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LECTURES ON DUAL RESONANCE MODELS - P. H. Frampton

PART FIVE: SPIN*

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INTRODUCTION

Here we study the introduction of extra spin degrees of freedom into the model. First we discuss a multiplicative approach to spin, analogous to that used successfully for isospin, and show that it leads to both parity doubling and to spin ghosts.

An approach to half-integer spin of applying a correspondence principle to the Dirac equation leads to a dual theory of free fermions with sufficient gauges to eliminate spin ghosts, and involving both commuting and anticommuting harmonic-oscillator-like operators.

Using similar sets of oscillators, a multipion amplitude is constructed with an enlarged gauge algebra and with different families of trajectories for even and odd G-parity. The gauges are adequate to remove all negative probability states. Although the trajectories have unphysical intercepts (a unit intercept rho trajectory persists) the spectrum of low-lying states is qualitatively similar to that observed for strangeness-zero mesons.

The pion model and the fermion model can be unified to give a consistent set of Born amplitudes with external fermion legs; this shows that there is no essential difficulty to introducing half-integer spins into dual theories.

Consideration of simple bilinear realisations of the generalised projective algebra reveals that only two such realisations are possible, corresponding to generalised projective spin $S=0$ and commuting operators, and $S=-1/2$ and anticommuting operators. Combining these additively

gives a group-theoretical derivation of the dual pion model within the operator formalism.

Finally, a non-planar extension of the dual pion model, analogous to the Shapiro-Virasoro model, is discussed.

5a. Multiplicative Spin Factor

In view of the success in dealing with an exact internal symmetry, namely isospin, in a simple multiplicative fashion, it is natural to attempt to introduce half-integer spin through a similar method. We shall therefore begin our discussion of spin by describing such a direct approach, as proposed by Mandelstam¹ and by Bardakci and Halpern², since the difficulties which arise here will strongly motivate the later more successful developments.

We write the amplitude for the planar quark diagram of Fig.1 as

$$A_{2n} = \Gamma_{2n} B_{2n} \quad (5.1)$$

where

$$\Gamma_{2n} = \bar{u}(2n)u(1)\bar{u}(2)u(3)\dots\bar{u}(2n-2)u(2n-1) \quad (5.2)$$

where the $u(i)$ are Dirac spinors describing the external spin 1/2 quarks. Next we make a Fierz transformation

$$u(i)\bar{u}(j) = \frac{1}{4} (S_{ji}1 + P_{ji}\gamma^5 + V_{ji}^\mu \gamma^\mu - A_{ji}^\mu \gamma^5 \gamma_\mu + \frac{1}{2} \Sigma_{ji}^{\mu\nu} \sigma_{\mu\nu}) \quad (5.3)$$

in which

$$S_{ji} = \bar{u}(j)u(i) \quad (5.4)$$

$$P_{ji} = \bar{u}(j)\gamma^5 u(i) \quad (5.5)$$

$$V_{ji}^\mu = \bar{u}(j)\gamma^\mu u(i) \quad (5.6)$$

$$A_{ji}^\mu = \bar{u}(j)\gamma^5 \gamma^\mu u(i) \quad (5.7)$$

$$\Sigma_{ji}^{\mu\nu} = \bar{u}(j)\sigma_{\mu\nu} u(i) \quad (5.8)$$

with

$$\sigma_{\mu\nu} = \frac{i}{2} [\gamma_\mu, \gamma_\nu] = -\sigma_{\nu\mu} \quad (5.9)$$

These quantities are 4x4 Dirac matrices and we may now re-write, inserting the spinor indices

$$\Gamma_{2n} = (\bar{u}(2n))_a (u(1)\bar{u}(2))_{ab} (u(3)\bar{u}(4))_{bc} \dots (u(2n-3)\bar{u}(2n-1))_{yz} u(2n-1)_z \quad (5.10)$$

$$= (u(1)\bar{u}(2))_{ab} (u(3)\bar{u}(4))_{bc} \dots (u(2n-1)\bar{u}(2n))_{za} \quad (5.11)$$

$$= \frac{1}{4} \text{Tr} [u(1)\bar{u}(2), u(3)\bar{u}(4), \dots, u(2n-1)\bar{u}(2n)] \quad (5.12)$$

$$= \frac{1}{4} \text{Tr} [S_{(1)}^1 + P_{(1)}\gamma_5 + V_{(1)}^\mu \gamma_\mu - A_{(1)}^\mu \gamma_5 \gamma_\mu + \frac{1}{2} \Sigma_{(1)}^{\mu\nu} \sigma_{\mu\nu}, \dots,]$$

$$S_{(n)}^1 + P_{(n)}\gamma_5 + V_{(n)}^\mu \gamma_\mu - A_{(n)}^\mu \gamma_5 \gamma_\mu + \frac{1}{2} \Sigma_{(n)}^{\mu\nu} \sigma_{\mu\nu}] \quad (5.13)$$

where we have adopted the notation

$$S_{(i)} = \bar{u}(2i) u(2i-1) \quad (5.14)$$

$$P_{(i)} = \bar{u}(2i) \gamma_5 u(2i-1) \quad (5.15)$$

$$V_{(i)}^\mu = \bar{u}(2i) \gamma^\mu u(2i-1) \quad (5.16)$$

$$A_{(i)}^\mu = \bar{u}(2i) \gamma^5 \gamma^\mu u(2i-1) \quad (5.17)$$

$$\Sigma_{(i)}^{\mu\nu} = \bar{u}(2i) \sigma_{\mu\nu} u(2i-1) \quad (5.18)$$

to identify with the n hadronic $q\bar{q}$ channels of the diagram. [Notice that the strictly analogous treatment of isospin, starting with $2n$

two-component spinors, would lead to the usual isospin trace factor.]

This spin factor, written as a trace, is manifestly cyclic symmetric. To discuss the factorisation on an interanl hadron state we must recognize that for any two Dirac matrices A and B there is the completeness relation

$$\begin{aligned} \text{Tr}(AB) = & \text{Tr}(A) \text{Tr}(B) + \text{Tr}(A\gamma_5) \text{Tr}(\gamma_5 B) + \text{Tr}(A\gamma_\mu) \text{Tr}(\gamma_\mu B) \\ & - \text{Tr}(A\gamma_5\gamma_\mu) \text{Tr}(\gamma_5\gamma_\mu B) + 1/2 \text{Tr}(A\sigma_{\mu\nu}) \text{Tr}(\sigma_{\mu\nu} B) \end{aligned} \quad (5.19)$$

Taking the momenta of the two subsystems as $q_A^\mu = -q_B^\mu$ we now decompose these terms into pure spin states.

Defining

$$\gamma_A^\mu = \gamma^\mu - \frac{\not{q}_A q_A^\mu}{(q_A^2)} \quad (5.20)$$

$$\gamma_B^\mu = \gamma^\mu - \frac{\not{q}_B q_B^\mu}{(q_B^2)} \quad (5.21)$$

and using

$$\sigma_{\mu\nu} = \frac{1}{q^2} (q_\mu \sigma_{\lambda\nu} q_\lambda - \sigma_{\mu\lambda} q_\lambda q_\nu + i \epsilon_{\mu\nu\lambda\kappa} q_\lambda \gamma_5 \sigma_{\kappa\rho} q_\rho) \quad (5.22)$$

for $q_\mu = q_A^\mu$ or q_B^μ we find eventually that (using repeatedly

$$q_A^\mu = -q_B^\mu)$$

$$\begin{aligned}
\text{Tr}(AB) = & \text{Tr}(A) \text{Tr}(B) + \text{Tr}(A\gamma_5) \text{Tr}(\gamma_5 B) \\
& + \text{Tr}(A\gamma_A^\mu) \text{Tr}(\gamma_B^\mu) - \text{Tr}\left(\frac{A \not{q}_A}{\sqrt{q_A^2}}\right) \text{Tr}\left(\frac{\not{q}_B^B}{\sqrt{q_B^2}}\right) \\
& - \text{Tr}\left(\frac{A\gamma_5 \gamma_\mu^A}{\sqrt{q_A^2}}\right) \text{Tr}\left(\frac{\gamma_5 \gamma_\mu^B}{\sqrt{q_B^2}}\right) \\
& + \text{Tr}\left(\frac{A\gamma_5 \not{q}_A}{\sqrt{q_A^2}}\right) \text{Tr}\left(\frac{\gamma_5 \not{q}_B^B}{\sqrt{q_B^2}}\right) \\
& - \text{Tr}\left(\frac{A\sigma_{\mu\nu} q_A^2}{\sqrt{q_A^2}}\right) \text{Tr}\left(\frac{\sigma^{\mu\nu} q_\lambda^B}{\sqrt{q_B^2}}\right) \\
& - \text{Tr}\left(\frac{A\gamma_5 \sigma_{\mu\nu} q_A^2}{\sqrt{q_A^2}}\right) \text{Tr}\left(\frac{\gamma_5 \sigma^{\mu\lambda} q_\lambda^B}{\sqrt{q_B^2}}\right)
\end{aligned} \tag{5.23}$$

To arrive at this result we have used

$$\text{Tr}(A\gamma^\mu) \text{Tr}(\gamma^\mu B) = \text{Tr}\left(A\gamma_A^\mu + \frac{A \not{q}_A q_A^\mu}{q_A^2}\right) \text{Tr}\left(\gamma_B^\mu + \frac{\not{q}_B q_B^\mu}{q_B^2}\right) \tag{5.24}$$

$$= \text{Tr}(A\gamma_A^\mu) \text{Tr}(\gamma_B^\mu) - \text{Tr}\left(\frac{A \not{q}_A}{\sqrt{q_A^2}}\right) \text{Tr}\left(\frac{\not{q}_B^B}{\sqrt{q_B^2}}\right) \tag{5.25}$$

since $q_A^\mu \gamma_B^\mu = 0$, $q_A^\mu = -q_B^\mu$. Similarly we treat the axial vector piece and finally for the tensor piece we use

$$\frac{1}{2} \text{Tr}(A\sigma_{\mu\nu}) \text{Tr}(\sigma_{\mu\nu} B) = \frac{1}{2q_A^2 q_B^2} \left[\text{Tr}(A q_\mu^A \sigma_{\lambda\nu} q_\lambda^A) \text{Tr}(q_\mu^B \sigma_{\kappa\nu} q_\kappa^B) + \right.$$

(cont'd)

$$\begin{aligned}
 & + \text{Tr}(A q_\nu^A \sigma_{\lambda\mu} q_\mu^A) \text{Tr}(q_\nu^B \sigma_{\kappa\mu} q_\mu^B B) \\
 & + 2(g_{\lambda\lambda}, g_{\kappa\kappa}, -g_{\lambda\kappa}, g_{\kappa\lambda},) \text{Tr}(A q_\lambda^A \gamma_5 \sigma_{\kappa\rho} q_\rho^A) \text{Tr}(q_\lambda^B \gamma_5 \sigma_{\kappa\rho} q_\rho^B B) \Bigg| \quad (5.26)
 \end{aligned}$$

$$\begin{aligned}
 & = -\text{Tr}\left(\frac{A \sigma_{\mu\nu} q_A^2}{\sqrt{q_A^2}}\right) \text{Tr}\left(\frac{\sigma_{\mu\lambda} q_B^\lambda}{\sqrt{q_B^2}}\right) \\
 & - \text{Tr}\left(\frac{A \gamma_5 \sigma_{\mu\nu} q_A^2}{\sqrt{q_A^2}}\right) \text{Tr}\left(\frac{\gamma_5 \sigma_{\mu\lambda} q_B^\lambda}{\sqrt{q_B^2}}\right) \quad (5.27)
 \end{aligned}$$

since

$$\epsilon_{\mu\nu\kappa\lambda} \epsilon_{\mu\nu\kappa'\lambda'} = -2(g_{\kappa\kappa'} g_{\lambda\lambda'} + g_{\kappa\lambda'} g_{\lambda\kappa'}). \quad (5.28)$$

This shows clearly that eight independent spin states are present corresponding to two each of $J^P = 0^+, 0^-, 1^+, 1^-$. Superimposing this result on the known spectrum of the B_{2n} function, we see that all levels exhibit parity-doubling.

Next we study the hermiticity of the couplings, by comparison with the usual Feynman rules.³ We are concerned with the quantities Γ satisfying

$$\gamma_0 \Gamma^\dagger \gamma_0 = \Gamma \quad (5.29)$$

Since for these quantities one knows that

$$\bar{u}(q_2) \Gamma u(q_1) = [\bar{u}(q_1) \Gamma u(q_2)]^\dagger \quad (5.30)$$

as required for hermiticity. With the usual definitions and conventions

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (5.31)$$

$$\underline{\gamma} = \begin{pmatrix} 0 & \underline{\sigma} \\ -\underline{\sigma} & 0 \end{pmatrix} \quad (5.32)$$

$$\gamma^5 = i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (5.33)$$

then the following are hermitian couplings

$$\Gamma = 1, i\gamma^5, \gamma^\mu, i\gamma^5\gamma^\mu, \sigma_{\mu\nu}, i\gamma^5\sigma_{\mu\nu} \quad (5.34)$$

Taking into account the factors $+i$ and $-ig^{\mu\nu}$ associated with the spin zero and spin one propagators, we find the appropriate vertex factors to be

$$J^P = 0^+ \quad -i ; \quad \not{q}/\sqrt{q^2} \quad (5.35)$$

$$J^P = 0^- \quad \gamma_5 ; \quad \gamma_5 \not{q}/\sqrt{q^2} \quad (5.36)$$

$$J^P = 1^- \quad -i\gamma_\mu ; \quad \sigma_{\mu\nu} q^\nu/\sqrt{q^2} \quad (5.37)$$

$$J^P = 1^+ \quad -i\gamma_5\gamma_\mu ; \quad -i\gamma_5\sigma_{\mu\nu} q^\nu/\sqrt{q^2} \quad (5.38)$$

For the second set of couplings, we have noted that $q_\mu \rightarrow -q_\mu$ under the hermitian conjugation.

Comparison of these correct couplings with those occurring in the completeness relation reveals that precisely half the states, namely the scalars and axial vectors, are ghosts with non-hermitian coupling. These are called spin ghosts and in the later developments we shall overcome this problem (although not the other problem of parity-doubling).

Why do we obtain ghosts with a spin multiplicative factor and not with the SU_2 or SU_3 internal symmetry factors? The reason is that the spin group associated with leaving invariant the Lorentz scalar

$$u^+ \gamma^0 u \quad \gamma^0 = \begin{pmatrix} +1 & & & \\ & +1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \quad (5.39)$$

is a non-compact group $u(2,2)$ and any finite-dimensional representation must be non-unitary. The groups SU_2 and SU_3 are compact and finite dimensional UIR are possible.

From another viewpoint, the negative signs in γ^0 correspond to the negative energy solutions of the Dirac equation, and are connected to the well-known re-interpretation in terms of positive-energy holes in the Dirac sea.

We can of course, combine the two types of multiplicative factor; combining the spin factor with SU_2 , SU_3 gives rise to the larger invariances contained in

$$U(4,4) \supset u(2,2) \times SU_2 \quad (5.40)$$

and

$$U(6,6) \supset u(2,2) \times SU_3 \quad (5.41)$$

respectively. [c.f. Salam, Delbourgo and Strathee, Ref.4; reprinted in Ref.5]. Half of the states are spin-ghosts which are theoretically unacceptable. The parity-doubling, although not objectional on purely theoretical grounds, is physically unacceptable since it is not observed.

Concerning this parity-doubling problem there is a more general discussion that can be made as follows. Let us multiply a dual amplitude B_N by a factor dependent on the external momenta

$$A_N = F_N(p_i) B_N \quad (5.42)$$

Consider the case of $N=6$ pseudoscalars and the factorisation at the internal three-particle ground-state pseudoscalar. We know that B_6 factorises correctly and hence we must have

$$F_6 = P(s_{12}, s_{23}) P(s_{45}, s_{56}) + \alpha_{123} \tilde{F}_6^{(123)} \quad (5.43)$$

$$= P(s_{23}, s_{34}) P(s_{56}, s_{61}) + \alpha_{234} \tilde{F}_6^{(234)} \quad (5.44)$$

where $S_{ij} = (p_i + p_j)^2$, $\alpha_{ijk} = \alpha((p_i + p_j + p_k)^2)$. Now put $\alpha_{123} = \alpha_{234} = 0$ and realise that all six energies $s_{i,i+1}$ ($i=1,2,3,4,5,6$) are independent variables to see that only a trivial solution $P=\text{constant}$ survives. Hence we cannot modify B_N by a non-trivial multiplicative factor and keep only a pseudoscalar in the 3π channel. By extending the argument (Reference 6) it can be shown that both 1^+ and 1^- must be present; and hence parity-doubling is inevitable.

The general conclusion of this subsection is that any simple multiplicative spin factor with the generalised Euler B function is unsuccessful and certainly cannot lead to a physically-acceptable Born amplitude. A less trivial introduction of spin becomes essential and we shall devote the rest of our general discussion

of spin to such an attempt, the Ramond-Neveu-Schwarz (RNS) theory, which has high internal consistency. It will succeed in eliminating spin ghosts although not parity doubling; it will not describe the real world but it does provide a very stimulating development.

5b. Free Fermions (Ramond)

To introduce a dual theory for free fermions we first present a different viewpoint⁷ on the bosonic case. We introduce a hamiltonian

$$H = \sum_{n=0}^{\infty} \frac{1}{2} (p_{\mu}^{(n)} p_{\mu}^{(n)} + n^2 q^{(n)} \cdot q^{(n)}) \quad (5.45)$$

at each point x_{μ} , p_{μ} of Lorentz phase space, with the commutators

$$[q_{\mu}^{(m)}, p_{\nu}^{(n)}] = -ig_{\mu\nu} \delta_{mn} \quad (5.46)$$

$$[q_{\mu}^{(m)}, q_{\nu}^{(n)}] = [p_{\mu}^{(m)}, p_{\nu}^{(n)}] = 0 \quad (5.47)$$

Defining generalised momentum and position operators through

$$P_{\mu} = \sqrt{2} \sum_{n=0}^{\infty} p_{\mu}^{(n)} \quad (5.48)$$

$$Q_{\mu} = \frac{1}{\sqrt{2}} \sum_{n=0}^{\infty} q_{\mu}^{(n)} \quad (5.49)$$

we introduce a τ -variable through the Heisenberg equations

$$P_{\mu}(\tau) = e^{i\tau H} P_{\mu} e^{-i\tau H} \quad (5.50)$$

$$Q_{\mu}(\tau) = e^{i\tau H} Q_{\mu} e^{-i\tau H} \quad (5.51)$$

Defining

$$a_{\mu}^{(n)+} = \frac{1}{\sqrt{2}} [\sqrt{n} q_{\mu}^{(n)} - \frac{i}{\sqrt{n}} p_{\mu}^{(n)}] \quad (5.52)$$

$$a_{\mu}^{(n)} = \frac{1}{\sqrt{2}} [\sqrt{n} q_{\mu}^{(n)} + \frac{i}{\sqrt{n}} p_{\mu}^{(n)}] \quad (5.53)$$

$$p_{\mu}^{(n)} = i \sqrt{\frac{n}{2}} (a_{\mu}^{(n)+} - a_{\mu}^{(n)}) \quad (5.54)$$

$$q_{\mu}^{(n)} = \frac{1}{\sqrt{2n}} (a_{\mu}^{(n)+} + a_{\mu}^{(n)}) \quad (5.55)$$

$$[a_{\mu}^{(n)}, a_{\nu}^{(m)+}] = -g_{\mu\nu} \delta_{mn} \quad (5.56)$$

then

$$p^{(n)} \cdot p^{(n)} + n^2 q^{(n)} \cdot q^{(n)} = 2na^{(n)+} a^{(n)-} + 4n \quad (5.57)$$

and

$$p_{\mu}(\tau) = \sqrt{2} p_{\mu}^{(0)} + i \sum_{n=1}^{\infty} \sqrt{n} e^{i\tau na^{(n)+} a^{(n)}} (a_{\mu}^{(n)+} - a_{\mu}^{(n)}) e^{-i\tau na^{(n)+} a^{(n)}} \quad (5.58)$$

$$= \sqrt{2} p_{\mu}^{(0)} - i \sum_{n=1}^{\infty} \sqrt{n} [a_{\mu}^{(n)} e^{-in\tau} - a_{\mu}^{(n)+} e^{in\tau}] \quad (5.59)$$

We may consider τ as a cyclic variable, and physical quantities are taken to be averages over $-\pi \leq \tau \leq +\pi$. With the notation

$$\langle A(\tau) \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\tau A(\tau) \quad (5.60)$$

we see that

$$\frac{1}{\sqrt{2}} \langle P_\mu(\tau) \rangle = p_\mu^{(0)} \quad (5.61)$$

$$\sqrt{2} \langle Q_\mu(\tau) \rangle = q_\mu^{(0)} \quad (5.62)$$

For products of operators, we adopt a prescription that a product of averages is replaced by an average over the product, suitably normal ordered. Thus

$$p_\mu p_\mu = \frac{1}{2} \langle P_\mu(\tau) \rangle \langle P_\mu(\tau) \rangle \quad (5.63)$$

$$\rightarrow \frac{1}{2} \langle : P^2(\tau) : \rangle \quad (5.64)$$

With such a correspondence principle the Klein-Gordon equation becomes

$$\left(\frac{1}{2} \langle : P^2(\tau) : \rangle - m^2 \right) |\phi\rangle = 0 \quad (5.65)$$

or

$$\left(p^2 + \sum_{n=1}^{\infty} n a^{(n)+} \cdot a^{(n)} - m^2 \right) |\phi\rangle = 0 \quad (5.66)$$

that is

$$(L_0 + m^2) |\phi\rangle = 0 \quad (5.67)$$

which is the familiar equation for the spectrum of dual models.

We can go further to identify the gauge conditions. To eliminate ghosts we need subsidiary conditions of the form

$$p_\mu a_\mu^{(n)} |\phi\rangle = 0 \quad (5.68)$$

that is

$$\langle P_\mu(\tau) \rangle \langle e^{i n \tau} P_\mu(\tau) \rangle | \phi \rangle = 0 \quad (5.69)$$

Through the correspondence principle this becomes

$$\langle e^{i n \tau} : P(\tau)^2 : \rangle | \phi \rangle = 0 \quad (5.70)$$

and we know from earlier discussions that this is equivalent to the Virasoro gauges.

$$\begin{aligned} [i \sqrt{2n} p \cdot a^{(n)} - \sum_{m=1}^{\infty} \sqrt{m(m+n)} a^{(m)} \cdot a^{(m+n)} \\ + \frac{1}{2} \sum_{m=1}^{n-1} \sqrt{m(n-m)} a^{(m)} \cdot a^{(n-m)}] | \phi \rangle = 0 \end{aligned} \quad (5.71)$$

These results suggest that we should attempt a similar treatment of the Dirac equation⁸. To do so we need a generalisation $\Gamma_\mu(\tau)$ of the Dirac gamma matrix γ_μ . We require that $\Gamma_\mu(\tau)$ satisfy the following three requirements.

$$i) \quad \langle \Gamma_\mu(\tau) \rangle = \gamma_\mu \quad (5.72)$$

$$ii) \quad \gamma_0 \Gamma_\mu(z)^+ \gamma_0 = \Gamma_\mu(z) \quad (5.73)$$

$$iii) \quad \{ \Gamma_\mu(\tau), \Gamma_\nu(\tau') \}_+ = 2 g_{\mu\nu} \delta\left(\frac{1}{2\pi} (\tau - \tau')\right) \quad (5.74)$$

These properties will generalise the well-known

$$\gamma_0 \gamma_\mu^+ \gamma_0 = \gamma_\mu \quad (5.75)$$

$$\{ \gamma_\mu, \gamma_\nu \}_+ = 2 g_{\mu\nu} \quad (5.76)$$

Concerning the argument in the delta function of $\{\Gamma_\mu(\tau), \Gamma_\nu(\tau')\}_+$ we may motivate it by noticing that

$$[Q_\mu(\tau), P_\nu(\tau')] = -i g_{\mu\nu} \delta\left(\frac{1}{2\pi} (\tau - \tau')\right) \quad (5.77)$$

To satisfy the three requirements we find

$$\Gamma_\mu(\tau) = \gamma_\mu + i\sqrt{2} \gamma_5 \sum_{n=1}^{\infty} (b_\mu^{(n)+} e^{in\tau} + b_\mu^{(n)} e^{-in\tau}) \quad (5.78)$$

where

$$\{b_\mu^{(m)}, b_\nu^{(n)+}\}_+ = -g_{\mu\nu} \delta_{mn} \quad (5.79)$$

Property (i) is immediately satisfied; property (ii) follows from

$$\gamma^0 \gamma_\mu^+ \gamma^0 = \gamma_\mu \quad (5.80)$$

$$\gamma^0 \gamma_5^+ \gamma^0 = \gamma_5 \quad (5.81)$$

and the anticommutator (iii) is

$$\begin{aligned} \{\Gamma_\mu(\tau), \Gamma_\nu(\tau')\}_+ &= \{\gamma_\mu, \gamma_\nu\}_+ - 2 \sum_{n=1}^{\infty} [\{\gamma_5 b_\mu^{(n)+}, \gamma_5 b_\nu^{(n)}\}_+ e^{in(\tau-\tau')} \\ &\quad + \{\gamma_5 b_\mu^{(n)}, \gamma_5 b_\nu^{(n)+}\}_+ e^{-in(\tau-\tau')}] \end{aligned} \quad (5.82)$$

$$= 2g_{\mu\nu} \sum_{n=-\infty}^{\infty} e^{in(\tau-\tau')} \quad (5.83)$$

$$= 2g_{\mu\nu} \delta\left(\frac{1}{2\pi} (\tau - \tau')\right) \quad (5.84)$$

as required.

We are now ready to apply the correspondence principle to the Dirac equation through the steps

$$(\gamma \cdot p - m) |\phi\rangle = 0 \quad (5.85)$$

$$\left(\frac{1}{\sqrt{2}} \langle \Gamma_\mu(\tau) \rangle \langle P_\mu(\tau) \rangle - m \right) |\phi\rangle = 0 \quad (5.86)$$

$$\rightarrow \left(\frac{1}{\sqrt{2}} \langle \Gamma_\mu(\tau) \cdot P_\mu(\tau) \rangle - m \right) |\phi\rangle = 0 \quad (5.87)$$

Hence

$$[\gamma \cdot p - m - \gamma_5 \sum_{n=1}^{\infty} \sqrt{n} (a^{(n)+} \cdot b^{(n)} - a^{(n)} \cdot b^{(n)+})] |\phi\rangle = 0 \quad (5.87)$$

By squaring the dual fermion equation, we can arrive eventually at a form exhibiting linear Regge trajectories, as follows.

Start from

$$\left(\frac{1}{2} \langle \Gamma \cdot P \rangle \langle \Gamma \cdot P \rangle - m^2 \right) |\phi\rangle = 0 \quad (5.89)$$

Now since

$$\begin{aligned} \frac{1}{2} \langle \Gamma \cdot P \rangle \langle \Gamma \cdot P \rangle &= \frac{1}{2} \langle \langle (-\Gamma_\nu(\tau') \Gamma_\mu(\tau) + 2g_{\mu\nu} \delta(\frac{1}{2\pi}(\tau - \tau'))) \\ &\quad (P_\nu(\tau') P_\mu(\tau) - ig_{\mu\nu} \frac{d}{d\tau} \delta(\frac{1}{2\pi}(\tau - \tau'))) \rangle \rangle \end{aligned} \quad (5.90)$$

$$= -\frac{1}{2} \langle \Gamma \cdot P \rangle \langle \Gamma \cdot P \rangle + \langle P^2 \rangle - \frac{i}{2} \langle \Gamma \cdot \dot{\Gamma} \rangle \quad (5.91)$$

with

$$\dot{\Gamma}(\tau) = \frac{d}{d\tau} \Gamma(\tau) \quad (5.92)$$

$$= -\sqrt{2} \gamma_5 \sum_{n=1}^{\infty} n (b_\mu^{(n)+} e^{in\tau} - b_\mu^{(n)} e^{-in\tau}) \quad (5.93)$$

and since

$$-\frac{1}{4} i \langle \Gamma \cdot \dot{\Gamma} \rangle = \sum_{n=1}^{\infty} n b_{\mu}^{(n)+} b_{\mu}^{(n)} \quad (5.94)$$

we arrive at

$$\left(\frac{1}{2} \langle \Gamma \cdot P \rangle \langle \Gamma \cdot P \rangle - m^2 \right) |\phi\rangle = \left(\frac{1}{2} \langle P^2 \rangle - \frac{1}{4} i \langle \Gamma \cdot \dot{\Gamma} \rangle - m^2 \right) |\phi\rangle \quad (5.95)$$

$$= [p^2 - m^2 + \sum_{n=1}^{\infty} n (a^{(n)+} \cdot a^{(n)} + b^{(n)+} \cdot b^{(n)})] |\phi\rangle = 0 \quad (5.96)$$

from which the general nature of the spectrum is manifest: namely, linear trajectories.

For consistency we require the term $-\frac{i}{4} \langle \Gamma \cdot \dot{\Gamma} \rangle$ to generate τ translations also. By using

$$[AB, C]_- = A\{B, C\}_+ - \{A, C\}_+ B \quad (5.97)$$

we can check that

$$\begin{aligned} \left[-\frac{1}{4} i \langle \Gamma \cdot \dot{\Gamma} \rangle, \Gamma_{\mu}(\tau) \right] &= -\frac{1}{4} i \langle \Gamma_{\mu}(\tau') \{ \dot{\Gamma}_{\nu}(\tau'), \Gamma_{\mu}(\tau) \}_+ \\ &\quad - \{ \Gamma_{\mu}(\tau'), \Gamma_{\mu}(\tau) \}_+ \dot{\Gamma}_{\nu}(\tau') \rangle \end{aligned} \quad (5.98)$$

$$\begin{aligned} &= -\frac{1}{4} i \langle \Gamma_{\mu}(\tau) 2 \frac{d}{d\tau} \delta\left(\frac{1}{2\pi} (\tau - \tau')\right) \\ &\quad - 2 \dot{\Gamma}_{\mu}(\tau) \delta\left(\frac{1}{2\pi} (\tau - \tau')\right) \rangle \end{aligned} \quad (5.99)$$

$$= i \frac{d}{d\tau} \Gamma_{\mu}(\tau) \quad (5.100)$$

as expected.

Putting $Z = e^{-i\tau}$, we introduce the operators

$$L_n^\Gamma = + \frac{1}{4} i \langle Z^{-n} : \Gamma(Z) \cdot \dot{\Gamma}(Z) : \rangle \quad (5.101)$$

$$\begin{aligned} &= - \frac{1}{2} \sum_{m=1}^{\infty} (n+2m) b^{(m)+} \cdot b^{(m+n)} - \frac{1}{4} \sum_{m=1}^{n-1} (n-2m) b^{(m)+} \cdot b^{(n-m)} \\ &\quad + \frac{\sqrt{2}}{4} i n \gamma \cdot b^{(n)} \gamma^5 \end{aligned} \quad (5.102)$$

Aside from possible anomaly terms at $m+n=0$ we can find by careful use of

$$[AB, CD] = A\{B, C\}D - \{A, C\} + BD + CA\{B, D\}_+ - C\{A, D\}_+ B \quad (5.103)$$

that these operators satisfy the generalised projective algebra

$$[L_m^\Gamma, L_n^\Gamma] = (m-n) L_{m+n}^\Gamma (+ \text{anomaly term}) \quad (5.104)$$

The anomaly for $n=-m$ arises from two terms: firstly the quadratic in $b^{(n)}$ piece gives rise to

$$\frac{d}{16} \sum_{m=1}^{m-1} 2(m-2n)^2 = \frac{d}{24} (m^3 - 3m^2 + 2m) \quad (5.105)$$

Secondly there is the term

$$\left[\frac{\sqrt{2} i m \gamma \cdot b^{(m)} \gamma^5}{4}, \frac{\sqrt{2} i m \gamma \cdot b^{(m)+} \gamma^5}{4} \right] \quad (5.106)$$

which contains $(m^2 d/8)$. This gives altogether

$$[L_m^\Gamma, L_n^\Gamma] = (m-n) L_{m+n}^\Gamma + \frac{d}{24} m(m^2+2) \delta_{m+n,0} \quad (5.107)$$

suggesting that we re-define

$$L_0^{\Gamma'} = L_0^\Gamma + d/16, \quad L_n^{\Gamma'} = L_n^\Gamma \quad (n \neq 0) \quad (5.108)$$

whereupon

$$[L_m^{\Gamma'}, L_n^{\Gamma'}] = (m-n) L_{m+n}^{\Gamma'} + \frac{d}{24} m(m^2-1) \delta_{m+n,0} \quad (5.109)$$

Combining these operators into

$$L_n^f = L_n^{\Gamma'} + L_n^a \quad (5.110)$$

$$= -\frac{1}{2} \langle Z^{-n} : (P^2 - \frac{i}{2} \Gamma \cdot \dot{\Gamma}) : \rangle + \frac{d}{16} \delta_{n0} \quad (5.111)$$

we arrive at

$$[L_m^f, L_n^f] = (m-n) L_{m+n}^f + \frac{d}{8} m(m^2-1) \delta_{m+n,0} \quad (5.112)$$

by bearing in mind the anomaly $\frac{d}{12} m(m^2-1) \delta_{m+n,0}$ of the L_n^a algebra.

The gauge condition

$$L_n^f |\phi\rangle = 0 \quad (5.113)$$

is compatible with the spectrum equation

$$(L_0^f + m^2 - \frac{d}{16}) |\phi\rangle = 0 \quad (5.114)$$

The crucial point is that we may introduce further gauges, to eliminate spin ghosts, in the form

$$F_n = \frac{1}{\sqrt{2}} \langle Z^{-n}; \Gamma(z)_\mu P_\mu(z) : \rangle \quad (5.115)$$

$$= \gamma \sum_{m=1}^{\infty} \sqrt{m} (a_m \cdot b_{m-n}^+ - a_m^+ \cdot b_{m+n}) \\ + i\sqrt{2} p_\mu \gamma_5 b_\mu^{(n)} - i \sqrt{n/2} \gamma \cdot a^{(n)} \quad (5.116)$$

which satisfy

$$[L_m^f, F_n] = (\frac{m}{2} - n) F_{m+n} \quad (5.117)$$

$$\{F_m, F_n\}_+ = -2 L_{m+n}^f - \frac{d}{2} (m - \frac{1}{4}) \delta_{m+n,0} \quad (5.118)$$

To arrive at the $[L_m^f, F_n]$ commutator we use

$$[P^2(z), P_\mu(z')] = 4\pi i P_\mu(z) \frac{d}{d\tau} \delta(\tau - \tau') \quad (5.119)$$

$$[\Gamma(z) \cdot \dot{\Gamma}(z), \Gamma_\mu(z')] = [4\pi \Gamma_\mu(z) \frac{d}{d\tau} \delta(\tau - \tau') - 4\pi \dot{\Gamma}_\mu(z) \delta(\tau - \tau')] \quad (5.120)$$

To arrive at the anticommutator $\{F_m, F_n\}_+$ we first use

$$\{P(z) \cdot \Gamma(z), P(z') \cdot \Gamma(z')\}_+ = 2\pi i \Gamma(z) \cdot \Gamma(z) \delta'(\tau - \tau') \\ + 4\pi P(z) \cdot P(z) \delta(\tau - \tau') \quad (5.121)$$

and then to obtain the anomaly term, we find two pieces, namely

$$-d \sum_{p=1}^{m-1} p = -\frac{d}{2} (m^2 - m) \quad (5.122)$$

$$\text{and} \quad -\frac{m}{2} \gamma_\mu \gamma_\mu = -\frac{dm}{2} \quad (5.123)$$

coming from the quadratic term and from the $\gamma.a^{(n)}$ term respectively. The sum gives $(-\frac{d}{2} m^2) \delta_{m+n,0}$ and hence the required result when we use $L_0^{\Gamma'} = (L_0^f + d/16)$.

To summarise the gauge algebra, it is (for the Ramond theory)

$$[L_m^f, L_n^f] = (m-n) L_{m+n}^f + \frac{d}{8} m(m^2-1) \delta_{m+n,0} \quad (5.124)$$

$$[L_m^f, F_n] = (\frac{m}{2}-n) F_{m+n} \quad (5.125)$$

$$\{F_m, F_n\} = -2L_{m+n}^f - \frac{d}{2} (m^2 - \frac{1}{4}) \delta_{m+n,0} \quad (5.126)$$

The F_n gauges now provide further subsidiary conditions since we know that

$$0 = F_n (F_0 - m) |\phi\rangle \quad (5.127)$$

$$= [2L_n^f - (F_0 + m) F_n] |\phi\rangle \quad (5.128)$$

and therefore

$$F_n(\phi) = 0 \quad (5.129)$$

It is amusing that these are the extension, through the correspondence principle, of the well-known Rarita-Schwinger subsidiary condition in the forms

$$p.b^{(n)} |\phi\rangle = 0 \quad (5.130)$$

$$\gamma.a^{(n)} |\phi\rangle = 0 \quad (5.131)$$

We shall discuss the spectrum later, but some general features are

- i) The parent trajectory is doubly degenerate since we

may take either of the forms, for the parent states

$$(a_{\mu}^{(1)+})^J |0\rangle \quad (5.132)$$

$$(a_{\mu}^{(1)+})^{J-1} (b_{\nu}^{(1)+}) |0\rangle \quad (5.133)$$

ii) All levels but the groundstate are parity-doubled.

5C Dual Pion Model (Neveu and Schwarz)

We now introduce a bosonic multiparticle amplitude, due to Neveu and Schwarz⁹, that involves anticommuting operators similar, but not identical, to those introduced in the previous section. For the moment we regard this development as completely separate from that for fermions, but at a later stage we shall couple the two together in the full RNS theory of bosons and fermions.

We introduce operators satisfying*

$$\{b_{\mu}^{(r)}, b_{\nu}^{(s)+}\}_+ = -g_{\mu\nu} \delta_{rs} \quad (5.134)$$

where $r, s = 1/2, 3/2, 5/2, \dots$ are now half-odd integers. Also we introduce a field

$$H_{\mu}(\tau) = \sum_{r=-\infty}^{\infty} b_{\mu}^{(r)} e^{-ir\tau} \quad (5.135)$$

*We adopt consistently the convention throughout that indices m, n on $b_{\mu}^{(m)+}, b_{\mu}^{(n)+}$, have integer values appropriate to the fermion sector, whereas indices r, s on $b_{\mu}^{(r)+}, b_{\mu}^{(s)+}$ have half-odd-integer values appropriate to the boson sector.

whereupon

$$\{H_\mu(\tau), H_\nu(\tau')\}_+ = -2\pi g_{\mu\nu} \delta(\tau-\tau') \quad (5.136)$$

Now we define

$$L_n^b = -\frac{1}{2} i \langle e^{in\tau} : H(\tau) \cdot \dot{H}(\tau) : \rangle \quad (5.137)$$

where

$$\dot{H}_\mu(\tau) = \frac{d}{d\tau} H_\mu(\tau) \quad (5.138)$$

It follows that

$$L_n^b = -\frac{1}{2} \sum_{r=1/2}^{\infty} (n+2r) b_r^+ \cdot b_{n+r} - \frac{1}{4} \sum_{r=1/2}^{n-1/2} (n-2r) b_r \cdot b_{n-r} \quad (5.139)$$

and these operators satisfy the algebra

$$[L_m^b, L_n^b] = (m-n) L_{m+n}^b + \frac{d}{24} m(m^2-1) \delta_{m+n,0} \quad (5.140)$$

The anomaly term arises from the piece

$$\frac{1}{16} \sum_{r=1/2}^{m-1/2} \sum_{s=1/2}^{n-1/2} (m-2r)(m-2s) [b_r \cdot b_{m-r}, b_{n-s}^+ \cdot b_s^+] \quad (5.141)$$

and using

$$[AB, CD] = A \{BC\}_+ D - \{AC\}_+ BD + CA \{BD\}_+ - C \{AD\}_+ B \quad (5.142)$$

we find, after normal ordering the extra term.

$$\frac{d}{8} \sum_{r=1/2}^{m-1/2} (m-2r)^2 = \frac{d}{24} m(m^2-1) \quad (5.143)$$

as required.

Adding*

$$L_n = L_n^a + L_n^b \quad (5.144)$$

then gives

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{d}{8} m(m^2-1) \delta_{m+n,0} \quad (5.145)$$

In terms of the vertex $V_0(p,z)$ defined for the generalised Veneziano model through

$$V_0(p,z) = : \exp(\sqrt{2} ip \cdot Q(z)) : \quad (5.146)$$

we define the Neveu-Schwarz vertex

$$V(p,z) = p \cdot H(z) V_0(p,z) \quad (5.147)$$

Now we need the commutator

$$[L_n, H_\mu(z)] = -\frac{1}{2} i \langle z'^{-n} [H_\mu(z') \dot{H}_\mu(z'), H_\mu(z)] \rangle \quad (5.148)$$

$$= -\frac{1}{2} i \langle z'^{-n} \{ H_\mu(z') \cdot -2\pi \frac{d}{d\tau}, \delta(\tau-\tau') \} - \dot{H}_\mu(z') \cdot 2\pi \delta(\tau-\tau') \rangle \quad (5.149)$$

$$= -\frac{1}{2} n z^{-n} H_\mu(z) - i z^{-n} \frac{d}{d\tau} H(z) \quad (5.150)$$

*For the full gauge operator (commuting part plus anticommuting part) we write consistently L_n with no superscript for the boson sector, to distinguish from L_n^f for the fermion sector. For the anticommuting part alone we use L_n^b for bosons, L_n^f for fermions; the commuting part L_n^a is the same for both sectors.

Now

$$z = e^{-i\tau} \quad (5.151)$$

$$z \frac{d}{dz} = i \frac{d}{d\tau} \quad (5.152)$$

$$\therefore [L_n, H_\mu(z)] = -z^{-n} \left(z \frac{d}{dz} - \frac{n}{2} \right) H_\mu(z) \quad (5.153)$$

Combining this result with

$$[L_n, V_0(p, z)] = -z^{-n} \left(z \frac{d}{dz} + np^2 \right) V_0(p, z) \quad (5.154)$$

we find

$$[L_n, V(p, z)] = -z^{-n} \left(z \frac{d}{dz} + n(p^2 - \frac{1}{2}) \right) V(p, z) \quad (5.155)$$

so that $V(p, z)$ is a generalised projective vector ($S=-1$) provided that

$$p^2 = -\frac{1}{2} \quad (5.156)$$

This ensures that the commutator is a perfect differential

$$[L_n, V(p, z)] = -z \frac{d}{dz} (z^{-n} V(p, z)) \quad (5.157)$$

as necessary for the generalised projective gauge conditions.

We are now ready to write the amplitude for N pions which is

$$A_N = \int \prod_{i=1}^N d_{zi} (dV_3)^{-1} \prod_{i=1}^N z_i^{-1/2} \langle 0 | V(p_1, z_1) V(p_2, z_2) \dots V(p_N, z_N) | 0 \rangle \quad (5.158)$$

First we may confirm projective invariance under

$$z_i \rightarrow z'_i = \left(\frac{az_i + b}{cz_i + d} \right) = \Lambda z \quad (5.159)$$

$$\Lambda = e^{i \underline{\xi} \cdot \underline{L}} \quad (5.160)$$

since we find

$$\Lambda \left[\frac{V(p, z)}{\sqrt{z}} \right] \Lambda^{-1} = (a - cz')^2 \left[\frac{V(p, z')}{\sqrt{z'}} \right] \quad (5.161)$$

similarly to the earlier result for $V_0(p, z)$, since in general a field on z with generalised projective spin S picks up a factor $(a - cz')^{-2S}$ under such a transformation.

This observation, together with the fact that

$$dz_i = \frac{dz'_i}{(a - cz'_i)^2} \quad (5.162)$$

enables us to confirm projective invariance.

Cyclic symmetry is proved by a straight forward extension of the earlier case; now we use both

$$V_0(p, z) V_0(p', z') = V_0(p', z') V_0(p, z) (-1)^{-2p \cdot p'} \quad (5.163)$$

and

$$p.H(z) p'.H(z') = -p'.H(z') p.H(z) - 2\pi p \cdot p' \delta(\tau - \tau') \quad (5.164)$$

The delta function term in the second equation gives zero contribution to the integral, and the new phase $(-1)^{N-1} = -1$ from $H_\mu(z)$ anticommutation is compensated by the factor

$$(-1)^{-2p_1 \cdot \sum_{i=1}^N p_i} = (-1)^{2p_1^2} = -1 \quad (5.165)$$

arising in the $V_0(p, z)$ commutation.

To see the factorisation properties explicitly we use

$$z^{-L_0} V(p, 1) z^{L_0} = V(p, z) \sqrt{z} \quad (5.166)$$

$$\lim_{z_1 \rightarrow 0} [\langle 0 | V(p_1, z_1)] = \langle 0 | e^{ip_1 \cdot Q} p_1 \cdot b_{1/2} \quad (5.167)$$

$$\lim_{z_N \rightarrow \infty} [V(p_N, z_N) | 0 \rangle] = p_N \cdot b_{1/2}^+ e^{ip_N \cdot Q} | 0 \rangle \quad (5.168)$$

and make the change of variables

$$U_{1j} = \frac{z_j}{z_{j+1}} \quad J = 2, 3, \dots, N-2 \quad (5.169)$$

precisely as in our earlier discussion, to arrive eventually at

$$A_N = \langle 0, p_1 | p_1 \cdot b_{1/2} V(p_2, 1) \frac{1}{L_0 - 1} V(p_3, 1) \dots \dots \frac{1}{L_0 - 1} V(p_{N-1}, 1) p_N \cdot b_{1/2}^+ | 0, p_N \rangle \quad (5.170)$$

Where we have taken a limit $z_1 \rightarrow 0$, $z_{N-1} \rightarrow 1$, $z_N \rightarrow \infty$.

The generalised projective gauges follow from

$$(L_0 - L_n - 1) \frac{1}{L_0 - 1} = \frac{1}{L_0 + n - 1} (L_0 + n - 1 - L_n) \quad (5.171)$$

$$(L_0 + n - 1 - L_n) V(p, 1) = V(p, 1) (L_0 - L_n - 1) \quad (5.172)$$

together with

$$(L_0 - L_n - 1) p.b^{(1/2)+} |0\rangle = 0 \quad (5.173)$$

implying

$$L_n |\phi\rangle = 0 \quad (5.174)$$

Where $|\phi\rangle$ is a physical state coupling to pions

$$|\phi\rangle = \frac{1}{L_0 - 1} V(p_N, 1) \frac{1}{L_0 - 1} V(p_{N-1}, 1) \dots V(p_2, 1) p_1.b^{(1/2)+} |0\rangle \quad (5.175)$$

As a preliminary remark to the study of the spectrum, note that since $V(p, z)$ is linear in $b_\mu^{(n)}$ it follows that A_N vanishes for N odd, and that we may define G-parity by

$$G = (-1)^{\sum b^{(r)+}.b^{(r)}} \quad (5.176)$$

so that $G = -1$ for the pion state

$$|\pi\rangle = p.b^{(1/2)+} |0\rangle \quad (5.177)$$

The twisting operator for the model requires a phase dependent on the G-parity as follows

$$\Omega = (-1)^{R_a + R_b} e^{-L_-} (-1)^{-1/4(1-G)} \quad (5.178)$$

where

$$R_a + R_b = - \sum_{n=1}^{\infty} n a^{(n)+} a^{(n)} - \sum_{r=1/2}^{\infty} n b^{(r)+}.b^{(r)} \quad (5.179)$$

Since the eigenvalue of Ω acting on a physical state is the charge conjugation C we have

$$C = (-1)^{M^2+1-1/4(1-G)} \quad (5.180)$$

When we use a Chan-Paton multiplicative factor for isospin, we deduce that the isospins of the levels in the model are

$$G = +1 \quad M^2 = 0, 2, 4, 6, \dots \quad T = 1 \quad (5.181)$$

$$M^2 = 1, 3, 5, 7, \dots \quad T = 0 \quad (5.182)$$

$$G = -1 \quad M^2 = 1/2, 5/2, 9/2, \dots \quad T = 0 \quad (5.183)$$

$$M^2 = 3/2, 7/2, 11/2, \dots \quad T = 1 \quad (5.184)$$

A cursory first examination of the spectrum

$$(L_0 - 1) |\phi\rangle = \left(- \sum_{n=1}^{\infty} n a^{(n)\dagger} a^{(n)} - \sum_{r=1/2}^{\infty} n b^{(r)\dagger} \cdot b^{(r)} - p^2 - 1 \right) |\phi\rangle \quad (5.185)$$

$$= 0 \quad (5.186)$$

reveals the apparent presence of all states ($R_b=0$) of the unit-intercept Veneziano model plus additional states ($R_b \neq 0$) lying on trajectories integer and half-integer spaced relative to the orbital ($R_b=0$) ones. Note in particular that acting with $b_{\mu}^{(1/2)\dagger}$ on the orbital $\alpha(0)=1$ parent trajectory gives rise to an apparent ancestor $\alpha_A(0)=3/2$ trajectory. We shall see shortly that both this ancestor trajectory and the $\alpha=0$ tachyon on the $\alpha_p(0)=1$ the trajectory are spurious.

Before examination of the spectrum, we calculate the four-pion amplitude

$$A_4 = \langle 0 | p_1 \cdot b^{(1/2)} V(p_2, 1) \frac{1}{L_0 - 1} V(p_3, 1) p_4 \cdot b^{(1/2)+} | 0 \rangle \quad (5.187)$$

$$= \int_0^1 dx x^{-s-2} \langle_a 0 | V_0(p_2, 1) x^{R_a} V_0(p_3, 1) | 0_a \rangle.$$

$$\langle_a 0 | p_1 \cdot b^{(1/2)} p_2 \cdot H(1) x^{R_b} p_3 \cdot H(1) p_4 \cdot b^{(1/2)+} | 0_b \rangle \quad (5.188)$$

$$= \int_0^1 dx x^{-2-s} (1-x)^{-1-t}$$

$$[(p_1 \cdot p_2)(p_3 \cdot p_4) + (p_2 \cdot p_3)(p_1 \cdot p_4) \left(\frac{x}{1-x}\right) - (p_1 \cdot p_3)(p_2 \cdot p_4)x] \quad (5.189)$$

$$= \frac{1}{4} [\alpha_s^2 B(-\alpha_s, 1-\alpha_t) + \alpha_t^2 B(1-\alpha_s, -\alpha_t) - \alpha_u^2 B(1-\alpha_s, 1-\alpha_t)] \quad (5.190)$$

$$= -\frac{1}{4} \frac{\Gamma(1-\alpha_s)\Gamma(1-\alpha_t)}{(1-\alpha_s-\alpha_t)} \quad (5.191)$$

where in the final step we used $\alpha_s + \alpha_t + \alpha_u = +1$.

Thus the present model gives a fully-factorised multiparticle extension of the Lovelace-Shapiro four-pion amplitude,^{10,11} for the particular intercept values $\alpha_\rho(0)=1$, $\alpha_\pi(0) = 1/2$.

Going back to the integral representation for A_N we may write

$$A_N = \int \prod_{i=1}^N dz_i (dV_3)^{-1} \prod_{i \neq j} (z_i - z_j)^{-p_i \cdot p_j} \prod_{i=1}^N z_i^{-1/2} {}_b \langle 0 | p_1 \cdot H(z_1) p_2 \cdot H(z_2) \dots$$

$$\dots p_N \cdot H(z_N) | 0 \rangle_b \quad (5.192)$$

and, evaluating the b vacuum expectation value gives (12,13)

$$A_N = \int \prod_{i=1}^N dz_i (dV_3)^{-1} \prod_{i \neq j} (z_i - z_j)^{-p_i \cdot p_j} \sum_{\{q_1 \dots q_N\}} (-1)^P \frac{p_{q_1} \cdot p_{q_2}}{(z_{q_1} - z_{q_2})} \frac{p_{q_3} \cdot p_{q_4}}{(z_{q_3} - z_{q_4})} \dots$$

$$\dots \frac{p_{q_{N-1}} \cdot p_{q_N}}{(z_{q_{N-1}} - z_{q_N})} \quad (5.193)$$

where the sum is over all permutations of $\{q_1 q_2 \dots q_N\} = 1, 2, 3, \dots, N$ and P is the parity of the permutation.

Note that, just as for the unit intercept conventional model, the integrand is invariant under arbitrary permutations of the N argument pairs $\{p_i, z_i\}$ $i=1, 2, \dots, N$; again we are led to associate this symmetric group (S_N) invariance, for all N , with the existence of the L_n algebra.

A slightly different form, treating all z -differences as linearly independent, is

$$A_N = \int \prod_{i=1}^N dz_i (dV_3)^{-1}$$

$$\sum_{\{q_1 \dots q_N\}} (-1)^P \frac{\partial}{\partial (z_{q_1} - z_{q_2})} \frac{\partial}{\partial (z_{q_3} - z_{q_4})} \dots \frac{\partial}{\partial (z_{q_{N-1}} - z_{q_N})}$$

$$\left[\prod_{i \neq j} (z_i - z_j)^{-p_i \cdot p_j} \right] \quad (5.194)$$

5d Spectrum

To fully analyse the spectrum, and to prove the absence of ghosts, we need to introduce⁽¹⁴⁾ G gauges defined by

$$G_r = \langle z^{-r} : P(z).H(z) : \rangle \quad (5.195)$$

$$= \sqrt{2} p \cdot b^{(r)} - i \sum_{m=1}^{\infty} \sqrt{m} (a^{(m)} \cdot b^{(m-r)+} - a^{(m)+} \cdot b^{(m+r)}) \quad (5.196)$$

These operators satisfy

$$[L_m, G_r] = \left(\frac{m}{2} - r\right) G_{m+r} \quad (5.197)$$

$$\{G_r, G_s\}_+ = 2L_{r+s} + \frac{d}{2} (r^2 - 1/4) \delta_{r+s,0} \quad (5.198)$$

The proofs are so close to those for the fermion model that we need not elaborate; we remark only that the anomaly term in the anticommutator is from

$$- \sum_{p=1}^{r-1/2} \sum_{q=1}^{s-1/2} \sqrt{pq} \left\{ a^{(p)} \cdot b^{(r-p)}, a^{(q)+} \cdot b^{(s-q)+} \right\}_+ \quad (5.199)$$

which contains a piece

$$d \sum_{p=1}^{r-1/2} p = \frac{d}{2} (r^2 - 1/4) \quad (5.200)$$

Consider now the commutator

$$[G_r, V_0(p, 1)] = \langle z^{-r} H_\mu(z) [P_\mu(z), e^{\sqrt{2} i p \cdot Q(z)}] \rangle \quad (5.201)$$

$$= -\sqrt{2} p \cdot H(1) V_0(p, 1) \quad (5.202)$$

$$= -\sqrt{2} V(p, 1) \quad (5.203)$$

It follows that

$$-\sqrt{2} \{G_r, V(p, 1)\}_+ = \{G_r, [G_r, V_0(p, 1)]\}_+ \quad (5.204)$$

$$= [L_{2r}, V_0(p, 1)] \quad (5.205)$$

$$= (L_0 + r - 1/2) V_0(p, 1) - V_0(p, 1) (L_0 - 1/2) \quad (5.206)$$

where we have used

$$[L_{2r} - L_0, V_0(p, 1)] = r V_0(p, 1) \quad (5.207)$$

Now we observe that

$$G_{-\frac{1}{2}} |0\rangle = \sqrt{2} p \cdot b^{(\frac{1}{2})+} |0\rangle = \sqrt{2} |\pi\rangle \quad (5.208)$$

and hence we may write the N-pion amplitude

$$A_N = \langle 0 | G_{\frac{1}{2}} V(p_2, 1) \frac{1}{L_0 - 1} V(p_3, 1) \dots \frac{1}{L_0 - 1} V(p_{N-1}, 1) G_{-\frac{1}{2}} |0\rangle \quad (5.209)$$

Now commute $G_{-1/2}$ to the left, making use of

$$\begin{aligned} V(p,1) G_{-\frac{1}{2}} &= - G_{-\frac{1}{2}} V(p,1) - \frac{1}{\sqrt{2}} (L_0 - 1) V_0(p,1) \\ &\quad + \frac{1}{\sqrt{2}} V_0(p,1) (L_0 - \frac{1}{2}) \end{aligned} \quad (5.210)$$

The second and third of these three terms give vanishing contribution. We need also

$$\frac{1}{L_0 - 1} G_{-\frac{1}{2}} = G_{-\frac{1}{2}} \frac{1}{L_0 - \frac{1}{2}} \quad (5.211)$$

which follows from

$$L_0 G_{-\frac{1}{2}} = G_{-\frac{1}{2}} (L_0 + \frac{1}{2}) \quad (5.212)$$

$$\therefore (L_0 - 1) G_{-\frac{1}{2}} = G_{-\frac{1}{2}} (L_0 - \frac{1}{2}) \quad (5.213)$$

Hence we easily arrive at

$$\begin{aligned} A_N &= \langle 0 | G_{\frac{1}{2}} G_{-\frac{1}{2}} V(p_1,1) \frac{1}{L_0 - \frac{1}{2}} V(p_1,1) \frac{1}{L_0 - \frac{1}{2}} \dots \\ &\quad \dots \frac{1}{L_0 - \frac{1}{2}} V(p_{N-1},1) | 0 \rangle \end{aligned} \quad (5.214)$$

Now use

$$\langle 0 | G_{\frac{1}{2}} G_{-\frac{1}{2}} = \langle 0 | \{ G_{\frac{1}{2}}, G_{-\frac{1}{2}} \}_+ \quad (5.215)$$

$$= \langle 0 | 2L_0 = \langle 0 | \quad (5.216)$$

to arrive at the final form

$$A_N = \langle 0 | V(p_2, 1) \frac{1}{L_0 - 1/2} V(p_3, 1) \frac{1}{L_0 - 1/2} \dots V(p_{N-1}, 0) | 0 \rangle \quad (5.217)$$

This enables us to abandon the original Fock space ($\mathcal{F}_1$) where the ground state is $|0\rangle$ in favor of a second Fock space ($\mathcal{F}_2$) built on the pion state $|\pi\rangle$. This greatly simplifies the analysis of the spectrum. Firstly, we see that the rho trajectory tachyon and the ancestor $\alpha_A(0)=3/2$ trajectory are outside of $\mathcal{F}_2$.

Next we should outline the no-ghost proof for this case.⁽¹⁵⁻¹⁷⁾ In $\mathcal{F}_2$ we have the mass-shell condition

$$(L_0 - \frac{1}{2}) |\phi\rangle = 0 \quad (5.218)$$

and gauges

$$G_r |\phi\rangle = 0 \quad (5.219)$$

$r = 1/2, 3/2, 5/2, \dots$. The G-gauge conditions imply the L-gauges through the gauge algebra; they follow from the consideration of

$$\langle \psi, L_0 = \frac{1}{2} - r | G_r V(p_1, 1) \frac{1}{L_0 - 1/2} V(p_2, 1) \dots V(p_{N-1}, 1) | 0 \rangle = 0 \quad (5.220)$$

The passage from the gauge conditions to a no-ghost theorem closely parallels that of the conventional model. Adopting again the notation $\mathcal{S}_{\ell_0}$ for the spurious subspace at the level $L_0 = \ell_0$ we write a general spurious state as

$$|s, n\rangle = G_{\frac{1}{2}}^\dagger |\phi, n-\frac{1}{2}\rangle + \tilde{G}_{\frac{3}{2}}^\dagger |\phi, n-\frac{3}{2}\rangle \in \mathcal{S}_{\frac{1}{2}} \quad (5.221)$$

with

$$\tilde{G}_{\frac{3}{2}} = \alpha L_1 G_{\frac{1}{2}} + \beta G_{3/2} \quad (5.222)$$

We can now try to show that

$$G_{\frac{1}{2}} |s, n\rangle \in \mathcal{S}_0 \quad (5.223)$$

$$G_{\frac{3}{2}} |s, n\rangle \in \mathcal{S}_{-1} \quad (5.224)$$

When this is done, it is possible to show that any on-shell state satisfying

$$G_{\frac{1}{2}} |\phi\rangle = G_{\frac{3}{2}} |\phi\rangle = 0 \quad (5.225)$$

(and thence all other gauge conditions) must be a positive-norm transverse state, modulo the addition of a possible null spurious state [we refer to the literature for the detailed proof, References 15,16,17]

Firstly we find that

$$\begin{aligned} G_{\frac{1}{2}} |s, n\rangle &= G_{\frac{1}{2}} \left[G_{\frac{1}{2}}^{\dagger} |\phi, n-\frac{1}{2}\rangle + \tilde{G}_{\frac{3}{2}}^{\dagger} |\phi, n-\frac{3}{2}\rangle \right] \\ &= - G_{\frac{1}{2}}^{\dagger} G_{\frac{1}{2}} |\phi, n-\frac{1}{2}\rangle \end{aligned} \quad (5.226)$$

$$+ \left\{ \alpha (L_1^{\dagger} L_1 + G_{\frac{1}{2}}^{\dagger} G_{\frac{1}{2}}) + \beta (-G_{\frac{3}{2}}^{\dagger} G_{\frac{1}{2}} + 2L_1^{\dagger}) \right\} |\phi, n-\frac{3}{2}\rangle \quad (5.227)$$

$$\in \mathcal{L}_0 \quad (5.228)$$

for any α, β , and space-time dimensionality d . Next we use

$$\begin{aligned} \{ \tilde{G}_{\frac{3}{2}}, G_{\frac{1}{2}}^{\dagger} \}_+ &= \alpha (-L_1 G_{\frac{1}{2}}^{\dagger} G_{\frac{1}{2}} + 2L_1 L_0 + G_{\frac{1}{2}}^{\dagger} L_1 G_{\frac{1}{2}}) \\ &\quad + 2\beta L_1 \end{aligned} \quad (5.229)$$

$$\begin{aligned} \{ \tilde{G}_{\frac{3}{2}}, \tilde{G}_{\frac{3}{2}}^{\dagger} \}_+ &= \alpha^2 [2L_1 L_0 L_1^{\dagger} - L_1 L_1^{\dagger} - G_{\frac{1}{2}}^{\dagger} L_1 G_{\frac{1}{2}} \\ &\quad - 2G_{\frac{1}{2}}^{\dagger} L_0 G_{\frac{1}{2}}] \\ &\quad + 4\alpha\beta (L_1 L_1^{\dagger} + G_{\frac{1}{2}}^{\dagger} G_{\frac{1}{2}}) \\ &\quad + \beta^2 (2L_0 + d) \end{aligned} \quad (5.230)$$

Acting on $|s, n\rangle$ we find

$$\tilde{G}_{\frac{3}{2}} (G_{\frac{1}{2}}^{\dagger} |\phi, n-\frac{1}{2}\rangle) = (-\alpha + 2\beta) L_1 |\phi, n-\frac{1}{2}\rangle \quad (5.231)$$

so that we must have $\alpha = 2\beta = 1$, say. Next we find

$$\tilde{G}_{\frac{3}{2}} (G_{\frac{3}{2}}^\dagger |\phi, n-3/2\rangle) = [8 G_{\frac{1}{2}}^\dagger G_{\frac{1}{2}} + (d-10)] |\phi, n-3/2\rangle \quad (5.232)$$

$$\in \mathcal{L}_-$$

(5.233)

provided that $d=10$, the new critical space-time dimensionality.

There are no ghosts in the spectrum, provided $d \leq d_c = 10$.

Having established such a theoretical consistency, we may look at the lowest lying levels in the spectrum. The positive norm physical states are indicated in the Chew-Frautschi plot in Fig.2. This should be compared with the spectrum of strangeness-zero mesons observed in Nature, Fig.3. We see that the states with normal parity couplings to pions i.e. with $P=G(-1)^S$, where $S=\text{spin}$, occur too far left by an amount $\Delta(M^2)=1/2 \alpha'$, that is

$$M^2 = M_{\text{expt}}^2 - 1/2 \quad (5.234)$$

On the other hand, for abnormal states with $P=-G(-1)^S$;

$$M^2 = M_{\text{expt}}^2 \quad (5.235)$$

The general agreement in number and quantum number of states between the model and Nature is striking.

Returning to the A_N integral representation, the $\mathcal{F}_2$ reformulation means that we can re-write

$$\begin{aligned}
 A_N &= \int \prod_{i=1}^N dz_i (dV_3)^{-1} \prod_{i \neq j} (z_i - z_j)^{-p_i \cdot p_j} \\
 &\quad \prod_{i=1}^N z_i^{-1/2} \frac{1}{(z_1 - z_N)} \langle 0 | p_2 \cdot H(z_2) p_3 \cdot H(z_3) \dots p_{N-1} \cdot H(z_{N-1}) | 0 \rangle \quad (5.236) \\
 &= \int \prod_{i=1}^N dz_i (dV_3)^{-1} \prod_{i \neq j} (z_i - z_j)^{-p_i \cdot p_j} \frac{1}{(z_1 - z_N)} \\
 &\quad \sum_{\{q_2, q_3, \dots, q_{N-1}\}} (-1)^P \frac{p_{q_2} \cdot p_{q_3} \quad p_{q_4} \cdot p_{q_5} \quad \dots \quad p_{q_{N-2}} \cdot p_{q_{N-1}}}{(z_{q_2} - z_{q_3}) (z_{q_4} - z_{q_5}) \dots (z_{q_{N-2}} - z_{q_{N-1}})} \quad (5.237)
 \end{aligned}$$

or, more usefully

$$\begin{aligned}
 A_N &= \int \prod_{i=1}^N dz_i (dV_3)^{-1} \frac{1}{(z_1 - z_N)} \\
 &\quad \sum_{\{q_2, \dots, q_{N-1}\}} (-1)^P \frac{\partial}{\partial (z_{q_2} - z_{q_3})} \cdot \frac{\partial}{\partial (z_{q_4} - z_{q_5})} \dots \frac{\partial}{\partial (z_{q_{N-2}} - z_{q_{N-1}})} \cdot \\
 &\quad \cdot \left[\prod_{i \neq j} (z_i - z_j)^{-p_i \cdot p_j} \right] \quad (5.238)
 \end{aligned}$$

In these two forms the sum is over all permutations of $\{q_2, q_3, \dots, q_{N-1}\} = 2, 3, \dots, (N-1)$.

Actually, because of the permutational invariance of the integrand (in its $\mathcal{H}_1$ form) it is clear that the choice of z_1, z_N as neighboring points is unnecessary and we may write finally

$$A_N = \int \prod_{i=1}^N dz_i (dv_3)^{-1} \frac{1}{(z_a - z_b)} \sum_{\{q_1, q_2, \dots, q_{N-2}\}} (-1)^p \frac{\partial}{\partial(z_{q_1} - z_{q_2})} \frac{\partial}{\partial(z_{q_3} - z_{q_4})} \dots \frac{\partial}{\partial(z_{q_{N-3}} - z_{q_{N-2}})} \cdot \left[\prod_{i \neq j} (z_i - z_j)^{-p_i \cdot p_j} \right] \quad (5.239)$$

where a, b are chosen arbitrarily and the sum is over $\{q_1, q_2, \dots, q_{N-2}\} = 1, 2, 3, \dots, N$ (excluding a, b). This re-writing is the reflection of the G-gauge algebra in the integral representation, and will be useful in the realistic amplitudes later.

So far we have examined the boson amplitude in isolation, although the similarity of its gauge algebra to that of the free fermion model is evident. Before discussing the fermion-boson couplings, we should mention that a no-ghost theorem holds for the free fermion spectrum satisfying

$$(F_0 - m) |\phi\rangle = 0$$

(5.240)

$$(L_0 + m^2) |\phi\rangle = 0$$

(5.241)

$$F_n |\phi\rangle = 0$$

(5.242)

$$L_n |\phi\rangle = 0$$

(5.243)

since it can be shown that the transverse states are complete for $d=10$, provided^{18,19,20} that $m=0$. The last condition implies that the leading fermion trajectory has intercept $\alpha_f(0)=1/2$.

Perhaps the simplest way²¹ to see the $m=0$ condition is to count states at the first excited level. In d space-time dimensions we may make

$$\left. \begin{array}{l} a_\mu^{(\dagger)} |0\rangle \\ b_\mu^{(\dagger)} |0\rangle \end{array} \right\} \quad 2d \text{ states} \quad (5.244)$$

Of these, two are spurious

$$F_1^+ |0\rangle \quad (5.245)$$

$$L_1^+ |0\rangle \quad (5.246)$$

For the transverse states to be complete the two spurious states should be null (i.e. should also be physical) so that their conjugate states also decouple leaving only $(d-4)$ positive-norm states.

Consider the combinations of the spurious states

$$|\psi_1\rangle = (F_0 + m) L_1^\dagger |0\rangle \quad (5.247)$$

$$|\psi_2\rangle = (F_0 + m) F_1^\dagger |0\rangle \quad (5.248)$$

so that

$$(L_0 + m^2) |\psi_i\rangle = 0 \quad (5.249)$$

$$(F_0 - m) |\psi_i\rangle = 0 \quad (5.250)$$

Now require annihilation by L_1 through

$$L_1 |\psi_2\rangle = L_1 (F_0 + m) F_1^\dagger |0\rangle = \left(\frac{5}{2} F_0^2 - \frac{d}{4} + \frac{3}{2} m F_0 \right) |0\rangle \quad (5.251)$$

$$= \left[\frac{5}{2} (m^2 + 1) - \frac{d}{4} + 3m F_0 \right] |0\rangle \quad (5.252)$$

$$= 0 \quad (5.253)$$

provided that $d=10$ and $m=0$. With these conditions satisfied one can confirm that

$$L_1 |\psi_1\rangle = F_1 |\psi_1\rangle = F_1 |\psi_2\rangle = 0 \quad (5.254)$$

and thus only $(2d-4)$ states remain, as required. Thus $m=0$ is necessary for the completeness of the transverse states.

5e Boson-Fermion Couplings

Consider the diagram, Fig.4(a), where a fermion line emits (N-1) ground-state mesons with momenta $p_1, p_2, \dots, p_{N-1}$. The fermion propagator is

$$\frac{1}{F_0} = - \frac{F_0}{L_0^f} \quad (5.255)$$

Now consider the identity²²

$$\begin{aligned} & \left[F_0 - F_n + \frac{2}{\sqrt{n}} (L_n^f - L_0^f) \right] \frac{1}{F_0} \\ &= \frac{1}{F_0 - \sqrt{n}} \left[F_n - F_0 + \frac{2}{\sqrt{n}} (L_n^f - L_0^f - \frac{n}{2}) \right] \end{aligned} \quad (5.256)$$

which will be useful to construct the emission vertex. This identity is checked by using

$$\begin{aligned} & (F_0 - \sqrt{n}) \left[F_0 - F_n + \frac{2}{\sqrt{n}} (L_n^f - L_0^f) \right] = \\ &= -L_0^f - F_0 F_n + \frac{2}{\sqrt{n}} (F_0 L_n^f - F_0 L_0^f) - F_0 \sqrt{n} + F_n \sqrt{n} - 2(L_n^f - L_0^f) \\ &= -F_0 F_n - 2L_n^f + L_0^f + \frac{2}{\sqrt{n}} (F_0 L_n^f + \frac{n}{2} F_n - F_0 L_0^f - F_0 \frac{n}{2}) \end{aligned} \quad (5.257)$$

$$= \left[F_n - F_0 + \frac{2}{\sqrt{n}} (L_n^f - L_0^f - \frac{n}{2}) F_0 \right] \quad (5.258)$$

$$(5.259)$$

as required

Now the conventional vertex has the properties

$$[F_0 - F_n, V_0(p, 1)] = 0 \quad (5.260)$$

$$[L_0^f - L_n^f, V_0(p, 1)] = n p^2 V_0(p, 1) \quad (5.261)$$

We need to multiply $V_0(p, 1)$ by an expression that anticommutes with $(F_0 - F_n)$. The simplest choice (more complicated choices correspond to emission of higher meson states) is²²

$$\Gamma_5 = \gamma_5 (-1)^{\sum b^{(n)+} b^{(n)}} \quad (5.262)$$

since

$$F_n = \frac{1}{\sqrt{2}} \langle z^{-n} : \Pi \cdot P : \rangle \quad (5.263)$$

$$\Gamma_\mu = \gamma_\mu + i\sqrt{2} \gamma_5 \sum_{n=1}^{\infty} (b^{(n)} z^n + b^{(n)+} z^{-n}) \quad (5.264)$$

$$\{\gamma_5, \gamma_\mu\}_+ = 0 \quad (5.265)$$

$$\left\{ (-1)^{\sum b^{(n)+} b^{(n)}}, b_\mu^{(n)+} \right\}_+ = 0 \quad (5.266)$$

and hence

$$\{\Gamma_5, \Gamma_\mu(z)\}_+ = 0 \quad (5.267)$$

$$\{\Gamma_5, F_n - F_0\}_+ = 0 \quad (5.268)$$

It follows now that

$$\begin{aligned}
 & \left[F_0 - F_n + \frac{2}{\sqrt{n}} (L_n^f - L_0^f) \right] \frac{1}{F_0} \Gamma_5 V_0(p,1) = \\
 & = \frac{1}{F_0 - \sqrt{n}} \left[F_n - F_0 + \frac{2}{\sqrt{n}} (L_n^f - L_0^f - \frac{n}{2}) \right] \Gamma_5 V_0(p,1)
 \end{aligned} \tag{5.269}$$

$$= \frac{1}{F_0 - \sqrt{n}} \Gamma_5 V_0(p,1) \left[F_0 - F_n + \frac{2}{\sqrt{n}} (L_n^f - L_0^f - \frac{n}{2} - np^2) \right] \tag{5.270}$$

so if $p^2 = -1/2$, as is appropriate to the pion state, we have gauge relations.

The amplitude for the diagram of Fig.4(a) is therefore

$$\begin{aligned}
 A = & \bar{u}(q_1) \langle 0 | \Gamma_5 V_0(p,1) \frac{1}{F_0} \Gamma_5 V_0(p,1) \dots \frac{1}{F_0} \Gamma_5 V_0(p_{N-1},1) | 0 \rangle u(q_2) \\
 & \tag{5.271}
 \end{aligned}$$

$$= \bar{u}(q_1) \langle 0 | \Gamma_5 V_0(p,1) \frac{F_0}{L_0^f} \Gamma_5 V_0(p,1) \dots \frac{F_0}{L_0^f} \Gamma_5 V_0(p_{N-1},1) | 0 \rangle u(q_2)$$

$$\begin{aligned}
 & \tag{5.272} \\
 = & (-1)^{\frac{N}{2}+1} \bar{u}(q_1) \gamma_5 \langle 0 | V_0(p,1) \frac{F_0}{L_0^f} V_0(p,1) \dots \frac{F_0}{L_0^f} V_0(p_{N-1},1) | 0 \rangle u(q_2) \\
 & \tag{5.273}
 \end{aligned}$$

where we have used

$$[\Gamma_5, L_0^f] = 0 \quad (5.274)$$

$$\{\Gamma_5, F_0\}_+ = 0 \quad (5.275)$$

$$\langle 0 | \Gamma_5 = \gamma_5 \langle 0 | \quad (5.276)$$

and have noted that the Γ_5 anticommutation gives $(N-2)/2$ sign changes.

Now since

$$[F_0, V_0(p, 1)] = \frac{1}{\sqrt{2}} \langle \Gamma_\mu(z) [P_\mu(z), e^{\sqrt{2}ip \cdot Q(1)}] \rangle \quad (5.277)$$

$$= -p \cdot \Gamma(1) V_0(p, 1) \quad (5.278)$$

we may take the F_0 's to the right, dropping terms with cancelled propagators to obtain

$$A = -\bar{u}(q_1) \gamma_5 \langle 0 | V_0(p_1, 1) \frac{1}{L_0^f} p_2 \cdot \Gamma(1) V_0(p_2, 1) \frac{1}{L_0^f} \dots \dots \frac{1}{L_0^f} p_{N-1} \cdot \Gamma(1) V_0(p_{N-1}, 1) | 0 \rangle u(q_2) \quad (5.279)$$

We may write now

$$\frac{1}{L_0^f} = \int_0^1 dz \quad z^{R_a + R_b - p^2 - 1} \quad (5.280)$$

whereupon the a modes give precisely the generalised conventional integrand. To deal with the b modes, we would like to make a cyclic change of variables to Fig.4(b). With the integration variables defined as implied by the figure 4, the appropriate transformation is

$$X_i = \left(\frac{1 - z_1 z_2 \dots z_i}{1 - z_1 z_2 \dots z_{i+1}} \right) ; 1 \leq i \leq (N-3) \quad (5.281)$$

$$y = 1 - \prod_{j=1}^{N-2} z_j \quad (5.282)$$

By going to the pion pole at $(q_1 - q_2)^2 = -1/2$ it is possible to show that

$$A \sim \frac{\bar{u}(q_1) \gamma_5 u(q_2)}{(q_1 - q_2)^2 \rightarrow -\frac{1}{2}} \frac{1}{(q_1 - q_2)^2 + \frac{1}{2}} A_N^{(NS)}(p_1, p_2, \dots, p_N) \quad (5.283)$$

with $A_N^{(NS)}$ the N-pion boson amplitude defined by

$$A_N^{(NS)} = \langle 0 | p_2 \cdot H(1) V_0(p_2, 1) \frac{1}{L_0 - 1/2} p_3 \cdot H(1) V_0(p_3, 1) \dots \frac{1}{L_0 - 1/2} p_{N-1} \cdot H(1) V_0(p_{N-1}, 1) | 0 \rangle \quad (5.284)$$

This shows that at the ground-state meson pole the RNS theory factorises correctly.

Making the cyclic change of variables carefully, in particular converting from the integer fermion modes to half-integer boson modes, it was shown (by Thorn, Reference 23) that

$$A = \langle 0 | p_2 \cdot H(1) V_0(p_2, 1) \frac{1}{L_0^{-1/2}} p_3 \cdot H(1) V_0(p_3, 1) \dots$$

$$\dots \frac{1}{L_0^{-1/2}} p_{N-1} \cdot H(1) V(p_{N-1}, 1) \frac{1}{L_0^{-1/2}} \bar{u}(q_1) V_0(q_1, 1) e^T | 0 \rangle \cdot$$

$$\cdot \gamma_5 u(q_2) \quad (5.285)$$

in which T is given by

$$T = -\frac{i}{\sqrt{2}} \gamma_5 \gamma_\mu \sum_{r=1/2}^{\infty} b^{(r)\dagger} \begin{pmatrix} -1/2 \\ r-1/2 \end{pmatrix} (-1)^{r-1/2}$$

$$+ \sum_{r,s=1/2}^{\infty} \frac{1}{4} (-1)^{r+s} C_{r-1/2, s-1/2} b^{(r)\dagger} b^{(s)\dagger}$$

(5.286)

with

$$C_{r-1/2, s-1/2} = \sum_{u=0}^{r-1/2} \begin{pmatrix} 1/2 \\ r+s-u \end{pmatrix} \begin{pmatrix} -1/2 \\ u \end{pmatrix} - \sum_{v=0}^{s-1/2} \begin{pmatrix} 1/2 \\ r+s-v \end{pmatrix} \begin{pmatrix} -1/2 \\ v \end{pmatrix}$$

(5.287)

$$= -C_{s-1/2, r-1/2}$$

(5.288)

Thus the vertex for emitting a general meson state $(|\lambda_a, \lambda_b\rangle$ from a ground-state fermion-antifermion pair is

$$\begin{aligned} & \langle \lambda_a | V_0(q_2, 1) | 0 \rangle_a \\ & \cdot \langle \lambda_b | \bar{u}(q_1) e^\tau \gamma_5 u(q_2) | 0 \rangle_b \end{aligned} \quad (5.289)$$

Note that, for the first time, the vertex contains an exponent quadratic in creation operators and this renders the further development within the operator formalism technically cumbersome (although possible) and a more powerful method is desirable.

To write a general tree involving several fermion lines (Fig.5) we need a vertex for ground-state fermion emission from excited meson-excited fermion; such a vertex has been constructed consistent with both the boson and the fermion gauge conditions (References 24, 25, 26)

A very interesting situation is that for four external ground-state fermions, Figs.5(a) and 5(b), which we expect to be related by duality plus a Fierz transformation. The appropriate propagator, avoiding spurious meson states is, according to Olive and Scherk²⁷, not $(L_0 - 1/2)^{-1}$ but

$$P(p^2) = \int_0^1 \frac{dx}{x} \frac{x^{L_0 - 1/2}}{\Delta(x)} \quad (5.290)$$

where

$$\Delta(x) = \det \parallel 1 - A(x)^2 \parallel \quad (5.291)$$

$$A(x)_{rs} = \frac{(-1)^{r+s+1} x^r}{2(r+s)} \begin{pmatrix} -3/2 \\ r-1/2 \end{pmatrix} \begin{pmatrix} -1/2 \\ s-1/2 \end{pmatrix} \quad (5.292)$$

with $r, s = 1/2, 3/2, 5/2, \dots$

Thus we would expect that (Fig.6)

$$A_4 = \langle 0 | \bar{u}(p_4) \gamma_5 e^{T^\dagger} u(q_1) V_0(-p_4, 1) \\ P(t) V_0(p_3, 1) \bar{u}(p_2) e^T \gamma_5 u(p_3) | 0 \rangle \quad (5.293)$$

$$= \langle 0 | \bar{u}(p_4) \gamma_5 e^{T^\dagger} u(q_3) V_0(-p_4, 1) \\ P(s) V_0(p_1, 1) \bar{u}(p_2) e^T \gamma_5 u(p_1) | 0 \rangle \quad (5.294)$$

Only after labyrinthine algebra, can this relationship be rigorously confirmed in the operator formalism²⁸; fortunately, however, a more powerful method is at hand. Using functional integration techniques (starting from the generalised rubber string of Kikkawa and Iwasaki, (29), Mandelstam (Refs.30-32) has shown that the GGNW amplitudes^{33*} for the process of Fig.6 are given as

$$\phi_1 = f_{++}^s = s B(-s-1/2, -t) \quad (5.295)$$

$$\phi_2 = f_{+-}^s = -\frac{1}{2}(s-t) (s+t+1)^{-1} B(-s-\frac{1}{2}, -t-\frac{1}{2}) \quad (5.296)$$

$$\phi_3 = f_{+-}^s = (s+t) B(-s, -t) \quad (5.297)$$

$$\phi_4 = f_{+-}^s = t B(-s, -t-1/2) \quad (5.298)$$

$$\phi_5 = f_{+-}^s = 0 \quad (5.299)$$

* A four-dimensional subspace of the full $d=10$ space-time has been selected by suitably restricting the excitations.

and that the correct crossing properties are maintained.

Concerning the RNS theory we should add the remarks:

- i) Although multifermion Born Amplitudes have not yet been studied carefully, it seems likely that the RNS theory is fully consistent for $d=10$ dimensions, leading meson intercepts $\alpha_\rho(0)=1$, $\alpha_\pi(0)=1/2$ and leading fermion intercept $\alpha_f(0)=1/2$. There are no ghosts, but all fermions except the ground state are parity-doubled. The main importance of the theory is that it shows that the introduction of half-integer spin presents no essential difficulty in dual theories.
- ii) Note that although the pion parity was undetermined in the NS model alone, the Γ_5 coupling of the RNS theory requires that it be pseudoscalar.
- iii) The meson spectrum has some qualitative similarity to Nature, but the fermion spectrum is more obscure. In view of this lack of realism, it is rather academic to discuss whether these fermions are more like nucleons or quarks. We should mention, however, that the fermions certainly have essentially different duality properties (i.e. topology of dual diagrams) from three-quark baryons since non-planar tree diagrams are absent.
- iv) At a purely technical level, for the computation of multifermion amplitudes the functional integration method appears to be much superior to the operator formalism approach.

5f Generalised Projective Algebra

For the UIR of $SU(1,1)$ we have previously derived, for the $J=-k$ representations, that

$$D_{mn}^{(J+)}(L_+) = \sqrt{n(n-1-2J)} \delta_{m,n-1} \quad (5.300)$$

$$D_{mn}^{(J+)}(L_-) = \sqrt{(n+1)(n-2J)} \delta_{m,n+1} \quad (5.301)$$

$$D_{mn}^{(J+)}(L_0) = (m-J) \delta_{mn} \quad (5.302)$$

Now, our gauge operators for the Neveu-Schwarz amplitude can be written

$$L_0^b = - \sum_{m=0}^{\infty} (m+\frac{1}{2}) b_{m+\frac{1}{2}}^+ \cdot b_{m+\frac{1}{2}} \quad (5.303)$$

$$L_1^b = - \sum_{m=0}^{\infty} (m+1) b_{m+\frac{1}{2}}^+ \cdot b_{m+\frac{1}{2}+1} \quad (5.304)$$

$$L_{-1}^b = - \sum_{m=0}^{\infty} (m+1) b_{m+\frac{1}{2}+1}^+ \cdot b_{m+\frac{1}{2}} \quad (5.305)$$

corresponding to $J=-1/2$ in the general formulae, as expected.

Consider now the multiregion vertex in the third-quantised form (discussed earlier)

$$A_N = \langle 0 | \langle 0_{j_1} \dots 0_{j_N} | : \prod_{i=1}^N e^{\sum_{k=0}^{\infty} Q_k^{(i)} b_{ik}} : | 0 \rangle_c \quad (5.306)$$

where the operators b_{ik} are associated with the leg i (k is the mode number) and satisfy anticommutation rules

$$\{b_{ik}^\mu, b_{jl}^{\nu\dagger}\}_+ = -g^{\mu\nu} \delta_{ij} \delta_{kl} \quad (5.307)$$

Let us consider the model³⁴ where this vertex is multiplied by the conventional (orbital) vertex. Taking the ground state to be

$$e^{ip \cdot Q} |p, b_0^\dagger|0\rangle \quad (5.308)$$

we find a new contribution from the anticommuting part which is

$$\langle 0 | \prod_{j=1}^M (-p_j \cdot Q_0^{(j)}) | 0 \rangle_c \quad (5.309)$$

the negative sign being from the $(-g^{\mu\nu})$ of the definition. Here

$$Q_{0\mu}^{(i)} = i \sum_{n=0}^{\infty} \left[c_{n\mu}^\dagger D_{n0}^{(-\frac{1}{2},+)}(y_j) + c_{n\mu} D_{n0}^{(-\frac{1}{2},+)}(\bar{\Gamma} y_j) \right] \quad (5.310)$$

Now recall that for the transformation

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (5.311)$$

with $ad-bc=1$ we have

$$D_{n0}^{(j+)}(A) = \frac{N_n^j}{N_0^j} \left(\frac{b}{d}\right)^n d^{2j} \quad (5.312)$$

and, since

$$\Gamma A = \begin{pmatrix} c & d \\ a & b \end{pmatrix} \quad (5.313)$$

we have

$$D_{n_0}^{(J+)} (\Gamma A) = \frac{N_n^J}{N_0^J} \left(\frac{d}{b} \right)^n b^{2J} \quad (5.314)$$

Substituting

$$b = \frac{z_i (z_{i-1} - z_{i+1})}{\sqrt{(z_{i-1} - z_{i+1}) (z_i - z_{i+1}) (z_i - z_{i-1})}} \quad (5.315)$$

$$d = \frac{(z_{i-1} - z_{i+1})}{\sqrt{(z_{i-1} - z_{i+1}) (z_i - z_{i+1}) (z_i - z_{i-1})}} \quad (5.316)$$

as appropriate to Y_i we find

$$Q_{op}^{(i)} = i \left[\frac{z_i (z_{i-1} - z_{i+1})}{(z_i - z_{i+1}) (z_i - z_{i-1})} \right]^J H_\mu^{(J)}(z_i) \quad (5.317)$$

with

$$H_\mu^{(J)} = \sum_{n=0}^{\infty} \frac{N_n^J}{N_0^J} (C_{n\mu}^+ z^{n-J} + C_{n\mu} z^{-n+J}) \quad (5.318)$$

$$= \sum_{n=0}^{\infty} \sqrt{\frac{\Gamma(n-2J)}{\Gamma(n+1) \Gamma(-2J)}} (C_{n\mu}^+ z^{n-J} + C_{n\mu} z^{-n+J}) \quad (5.319)$$

where we use

$$N_n^J = \sqrt{\frac{\Gamma(n-2J)}{n!}} \quad (5.320)$$

In particular, for $J=-1/2$ we find

$$H_\mu(z) = \sum_{r=1/2}^{\infty} (b_{r\mu} z^r + b_{r\mu}^\dagger z^{-r}) \quad (5.321)$$

in which we define $b_{r\mu}^\dagger = c_{r-1/2,\mu}$; this is precisely the Neveu-Schwarz field.

The contribution from the b oscillators to the N point function is therefore (within a phase)

$$\prod_{i=1}^N \left[\frac{z_i (z_{i-1} - z_{i+1})}{(z_i - z_{i+1})^2} \right]^{-1/2} \langle 0 | \prod_{i=1}^N p_i \cdot H(z_i) | 0 \rangle_c \quad (5.322)$$

Combining this with the contribution, already calculated, from the a oscillators, we obtain

$$A_N = \int \prod_{i=1}^N dz_i (dV_3)^{-1} \prod_{i \neq j} (z_i - z_j)^{-p_i \cdot p_j} \prod_{i=1}^N z_i^{-1/2} \langle 0 | p_1 \cdot H(z_1) p_2 \cdot H(z_2) \dots p_N \cdot H(z_N) | 0 \rangle \quad (5.323)$$

which is the Neveu-Schwarz amplitude.

It turns out that the choice of projective spin is severely restricted by the requirement that the extension to a generalized projective algebra exist. Let us write, rather generally,

$$L_n = \sum_{m=-\infty}^{\infty} \gamma_m^{n,J} a^{(n)†} a^{(m+n)} \quad (5.324)$$

Then the hermiticity

$$L_{-n} = (L_n)^\dagger \quad (5.325)$$

implies

$$\gamma_p^{-n,J} = \gamma_{p-n}^{n,J} \quad (5.326)$$

The algebra

$$[L_m, L_n] = (m-n) L_{m+n} \quad (5.327)$$

implies (for a operators commuting or anticommuting)

$$\gamma_{p+n}^{m,J} \gamma_p^{n,J} - \gamma_p^{m,J} \gamma_{p+n}^{n,J} = (m-n) \gamma_p^{m+n,J} \quad (5.328)$$

For the SU(1,1) sub-algebra we know that

$$- \gamma_m^{0,J} = (m-J) \quad (5.329)$$

$$- \gamma_m^{1,J} = \sqrt{(m+1)(m-2J)} \quad (5.330)$$

Putting $m=2$, $n=-2$ in the coefficient relation, we obtain

$$(\gamma_p^{2,J})^2 - (\gamma_{p-2}^{2,J})^2 = 4(p-J) \quad (5.331)$$

Putting $m=2$, $n=-1$ gives

$$\begin{aligned}
 -\gamma_{p-1}^{2,J} \sqrt{p(p-1-2J)} + \gamma_p^{2,J} \sqrt{(p+2)(p+1-2J)} \\
 = -3\sqrt{(p+1)(p-2J)}
 \end{aligned}
 \tag{5.332}$$

Putting successively $p=0,1,2$ we obtain

$$\gamma_0^{2,J} = \frac{-3\sqrt{-2J}}{\sqrt{2(1-2J)}}
 \tag{5.333}$$

$$\gamma_1^{2,J} = \frac{3(3J-1)}{\sqrt{3(1-J)(1-2J)}}
 \tag{5.334}$$

$$\gamma_2^{2,J} = \frac{3(6J-4)}{\sqrt{6(3-2J)(1-J)}}
 \tag{5.335}$$

Substituting these results into the earlier $m=2$, $n=-2$ relation gives

$$J(J+1)(2J+1)(8J-7) = 0
 \tag{5.336}$$

or

$$J = 0, -\frac{1}{2}, -1, +7/8$$

Going to $p=3$

$$\gamma_3^{2,J} = \frac{30(J-1)}{\sqrt{5(4-2J)(3-2J)}}
 \tag{5.337}$$

and one finds

$$(\gamma_3^{2,J})^2 - (\gamma_1^{2,J})^2 = 4(3-J)
 \tag{5.338}$$

holds for $J=0, -1/2, -1$ but not for $J=7/8$ which is thereby eliminated.

For the remaining three solutions we can find explicit solutions

$$\gamma_p^{n,0} = -\sqrt{p(p+n)} \quad (5.339)$$

$$\gamma_p^{n,-1/2} = -\left(p + \frac{1}{2} + \frac{n}{2}\right) \quad (5.340)$$

$$\gamma_p^{n,-1} = -\sqrt{(p+1)(p+n+1)} \quad (5.341)$$

To avoid vanishing L_n it is essential that the a -operators commute for $J=0, -1$ and anticommute for $J=-1/2$. For example, with $J=0$ we have

$$L_n = - \sum_{p=0}^{\infty} \sqrt{p(p+n)} a^{(p)} \cdot a^{(p+n)} \quad (5.342)$$

$$= - \sum_{P>0} \sqrt{\left(P+\frac{n}{2}\right)\left(P-\frac{n}{2}\right)} a_{P-\frac{n}{2}}^{\dagger} \cdot a_{P+\frac{n}{2}} \quad (5.343)$$

$$= - \sum_{P>0} \sqrt{\left(P+\frac{n}{2}\right)\left(P-\frac{n}{2}\right)} a_{P+\frac{n}{2}} \cdot a_{P-\frac{n}{2}}^{\dagger} \quad (5.344)$$

by substituting $P=(p-n/2)$ and $P \rightarrow -P$. Thus unless the a 's commute there is an inconsistency. Similar arguments relate "spin and statistics" of the oscillators when $J=-1/2, -1$.

Finally we note that the $J=-1$ solution is simply a re-labelling of the $J=0$ one.

In summary, if we assume simple bilinear representations of the L_n operators, we are restricted to the two inequivalent cases^{34,35}

$$J=0 \quad \text{commuting operators} \quad (5.345)$$

$$J=-1/2 \quad \text{anticommuting operators} \quad (5.346)$$

The simplest additive combination gives the Neveu-Schwarz amplitude, which can thus be derived group-theoretically within the operator formalism.

We remark here that the Neveu-Schwarz spectrum can also be obtained by quantising a modified classical rubber string on which Lorentz vector quantities are continuously distributed; see Iwasaki and Kikkawa²⁹ and related works.^{36,37}

5g. Non-Planar Extension

It was pointed out by Aldrovandi and Neveu³⁸ and independently by Schwarz¹⁵, that there exists a non-planar extension of the dual pion model analogous to that of Shapiro and Virasoro for the conventional model.

We introduce oscillators

$$[a_\mu^{(n)}, a_\nu^{(n)+}] = -g_{\mu\nu} \delta_{nn} \quad (5.347)$$

$$[\bar{a}_\mu^{(n)}, \bar{a}_\nu^{(n)+}] = -g_{\mu\nu} \delta_{nn} \quad (5.348)$$

$$\{b_\mu^{(r)}, b_\nu^{(s)+}\}_+ = -g_{\mu\nu} \delta_{rs} \quad (5.349)$$

$$\{\bar{b}_\mu^{(r)}, \bar{b}_\nu^{(s)+}\}_+ = -g_{\mu\nu} \delta_{rs} \quad (5.350)$$

and define, in an obvious notation $L_n, \bar{L}_n$. Then the non-planar model has propagator and vertex defined by the replacements

$$(L_0 - 1)^{-1} \rightarrow (L_0 + \bar{L}_0 - 2)^{-1} \quad (5.351)$$

$$p \cdot H(1) V_0(p, 1) \rightarrow p \cdot H(1) p \cdot \bar{H}(1) V'_0(p, 1) \bar{V}'_0(p, 1) \quad (5.352)$$

where the barred operators are constructed from barred oscillators, and where $V'_0(p, z) = : \exp(ip \cdot Q(z, 1)) :$ compared to $V_0(p, z) = : \exp(\sqrt{2} ip \cdot Q(z, 1)) :$. This completely defines the model.

We can go into the equivalent of an $\mathcal{F}_2$ representation by redefining the vacuum as

$$|0'\rangle = G_{\frac{1}{2}}^+ \bar{G}_{\frac{1}{2}}^+ |0\rangle \quad (5.353)$$

and using the propagator $(L_0 + \bar{L}_0 - 1)^{-1}$. In this $\mathcal{F}_2$ reformulation it becomes clear that the $\alpha=0$ tachyon on the leading trajectory of intercept $\alpha(0)=2$ is missing.

As an example, we calculate the four-point amplitude in the $\mathcal{F}_2$ formulation. It is (putting $z_1=0, z_3=1, z_4=\infty$)

$$A_4 = \int d^2z |z|^{-s-3} \langle 0' | p_2 \cdot H p_2 \cdot \bar{H} V'_0(p_2, 1) \bar{V}'_0(p_2, 1)$$

$$z^R \bar{z}^{\bar{R}} p_3 \cdot H p_3 \cdot \bar{H} V'_0(p_3, 1) \bar{V}'_0(p_3, 1) |0'\rangle$$

(5.354)

where we have re-written

$$2\pi (L_0 + \bar{L}_0 - 1)^{-1} = \int d^2z \quad z^{L_0-3/2} \bar{z}^{L_0-3/2} \quad (5.355)$$

Noting that

$$\begin{aligned} & \langle 0' | p_2 \cdot H(z) p_2 \cdot \bar{H}(\bar{z}) p_3 \cdot H(1) p_3 \cdot \bar{H}(\bar{1}) | 0' \rangle \\ &= \left| \langle 0' | p_2 \cdot H(z) p_3 \cdot H(1) | 0' \rangle \right|^2 \end{aligned} \quad (5.356)$$

$$= (p_2 \cdot p_3)^2 \frac{|z|}{|1-z|^2} \quad (5.357)$$

we arrive at (using $p_i^2 = -1$, $s+2 = 2p_1 \cdot p_2$, etc.)

$$A_4 = \int d^2z \quad |z|^{-2p_1 \cdot p_2} |1-z|^{-2p_2 \cdot p_3 - 2} (p_2 \cdot p_3)^2 \quad (5.358)$$

This is obtained from our earlier integral for the Shapiro-Virasoro model through $p_2 \cdot p_3 \rightarrow p_2 \cdot p_3 + 1$. Hence in terms of gamma functions it is (using $p_2 \cdot p_3 = 1/2 \alpha_t, \alpha_s + \alpha_t + \alpha_u = 2$)

$$A_4 = \pi \left(\frac{1}{2} \alpha_t \right)^2 \frac{\Gamma(1 - \frac{\alpha_s}{2}) \Gamma(-\frac{\alpha_t}{2}) \Gamma(\frac{\alpha_s + \alpha_t}{2})}{\Gamma(1 - \frac{\alpha_s + \alpha_t}{2}) \Gamma(\frac{\alpha_t}{2}) \Gamma(1 + \frac{\alpha_t}{2})} \quad (5.359)$$

$$= -\pi \frac{\Gamma(1 - \frac{\alpha_s}{2}) \Gamma(1 - \frac{\alpha_t}{2}) \Gamma(1 - \frac{\alpha_u}{2})}{\Gamma(1 - \frac{\alpha_s + \alpha_t}{2}) \Gamma(1 - \frac{\alpha_t + \alpha_u}{2}) \Gamma(1 - \frac{\alpha_u + \alpha_s}{2})} \quad (5.360)$$

which is related to the four-point Shapiro-Virasoro formula by the addition of one in the arguments, similarly to the way the Lovelace-Shapiro formula is related to the Euler B function.

SUMMARY

The step from the conventional multiparticle dual model, taken with unit intercept, to the Ramond-Neveu-Schwarz (RNS) model represents an improvement in several properties.

- i) In the boson sector, we now have introduced G-parity with different families of trajectories in the even and odd G-parity channels.
- ii) The rho trajectory has positive intercept and no tachyon.
- iii) Extra degrees of freedom have been added without introducing ghost states.
- iv) Most encouragingly, there is some qualitative similarity between the low-mass boson spectrum and experiment.
- v) There is now a fermion sector, hitherto absent, and here again extra anticommuting modes lead to no added ghosts.

To be fair one should emphasize the shortcomings (which are reasonably obvious): the leading intercepts $\alpha_\rho(0)=1$, $\alpha_\pi(0)=1/2$ and $\alpha_f(0)=1/2$ are all unphysical and all fermions above the ground state are parity doubled.

In spite of these difficulties, the importance of RNS theory is firstly that it shows there is no essential theoretical difficulty in introducing fermions, and secondly that it provides food for thought towards the construction of a realistic theory.

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FIGURE 1

FIGURE 2

FIGURE 3

FIGURE 4

--- BOSON
—> FERMION

FIGURE 5

FIGURE 6