \times 10^{-2} = -0.0123456 ,$$

$$M_{r2} = 3.79259 1 = 3.79259 \times 10 = 37.9259 .$$

Table 1A

Values of the Functions Appearing in the Formulas for M_p and σ_{Bp}

$$M_p = x^2 (C_1 M_{p1} + C_2 M_{p2} + k M_{pk} + p M_{pd})$$

$$\sigma_{Bp} = \frac{\pm 6}{(mc)^2} (C_1 M_{p1} + C_2 M_{p2} + k M_{pk} + p M_{pd})$$

$\bar{x}$	M_{p1}	M_{p2}	M_{pk}	M_{pd}
0.0125	-1.78027	1.29077	6.37303	6.37403
0.0250	-7.30501	3.25405	1.58618	1.58719
0.0375	-1.68660	1.45845	7.01713	7.02733
0.0500	-3.07784	8.27337	3.92814	3.93841
0.0625	-4.93824	5.34000	2.50140	2.51173
0.0750	-7.30454	3.73997	1.72800	1.73840
0.0875	-1.02165	2.77124	1.26264	1.27310
0.1000	-1.37173	2.13992	9.61222	9.61751
0.1125	-1.78536	1.70532	7.54987	7.65580
0.1250	-2.26757	1.39319	6.07768	6.18426
0.1375	-2.82387	1.16131	4.99058	5.09780
0.1500	-3.46020	9.84236	4.16537	4.27323
0.1625	-4.18306	8.45863	3.52440	3.63290
0.1750	-4.99948	7.35618	3.01675	3.12589
0.1875	-5.91715	6.46311	2.60797	2.71774
0.2000	-6.94444	5.72916	2.27400	2.38441
0.2125	-8.09048	5.11832	1.99769	2.10874
0.2250	-9.36524	4.60425	1.76653	1.87820
0.2375	-1.07796	4.16730	1.57120	1.68351
0.2500	-1.23456	3.79259	1.40469	1.51763
0.2625	-1.40764	3.46865	1.26160	1.37516
0.2750	-1.59862	3.18655	1.13774	1.25192
0.2875	-1.80910	2.93923	1.02982	1.14460
0.3000	-2.04081	2.72108	9.35209	1.05060
0.3125	-2.29568	2.52755	8.51805	9.67800
0.3250	-2.57582	2.35495	7.77900	8.94489
0.3375	-2.88358	2.20025	7.12101	8.29274
0.3500	-3.22156	2.06094	6.53256	7.71004
0.3625	-3.59263	1.93492	6.00413	7.18723
0.3750	-4.00000	1.82044	5.52774	6.71632
0.3875	-4.44722	1.71601	5.09668	6.29058
0.4000	-4.93827	1.62037	4.70529	5.90433
0.4125	-5.47759	1.53242	4.34875	5.55273
0.4250	-6.07015	1.45123	4.02295	5.23161
0.4375	-6.72153	1.37599	3.72435	4.93742
0.4500	-7.43801	1.30598	3.44989	4.66706
0.4625	-8.22666	1.24058	3.19694	4.41783
0.4750	-9.09549	1.17922	2.96319	4.18740
0.4875	-1.00535	1.12140	2.74664	3.97367
0.5000	-1.11111	1.06666	2.54551	3.77483

Table 1A (continued)

x	M_{11}	M_{12}	M_{13}	M_{14}
0.5125	-1.22799	-1.01460	2.35827	0
0.5250	-1.35734	9.64818	2.18353	0
0.5375	-1.50069	9.16953	2.02008	0
0.5500	-1.65980	8.70659	1.86685	0
0.5625	-1.83673	8.25598	1.72285	0
0.5750	-2.03383	7.81435	1.58724	0
0.5875	-2.25385	7.37833	1.45923	0
0.6000	-2.50000	6.94444	1.33811	0
0.6125	-2.77604	6.50902	1.22325	0
0.6250	-3.08642	6.06814	1.11407	0
0.6375	-3.43638	5.61752	1.01003	0
0.6500	-3.83219	5.15235	9.10665	1
0.6625	-4.28135	4.66721	8.15501	1
0.6750	-4.79289	4.15581	7.24126	1
0.6875	-5.37777	3.61079	6.36148	1
0.7000	-6.04938	3.02343	5.51194	1
0.7125	-6.82419	2.38316	4.68891	1
0.7250	-7.72268	1.67713	3.88861	1
0.7375	-8.77047	8.89397	3.11006	1
0.7500	-1.00000	2.25000	2.34721	1
0.7625	-1.14527	-1.01641	1.59773	1
0.7750	-1.31824	-2.19250	8.58265	2
0.7875	-1.52595	-3.57092	1.25278	2
0.8000	-1.77777	-5.20833	-6.04973	2
0.8125	-2.08641	-7.18123	-1.33656	1
0.8250	-2.46938	-9.59502	-2.07404	1
0.8375	-2.95134	-1.25979	-2.82259	1
0.8500	-3.56790	-1.64039	-3.58833	1
0.8625	-4.37190	-2.13303	-4.37861	1
0.8750	-5.44444	-2.78639	-5.20258	1
0.8875	-6.91495	-3.67815	-6.07199	1
0.9000	-9.00000	-4.93827	-7.00246	1
0.9125	-1.20839	-6.79735	-8.001565	1
0.9250	-1.69012	-9.69618	-9.14305	1
0.9375	-2.50000	-1.45635	-1.04333	0
0.9500	-4.01111	-2.636380	-1.19683	0
0.9625	-7.31975	-4.34974	-1.39013	0
0.9750	-1.69000	-1.60086	-1.65793	0
0.9875	-6.93444	-4.15659	-2.11161	0
0.5125	-1.22799	1.01460	2.35827	0
0.5250	-1.35734	9.64818	2.18353	0
0.5375	-1.50069	9.16953	2.02008	0
0.5500	-1.65980	8.70659	1.86685	0
0.5625	-1.83673	8.25598	1.72285	0
0.5750	-2.03383	7.81435	1.58724	0
0.5875	-2.25385	7.37833	1.45923	0
0.6000	-2.50000	6.94444	1.33811	0
0.6125	-2.77604	6.50902	1.22325	0
0.6250	-3.08642	6.06814	1.11407	0
0.6375	-3.43638	5.61752	1.01003	0
0.6500	-3.83219	5.15235	9.10665	1
0.6625	-4.28135	4.66721	8.15501	1
0.6750	-4.79289	4.15581	7.24126	1
0.6875	-5.37777	3.61079	6.36148	1
0.7000	-6.04938	3.02343	5.51194	1
0.7125	-6.82419	2.38316	4.68891	1
0.7250	-7.72268	1.67713	3.88861	1
0.7375	-8.77047	8.89397	3.11006	1
0.7500	-1.00000	2.25000	2.34721	1
0.7625	-1.14527	-1.01641	1.59773	1
0.7750	-1.31824	-2.19250	8.58265	2
0.7875	-1.52595	-3.57092	1.25278	2
0.8000	-1.77777	-5.20833	-6.04973	2
0.8125	-2.08641	-7.18123	-1.33656	1
0.8250	-2.46938	-9.59502	-2.07404	1
0.8375	-2.95134	-1.25979	-2.82259	1
0.8500	-3.56790	-1.64039	-3.58833	1
0.8625	-4.37190	-2.13303	-4.37861	1
0.8750	-5.44444	-2.78639	-5.20258	1
0.8875	-6.91495	-3.67815	-6.07199	1
0.9000	-9.00000	-4.93827	-7.00246	1
0.9125	-1.20839	-6.79735	-8.001565	1
0.9250	-1.69012	-9.69618	-9.14305	1
0.9375	-2.50000	-1.45635	-1.04333	0
0.9500	-4.01111	-2.636380	-1.19683	0
0.9625	-7.31975	-4.34974	-1.39013	0
0.9750	-1.69000	-1.60086	-1.65793	0
0.9875	-6.93444	-4.15659	-2.11161	0

Table 2A

Values of the Functions Appearing in the Formulas for M_θ and $\sigma_{B\theta}$

$$M_\theta = x^2(C_1 M_{\theta 1} + C_2 M_{\theta 2} + k M_{\theta k} + \rho M_{\theta \rho})$$

$$\sigma_{B\theta} = \frac{\pm 6}{(mc)^2} (C_1 M_{\theta 1} + C_2 M_{\theta 2} + k M_{\theta k} + \rho M_{\theta \rho})$$

x	$M_{\theta 1}$	$M_{\theta 2}$	$M_{\theta k}$	$M_{\theta \rho}$
0.0125	5.64359 -3	4.37550 3	2.15987 3	2.16026 3
0.0250	1.14689 -2	1.12208 3	5.46456 2	5.46890 2
0.0375	1.74845 -2	5.11739 2	2.45690 2	2.46162 2
0.0500	2.36994 -2	2.95477 2	1.39744 2	1.40252 2
0.0625	3.01233 -2	1.94181 2	9.03932 1	9.09351 1
0.0750	3.67663 -2	1.38517 2	6.34134 1	6.39875 1
0.0875	4.36395 -2	1.04575 2	4.70405 1	4.76458 1
0.1000	5.07543 -2	8.23048 1	3.63445 1	3.69802 1
0.1125	5.81233 -2	6.68754 1	2.89629 1	2.96284 1
0.1250	6.57596 -2	5.57278 1	2.36476 1	2.43423 1
0.1375	7.36773 -2	4.74007 1	1.96881 1	2.04116 1
0.1500	8.18915 -2	4.10098 1	1.66556 1	1.74076 1
0.1625	9.04185 -2	3.59941 1	1.42788 1	1.50593 1
0.1750	9.92755 -2	3.19834 1	1.23793 1	1.31879 1
0.1875	1.08481 -1	2.87249 1	1.08354 1	1.16722 1
0.2000	1.18055 -1	2.60416 1	9.56227 0	1.04272 1
0.2125	1.28019 -1	2.38061 1	8.49882 0	9.39204 0
0.2250	1.38397 -1	2.19250 1	7.60045 0	8.52202 0
0.2375	1.49213 -1	2.03283 1	6.83384 0	7.78392 0
0.2500	1.60493 -1	1.89629 1	6.17372 0	7.15253 0
0.2625	1.72268 -1	1.77879 1	5.60060 0	6.60841 0
0.2750	1.84568 -1	1.67713 1	5.09931 0	6.13643 0
0.2875	1.97428 -1	1.58877 1	4.65783 0	5.72464 0
0.3000	2.10884 -1	1.51171 1	4.26658 0	5.36349 0
0.3125	2.24977 -1	1.44431 1	3.91782 0	5.04532 0
0.3250	2.39750 -1	1.38527 1	3.60526 0	4.76388 0
0.3375	2.55250 -1	1.33348 1	3.32374 0	4.51406 0
0.3500	2.71531 -1	1.28808 1	3.06896 0	4.29164 0
0.3625	2.88649 -1	1.24833 1	2.83738 0	4.09313 0
0.3750	3.06666 -1	1.21362 1	2.62599 0	3.91560 0
0.3875	3.25651 -1	1.18345 1	2.43227 0	3.75660 0
0.4000	3.45679 -1	1.15740 1	2.25407 0	3.61406 0
0.4125	3.66832 -1	1.13512 1	2.08957 0	3.48624 0
0.4250	3.89203 -1	1.11633 1	1.93717 0	3.37166 0
0.4375	4.12894 -1	1.10079 1	1.79553 0	3.26905 0
0.4500	4.38016 -1	1.08832 1	1.66346 0	3.17736 0
0.4625	4.64695 -1	1.07876 1	1.53994 0	3.09567 0
0.4750	4.93071 -1	1.07202 1	1.42406 0	3.02322 0
0.4875	5.23299 -1	1.06800 1	1.31504 0	2.95937 0
0.5000	5.55555 -1	1.06666 1	1.21218 0	2.90360 0

Table 2A (continued)

x	$M_{\theta 1}$	$M_{\theta 2}$	$M_{\theta k}$	$M_{\theta p}$
0.5125	5.90035	1.06800	1.11485	2.85545
0.5250	6.26962	1.07202	1.02250	2.81460
0.5375	6.66585	1.07876	9.634653	2.78079
0.5500	7.09190	1.08832	8.650844	2.675382
0.5625	7.55102	1.10079	7.670680	2.673361
0.5750	8.04690	1.11633	6.93799	2.672012
0.5875	8.58381	1.13512	6.19868	2.671341
0.6000	9.16666	1.15740	5.48584	2.671362
0.6125	9.80113	1.18345	4.679666	2.672099
0.6250	1.04938	1.21362	4.12853	2.673584
0.6375	1.12524	1.24833	3.47902	2.675863
0.6500	1.20861	1.28808	2.84582	2.678992
0.6625	1.30056	1.33348	2.222676	2.683045
0.6750	1.40236	1.38527	1.61973	2.688114
0.6875	1.51555	1.44431	1.022269	2.694313
0.7000	1.64197	1.51171	4.33641	3.01783
0.7125	1.78386	1.58877	1.49425	3.10699
0.7250	1.94398	1.67713	7.28521	3.21281
0.7375	2.12572	1.77879	1.30571	3.33802
0.7500	2.33333	1.89629	1.88316	3.48607
0.7625	2.57217	2.03283	2.46314	3.66135
0.7750	2.84910	2.19250	3.04812	3.86950
0.7875	3.17301	2.38061	3.64078	4.11786
0.8000	3.55555	2.60416	4.24416	4.41608
0.8125	4.01234	2.87249	4.86168	4.77711
0.8250	4.56462	3.19834	5.49732	5.21860
0.8375	5.24194	3.59941	6.15581	5.76512
0.8500	6.08641	4.10098	6.84285	6.45172
0.8625	7.15977	4.74007	7.56551	7.32981
0.8750	8.55555	5.57278	8.33273	8.47728
0.8875	1.04211	6.68754	9.15616	1.00169
0.9000	1.30000	8.23045	1.00514	1.21515
0.9125	1.67188	1.04574	1.10405	1.52368
0.9250	2.23827	1.38516	1.21553	1.99434
0.9375	3.16666	1.94180	1.34452	2.66674
0.9500	4.85555	2.95475	1.49932	4.17293
0.9625	8.46049	5.11734	1.69551	7.17587
0.9750	1.86333	1.12206	1.696817	1.56521
0.9875	7.28555	4.37536	2.422963	6.08362

Table 3A

Values of the Functions Appearing in the Formula for dw/dr

$$\frac{dw}{dr} = \frac{32}{3m(mc)^2 E} (C_1^2 W_1^2 + C_2^2 W_2^2 + kW_k^2 + pW_p^2)$$

x	W_1^2	W_2^2	W_k^2	W_p^2
0.0125	5.02109	6.48121	3.15765	3.16255
0.0250	5.04273	1.64105	7.77907	7.82126
0.0375	5.06493	7.38824	3.40320	3.44144
0.0500	5.08771	4.21055	1.88237	1.91783
0.0625	5.11111	2.73068	1.18346	1.21677
0.0750	5.13513	1.92193	8.06575	8.13814
0.0875	5.15981	1.43137	5.81026	6.11124
0.1000	5.18518	1.11111	4.35764	4.64595
0.1125	5.21126	8.90279	3.36959	3.64678
0.1250	5.23809	7.31428	2.66862	2.93590
0.1375	5.26570	6.13247	2.15432	2.41268
0.1500	5.29411	5.22875	1.76653	2.01677
0.1625	5.32338	4.52176	1.46742	1.71022
0.1750	5.35353	3.95794	1.23226	1.46819
0.1875	5.38461	3.50085	1.04433	1.27389
0.2000	5.41666	3.12500	8.92021	1.11565
0.2125	5.44973	2.81210	7.67051	9.85126
0.2250	5.48387	2.54878	6.63406	8.76264
0.2375	5.51912	2.32505	5.76624	7.84568
0.2500	5.55555	2.13333	5.03345	7.06645
0.2625	5.59322	1.96779	4.40998	6.39900
0.2750	5.63218	1.82388	3.87591	5.82317
0.2875	5.67251	1.69800	3.41561	5.32315
0.3000	5.71428	1.58730	3.01668	4.88639
0.3125	5.75757	1.48945	2.66922	4.50282
0.3250	5.80246	1.40258	2.36519	4.16429
0.3375	5.84905	1.32515	2.09807	3.86416
0.3500	5.89743	1.25588	1.86248	3.59697
0.3625	5.94771	1.19372	1.65399	3.35819
0.3750	6.00000	1.13777	1.46890	3.14405
0.3875	6.05442	1.08730	1.30412	2.95139
0.4000	6.11111	1.04166	1.15703	2.77754
0.4125	6.17021	1.00033	1.02254	2.62025
0.4250	6.23188	9.62840	9.10743	2.47759
0.4375	6.29629	9.28798	8.01443	2.34791
0.4500	6.36363	8.97867	7.06064	2.22980
0.4625	6.43410	8.69757	6.20111	2.12204
0.4750	6.50793	8.44215	5.42551	2.02357
0.4875	6.58536	8.21025	4.72487	1.93348
0.5000	6.66666	8.00000	4.09137	1.85098

Table 3A (continued)

x	W ¹	W ²	W ³	W ⁴
0.5125	6.75213 -1	7.80975 0	3.51816 -1	1.77538 0
0.5250	6.84210 -1	7.63814 0	2.69926 -1	1.70606 0
0.5375	6.93693 -1	7.48395 0	2.52935 -1	1.64250 0
0.5500	7.03703 -1	7.34618 0	2.10379 -1	1.58425 0
0.5625	7.14285 -1	7.22398 0	1.71845 -1	1.53090 0
0.5750	7.25490 -1	7.11664 0	1.36966 -1	1.48210 0
0.5875	7.37373 -1	7.02360 0	1.05418 -1	1.43756 0
0.6000	7.50000 -1	6.94444 0	7.69104 -2	1.39700 0
0.6125	7.63440 -1	6.87885 0	5.11846 -2	1.36022 0
0.6250	7.77777 -1	6.82666 0	2.80098 -2	1.32703 0
0.6375	7.93103 -1	6.78784 0	7.17902 -3	1.29729 0
0.6500	8.09523 -1	6.76246 0	-1.14930 -2	1.27086 0
0.6625	8.27160 -1	6.75078 0	-2.81726 -2	1.24768 0
0.6750	8.46153 -1	6.75319 0	-4.30094 -2	1.22768 0
0.6875	8.66666 -1	6.77024 0	-5.61374 -2	1.21086 0
0.7000	8.88888 -1	6.80272 0	-6.76771 -2	1.19724 0
0.7125	9.13043 -1	6.85160 0	-7.77370 -2	1.18686 0
0.7250	9.39393 -1	6.91817 0	-8.64141 -2	1.17984 0
0.7375	9.68253 -1	7.00400 0	-9.37954 -2	1.17634 0
0.7500	1.00000 0	7.11111 0	-9.99587 -2	1.17657 0
0.7625	1.03508 0	7.24196 0	-1.04973 -1	1.18081 0
0.7750	1.07407 0	7.39969 0	-1.08899 -1	1.18947 0
0.7875	1.11764 0	7.58822 0	-1.11791 -1	1.20303 0
0.8000	1.16666 0	7.81250 0	-1.13695 -1	1.22213 0
0.8125	1.22222 0	8.07889 0	-1.14650 -1	1.24764 0
0.8250	1.28571 0	8.39564 0	-1.14688 -1	1.28065 0
0.8375	1.35897 0	8.77358 0	-1.13833 -1	1.32264 0
0.8500	1.44444 0	9.22722 0	-1.12104 -1	1.37559 0
0.8625	1.54545 0	9.77640 0	-1.09509 -1	1.44223 0
0.8750	1.66666 0	1.04489 1	-1.06047 -1	1.52640 0
0.8875	1.81481 0	1.12852 1	-1.01708 -1	1.63368 0
0.9000	2.00000 0	1.23456 1	-9.64666 -2	1.77244 0
0.9125	2.23809 0	1.37254 1	-9.02784 -2	1.95589 0
0.9250	2.55555 0	1.55831 1	-8.30762 -2	2.20602 0
0.9375	3.00000 0	1.82044 1	-7.47556 -2	2.56250 0
0.9500	3.66666 0	2.21606 1	-6.51541 -2	3.10466 0
0.9625	4.77777 0	2.87850 1	-5.40060 -2	4.01766 0
0.9750	7.00000 0	4.20775 1	-4.08327 -2	5.85704 0
0.9875	1.36666 1	8.20381 1	-2.45757 -2	1.14005 1

Table 4A

Values of the Functions Appearing in the Formula for w

$$w = \frac{32}{3m^3 c E} (C_1^1 w_1 + C_2^2 w_2 + k w_k + p w_p + C_3^3)$$

x	w_1	w_2	w_k	w_p
0.0125	6.26302 -3	-8.43707 1	3.76798 1	-3.77728 1
0.0250	1.25528 -2	-4.36638 1	1.80541 1	-1.80906 1
0.0375	1.88701 -2	-2.99120 1	-1.16150 1	-1.16015 1
0.0500	2.52154 -2	-2.29445 1	-8.44840 0	-8.38885 0
0.0625	3.15896 -2	-1.87081 1	6.58130 0	-6.47882 0
0.0750	3.79934 -2	-1.58456 1	-5.35923 0	-5.21623 0
0.0875	4.44277 -2	-1.37731 1	-4.50300 0	-4.32149 0
0.1000	5.08933 -2	-1.21972 1	-3.87368 0	-3.65535 0
0.1125	5.73911 -2	-1.09543 1	-3.39444 0	-3.14078 0
0.1250	6.39218 -2	-9.94591 0	-3.01941 0	-2.73174 0
0.1375	7.04866 -2	-9.10893 0	-2.71955 0	-2.39904 0
0.1500	7.70864 -2	-8.40126 0	-2.47559 0	-2.12329 0
0.1625	8.37223 -2	-7.79358 0	-2.27424 0	-1.89113 0
0.1750	9.03953 -2	-7.26488 0	-2.010608 0	-1.69306 0
0.1875	9.71065 -2	-6.79967 0	-1.96421 0	-1.52210 0
0.2000	1.03857 -1	-6.38629 0	-1.84351 0	-1.37308 0
0.2125	1.10648 -1	-6.01580 0	-1.74007 0	-1.24204 0
0.2250	1.17482 -1	-5.68120 0	-1.65086 0	-1.12590 0
0.2375	1.24358 -1	-5.37696 0	-1.57352 0	-1.02226 0
0.2500	1.31280 -1	-5.09861 0	-1.50614 0	-9.29185 -1
0.2625	1.38248 -1	-4.84253 0	-1.44722 0	-8.45131 -1
0.2750	1.45263 -1	-4.60576 0	-1.39552 0	-7.68829 -1
0.2875	1.52329 -1	-4.38581 0	-1.35002 0	-6.99236 -1
0.3000	1.59445 -1	-4.18063 0	-1.30988 0	-6.35486 -1
0.3125	1.66615 -1	-3.98845 0	-1.27439 0	-5.76855 -1
0.3250	1.73840 -1	-3.80781 0	-1.24296 0	-5.22728 -1
0.3375	1.81122 -1	-3.63741 0	-1.21510 0	-4.722587 -1
0.3500	1.88463 -1	-3.47618 0	-1.19038 0	-4.25987 -1
0.3625	1.95866 -1	-3.32315 0	-1.16843 0	-3.82545 -1
0.3750	2.03333 -1	-3.17749 0	-1.14893 0	-3.41929 -1
0.3875	2.10867 -1	-3.03847 0	-1.13162 0	-3.03854 -1
0.4000	2.18470 -1	-2.90546 0	-1.11626 0	-2.68066 -1
0.4125	2.26146 -1	-2.77788 0	-1.10263 0	-2.34346 -1
0.4250	2.33897 -1	-2.65522 0	-1.09056 0	-2.02499 -1
0.4375	2.41727 -1	-2.53702 0	-1.07990 0	-1.72352 -1
0.4500	2.49639 -1	-2.42289 0	-1.07048 0	-1.43753 -1
0.4625	2.57637 -1	-2.31244 0	-1.06221 0	-1.1616564 -1
0.4750	2.65726 -1	-2.20534 0	-1.05495 0	-9.06629 -2
0.4875	2.73909 -1	-2.10129 0	-1.04861 0	-6.59395 -2
0.5000	2.82191 -1	-2.00000 0	-1.04311 0	-4.22941 -2

Table 4A (continued)

x	w_1	w_2	w_b	w_o
0.5125	2.90577 -1	-1.90120 0	-1.03836 0	-1.96361 -2
0.5250	2.99073 -1	-1.80467 0	-1.03429 0	2.11665 -3
0.5375	3.07684 -1	-1.71018 0	-1.03084 0	2.830395 -2
0.5500	3.16417 -1	-1.61751 0	-1.02795 0	4.832014 -2
0.5625	3.25279 -1	-1.52646 0	-1.02556 0	6.26662 -2
0.5750	3.34277 -1	-1.43684 0	-1.02364 0	8.14930 -2
0.5875	3.43419 -1	-1.34848 0	-1.02212 0	9.97366 -2
0.6000	3.52715 -1	-1.26120 0	-1.02099 0	1.17449 -1
0.6125	3.62173 -1	-1.17482 0	-1.02019 0	1.834678 -1
0.6250	3.71804 -1	-1.08917 0	-1.01970 0	1.51469 -1
0.6375	3.81621 -1	-1.00409 0	-1.01948 0	1.67868 -1
0.6500	3.91637 -1	-9.619422 -1	-1.01951 0	1.83915 -1
0.6625	4.01864 -1	-8.634978 -1	-1.01976 0	1.99653 -1
0.6750	4.12321 -1	-7.650594 -1	-1.02021 0	2.15121 -1
0.6875	4.23025 -1	-6.66088 -1	-1.02083 0	2.30359 -1
0.7000	4.33995 -1	-5.681273 -1	-1.02160 0	2.45406 -1
0.7125	4.45255 -1	-4.695951 -1	-1.02251 0	2.60303 -1
0.7250	4.56830 -1	-4.09909 -1	-1.02354 0	2.75091 -1
0.7375	4.68750 -1	-3.622917 -1	-1.02467 0	2.89814 -1
0.7500	4.81049 -1	-2.634721 -1	-1.02588 0	3.04516 -1
0.7625	4.93764 -1	-1.645040 -1	-1.02716 0	3.619245 -1
0.7750	5.06942 -1	-5.835600 -2	-1.02850 0	3.834054 -1
0.7875	5.20635 -1	4.00801 -2	-1.02988 0	3.649002 -1
0.8000	5.34906 -1	1.636294 -1	-1.03129 0	3.64153 -1
0.8125	5.49829 -1	2.35567 -1	-1.03272 0	3.679582 -1
0.8250	5.65494 -1	3.8476 -1	-1.03415 0	3.95376 -1
0.8375	5.82012 -1	4.45713 -1	-1.03558 0	4.11636 -1
0.8500	5.99519 -1	5.58130 -1	-1.03699 0	4.28487 -1
0.8625	6.18188 -1	6.76790 -1	-1.03838 0	4.46083 -1
0.8750	6.38240 -1	8.03053 -1	-1.03973 0	4.64616 -1
0.8875	6.59967 -1	9.38694 -1	-1.04103 0	4.84338 -1
0.9000	6.83764 -1	1.08611 0	-1.04227 0	5.05588 -1
0.9125	7.10186 -1	1.24865 0	-1.04344 0	5.28835 -1
0.9250	7.40044 -1	1.43122 0	-1.04452 0	5.54761 -1
0.9375	7.74598 -1	1.64138 0	-1.04551 0	5.84423 -1
0.9500	8.15955 -1	1.89180 0	-1.04638 0	6.19584 -1
0.9625	8.68069 -1	2.20623 0	-1.04713 0	6.663544 -1
0.9750	9.39813 -1	2.63792 0	-1.04772 0	7.23704 -1
0.9875	1.05950 0	3.35678 0	-1.04814 0	8.23665 -1

Table 5A

Values of the Functions Appearing in the Formulas for N_r , σ_{mr} , N_{θ} and $\sigma_{m\theta}$

$$N_r = mcx(A_1 N_{r1} + A_2 N_{r2})$$

$$N_{\theta} = mcx(A_1 N_{\theta1} + A_2 N_{\theta2})$$

$$\sigma_{mr} = A_1 N_{r1} + A_2 N_{r2}$$

$$\sigma_{m\theta} = A_1 N_{\theta1} + A_2 N_{\theta2}$$

x	N_{r1}	N_{r2}	$N_{\theta1}$	$N_{\theta2}$
0.0125	6.39141 -3	-8.03249 1	1.01572 0	-3.19114 1
0.0250	-1.30758 -2	-4.57117 1	1.03012 0	-1.90448 1
0.0375	-2.00697 -2	-3.06988 1	1.04593 0	-1.36761 1
0.0500	-2.73903 -2	-2.31876 1	1.06228 0	-1.09213 1
0.0625	-3.50563 -2	-1.86776 1	1.07918 0	-9.22818 0
0.0750	-4.30876 -2	-1.56685 1	1.09668 0	-8.07369 0
0.0875	-5.15055 -2	-1.35171 1	1.11479 0	-7.23134 0
0.1000	-6.03328 -2	-1.19021 1	1.13354 0	-6.58674 0
0.1125	-6.95940 -2	-1.06446 1	1.15299 0	-6.07572 0
0.1250	-7.93154 -2	-9.63749 0	1.17315 0	-5.65942 0
0.1375	-8.95254 -2	-8.81254 0	1.19407 0	-5.31287 0
0.1500	-1.00254 -1	-8.12424 0	1.21580 0	-5.01928 0
0.1625	-1.11535 -1	-7.54107 0	1.23837 0	-4.76693 0
0.1750	-1.23403 -1	-7.04053 0	1.26183 0	-4.54736 0
0.1875	-1.35896 -1	-6.60611 0	1.28624 0	-4.35432 0
0.2000	-1.49056 -1	-6.22542 0	1.31165 0	-4.18309 0
0.2125	-1.62926 -1	-5.88899 0	1.33812 0	-4.03004 0
0.2250	-1.77557 -1	-5.58945 0	1.36571 0	-3.89230 0
0.2375	-1.92999 -1	-5.32096 0	1.39450 0	-3.76760 0
0.2500	-2.09310 -1	-5.07888 0	1.42456 0	-3.65412 0
0.2625	-2.26552 -1	-4.85943 0	1.45597 0	-3.55036 0
0.2750	-2.44791 -1	-4.65952 0	1.48881 0	-3.45510 0
0.2875	-2.64103 -1	-4.47660 0	1.52320 0	-3.36732 0
0.3000	-2.84566 -1	-4.30853 0	1.55922 0	-3.28616 0
0.3125	-3.06269 -1	-4.15352 0	1.59699 0	-3.21091 0
0.3250	-3.29307 -1	-4.01006 0	1.63664 0	-3.14094 0
0.3375	-3.53787 -1	-3.87684 0	1.67829 0	-3.07573 0
0.3500	-3.79824 -1	-3.75277 0	1.72211 0	-3.01484 0
0.3625	-4.07545 -1	-3.63687 0	1.76824 0	-2.95788 0
0.3750	-4.37091 -1	-3.52832 0	1.81686 0	-2.90451 0
0.3875	-4.68618 -1	-3.42638 0	1.86818 0	-2.85443 0
0.4000	-5.02297 -1	-3.33041 0	1.92239 0	-2.80740 0
0.4125	-5.38320 -1	-3.23984 0	1.97975 0	-2.76318 0
0.4250	-5.76898 -1	-3.15418 0	2.04051 0	-2.72160 0
0.4375	-6.18268 -1	-3.07297 0	2.10497 0	-2.68248 0
0.4500	-6.62694 -1	-2.99581 0	2.17343 0	-2.64567 0
0.4625	-7.10471 -1	-2.92233 0	2.24628 0	-2.61105 0
0.4750	-7.61931 -1	-2.85222 0	2.32390 0	-2.57850 0
0.4875	-8.17446 -1	-2.78515 0	2.40675 0	-2.54795 0
0.5000	-8.77437 -1	-2.72087 0	2.49533 0	-2.51930 0

Table SA (continued)

x	N ¹	N ²	N ³	N ⁴
0.5125	-9.42381	-2.65911	2.59021	-2.49250
0.5250	-1.01282	-2.59962	2.69204	-2.46750
0.5375	-1.08936	-2.54219	2.80154	-2.44427
0.5500	-1.17272	-2.48660	2.91954	-2.42279
0.5625	-1.26369	-2.43264	3.04699	-2.40305
0.5750	-1.36321	-2.38011	3.18498	-2.38507
0.5875	-1.47235	-2.32880	3.33474	-2.36886
0.6000	-1.59234	-2.27852	3.49771	-2.35448
0.6125	-1.72465	-2.22905	3.67558	-2.34199
0.6250	-1.87096	-2.18017	3.87028	-2.33147
0.6375	-2.03328	-2.13167	4.08410	-2.32305
0.6500	-2.21397	-2.08329	4.31972	-2.31686
0.6625	-2.41584	-2.03477	4.58033	-2.31309
0.6750	-2.64227	-1.98579	4.86972	-2.31197
0.6875	-2.89731	-1.93600	5.19246	-2.31377
0.7000	-3.18590	-1.88502	5.55404	-2.31885
0.7125	-3.51408	-1.83236	5.96118	-2.32764
0.7250	-3.88929	-1.77745	6.42211	-2.34069
0.7375	-4.32081	-1.71961	6.94707	-2.35867
0.7500	-4.82031	-1.65798	7.54884	-2.38244
0.7625	-5.40262	-1.59150	8.24361	-2.41311
0.7750	-6.08683	-1.51880	9.05214	-2.45207
0.7875	-6.89784	-1.43814	1.00014	-2.50117
0.8000	-7.86860	-1.34721	1.11268	-2.56284
0.8125	-9.04349	-1.24290	1.24761	-2.64035
0.8250	-1.04833	-1.12099	1.41141	-2.73816
0.8375	-1.22732	-9.75582	1.61315	-2.86250
0.8500	-1.45355	-7.98209	1.86576	-3.02229
0.8625	-1.74507	-5.76396	2.18828	-3.23063
0.8750	-2.12941	-2.91121	2.60960	-3.50745
0.8875	-2.65027	8.77799	3.17538	-3.88424
0.9000	-3.38039	6.10499	3.96131	-4.41318
0.9125	-4.44808	1.36557	5.10031	-5.18596
0.9250	-6.09730	2.52130	6.84399	-6.37611
0.9375	-8.83926	4.43032	9.71732	-8.34575
0.9500	-1.38992	7.93782	1.49730	-1.19589
0.9625	-2.48572	1.55133	2.62549	-1.97292
0.9750	-5.62390	3.71764	5.82796	-4.18077
0.9875	-2.26095	1.54357	2.30049	-1.60280

Values of the Functions Appearing in the Formula for z

$$z = \frac{E}{c} (A_1 u_1 + A_2 u_2)$$

x	u_1	u_2
0.0125	1.00424	-4.30643
0.0250	1.00862	-3.71235
0.0375	1.01315	-3.31410
0.0500	1.01784	-3.03251
0.0625	1.02269	-2.81467
0.0750	1.02771	-2.63706
0.0875	1.03291	-2.48714
0.1000	1.03829	-2.35745
0.1125	1.04387	-2.24318
0.1250	1.04964	-2.14106
0.1375	1.05563	-2.04874
0.1500	1.06183	-1.96452
0.1625	1.06827	-1.88709
0.1750	1.07495	-1.81542
0.1875	1.08187	-1.74873
0.2000	1.08907	-1.68636
0.2125	1.09653	-1.62779
0.2250	1.10429	-1.57259
0.2375	1.11236	-1.52038
0.2500	1.12075	-1.47087
0.2625	1.12947	-1.42378
0.2750	1.13855	-1.37890
0.2875	1.14800	-1.33602
0.3000	1.15785	-1.29499
0.3125	1.16812	-1.25565
0.3250	1.17882	-1.21787
0.3375	1.19000	-1.18154
0.3500	1.20166	-1.14655
0.3625	1.21385	-1.11281
0.3750	1.22660	-1.08025
0.3875	1.23993	-1.04879
0.4000	1.25390	-1.01836
0.4125	1.26853	-9.88903
0.4250	1.28387	-9.60371
0.4375	1.29997	-9.32713
0.4500	1.31688	-9.05887
0.4625	1.33467	-8.79853
0.4750	1.35339	-8.54576
0.4875	1.37311	-8.30025
0.5000	1.39391	-8.06171

x	u_1	u_2
0.5125	1.41587	0
0.5250	1.43908	0
0.5375	1.46366	0
0.5500	1.48970	0
0.5625	1.51735	0
0.5750	1.54674	0
0.5875	1.57803	0
0.6000	1.61140	0
0.6125	1.64706	0
0.6250	1.68523	0
0.6375	1.72617	0
0.6500	1.77020	0
0.6625	1.81764	0
0.6750	1.86891	0
0.6875	1.92445	0
0.7000	1.98480	0
0.7125	2.05061	0
0.7250	2.12260	0
0.7375	2.20168	0
0.7500	2.28890	0
0.7625	2.38557	0
0.7750	2.49325	0
0.7875	2.61389	0
0.8000	2.74995	0
0.8125	2.90449	0
0.8250	3.08150	0
0.8375	3.28617	0
0.8500	3.52543	0
0.8625	3.80871	0
0.8750	4.14926	0
0.8875	4.56618	0
0.9000	5.08812	0
0.9125	5.76015	0
0.9250	6.65735	0
0.9375	7.91488	0
0.9500	9.80311	0
0.9625	1.29529	1
0.9750	1.92568	1
0.9875	3.81780	1
0.5125	-7.82290	-1
0.5250	-7.60458	-1
0.5375	-7.38555	-1
0.5500	-7.17266	-1
0.5625	-6.96576	-1
0.5750	-6.76472	-1
0.5875	-6.56946	-1
0.6000	-6.37991	-1
0.6125	-6.19603	-1
0.6250	-6.01782	-1
0.6375	-5.84529	-1
0.6500	-5.67851	-1
0.6625	-5.51758	-1
0.6750	-5.36263	-1
0.6875	-5.21387	-1
0.7000	-5.07154	-1
0.7125	-4.93597	-1
0.7250	-4.80758	-1
0.7375	-4.68686	-1
0.7500	-4.57447	-1
0.7625	-4.47120	-1
0.7750	-4.37806	-1
0.7875	-4.29631	-1
0.8000	-4.22755	-1
0.8125	-4.17385	-1
0.8250	-4.13787	-1
0.8375	-4.12313	-1
0.8500	-4.13433	-1
0.8625	-4.17795	-1
0.8750	-4.26303	-1
0.8875	-4.40271	-1
0.9000	-4.61670	-1
0.9125	-4.93603	-1
0.9250	-5.41244	-1
0.9375	-6.13911	-1
0.9500	-7.30250	-1
0.9625	-9.633770	-1
0.9750	-1.635502	0
0.9875	-2.664674	0

Table 6A (continued)

Table 1B

Values of the Functions Appearing in the Formulas for M_r and σ_{Br}

$$M_r = x^2(C_1 M_{r1} + C_2 M_{r2} + k M_{rk} + \rho M_{rp})$$

$$\sigma_{Br} = \frac{\pm 6}{(mc)^2} (C_1 M_{r1} + C_2 M_{r2} + k M_{rk} + \rho M_{rp})$$

x	$r_1 M_{r1}$	M_{r2}	M_{rk}	M_{rp}
-0.0125	-1.69355 -5	1.26935 4	6.42596 3	6.42696 3
-0.0250	-6.60992 -5	3.14728 3	1.61281 3	1.61380 3
-0.0375	-1.45160 -4	1.38731 3	7.19509 2	7.20487 2
-0.0500	-2.51954 -4	7.73992 2	4.06185 2	4.07157 2
-0.0625	-3.84469 -4	4.91327 2	2.60859 2	2.61823 2
-0.0750	-5.40835 -4	3.38439 2	1.81754 2	1.82711 2
-0.0875	-7.19310 -4	2.46647 2	1.33959 2	1.34909 2
-0.1000	-9.18273 -4	1.87327 2	1.02876 2	1.03820 2
-0.1125	-1.13622 -3	1.46832 2	8.15240 1	8.24606 1
-0.1250	-1.37174 -3	1.17991 2	6.62204 1	6.71503 1
-0.1375	-1.62352 -3	9.67450 1	5.48756 1	5.57988 1
-0.1500	-1.89035 -3	8.06553 1	4.62304 1	4.71470 1
-0.1625	-2.17109 -3	6.81881 1	3.94897 1	4.03998 1
-0.1750	-2.46466 -3	5.83389 1	3.41312 1	3.50349 1
-0.1875	-2.77008 -3	5.04278 1	2.98003 1	3.06977 1
-0.2000	-3.08642 -3	4.39814 1	2.62493 1	2.71406 1
-0.2125	-3.41280 -3	3.86623 1	2.33012 1	2.41863 1
-0.2250	-3.74843 -3	3.42244 1	2.08263 1	2.17054 1
-0.2375	-4.09255 -3	3.04851 1	1.87281 1	1.96014 1
-0.2500	-4.44444 -3	2.73066 1	1.69338 1	1.78012 1
-0.2625	-4.80345 -3	2.45834 1	1.53870 1	1.62487 1
-0.2750	-5.16895 -3	2.22334 1	1.40441 1	1.49003 1
-0.2875	-5.54036 -3	2.01923 1	1.28707 1	1.37213 1
-0.3000	-5.91715 -3	1.84089 1	1.18392 1	1.26845 1
-0.3125	-6.29881 -3	1.68422 1	1.09277 1	1.17676 1
-0.3250	-6.68486 -3	1.54589 1	1.01180 1	1.09527 1
-0.3375	-7.07485 -3	1.42318 1	9.39559 0	1.02251 1
-0.3500	-7.46837 -3	1.31388 1	8.74821 0	9.57277 0
-0.3625	-7.86503 -3	1.21612 1	8.16581 0	8.98543 0
-0.3750	-8.26446 -3	1.12837 1	7.63993 0	8.45470 0
-0.3875	-8.66632 -3	1.04932 1	7.16346 0	7.97347 0
-0.4000	-9.07029 -3	9.77891 0	6.73038 0	7.53571 0
-0.4125	-9.47607 -3	9.13138 0	6.33555 0	7.13629 0
-0.4250	-9.88338 -3	8.54277 0	5.97458 0	6.77081 0
-0.4375	-1.02919 -2	8.00627 0	5.64369 0	6.43549 0
-0.4500	-1.07015 -2	7.51603 0	5.33961 0	6.12706 0
-0.4625	-1.11119 -2	7.06700 0	5.05951 0	5.84270 0
-0.4750	-1.15228 -2	6.65477 0	4.80093 0	5.57992 0
-0.4875	-1.19341 -2	6.27553 0	4.56170 0	5.33658 0
-0.5000	-1.23456 -2	5.92592 0	4.33994 0	5.11078 0

Table 1B (continued)

x	M^p_1	M^p_2	M^p_k	M^p_b
-0.65125	-1.27571 -2	5.60301 0	4.13398 0	4.90085 0
-0.5250	-1.31685 -2	5.30420 0	3.94235 0	4.70532 0
-0.5375	-1.35795 -2	5.02723 0	3.76375 0	4.52289 0
-0.5500	-1.39900 -2	4.77005 0	3.59701 0	4.35239 0
-0.5625	-1.43999 -2	4.53088 0	3.44110 0	4.19280 0
-0.5750	-1.48092 -2	4.30811 0	3.29511 0	4.04318 0
-0.5875	-1.52175 -2	4.10033 0	3.15820 0	3.90271 0
-0.6000	-1.56249 -2	3.90625 0	3.02965 0	3.77066 0
-0.6125	-1.60313 -2	3.72471 0	2.90877 0	3.64635 0
-0.6250	-1.64365 -2	3.55471 0	2.79496 0	3.52917 0
-0.6375	-1.68405 -2	3.39530 0	2.68770 0	3.41859 0
-0.6500	-1.72431 -2	3.24565 0	2.58647 0	3.31411 0
-0.6625	-1.76443 -2	3.10500 0	2.49084 0	3.21528 0
-0.6750	-1.80441 -2	2.97266 0	2.40040 0	3.12170 0
-0.6875	-1.84423 -2	2.84802 0	2.31478 0	3.03300 0
-0.7000	-1.88389 -2	2.73050 0	2.23365 0	2.94883 0
-0.7125	-1.92338 -2	2.61959 0	2.15669 0	2.86889 0
-0.7250	-1.96270 -2	2.51481 0	2.08362 0	2.79290 0
-0.7375	-2.00185 -2	2.41575 0	2.01418 0	2.72058 0
-0.7500	-2.04081 -2	2.32199 0	1.94814 0	2.65172 0
-0.7625	-2.07959 -2	2.23319 0	1.88528 0	2.58608 0
-0.7750	-2.11818 -2	2.14901 0	1.82540 0	2.52347 0
-0.7875	-2.15658 -2	2.06914 0	1.76831 0	2.46370 0
-0.8000	-2.19478 -2	1.99331 0	1.71384 0	2.40659 0
-0.8125	-2.23279 -2	1.92125 0	1.66183 0	2.35199 0
-0.8250	-2.27059 -2	1.85274 0	1.61214 0	2.29976 0
-0.8375	-2.30819 -2	1.78754 0	1.56463 0	2.24974 0
-0.8500	-2.34558 -2	1.72547 0	1.51917 0	2.20183 0
-0.8625	-2.38277 -2	1.66631 0	1.47566 0	2.15589 0
-0.8750	-2.41975 -2	1.60991 0	1.43397 0	2.11182 0
-0.8875	-2.45651 -2	1.55610 0	1.39401 0	2.06952 0
-0.9000	-2.49307 -2	1.50473 0	1.35568 0	2.02890 0
-0.9125	-2.52941 -2	1.45566 0	1.31890 0	1.98986 0
-0.9250	-2.56554 -2	1.40876 0	1.28359 0	1.95232 0
-0.9375	-2.60145 -2	1.36391 0	1.24966 0	1.91621 0
-0.9500	-2.63715 -2	1.32099 0	1.21705 0	1.88145 0
-0.9625	-2.67263 -2	1.27990 0	1.18568 0	1.84797 0
-0.9750	-2.70789 -2	1.24055 0	1.15550 0	1.81571 0
-0.9875	-2.74294 -2	1.20283 0	1.12645 0	1.78462 0

Table 2B

Values of the Functions Appearing in the Formulas for M_θ and $\sigma_{B\theta}$

$$M_\theta = x^2(C_1 M_{\theta 1} + C_2 M_{\theta 2} + k M_{\theta k} + \rho M_{\theta \rho})$$

$$\sigma_{B\theta} = \frac{\pm 6}{(mc)^2} (C_1 M_{\theta 1} + C_2 M_{\theta 2} + k M_{\theta k} + \rho M_{\theta \rho})$$

x	$M_{\theta 1}$	$M_{\theta 2}$	$M_{\theta k}$	$M_{\theta \rho}$
-0.0125	-5.47011 -3	4.16183 3	2.10646 3	2.10673 3
-0.0250	-1.07740 -2	1.01525 3	5.19873 2	5.20112 2
-0.0375	-1.59191 -2	4.40418 2	2.28044 2	2.28250 2
-0.0500	-2.09121 -2	2.41872 2	1.26579 2	1.26757 2
-0.0625	-2.57594 -2	1.51177 2	7.99267 1	8.00797 1
-0.0750	-3.04669 -2	1.02557 2	5.47536 1	5.48846 1
-0.0875	-3.50406 -2	7.36261 1	3.96776 1	3.97887 1
-0.1000	-3.94857 -2	5.50964 1	2.99597 1	3.00529 1
-0.1125	-4.38076 -2	4.25602 1	2.33433 1	2.34201 1
-0.1250	-4.80109 -2	3.37119 1	1.86438 1	1.87058 1
-0.1375	-5.21005 -2	2.72521 1	1.51914 1	1.52399 1
-0.1500	-5.60806 -2	2.24042 1	1.25845 1	1.26206 1
-0.1625	-5.99555 -2	1.86816 1	1.05706 1	1.05952 1
-0.1750	-6.37291 -2	1.57672 1	8.98438 0	8.99862 0
-0.1875	-6.74053 -2	1.34474 1	7.71425 0	7.71890 0
-0.2000	-7.09876 -2	1.15740 1	6.68259 0	6.67844 0
-0.2125	-7.44795 -2	1.00421 1	5.83411 0	5.82186 0
-0.2250	-7.78842 -2	8.77549 0	5.12855 0	5.10887 0
-0.2375	-8.12048 -2	7.71775 0	4.53609 0	4.50958 0
-0.2500	-8.44444 -2	6.82666 0	4.03424 0	4.00145 0
-0.2625	-8.76057 -2	6.06998 0	3.60578 0	3.56724 0
-0.2750	-9.06916 -2	5.42279 0	3.23738 0	3.19356 0
-0.2875	-9.37045 -2	4.86562 0	2.91858 0	2.86993 0
-0.3000	-9.66469 -2	4.38308 0	2.64107 0	2.58800 0
-0.3125	-9.95212 -2	3.96287 0	2.39819 0	2.34109 0
-0.3250	-1.02329 -1	3.59509 0	2.18458 0	2.12381 0
-0.3375	-1.05074 -1	3.27170 0	1.99584 0	1.93173 0
-0.3500	-1.07757 -1	2.98610 0	1.82837 0	1.76125 0
-0.3625	-1.10381 -1	2.73287 0	1.67919 0	1.60935 0
-0.3750	-1.12947 -1	2.50749 0	1.54583 0	1.47354 0
-0.3875	-1.15457 -1	2.30621 0	1.42619 0	1.35171 0
-0.4000	-1.17913 -1	2.12585 0	1.31852 0	1.24210 0
-0.4125	-1.20317 -1	1.96373 0	1.22133 0	1.14321 0
-0.4250	-1.22670 -1	1.81761 0	1.13336 0	1.05374 0
-0.4375	-1.24973 -1	1.68553 0	1.05353 0	9.72617 -1
-0.4500	-1.27229 -1	1.56584 0	9.80900 -1	8.98880 -1
-0.4625	-1.29438 -1	1.45711 0	9.14664 -1	8.31721 -1
-0.4750	-1.31603 -1	1.35811 0	8.54127 -1	7.70431 -1
-0.4875	-1.33724 -1	1.26778 0	7.98684 -1	7.14393 -1
-0.5000	-1.35802 -1	1.18518 0	7.47804 -1	6.63070 -1

Table 2B (continued)

x	$M_{\theta 1}$	$M_{\theta 2}$	$M_{\theta k}$	$M_{\theta p}$
-0.5125	-1.37839 -1	1.10950 0	7.01024 -1	6.15989 -1
-0.5250	-1.39836 -1	1.04004 0	6.57935 -1	5.72734 -1
-0.5375	-1.41795 -1	9.76161 -1	6.18179 -1	5.32939 -1
-0.5500	-1.43716 -1	9.17318 -1	5.81438 -1	4.96278 -1
-0.5625	-1.45600 -1	8.63025 -1	5.447430 -1	4.62465 -1
-0.5750	-1.47448 -1	8.12852 -1	5.15906 -1	4.31243 -1
-0.5875	-1.49261 -1	7.66417 -1	4.86642 -1	4.02382 -1
-0.6000	-1.51041 -1	7.23379 -1	4.59440 -1	3.75679 -1
-0.6125	-1.52788 -1	6.83434 -1	4.34121 -1	3.50950 -1
-0.6250	-1.54503 -1	6.46311 -1	4.10526 -1	3.28031 -1
-0.6375	-1.56187 -1	6.11766 -1	3.88510 -1	3.06773 -1
-0.6500	-1.57841 -1	5.79581 -1	3.67945 -1	2.87042 -1
-0.6625	-1.59465 -1	5.49558 -1	3.448713 -1	2.68718 -1
-0.6750	-1.61060 -1	5.21520 -1	3.20709 -1	2.51690 -1
-0.6875	-1.62627 -1	4.95308 -1	3.013838 -1	2.35861 -1
-0.7000	-1.64167 -1	4.70776 -1	2.98011 -1	2.21139 -1
-0.7125	-1.65681 -1	4.47793 -1	2.83152 -1	2.07442 -1
-0.7250	-1.67168 -1	4.26240 -1	2.69187 -1	1.94695 -1
-0.7375	-1.68630 -1	4.06008 -1	2.56051 -1	1.82830 -1
-0.7500	-1.70068 -1	3.86999 -1	2.43685 -1	1.71784 -1
-0.7625	-1.71481 -1	3.69122 -1	2.32033 -1	1.61501 -1
-0.7750	-1.72871 -1	3.52296 -1	2.21047 -1	1.51926 -1
-0.7875	-1.74238 -1	3.36446 -1	2.10678 -1	1.43013 -1
-0.8000	-1.75582 -1	3.21502 -1	2.00887 -1	1.34716 -1
-0.8125	-1.76905 -1	3.07401 -1	1.91633 -1	1.26995 -1
-0.8250	-1.78207 -1	2.94086 -1	1.82882 -1	1.19812 -1
-0.8375	-1.79487 -1	2.81503 -1	1.74600 -1	1.13132 -1
-0.8500	-1.80748 -1	2.69604 -1	1.666756 -1	1.06923 -1
-0.8625	-1.81988 -1	2.58343 -1	1.59324 -1	1.01155 -1
-0.8750	-1.83209 -1	2.47679 -1	1.52277 -1	9.58010 -2
-0.8875	-1.84411 -1	2.37573 -1	1.45591 -1	9.08349 -2
-0.9000	-1.85595 -1	2.27990 -1	1.39244 -1	8.662331 -2
-0.9125	-1.86761 -1	2.18897 -1	1.33215 -1	8.19735 -2
-0.9250	-1.87908 -1	2.10263 -1	1.27485 -1	7.80358 -2
-0.9375	-1.89039 -1	2.02061 -1	1.22037 -1	7.44008 -2
-0.9500	-1.90152 -1	1.94264 -1	1.16853 -1	7.10508 -2
-0.9625	-1.91249 -1	1.86847 -1	1.11919 -1	6.79691 -2
-0.9750	-1.92330 -1	1.79789 -1	1.07221 -1	6.51404 -2
-0.9875	-1.93395 -1	1.73069 -1	1.02744 -1	6.25503 -2

Table 3B

Values of the Functions Appearing in the Formula for dw/dr

$$\frac{dw}{dr} = \frac{32}{3m(mc)^2 E} (C_1 W_1' + C_2 W_2' + k W_3' + p W_4')$$

x	W_1'	W_2'	W_3'	W_4'
-0.0125	4.97942 -1	6.32078 3	3.23746 3	3.24232 3
-0.0250	4.95934 -1	1.56095 3	8.17902 2	8.22063 2
-0.0375	4.93975 -1	6.85400 2	3.67005 2	3.70751 2
-0.0500	4.92063 -1	3.80949 2	2.08264 2	2.11715 2
-0.0625	4.90196 -1	2.40939 2	1.34378 2	1.37600 2
-0.0750	4.88372 -1	1.65373 2	9.40258 1	9.70603 1
-0.0875	4.86590 -1	1.20102 2	6.95682 1	7.24443 1
-0.1000	4.84848 -1	9.09090 1	5.36148 1	5.63537 1
-0.1125	4.83146 -1	7.10223 1	4.26239 1	4.52421 1
-0.1250	4.81481 -1	5.68888 1	3.47255 1	3.722357 1
-0.1375	4.79853 -1	4.64989 1	2.88550 1	3.12679 1
-0.1500	4.78260 -1	3.86473 1	2.43705 1	2.66947 1
-0.1625	4.76702 -1	3.25761 1	2.08657 1	2.31086 1
-0.1750	4.75177 -1	2.77898 1	1.80733 1	2.02412 1
-0.1875	4.73684 -1	2.39532 1	1.58117 1	1.79098 1
-0.2000	4.72222 -1	2.08333 1	1.39536 1	1.59868 1
-0.2125	4.70790 -1	1.82641 1	1.24079 1	1.43804 1
-0.2250	4.69387 -1	1.61249 1	1.11080 1	1.30234 1
-0.2375	4.68013 -1	1.43260 1	1.00040 1	1.18656 1
-0.2500	4.66666 -1	1.28000 1	9.05830 0	1.08691 1
-0.2625	4.65346 -1	1.14950 1	8.24177 0	1.00045 1
-0.2750	4.64052 -1	1.03710 1	7.53180 0	9.24887 0
-0.2875	4.62783 -1	9.39673 0	6.91051 0	8.58414 0
-0.3000	4.61538 -1	8.54700 0	6.36363 0	7.99585 0
-0.3125	4.60317 -1	7.80190 0	5.87967 0	7.47234 0
-0.3250	4.59119 -1	7.14524 0	5.44927 0	7.00414 0
-0.3375	4.57943 -1	6.56385 0	5.06477 0	6.58342 0
-0.3500	4.56790 -1	6.04686 0	4.71982 0	6.20374 0
-0.3625	4.55657 -1	5.58531 0	4.40916 0	5.85972 0
-0.3750	4.54545 -1	5.17171 0	4.12835 0	5.54684 0
-0.3875	4.53453 -1	4.79980 0	3.87367 0	5.26129 0
-0.4000	4.52380 -1	4.46428 0	3.64195 0	4.99983 0
-0.4125	4.51327 -1	4.16067 0	3.43051 0	4.75969 0
-0.4250	4.50292 -1	3.88514 0	3.23703 0	4.53851 0
-0.4375	4.49275 -1	3.63442 0	3.05952 0	4.33423 0
-0.4500	4.48275 -1	3.40570 0	2.89626 0	4.14509 0
-0.4625	4.47293 -1	3.19654 0	2.74577 0	3.96954 0
-0.4750	4.46327 -1	3.00483 0	2.60672 0	3.80623 0
-0.4875	4.45378 -1	2.82874 0	2.47800 0	3.65399 0
-0.5000	4.44444 -1	2.66666 0	2.35860 0	3.51176 0

Table 3B (continued)

x	W_1	W_2	W_k	W_0
-0.5125	4.43526 -1	2.51719 0	2.24763 0	3.37865 0
-0.5250	4.42622 -1	2.37909 0	2.14431 0	3.25383 0
-0.5375	4.41734 -1	2.25127 0	2.04796 0	3.13658 0
-0.5500	4.40860 -1	2.13276 0	1.95796 0	3.02626 0
-0.5625	4.40000 -1	2.02271 0	1.87376 0	2.92229 0
-0.5750	4.39153 -1	1.92036 0	1.79487 0	2.82416 0
-0.5875	4.38320 -1	1.82503 0	1.72086 0	2.73141 0
-0.6000	4.37500 -1	1.73611 0	1.65132 0	2.64362 0
-0.6125	4.36692 -1	1.65305 0	1.58591 0	2.56041 0
-0.6250	4.35897 -1	1.57538 0	1.52430 0	2.48145 0
-0.6375	4.35114 -1	1.50265 0	1.46620 0	2.40642 0
-0.6500	4.34343 -1	1.43446 0	1.41136 0	2.33505 0
-0.6625	4.33583 -1	1.37046 0	1.35952 0	2.26708 0
-0.6750	4.32835 -1	1.31032 0	1.31048 0	2.20228 0
-0.6875	4.32098 -1	1.25374 0	1.26403 0	2.14044 0
-0.7000	4.31372 -1	1.20048 0	1.22000 0	2.08136 0
-0.7125	4.30656 -1	1.15026 0	1.17823 0	2.02486 0
-0.7250	4.29951 -1	1.10289 0	1.13855 0	1.97079 0
-0.7375	4.29256 -1	1.05815 0	1.10083 0	1.91899 0
-0.7500	4.28571 -1	1.01587 0	1.06495 0	1.86932 0
-0.7625	4.27895 -1	9.75868 -1	1.03079 0	1.82166 0
-0.7750	4.27230 -1	9.37990 -1	9.98234 -1	1.77588 0
-0.7875	4.26573 -1	9.02096 -1	9.67188 -1	1.73188 0
-0.8000	4.25925 -1	8.68055 -1	9.37559 -1	1.68956 0
-0.8125	4.25287 -1	8.35747 -1	9.09262 -1	1.64882 0
-0.8250	4.24657 -1	8.05061 -1	8.82219 -1	1.60958 0
-0.8375	4.24036 -1	7.75895 -1	8.56357 -1	1.57176 0
-0.8500	4.23423 -1	7.48152 -1	8.31608 -1	1.53527 0
-0.8625	4.22818 -1	7.21747 -1	8.07909 -1	1.50006 0
-0.8750	4.22222 -1	6.96598 -1	7.85202 -1	1.46605 0
-0.8875	4.21633 -1	6.72630 -1	7.63431 -1	1.43319 0
-0.9000	4.21052 -1	6.49772 -1	7.42547 -1	1.40141 0
-0.9125	4.20479 -1	6.27961 -1	7.22501 -1	1.37066 0
-0.9250	4.19913 -1	6.07135 -1	7.03250 -1	1.34090 0
-0.9375	4.19354 -1	5.87240 -1	6.84752 -1	1.31208 0
-0.9500	4.18803 -1	5.68222 -1	6.66968 -1	1.28415 0
-0.9625	4.18259 -1	5.50033 -1	6.49862 -1	1.25706 0
-0.9750	4.17721 -1	5.32627 -1	6.33401 -1	1.23080 0
-0.9875	4.17190 -1	5.15963 -1	6.17551 -1	1.20530 0

Table 4B

Values of the Functions Appearing in the Formula for w

$$w = \frac{32}{3m^3 c E} (C_1 W_1 + C_2 W_2 + k W_k + p W_p + C_3)$$

x	W_1	W_2	W_k	W_p
-0.0125	-6.23717	7.56042	4.23938	4.21537
-0.0250	-1.24488	3.62861	2.20757	2.17797
-0.0375	-1.86357	2.33463	1.52310	1.48858
-0.0500	-2.47984	1.69554	1.17764	1.13863
-0.0625	-3.09375	1.31667	9.68580	9.25403
-0.0750	-3.70535	1.06707	8.28099	7.81015
-0.0875	-4.31469	8.90854	7.27015	6.76240
-0.1000	-4.92183	7.60210	6.50683	5.96400
-0.1125	-5.52682	6.59747	5.90933	5.33303
-0.1250	-6.12971	5.80277	5.42845	4.82012
-0.1375	-6.73054	5.15976	5.03280	4.39370
-0.1500	-7.32936	4.62978	4.70134	4.03264
-0.1625	-7.92621	4.18619	4.41947	3.72224
-0.1750	-8.52113	3.81004	4.17674	3.45195
-0.1875	-9.11416	3.48750	3.96543	3.21399
-0.2000	-9.70535	3.20824	3.77977	3.00250
-0.2125	-1.02947	2.96438	3.61530	2.81300
-0.2250	-1.08823	2.74984	3.46855	2.64196
-0.2375	-1.14682	2.55984	3.33678	2.48658
-0.2500	-1.20523	2.39056	3.21779	2.34464
-0.2625	-1.26348	2.23892	3.10979	2.21431
-0.2750	-1.32157	2.10243	3.01130	2.09408
-0.2875	-1.37950	1.97902	2.92112	1.98271
-0.3000	-1.43727	1.86699	2.83823	1.87916
-0.3125	-1.49488	1.76491	2.76177	1.78254
-0.3250	-1.55235	1.67158	2.69102	1.69212
-0.3375	-1.60967	1.58597	2.62535	1.60724
-0.3500	-1.66684	1.50721	2.56423	1.52736
-0.3625	-1.72386	1.43456	2.50721	1.45200
-0.3750	-1.78075	1.36738	2.45388	1.38074
-0.3875	-1.83750	1.30510	2.40389	1.31321
-0.4000	-1.89412	1.24723	2.35694	1.24911
-0.4125	-1.95060	1.19336	2.31276	1.18813
-0.4250	-2.00695	1.14310	2.27110	1.13003
-0.4375	-2.06317	1.09613	2.23177	1.07460
-0.4500	-2.11927	1.05215	2.19456	1.02162
-0.4625	-2.17524	1.01090	2.15931	9.70913
-0.4750	-2.23109	9.72164	2.12586	9.22326
-0.4875	-2.28682	9.35720	2.09409	8.75711
-0.5000	-2.34244	9.01387	2.06388	8.30935

Table 4B (continued)

x	W_1	W_2	W_k	W_p
-0.5125	-2.39793 -1	8.69001 -1	2.03509 0	7.87879 -1
-0.5250	-2.45332 -1	8.38410 -1	2.00765 0	7.46434 -1
-0.5375	-2.50859 -1	8.09481 -1	1.98146 0	7.06501 -1
-0.5500	-2.56375 -1	7.82089 -1	1.95643 0	6.67991 -1
-0.5625	-2.61881 -1	7.56126 -1	1.93248 0	6.30818 -1
-0.5750	-2.67375 -1	7.31490 -1	1.90956 0	5.94909 -1
-0.5875	-2.72860 -1	7.08088 -1	1.88759 0	5.60192 -1
-0.6000	-2.78333 -1	6.85837 -1	1.86652 0	5.26603 -1
-0.6125	-2.83797 -1	6.64661 -1	1.84629 0	4.94082 -1
-0.6250	-2.89251 -1	6.44488 -1	1.82686 0	4.62575 -1
-0.6375	-2.94695 -1	6.25255 -1	1.80817 0	4.32030 -1
-0.6500	-3.00129 -1	6.06903 -1	1.79019 0	4.02399 -1
-0.6625	-3.05553 -1	5.89376 -1	1.77287 0	3.73639 -1
-0.6750	-3.10968 -1	5.72625 -1	1.75619 0	3.45709 -1
-0.6875	-3.16374 -1	5.56604 -1	1.74010 0	3.18569 -1
-0.7000	-3.21771 -1	5.41268 -1	1.72458 0	2.92186 -1
-0.7125	-3.27159 -1	5.26579 -1	1.70959 0	2.66525 -1
-0.7250	-3.32537 -1	5.12499 -1	1.69511 0	2.41554 -1
-0.7375	-3.37907 -1	4.98995 -1	1.68112 0	2.17245 -1
-0.7500	-3.43269 -1	4.86035 -1	1.66758 0	1.93570 -1
-0.7625	-3.48622 -1	4.73589 -1	1.65449 0	1.70504 -1
-0.7750	-3.53966 -1	4.61629 -1	1.64181 0	1.48021 -1
-0.7875	-3.59302 -1	4.50131 -1	1.62953 0	1.26099 -1
-0.8000	-3.64631 -1	4.39069 -1	1.61762 0	1.04717 -1
-0.8125	-3.69951 -1	4.28422 -1	1.60608 0	8.38537 -2
-0.8250	-3.75263 -1	4.18169 -1	1.59489 0	6.34902 -2
-0.8375	-3.80567 -1	4.08290 -1	1.58402 0	4.36082 -2
-0.8500	-3.85864 -1	3.98766 -1	1.57347 0	2.41906 -2
-0.8625	-3.91153 -1	3.89580 -1	1.56323 0	5.22098 -3
-0.8750	-3.96434 -1	3.80717 -1	1.55327 0	-1.33160 -2
-0.8875	-4.01708 -1	3.72160 -1	1.54359 0	-3.14351 -2
-0.9000	-4.06975 -1	3.63896 -1	1.53418 0	-4.91503 -2
-0.9125	-4.12235 -1	3.55912 -1	1.52503 0	-6.64747 -2
-0.9250	-4.17487 -1	3.48193 -1	1.51612 0	-8.34211 -2
-0.9375	-4.22733 -1	3.40729 -1	1.50744 0	-1.00001 -1
-0.9500	-4.27971 -1	3.33508 -1	1.49899 0	-1.16227 -1
-0.9625	-4.33203 -1	3.26520 -1	1.49076 0	-1.32109 -1
-0.9750	-4.38428 -1	3.19754 -1	1.48274 0	-1.47657 -1
-0.9875	-4.43646 -1	3.13201 -1	1.47493 0	-1.62882 -1

Table.5B

Values of the Functions Appearing in the Formulas for N_p , σ_{mp} , N_{θ} and $\sigma_{m\theta}$

$$N_p = mcx(A_1 N_{p1} + A_2 N_{p2})$$

$$N_{\theta} = mcx(A_1 N_{\theta1} + A_2 N_{\theta2})$$

$$\sigma_{mp} = A_1 N_{p1} + A_2 N_{p2}$$

$$\sigma_{m\theta} = A_1 N_{\theta1} + A_2 N_{\theta2}$$

x	N_{p1}	N_{p2}	$N_{\theta1}$	$N_{\theta2}$
-0.0125	6.11322 -3	8.92309 1	9.85645 -1	2.54380 1
-0.0250	1.19636 -2	4.42147 1	9.71731 -1	1.11602 1
-0.0375	1.75634 -2	2.92058 1	9.58241 -1	6.59615 0
-0.0500	2.29257 -2	2.16992 1	9.45156 -1	4.40867 0
-0.0625	2.80623 -2	1.71944 1	9.32457 -1	3.15249 0
-0.0750	3.29843 -2	1.41910 1	9.20129 -1	2.35218 0
-0.0875	3.77022 -2	1.20458 1	9.08156 -1	1.80672 0
-0.1000	4.22259 -2	1.04374 1	8.96523 -1	1.41693 0
-0.1125	4.65646 -2	9.18681 0	8.85216 -1	1.12848 0
-0.1250	5.07271 -2	8.18695 0	8.74223 -1	9.09247 -1
-0.1375	5.47218 -2	7.36949 0	8.63531 -1	7.39083 -1
-0.1500	5.85563 -2	6.68890 0	8.53127 -1	6.04759 -1
-0.1625	6.22381 -2	6.11365 0	8.43001 -1	4.97260 -1
-0.1750	6.57740 -2	5.62120 0	8.33141 -1	4.10251 -1
-0.1875	6.91709 -2	5.19502 0	8.23538 -1	3.39162 -1
-0.2000	7.24347 -2	4.82272 0	8.14182 -1	2.80626 -1
-0.2125	7.55715 -2	4.49479 0	8.05064 -1	2.32111 -1
-0.2250	7.85868 -2	4.20386 0	7.96175 -1	1.91687 -1
-0.2375	8.14860 -2	3.94409 0	7.87506 -1	1.57855 -1
-0.2500	8.42740 -2	3.71081 0	7.79050 -1	1.29441 -1
-0.2625	8.69557 -2	3.50024 0	7.70799 -1	1.05512 -1
-0.2750	8.95355 -2	3.30930 0	7.62746 -1	8.53218 -2
-0.2875	9.20177 -2	3.13541 0	7.54883 -1	6.82647 -2
-0.3000	9.44064 -2	2.97645 0	7.47205 -1	5.38482 -2
-0.3125	9.67056 -2	2.83062 0	7.39705 -1	4.16672 -2
-0.3250	9.89188 -2	2.69641 0	7.32377 -1	3.13864 -2
-0.3375	1.01050 -1	2.57252 0	7.25215 -1	2.27272 -2
-0.3500	1.03101 -1	2.45785 0	7.18213 -1	1.54560 -2
-0.3625	1.05077 -1	2.35144 0	7.11367 -1	9.37652 -3
-0.3750	1.06980 -1	2.25245 0	7.04672 -1	4.32276 -3
-0.3875	1.08813 -1	2.16017 0	6.98121 -1	1.53883 -4
-0.4000	1.10579 -1	2.07397 0	6.91712 -1	-3.25021 -3
-0.4125	1.12280 -1	1.99328 0	6.85439 -1	-5.99224 -3
-0.4250	1.13919 -1	1.91762 0	6.79298 -1	-8.16035 -3
-0.4375	1.15499 -1	1.84656 0	6.73285 -1	-9.83048 -3
-0.4500	1.17021 -1	1.77970 0	6.67396 -1	-1.10682 -2
-0.4625	1.18488 -1	1.71670 0	6.61627 -1	-1.19301 -2
-0.4750	1.19901 -1	1.65726 0	6.55975 -1	-1.24657 -2
-0.4875	1.21264 -1	1.60109 0	6.50437 -1	-1.27177 -2
-0.5000	1.22577 -1	1.54795 0	6.45008 -1	-1.27236 -2

Table 5B (continued)

x	N ₁	N ₂	N ₀₁	N ₀₂
-0.5125	1.23843 -1	1.49761 0	6.339685 -1	-1.625158 -2
-0.5250	1.25063 -1	1.44987 0	6.34467 -1	-1.621231 -2
-0.5375	1.26239 -1	1.40454 0	6.29348 -1	-1.15703 -2
-0.5500	1.27372 -1	1.36146 0	6.24328 -1	-1.08795 -2
-0.5625	1.28465 -1	1.32048 0	6.19402 -1	-1.00700 -2
-0.5750	1.29518 -1	1.28144 0	6.14569 -1	-9.15872 -3
-0.5875	1.30533 -1	1.24423 0	6.09825 -1	-8.16059 -3
-0.6000	1.31512 -1	1.20874 0	6.05169 -1	-7.08880 -3
-0.6125	1.32455 -1	1.17484 0	6.00597 -1	-5.95497 -3
-0.6250	1.33364 -1	1.14245 0	5.96108 -1	-4.76933 -3
-0.6375	1.34240 -1	1.11146 0	5.91699 -1	-3.54096 -3
-0.6500	1.35084 -1	1.08181 0	5.87368 -1	-2.62781 -3
-0.6625	1.35898 -1	1.05340 0	5.83114 -1	-9.86957 -4
-0.6750	1.36681 -1	1.02618 0	5.78933 -1	3.625381 -4
-0.6875	1.37436 -1	1.00007 0	5.74825 -1	1.65370 -3
-0.7000	1.38164 -1	9.75002 -1	5.70787 -1	2.699315 -3
-0.7125	1.38864 -1	9.50931 -1	5.66817 -1	4.33945 -3
-0.7250	1.39539 -1	9.27800 -1	5.62915 -1	5.68885 -3
-0.7375	1.40189 -1	9.05558 -1	5.59077 -1	7.03799 -3
-0.7500	1.40814 -1	8.84160 -1	5.55303 -1	8.38399 -3
-0.7625	1.41417 -1	8.63561 -1	5.51591 -1	9.72427 -3
-0.7750	1.41996 -1	8.43722 -1	5.47939 -1	1.10566 -2
-0.7875	1.42554 -1	8.24605 -1	5.44346 -1	1.23791 -2
-0.8000	1.43090 -1	8.06173 -1	5.40810 -1	1.36899 -2
-0.8125	1.43606 -1	7.88394 -1	5.37331 -1	1.49877 -2
-0.8250	1.44103 -1	7.71237 -1	5.33906 -1	1.62711 -2
-0.8375	1.44580 -1	7.54672 -1	5.30535 -1	1.75391 -2
-0.8500	1.45038 -1	7.38671 -1	5.27216 -1	1.87906 -2
-0.8625	1.45479 -1	7.23210 -1	5.23948 -1	2.00250 -2
-0.8750	1.45902 -1	7.08262 -1	5.20730 -1	2.12416 -2
-0.8875	1.46308 -1	6.93806 -1	5.17561 -1	2.24398 -2
-0.9000	1.46698 -1	6.79819 -1	5.14439 -1	2.36191 -2
-0.9125	1.47072 -1	6.66281 -1	5.11363 -1	2.47793 -2
-0.9250	1.47430 -1	6.53173 -1	5.08333 -1	2.59199 -2
-0.9375	1.47774 -1	6.40476 -1	5.05348 -1	2.70409 -2
-0.9500	1.48104 -1	6.28173 -1	5.02405 -1	2.81421 -2
-0.9625	1.48419 -1	6.16247 -1	4.99506 -1	2.92233 -2
-0.9750	1.48721 -1	6.04683 -1	4.96648 -1	3.02846 -2
-0.9875	1.49010 -1	5.93467 -1	4.93830 -1	3.13260 -2

Table 6B

Values of the Functions Appearing in the Formula for u

$$u = \frac{c}{E}(A_1 U_1 + A_2 U_2)$$

x	U_1	U_2
-0.0125	9.95902 -1	-4.35990 0
-0.0250	9.91937 -1	-3.66746 0
-0.0375	9.88102 -1	-3.25684 0
-0.0500	9.84390 -1	-2.96563 0
-0.0625	9.80797 -1	-2.74017 0
-0.0750	9.77319 -1	-2.55651 0
-0.0875	9.73952 -1	-2.40181 0
-0.1000	9.70693 -1	-2.26841 0
-0.1125	9.67536 -1	-2.15134 0
-0.1250	9.64479 -1	-2.04720 0
-0.1375	9.61518 -1	-1.95356 0
-0.1500	9.58650 -1	-1.86861 0
-0.1625	9.55871 -1	-1.79097 0
-0.1750	9.53180 -1	-1.71959 0
-0.1875	9.50572 -1	-1.65361 0
-0.2000	9.48045 -1	-1.59234 0
-0.2125	9.45597 -1	-1.53521 0
-0.2250	9.43225 -1	-1.48176 0
-0.2375	9.40926 -1	-1.43159 0
-0.2500	9.38699 -1	-1.38437 0
-0.2625	9.36540 -1	-1.33981 0
-0.2750	9.34448 -1	-1.29767 0
-0.2875	9.32422 -1	-1.25772 0
-0.3000	9.30457 -1	-1.21979 0
-0.3125	9.28554 -1	-1.18371 0
-0.3250	9.26710 -1	-1.14933 0
-0.3375	9.24924 -1	-1.11652 0
-0.3500	9.23193 -1	-1.08517 0
-0.3625	9.21516 -1	-1.05517 0
-0.3750	9.19891 -1	-1.02643 0
-0.3875	9.18317 -1	-9.98866 -1
-0.4000	9.16793 -1	-9.72402 -1
-0.4125	9.15317 -1	-9.46968 -1
-0.4250	9.13888 -1	-9.22500 -1
-0.4375	9.12504 -1	-8.98941 -1
-0.4500	9.11164 -1	-8.76237 -1
-0.4625	9.09867 -1	-8.54341 -1
-0.4750	9.08612 -1	-8.33206 -1
-0.4875	9.07398 -1	-8.12792 -1
-0.5000	9.06223 -1	-7.93061 -1

Table 6B (continued)

x	U_1	U_2
-0.5125	9.05087 -1	-7.73976 -1
-0.5250	9.03988 -1	-7.55505 -1
-0.5375	9.02926 -1	-7.37618 -1
-0.5500	9.01899 -1	-7.20286 -1
-0.5625	9.00907 -1	-7.03483 -1
-0.5750	8.99949 -1	-6.87182 -1
-0.5875	8.99024 -1	-6.71363 -1
-0.6000	8.98130 -1	-6.56002 -1
-0.6125	8.97268 -1	-6.41079 -1
-0.6250	8.96436 -1	-6.26575 -1
-0.6375	8.95634 -1	-6.12472 -1
-0.6500	8.94861 -1	-5.98753 -1
-0.6625	8.94116 -1	-5.85403 -1
-0.6750	8.93399 -1	-5.72405 -1
-0.6875	8.92709 -1	-5.59746 -1
-0.7000	8.92045 -1	-5.47413 -1
-0.7125	8.91406 -1	-5.35392 -1
-0.7250	8.90793 -1	-5.23672 -1
-0.7375	8.90203 -1	-5.12241 -1
-0.7500	8.89638 -1	-5.01088 -1
-0.7625	8.89096 -1	-4.90203 -1
-0.7750	8.88577 -1	-4.79577 -1
-0.7875	8.88080 -1	-4.69199 -1
-0.8000	8.87604 -1	-4.59062 -1
-0.8125	8.87150 -1	-4.49156 -1
-0.8250	8.86716 -1	-4.39474 -1
-0.8375	8.86303 -1	-4.30009 -1
-0.8500	8.85910 -1	-4.20751 -1
-0.8625	8.85536 -1	-4.11696 -1
-0.8750	8.85180 -1	-4.02836 -1
-0.8875	8.84844 -1	-3.94164 -1
-0.9000	8.84525 -1	-3.85676 -1
-0.9125	8.84224 -1	-3.77364 -1
-0.9250	8.83940 -1	-3.69223 -1
-0.9375	8.83673 -1	-3.61249 -1
-0.9500	8.83423 -1	-3.53435 -1
-0.9625	8.83189 -1	-3.45777 -1
-0.9750	8.82971 -1	-3.38271 -1
-0.9875	8.82768 -1	-3.30911 -1

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Part III

ILLUSTRATIVE EXAMPLES

EXAMPLE 1

The radii of a circular plate thicker at the inner radius are 3 and 8 in. The thicknesses are $\frac{1}{2}$ and $\frac{1}{4}$ in., respectively. The plate is simply supported along the outer edge when it is loaded with a vertical shearing force of Q_a lb/in. along the inner edge. Determine the circumferential bending stress at the inner edge.

Solution

Formulas i and ii, respectively, yield

$$m = \frac{1/2 - 1/4}{8 - 3} = \frac{1}{20} ,$$

$$c = \frac{8(1/2) - 3(1/4)}{1/2 - 1/4} = 13 .$$

Then by formulas vii and viii,

$$x_a = 1 - 3/13 = 0.76923 ,$$

$$x_b = 1 - 8/13 = 0.38462 .$$

The nearest values appearing in the tables are 0.7750 and 0.3875, respectively. Hence the adjusted values for a and b are

$$a = 13(1 - 0.7750) = 2.925 ,$$

$$b = 13(1 - 0.3875) = 7.9625 .$$

The first definition of k (formula iv), since $p = 0$, yields

$$k = 2.925Q_a .$$

The necessary boundary conditions for determining C_1 and C_2 are

$$(1) \quad M_r = 0 \text{ at } x = x_a = 0.7750 ,$$

$$(2) \quad M_r = 0 \text{ at } x = x_b = 0.3875 .$$

Formula 1 and the values in Table 1A yield the following set of simultaneous equations (the x^2 term is omitted because it will divide out):

$$-1.3182C_1 - 2.1925C_2 + 0.085826k = 0 ,$$

$$-0.044472C_1 + 17.160C_2 + 5.0967k = 0 ,$$

whose solution is

$$C_1 = 0.55670k ,$$

$$C_2 = -0.29557k .$$

Formula 4 and the values in Table 2A for $x = x_a = 0.7750$ yield

$$\begin{aligned}\sigma_{B\theta} &= \frac{6}{(13/20)^2} (2.8491C_1 + 21.925C_2 - 0.30481k) \\ &= -73.838k = -215.98Q_a,\end{aligned}$$

where the negative sign denotes compressive stress at the top surface. The adjusted thicknesses at the inner and outer radii, respectively, are

$$\begin{aligned}b_a &= mcx_a = (13/20) (0.7750) = 0.50375 \text{ in.}, \\ b_b &= mcx_b = (13/20) (0.3875) = 0.25187 \text{ in.}\end{aligned}$$

EXAMPLE 2

Determine the radial bending stress at the outer edge of the plate in example 1 when the outer edge is fixed.

Solution

Prior to the designation of the boundary conditions, the procedure for this example is the same as in example 1. The new boundary conditions are

$$(1) \quad M_r = 0 \text{ at } x = x_a = 0.7750 ,$$

$$(2) \quad \frac{dw}{dr} = 0 \text{ at } x = x_b = 0.3875 .$$

The set of simultaneous equations is

$$-1.3182C_1 - 2.1925C_2 + 0.085826k = 0 ,$$

$$0.60544C_1 + 10.873C_2 + 1.3041k = 0 ,$$

whose solution is

$$C_1 = 0.29159k ,$$

$$C_2 = -0.13617k .$$

Formula 3 for $x = x_b = 0.3875$ yields

$$\begin{aligned} \sigma_{Br} &= \frac{6}{(13/20)^2} (-0.04472C_1 + 17.160C_2 + 5.0967k) \\ &= 39.013k = 114.11Q_a , \end{aligned}$$

which is a tensile stress at the top surface.

EXAMPLE 3

A plate thicker at the outer radius has thicknesses of $b_a = \frac{1}{8}$ in. and $b_b = \frac{1}{4}$ in. at the radii $a = 7$ in. and $b = 9$ in., respectively. The plate is subjected to a uniform pressure of p psi. Determine the circumferential bending stress at the inner edge when the plate is simply supported at (A) the inner edge, (B) the outer edge.

Solution

A. Formulas i, ii, vii, and viii, respectively, yield

$$m = \frac{1/8 - 1/4}{9 - 7} = \frac{-1}{16},$$

$$c = \frac{9(1/8) - 7(1/4)}{1/8 - 1/4} = 5,$$

$$x_a = 1 - 7/5 = -0.4,$$

$$x_b = 1 - 9/5 = -0.8.$$

Then formula iii yields

$$\rho = \frac{5^2 p}{2} = 12.5p.$$

The vertical shearing force at the outer edge is zero, while at the inner edge it is

$$Q_a = \frac{a^2 - b^2}{2a} p = \frac{a^2 - b^2}{ac^2} \rho.$$

Hence either definition of k (formula iv) yields

$$(a) \quad k = -\frac{b^2 \rho}{c^2} = -3.2400\rho.$$

The boundary conditions are

$$(1) \quad M_r = 0 \text{ at } x = x_a = -0.4,$$

$$(2) \quad M_r = 0 \text{ at } x = x_b = -0.8.$$

Formula 1 and the values in Table 1B yield the following set of simultaneous equations:

$$(b) \quad -0.0090703C_1 + 9.7789C_2 + 6.7304k + 7.5357\rho = 0,$$

$$-0.021948C_1 + 1.9933C_2 + 1.7138k + 2.4066\rho = 0.$$

Substituting for k from Eq. a and solving yields

$$C_1 = -11.83\rho,$$

$$C_2 = 1.448\rho.$$

Then formula 4 and the values in Table 2B for $x = x_a = -0.4000$ yield

$$\begin{aligned} (c) \quad \sigma_{B\theta} &= \frac{6}{(-5/16)^2} (-0.11791C_1 + 2.1258C_2 + 1.3185k + 1.2421\rho) \\ &= 88.676\rho = 1108.4p, \end{aligned}$$

which is a tensile stress at the top surface.

B. When the plate is simply supported at the outer edge, only the value for k in solution A is changed. Its new value is

$$k = -\frac{a^2 \rho}{c^2} = -1.9600\rho$$

Substituting this value for k in Eqs. b and solving yields

$$C_1 = 9.972\rho,$$

$$C_2 = 0.5876\rho,$$

which, when substituted with k in Eq. c, yields

$$\sigma_{B\theta} = -77.96\rho = -974.5p,$$

where the negative sign denotes compressive stress at the top surface.

EXAMPLE 4

Determine the minimum required thickness at the inner edge (b_a) of a tapered plate thicker at the inner radius which is to be mounted on a rigid shaft of radius 8 in. A shearing force of 1000 lb/in. is to be applied along the outer edge at a radius of 12 in. The thickness at the inner edge is to be twice the thickness at the outer edge ($b_a/b_b = 2$). The maximum allowable design stress is 36,000 psi.

Solution

The data for this plate are

$$c = \frac{bh_a - ab_b}{b_a - b_b} = \frac{b(b_a/b_b) - a}{(b_a/b_b) - 1} = 16 ,$$

$$x_a = 0.5000 ,$$

$$x_b = 0.2500 .$$

Since the inner edge is fixed, the boundary conditions are

$$(1) \quad \frac{dw}{dr} = 0 \text{ at } x = x_a = 0.5000 ,$$

$$(2) \quad M_r = 0 \text{ at } x = x_b = 0.2500 .$$

Formulas 1 and 5 and the values in Tables 1A and 3A yield the following set of simultaneous equations:

$$\begin{aligned} 0.666666C_1 + 8.00000C_2 &= -0.409137k , \\ -0.012345C_1 + 37.9259C_2 &= -14.0469k , \end{aligned}$$

whose solution is

$$C_1 = 3.81587k ,$$

$$C_2 = -0.369132k .$$

Formulas 3 and 4 yield the results shown in the following table:

Radial and Circumferential Bending Stresses in Plate

x	$\frac{-\sigma_{Br}(mc)^2}{6k}$	$\frac{-\sigma_{B\theta}(mc)^2}{6k}$
0.25	0	0.2137
0.30	0.770	0.5089
0.35	1.198	0.6496
0.40	1.464	0.6992
0.45	1.655	0.6825
0.50	1.816	0.6053

The negative signs denote compressive stresses in the top surface, because the positive direction for Q_b is upward (see Fig. 1). The maximum stress in the plate is the radial bending stress at the inner edge:

$$\sigma_{\max} = \frac{6(1.816) k}{(mc)^2}$$

Since

$$k = bQ_b = 12,000 \text{ lb},$$

$$mc = 2b_a,$$

and

$$\sigma_{\max} = 36,000 \text{ psi},$$

then

$$b_a = 0.953 \text{ in.}$$

THE PARTIALLY TAPERED CIRCULAR PLATE PROBLEM

Occasionally in design the outer section of a flat circular plate is tapered. Theoretically this is considered as a flat circular plate which is surrounded by a tapered plate thicker at the inner radius. The problem then reduces to that of mathematically "connecting" the two plates by the compatibility conditions at their common junction. The equations necessary for doing this are listed below. Equations 1 and 2 are used during the bending analysis, and Ea. 3 is used during the membrane analysis. A concentrated load at the center of the flat plate (downward positive) is designated by F ; otherwise the symbols and subscripts are the same as those established in part I.

Compatibility Equations

$$(1) \quad k = \frac{F}{2\pi} ,$$

$$(2) \quad \left(\frac{dw}{dr} \right)_a = \frac{2.1a}{Eb_a^3} (4M_a - 2k - 0.5pa^2) ,$$

$$(3) \quad u_a = \frac{0.7aN_a}{Eb_a} .$$

Example

The outer section of a flat circular plate is tapered from a radius of 1 in. and a thickness of $\frac{1}{4}$ in. to its outer radius of 2 in. and a thickness of $\frac{3}{16}$ in. The plate is simply supported along the outer edge when it is subjected to a concentrated force of F pounds at the center. Determine the necessary set of simultaneous equations for determining the constants of integration in the bending analysis of the tapered plate.

Solution

The various constants for this plate are

$$\begin{aligned} b_a &= \frac{1}{4} , & b_b &= \frac{3}{16} , \\ a &= 1 , & b &= 2 , \\ m &= \frac{1}{16} , & c &= 5 , \\ x_a &= 0.8 , & x_b &= 0.6 . \end{aligned}$$

The boundary conditions are

$$(1) \quad \text{Eq. 2 at } x = x_a = 0.8 ,$$

$$(2) \quad M_r = 0 \text{ at } x = x_b = 0.6 .$$

Now $(dw/dr)_a$ and M_a are expressed in terms of C_1 and C_2 by the following relations:

$$\left. \begin{aligned} \left(\frac{dw}{dr} \right)_a &= \frac{dw}{dr} \text{ (formula 5)} \\ M_a &= M_r \text{ (formula 1)} \end{aligned} \right\} \text{ at } x = x_a = 0.8 .$$

Hence

$$\left(\frac{dw}{dr} \right)_a = \frac{1747.6}{E} (1.1667C_1 + 7.8125C_2 - 0.11370k) ,$$

$$M_a = 0.64(-1.7778C_1 - 5.2083C_2 - 0.060497k) .$$

When the above expressions are substituted in compatibility Eq. 2 (boundary condition 1), the following is obtained:

$$(a) \quad -2650.6C_1 - 15445.0C_2 = 90.92k .$$

Boundary condition 2 yields.

$$(b) \quad -0.25000C_1 + 6.9444C_2 = -1.3381k .$$

Since $k = F/2\pi$, Eqs. *a* and *b* are the required equations of the set.

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Part IV

DERIVATION OF THE PLATE FORMULAS

The symbol and sign convention established in part I is used in part IV. Additional symbols are designated as required.

BENDING ANALYSIS¹

The differential equation for the bending analysis is derived from the following sets of equations.²

Equations of Equilibrium

$$(1) \quad \frac{d(M_r r)}{dr} - M_\theta = rQ ,$$

$$(2) \quad rQ = k + \frac{pr^2}{2} ,$$

where $k = a(Q_a - ap/2) \equiv b(Q_b - bp/2)$, as defined in part I. Equation 1 is obtained by summing the moments about a radial edge of a free-body diagram of a differential element of the plate. Equation 2 is obtained by summing the forces in the axial direction of a free-body diagram of a finite portion of the plate (see Fig. 1).

Moment-Displacement Relations

$$(3) \quad M_r = D \left(\frac{d^2 w}{dr^2} + \frac{\gamma}{r} \frac{dw}{dr} \right) = D \left(\frac{d\phi}{dr} + \frac{\gamma\phi}{r} \right) ,$$

$$(4) \quad M_\theta = D \left(\frac{dw}{r dr} + \frac{\gamma}{dr^2} \frac{d^2 w}{dr^2} \right) = D \left(\frac{\phi}{r} + \frac{\gamma}{dr} \frac{d\phi}{dr} \right) ,$$

where $\phi = dw/dr$, γ is Poisson's ratio, and $D = Eb^3/12(1 - \gamma^2)$, the flexural rigidity of the plate.

Equations 1 through 4 hold true regardless of the thickness-radius relationship. The required differential equation is obtained by substituting Eqs. 2, 3, and 4 into Eq. 1. In order to do this, the thickness b must be expressed as a function of r . For a linear thickness-radius relationship,

$$(5) \quad b = m(c - r) ,$$

¹For a similar approach see H. D. Conway, *J. Appl. Mechanics* 15, 1 (1948); 18, 140-42 (1951).

²S. Timoshenko, *Theory of Plates and Shells*, 1st ed., chap. III, McGraw-Hill, New York, 1940. (In this report the positive directions for the bending moments are changed.)

and the plate rigidity D becomes the following function of r :

$$D = \frac{Em^3 (c-r)^3}{12(1-\gamma^2)} = D_1 (c-r)^3$$

Performing the above designated substitutions yields

$$(6) \quad D_1 (c-r)^2 \left[(c-r) r \frac{d^2 \phi}{dr^2} + (c-4r) \frac{d\phi}{dr} - \left(\frac{c-r}{r} + 3\gamma \right) \phi \right] = k + \frac{pr^2}{2}.$$

It is convenient to introduce a dimensionless coordinate x such that

$$(7) \quad r = c(1-x).$$

Then Eq. 6 transforms to

$$(8) \quad x(1-x)^2 \frac{d^2 \phi}{dx^2} + (3-4x)(1-x) \frac{d\phi}{dx} - [x + 3\gamma(1-x)] \phi = \frac{1}{D_1 c^2} \left[\frac{k(1-x)}{x^2} + \frac{\rho(1-x)^3}{x^2} \right],$$

where $\rho = c^2 p/2$.

When Poisson's ratio, γ , is taken equal to $1/3$, Eq. 8 may be written in the following form and then integrated directly:

$$(9) \quad (1-x)^2 \frac{d}{dx} \left\{ \left[\frac{1}{x(1-x)} \right] \frac{d[x^2(1-x)] \phi}{dx} \right\} = \frac{1}{D_1 c^2} \left[\frac{k(1-x)}{x^2} + \frac{\rho(1-x)^3}{x^2} \right].$$

Integrating Eq. 9 yields

$$(10) \quad \phi = \frac{1}{D_1 c^2} (C_1 W'_1 + C_2 W'_2 + k W'_k + \rho W'_\rho),$$

where C_1 and C_2 are constants of integration, and

$$(11) \quad \begin{aligned} W'_1 &= \frac{1}{6(1-x)} + \frac{1}{3}, \\ W'_2 &= \frac{1}{x^2(1-x)}, \\ W'_k &= \left[\frac{1}{6(1-x)} + \frac{1}{3} \right] \ln x + \frac{1}{6} \left(\frac{1}{x^2} + \frac{1}{x} - 2 \right) \ln(1-x) + \frac{1}{2x^2} - \frac{1}{3x}, \\ W'_\rho &= - \left[\frac{1}{6(1-x)} + \frac{1}{3} \right] \ln x + \frac{1-x}{2x^2} + \frac{5}{36(1-x)} + \frac{1}{9}. \end{aligned}$$

Since $\phi = dw/dr$, and $D_1 = 3Em^3/32$ (with $\gamma = 1/3$), Eq. 10 is actually formula 5 of part I. Numerical values for each of the four W' functions are given in Table 3.

In order that the remaining formulas of the bending analysis may be derived, Eqs. 3 and 4 are transformed to the coordinate x , and the expression for ϕ from Eq. 10 is substituted. In the derivations that follow, the subscript j is used in writing one equation to represent four ($j = 1, 2, k$, and ρ).

Equations 3 and 4, respectively, transform to

$$M_r = D_1 c^2 x^3 \left[\frac{-d\phi}{dx} + \frac{\phi}{3(1-x)} \right],$$

$$M_\theta = D_1 c^2 x^3 \left(\frac{\phi}{1-x} - \frac{d\phi}{3dx} \right).$$

Formulas 1 and 2,

$$M_r = x^2 (C_1 M_{r1} + C_2 M_{r2} + k M_{rk} + \rho M_{r\rho}),$$

$$M_\theta = x^2 (C_1 M_{\theta1} + C_2 M_{\theta2} + k M_{\theta k} + \rho M_{\theta\rho}),$$

are derived by substituting the expression for ϕ from Eq. 10 and defining

$$M_{rj} = x \left[\frac{-dW'_j}{dx} + \frac{W'_j}{3(1-x)} \right],$$

$$M_{\theta j} = x \left(\frac{W'_j}{1-x} - \frac{dW'_j}{3dx} \right).$$

Formulas 3 and 4 are derived by substituting formulas 1 and 2 in the following equations:

$$\sigma_{Br} = \frac{\pm 6M_r}{b^2} = \frac{\pm 6M_r}{(mcx)^2},$$

$$\sigma_{B\theta} = \frac{\pm 6M_\theta}{b^2} = \frac{\pm 6M_\theta}{(mcx)^2}.$$

Numerical values for each of the four M_{rj} and $M_{\theta j}$ functions are given in Tables 1 and 2, respectively.

The formula for the displacement w (formula 6) is obtained by integrating Eq. 10 with the aid of the coordinate transformation relationship:

$$w = \int \phi dr = -c \int \phi dx.$$

Defining

$$W_j = \int W'_j dx,$$

substituting from Eqs. 11 for W'_j , and integrating yields

$$W_1 = \frac{-\ln(1-x)}{6} + \frac{x}{3},$$

$$W_2 = \frac{-1}{x} + \ln \frac{x}{1-x},$$

$$W_k = \left[\frac{x}{3} - \frac{1}{2} - \frac{\ln(1-x)}{6} \right] \ln x + \left(\frac{1}{2} - \frac{x}{3} - \frac{1}{6x} \right) \ln(1-x) + \frac{1}{3} \int \frac{\ln(1-x)}{x} dx - \frac{1}{2x},$$

$$W_\rho = \left[\frac{-x}{3} - \frac{1}{2} + \frac{\ln(1-x)}{6} \right] \ln x - \frac{5}{36} \ln(1-x) - \frac{1}{6} \int \frac{\ln(1-x)}{x} dx + \frac{4x}{9} - \frac{1}{2x}.$$

The integral,

$$\int \frac{\ln(1-x)}{x} dx,$$

in the above expressions is evaluated by expanding $[\ln(1-x)]/x$ into a series and integrating the series termwise. This yields

$$\int \frac{\ln(1-x)}{x} dx = - \sum_{n=1, 2, 3}^{\infty} \frac{x^n}{n^2}$$

Numerical values for each of the four W_j functions are given in Table 4.

MEMBRANE ANALYSIS

The differential equation for the membrane analysis originates from the following set of equations:³

$$(12) \quad \frac{dN_r}{dr} + \frac{N_r - N_\theta}{r} = 0,$$

$$(13) \quad N_r = \frac{Eb}{1-\gamma^2} \left(\frac{du}{dr} + \frac{\gamma u}{r} \right),$$

$$(14) \quad N_\theta = \frac{Eb}{1-\gamma^2} \left(\frac{u}{r} + \frac{\gamma du}{dr} \right),$$

which holds true regardless of the thickness-radius relationship. Transforming to the coordinate x and substituting mcx for b yields

$$(12a) \quad \frac{-dN_r}{dx} + \frac{N_r - N_\theta}{1-x} = 0,$$

$$(13a) \quad N_r = \frac{Emx}{1-\gamma^2} \left(\frac{-du}{dx} + \frac{\gamma u}{1-x} \right),$$

³S. Timoshenko and J. N. Goodier, *Theory of Elasticity*, 2d ed., chap. 4, secs 26 and 28, McGraw-Hill, New York, 1951.

$$(14a) \quad N_{\theta} = \frac{Emx}{1-\gamma^2} \left(\frac{u}{1-x} - \frac{\gamma du}{dx} \right)$$

Substituting Eqs. 13a and 14a in Eq. 12a yields the following basic differential equation for the membrane analysis:

$$(15) \quad x(1-x)^2 \frac{d^2 u}{dx^2} + (1-2x)(1-x) \frac{du}{dx} - [\gamma(1-x) + x] u = 0$$

Equation 15 is solved more readily by substituting $\beta/(1-x)$ for u and then solving

$$(15a) \quad x(1-x) \frac{d^2 \beta}{dx^2} + \frac{d\beta}{dx} + (1-\gamma) \beta = 0$$

for β by a power series. Accordingly, the solution of Eq. 15 is obtained in the form

$$(16) \quad u = \frac{c}{E} (A_1 U_1 + A_2 U_2),$$

where A_1 and A_2 are constants of integration:

$$U_1 = \frac{1}{1-x} \left(1 + \sum_{n=1, 2, 3}^{\infty} a_n x^n \right),$$

$$U_2 = U_1 \ln x + \frac{1}{1-x} \sum_{n=1, 2, 3}^{\infty} b_n x^n,$$

$$a_n = \frac{\prod_{1, 2, 3}^n [(n-1)(n-2) - (1-\gamma)]}{(n!)^2},$$

$$b_n = a_n \left[-2S(n) + \sum_{1, 2, 3}^n \frac{2n-3}{(n-1)(n-2) - (1-\gamma)} \right],$$

$$S(n) = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}.$$

Equation 16 is formula 11. The constant factor c/E is introduced in Eq. 16 merely to simplify the remaining formulas in the membrane analysis. The formulas for N_r and N_{θ} (formulas 7 and 8) are derived by substituting Eq. 16 in Eqs. 13a and 14a, respectively. The first derivatives of U_1 and U_2 are

$$\frac{dU_1}{dx} = \frac{1}{(1-x)^2} \left(\gamma + \sum_{n=1, 2, 3}^{\infty} a'_n x^n \right),$$

$$\frac{dU_2}{dx} = \frac{dU_1}{dx} \ln x + \frac{1}{(1-x)^2} \left[\frac{1}{x} - (1+\gamma) + \sum_{n=1, 2, 3}^{\infty} b'_n x^n \right],$$

where

$$a'_n = a_n \left(\frac{1+\gamma}{n+1} - 1 \right),$$

$$b'_n = b_n \left(\frac{1+\gamma}{n+1} - 1 \right) - a_n \left[\frac{1+\gamma}{(n+1)^2} \right]:$$

Formula 7 is derived by substituting

$$\frac{du}{dx} = \frac{c}{E} \left(A_1 \frac{dU_1}{dx} + A_2 \frac{dU_2}{dx} \right)$$

and Eq. 16 in Eq. 13a and defining

$$N_{rj} = \frac{1}{1-\gamma^2} \left(\frac{-dU_j}{dx} + \frac{\gamma U_j}{1-x} \right) \quad (j = 1, 2).$$

Formula 8 is derived similarly by defining

$$N_{\theta j} = \frac{1}{1-\gamma^2} \left(\frac{U_j}{1-x} - \frac{\gamma dU_j}{dx} \right) \quad (j = 1, 2).$$

Numerical values for the two N_{rj} and the two $N_{\theta j}$ functions, with $\gamma = \frac{1}{3}$, are given in Table 5. Numerical values for the two U_j functions are given in Table 6. Formulas 9 and 10 are derived very readily, since

$$\sigma_{mr} = \frac{N_r}{b} = \frac{N_r}{mcx},$$

$$\sigma_{m\theta} = \frac{N_\theta}{b} = \frac{N_\theta}{mcx}.$$

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