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SPACE - AND TIME-DEPENDENT FLUX OSCILLATIONS (AND INSTABILITY IN THERMAL REACTORS DUE TO NONUNIFORM FORMATION AND DEPLETION OF XENON

August 1958

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SPACE- AND TIME-DEPENDENT FLUX OSCILLATIONS (AND INSTABILITY)
IN THERMAL REACTORS DUE TO NONUNIFORM FORMATION AND DEPLETION OF XENON

by

R. S. Wick

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TABLE OF CONTENTS

	Page
Abstract.....	1
I. Introduction.....	1
II. Flux Oscillations In the Interior of a Uniform Region.....	3
A. Descriptive Equations.....	3
B. Linearized Results.....	7
III. Flux Oscillations Due to Exterior Coupling Between Regions.....	15
A. Descriptive Equations.....	15
B. Linearized Results.....	17
IV. Combined Exterior and Interior Spatial Coupling.....	23
V. Energy Dependent Coupling.....	28
VI. Summary.....	32
References.....	33
Figures.....	34
Appendix.....	41

SPACE- AND TIME-DEPENDENT FLUX OSCILLATIONS (AND INSTABILITY) IN
THERMAL REACTORS DUE TO NONUNIFORM FORMATION AND DEPLETION OF XENON

ABSTRACT

A theoretical treatment of the space- and time-dependent flux oscillations (and instability) in an annular seed-blanket reactor due to nonuniform formation and depletion of xenon is developed by a linearized treatment. The distinction between coupling within the seed and through the central blanket is brought out in the analysis. These results are compared to a four-point model of the seed blanket reactor. In addition, the unity of various methods of analysis is shown by the use of the concept of a xenon "transfer function."

I. INTRODUCTION

Since the discovery that the distribution of power within a reactor may shift periodically, there has been considerable interest in the phenomena of xenon instability (Ref 1, 2, and 3).^{*}

Theoretical treatments of the xenon instability phenomena have usually been restricted to either the diffusion coupling of one region to another through a separating media or to diffusion coupling within a uniform region. This report presents an analysis of xenon instability in an annular seed-blanket type of reactor where there is coupling of various parts of the seed with each other due to diffusion within the seed proper and through the exterior blanket material. The underlying unity of the various treatments is shown readily by using the method of Laplace transforms. In addition, the effect of neutron flux energy coupling is investigated in a preliminary manner in order to estimate the effect of applying a one-group description to describe xenon instability phenomena.

The phenomena of flux oscillations within a uniform region is treated first, and it is shown that the linearized characteristic equation from which the stability criteria can be derived is

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^{*} See also "Diffusion-Coupled Oscillations With Xenon Reactivity Feedback," J. N. Gracc, M. A. Schultz, Presented at the American Nuclear Society Meeting, Los Angeles, California, June 2 - 5, 1958.

$$G + \frac{\phi(0)_{TF} [\bar{s}, \phi(0)]}{\Sigma_f^{25}} = 0 \quad (I-1)$$

The quantity $TF[\bar{s}, \phi(0)]$ is simply the transfer function relating the perturbations in xenon to the perturbations in flux considering this phenomena as an open loop circuit. The quantity G depends on the geometry and temperature coefficient of reactivity.

Next, the phenomena of flux oscillations between points coupled through a nonmultiplying exterior region is considered and the characteristic equation is shown to have the identical form as Eq (I-1). In the following section the general case where coupling takes place within the region proper in addition to exterior coupling is shown to yield a characteristic equation identical in form to Eq (I-1).

The seed blanket reactor being analysed herein is shown schematically in Fig. 1. It is assumed that the dimensions of the seed are such that a one-dimensional treatment along the seed as indicated in Fig. 1 is adequate to illustrate the important physical phenomena regarding internal diffusion coupling along the seed. The coupling of one portion of the seed to another will be treated by considering the leakage through the central region but still using a one-dimensional treatment along the seed. In a simplified treatment of exterior coupling, the four sides of the seed will be considered as separate. It will be assumed that the flux in each side is uniform and obtained by a spatial average over each side. A more exact treatment of the coupling through the central blanket is presented which does not separate the seed into a finite number of parts, but instead makes use of an integral over a coupling kernel to represent the leakage from any one portion of the seed in a continuous representation of the seed. A solution is thus obtained which allows the relative importance of interior and exterior coupling on the stability of the seed-blanket reactor to be estimated.

II. FLUX OSCILLATIONS IN THE INTERIOR OF A UNIFORM REGION

In this section the conditions necessary for divergent thermal flux oscillations to exist in uniform multiplying regions are investigated. A slightly different method from other authors (Ref 1 and 3) is used. The use of the Laplace transform and the concept of transfer functions is stressed. As an example, the seed of Fig. 1 (straightened out as an equivalent slab with no flux variation across it) will be analysed.

A. Descriptive Equations

It will be assumed that a one-group formulation of the neutron balance is sufficient to describe the physics involved in the phenomena under investigation. The neutron balance equation in terms of the slow flux ϕ is

$$\begin{aligned} \nabla \cdot D(y) \nabla \phi(y, t) - [\bar{\Sigma}_a(y) + \Sigma_x(y, t)] \phi(y, t) \\ + \epsilon(y) p(y) \nu \Sigma_f^{25}(y) \phi(y, t) + \delta \Sigma_a(y, t) \phi(y, t) = 0 \end{aligned} \quad (\text{II-1})$$

if it is assumed that the region is always critical. For the sake of discussion let the fission cross section $\Sigma_f^{25}(y)$, the fast fission factor $\epsilon(y)$, the resonance escape probability $p(y)$, the diffusion coefficient $D(y)$, and the total absorption cross section $\Sigma_a(y)$ [exclusive of that due to xenon $\Sigma_x(y)$] be independent of position y . The term $\delta \Sigma_a(y, t)$ represents a time and space dependent disturbance in the total absorption cross section (exclusive of xenon) which is imposed on the region to maintain criticality (i.e., motion of control rods). The xenon macroscopic cross section is represented by $\Sigma_x(y, t)$. The diffusion constant D is defined as

$$D = D_{\text{thermal}} \left[1 + \frac{\tau}{L_s^2} \right]$$

where τ is the age and L_s the diffusion length. Consequently, under steady state conditions when Eq (II-1) is applied to a bare reactor the condition for criticality is

$$1 = \frac{\nu p \Sigma_f^{25} / (\Sigma_a + \Sigma_x)}{1 + M^2 B^2} \quad (\text{II-3})$$

where $M^2 = L_s^2 + \tau$ and B^2 is the buckling. It will be shown later that the preceding approximation for D is not strictly true if one uses a two-group formulation of the problem. Eq (II-1) tacitly assumes that although the flux is changing with time, the neutron balance is maintained by the leakage from one region counteracting the changes in neutron absorption and formation.

Before including the effect of time dependent changes in xenon formation and its time dependent relation to the flux, the effect of the temperature coefficient of reactivity should be included. This is accomplished by assuming the linear approximation,

$$\frac{1}{k_{\text{eff}}(T, y, t)} = \frac{1}{k_{\text{eff}}(T_0, y, t)} + \frac{\partial}{\partial T} \left[\frac{1}{k_{\text{eff}}(T_0, y, t)} \right] \Delta T \quad (\text{II-4})$$

where T represents the average temperature at y at time t and ΔT is the change in average temperature due to the change in flux from the steady state value. The quantity $k_{\text{eff}}(T, y, t)$ is a measure of the local neutron balance at the point y as it is affected by the temperature. The quantity T_0 is the unperturbed average temperature. If $T_c(y, t)$ and $T_h(y, t)$ are the temperatures of cold water entering and the hot water leaving the region y (in the axial direction perpendicular to the plane containing the annulus), then

$$\frac{T_h(t, y) - T_c(t, y)}{T_h(0, y) - T_c(0, y)} = \frac{\phi(y, t)}{\phi(y, 0)} \quad (\text{II-5})$$

where $T_h(0, y) - T_c(0, y)$ is the unperturbed temperature rise at time zero when flux in the region is $\phi(y, 0)$. It follows from the assumption of a linear relationship between the temperature rise and the flux, if $T_c(y, t) = T_c(y, 0)$, that

$$\Delta T(y, t) = T(y, t) - T_0(y, 0) = \quad (\text{II-6})$$

$$\frac{1}{2} [T_h(y, t) - T_c(y, t)] - \frac{1}{2} [T_h(y, 0) - T_c(y, 0)] =$$

$$\frac{1}{2} [T_h(y, 0) - T_c(y, 0)] \left[\frac{\phi(y, t)}{\phi(y, 0)} - 1 \right]$$

With the local temperature coefficient defined as

$$\frac{\partial \rho}{\partial T}(T_o, y) = - \frac{\partial}{\partial T} \left[\frac{1}{k_{eff}(T_o, y, 0)} \right] \quad (II-7)$$

Eq (II-4) becomes

$$\frac{1}{k_{eff}(T, y, t)} = \frac{1}{k_{eff}(T_o, y, t)} - \frac{\partial \rho}{\partial T}(T_o, y) \Delta T \quad (II-8)$$

From Eq (II-1) it is apparent that

$$\frac{1}{k_{eff}(T_o, y, t)} = \frac{[\Sigma_a - \delta \Sigma_a(y, t) + \Sigma_x(y, t)] \phi(y, t) - DV^2 \phi(y, t)}{\epsilon p \nu \Sigma_f^{25} \phi(y, t)} \quad (II-9)$$

Since it is tacitly assumed that the reactor is critical, Eq (II-4) becomes

$$1 = \frac{[\Sigma_a - \delta \Sigma_a(y, t) + \Sigma_x(y, t)] \phi(y, t) - DV^2 \phi(y, t)}{\epsilon p \nu \Sigma_f^{25} \phi(y, t)} \quad (II-10)$$

$$- \frac{\partial \rho}{\partial T}(T_o, y) \frac{1}{2} [T_h(y, 0) - T_c(y, 0)] \left[\frac{\phi(y, t)}{\phi(y, 0)} - 1 \right]$$

because $k_{eff}(T, y, t)$ must be unity at all times.

To arrive at a stability criteria it is next necessary to relate the flux to the xenon cross section. It is assumed that the xenon is formed at the same position that fission occurs. The equations for I^{135} and Xe^{135} formation are

$$\frac{dI(y, t)}{dt} = - \lambda_I I(y, t) + \gamma_I \Sigma_f^{25} \phi(y, t) \quad (II-11)$$

and

$$\frac{dX(y, t)}{dt} = \lambda_I I(y, t) + \gamma_X \Sigma_f^{25} \phi(y, t) - \lambda_X X(y, t) - \sigma_X X(y, t) \phi(y, t) \quad (II-12)$$

where I , γ_I , and λ_I represent the concentration, direct fission yield and decay constant for I^{135} , X , γ_X , and λ_X being these quantities for Xe^{135} , and σ_X being the microscopic absorption cross section of Xe^{135} . Values of these quantities used in the calculations in this report are presented in Table I.

Table I. Physical Constants

<u>Iodine¹³⁵</u>	<u>Xenon¹³⁵</u>
$\gamma_I = 0.069$	$\gamma_X = 0.003$
$\lambda_I = 2.9 \times 10^{-5}/\text{sec}$	$\lambda_X = 2.1 \times 10^{-5}/\text{sec}$
	$\sigma_X = 3 \times 10^{-18} \text{ cm}$

when Eq (II-12) is multiplied by σ_X the product is an expression for the macroscopic xenon cross section. Eqs (II-11) and (II-12) can be combined to eliminate the variable I^{135}

$$\begin{aligned}
 & \frac{d^2 \Sigma_X(y,t)}{dt^2} + (\lambda_X + \lambda_I + \sigma_X \phi(y,t)) \frac{d \Sigma_X(y,t)}{dt} \\
 & + \sigma_X \frac{d \phi(y,t)}{dt} + \sigma_X \lambda_I \phi(y,t) + \lambda_I \lambda_X \Sigma_X(y,t) \quad (II-13) \\
 & = \sigma_X \gamma_X \Sigma_f^{25} \frac{d \phi(y,t)}{dt} + \sigma_X \Sigma_f^{25} \gamma \lambda_I \phi(y,t)
 \end{aligned}$$

where $\gamma \equiv \gamma_I + \gamma_X$.

Equations (II-13) and (II-10) describe the dynamic relationship between the flux and the xenon formation at a particular point in a multiplying region and the manner in which events at one portion of the region affect events in other portions of the region. It is immediately apparent that the time dependent relationship between the flux and xenon cross section affects the time and spatial behavior of the flux in the multiplying regions. If the reactor is assumed to be critical at all instants of time, Eq (II-10) must always be satisfied. Hence, if the region were critical for a particular value of $\Sigma_a + \delta \Sigma_a(y, t_0)$ at a particular instant of time t_0 , only a very particular value of the disturbance at time $t_0 + \Delta t$ would still allow the region to remain critical depending on the time dependence

of the temperature and cross sections. This situation corresponds to adding and subtracting control rod poison in a reactor. While it is possible to cause oscillating fluxes by control rod movements in a reactor, instability ordinarily will not result unless the external feedback loop between the reactor flux and the control rod drives is such as to cause it (Ref 4). However, if there is an "internal" feedback mechanism relating local fluxes to local values of xenon cross section, the possibility exists of an internal instability wherein the overall power level of the reactor is constant but local values of power are oscillating with a divergent amplitude. As far as "internal" stability is concerned, time- and space-wise variations depend on the relationship, Eq (II-13), between local fluxes and local xenon cross section. If Eq (II-13) was a first order equation with respect to time, excursions in flux would be possible but not oscillations in flux. The presence of the second order term in time is responsible for the possibility of flux oscillations. In mechanical terms, the second order term is equivalent to the inertia of a spring-mass system.

The nonlinear nature of Eqs (II-11) and (II-13) makes it difficult to obtain any general results. Consequently, these equations will be linearized in order to investigate the phenomenon of spatial and time-wise flux oscillations. In general, there is no assurance that any stability criteria obtained from the linearized equations is adequate to describe the nonlinear situation. However, linear stability criteria has been found to be adequate to describe the onset of instability for those xenon instability problems investigated by Henry and Germann (Ref 2). Hence it is assumed that the linearized criteria will be adequate to describe the onset of instability in the more complex cases described herein which cannot be solved exactly because of their nonlinearity.

B. Linearized Results

In order to obtain a linearized solution of Eq (II-1) which illustrates the nature of the effect of xenon formation and decay on flux oscillations, the method of small perturbations will be used. Let us, for simplicity, first consider a region infinite in two directions in which the disturbances are only a function of the coordinate y . In this case the fluxes and xenon will be functions only of the spatial coordinate, y , and time. Making the following substitution in Eq (II-11)

$$\phi(y,t) = \phi(y,0) + \pi(y,t) \quad (II-14)$$

where $\phi(y,0)$ is the steady-state value of the flux at time zero, and $\psi(y,t)$ is the flux perturbation limited by the inequality

$$\frac{\psi(y,t)}{\phi(y,0)} \ll 1 \quad (\text{II-15})$$

and then expressing the change in the amount of xenon as

$$\Sigma_x(y,t) = \delta \Sigma_x(y,t) + \Sigma_x(y,0) \quad (\text{II-16})$$

where

$$\frac{\delta \Sigma_x(y,t)}{\Sigma_x(y,0)} \ll 1 \quad (\text{II-17})$$

the equation of neutron conservation becomes

$$\begin{aligned} DV^2 \psi(y,t) + \frac{\partial \rho}{\partial T}(T_o, y) \frac{T_h(y,0) - T_c(y,0)}{2} \nu_{sp} \Sigma_f^{25} \psi(y,t) \\ + \psi(y,t) - \frac{DV^2 \phi(y,0)}{\phi(y,0)} - \phi(y,0) \delta \Sigma_x(y,t) = - \delta \Sigma_a(y,t) \phi(y,0) \end{aligned} \quad (\text{II-18})$$

since the unperturbed conditions

$$DV^2 \phi(y,0) - \Sigma_a \phi(y,0) - \Sigma_x(y,0) \phi(y,0) + \nu_{sp} \Sigma_f^{25} \phi(y,0) = 0 \quad (\text{II-19})$$

In arriving at Eq (II-19) second order quantities, i.e., the products of perturbed quantities have been neglected and the steady-state terms eliminated by means of Eq (II-19), Eq (II-13) relating xenon to flux can also be linearized by the substitution of Eqs (II-14) and (II-16) and the elimination of unperturbed terms to give

$$\begin{aligned} \frac{d^2 \delta \Sigma_x(y,t)}{dt^2} + (\lambda_x + \lambda_I + \sigma_x \phi(y,0)) \frac{d \delta \Sigma_x(y,t)}{dt} \\ + \Sigma_x(y,0) \sigma_x \frac{d \psi(y,t)}{dt} + \lambda_I \lambda_x \delta \Sigma_x(y,t) \\ + \sigma_x \lambda_I \Sigma_x(y,0) \psi(y,t) + \delta \Sigma_x(y,t) \phi(y,0) \\ = \sigma_x \gamma_x \Sigma_f^{25} \frac{d \psi(y,t)}{dt} + \sigma_x \Sigma_f^{25} \gamma \lambda_I \psi(y,t) \end{aligned} \quad (\text{II-20})$$

A convenient way of combining Eqs (II-18) and (II-20) and investigating stability is to take the Laplace transform of these equations, with respect to time, and use conventional stability criteria (Ref 5). Eq (II-20) becomes

$$\delta \Sigma_x(y, s) = \text{TF} [s, \phi(y, 0)] \psi(y, s) \quad (\text{II-21})$$

where s is defined by the definition of the Laplace transform (Ref 5) of a function $f(t)$ as

$$L \{ f(t) \} = \int_0^{\infty} e^{-st} f(t) dt \quad (\text{II-22})$$

and $\text{TF} [s, \phi(y, 0)]$, a transfer function relating perturbations in xenon formation to flux perturbation, is

$$\begin{aligned} \text{TF} [s, \phi(y, 0)] &= \frac{\lambda_I \sigma_x [\gamma \Sigma_f^{25} - \Sigma_x(0)] + \sigma_x [\gamma_x \Sigma_f^{25} - \Sigma_x(0)] s}{\sigma_x \lambda_I \phi(0) + \lambda_I \lambda_x + [\lambda_x + \lambda_I + \sigma_x \phi(0)] s + s^2} \quad (\text{II-23}) \\ &= \frac{\delta \Sigma_x(y, s)}{\psi(y, s)} \end{aligned}$$

It should be noted that this relation is a function of position and the initial steady-state values of flux and xenon. Substituting Eq (II-21) in the transformed Eq (II-18) we obtain

$$\begin{aligned} D V^2 \psi(y, s) + \frac{\partial \rho}{\partial T} (T_o, y) \frac{T_h(y, 0) - T_c(y, 0)}{2} \nu \epsilon \Sigma_I^{25} \psi(y, s) \\ + \psi(y, s) \left[- \frac{D V^2 \phi(y, 0)}{\phi(y, 0)} - \phi(y, 0) \text{TF} \{ s, \phi(y, 0) \} \right] \quad (\text{II-24}) \\ = - \phi(y, 0) \delta \Sigma_a(y, s) \end{aligned}$$

Eq (II-24) is a linear differential equation for $\psi(y, s)$ as a function of y for a disturbance $\delta \Sigma_a(y, s)$. Because the coefficients of $\psi(y, s)$ are dependent on position it is difficult to arrive at a general solution of Eq (II-24). If

initially the flux is assumed constant with respect to the variable y , Eq (II-24) becomes

$$DV^2\psi(y,s) + \frac{\partial \rho}{\partial T} \frac{[T_h(0) - T_c(0)]}{2} \nu \epsilon p \Sigma_f^{25} \psi(y,s) = -\psi(y,s) \phi(0) TF [s, \phi(0)] = -\phi(0) \delta \Sigma_a(y,s) \quad (II-25)$$

Under the restrictions of the above assumptions Eq (II-25) can be solved for the case of interest. Consider first a reactor with an annular seed of mean radius r and thickness m . For the present we have assumed that there is negligible coupling between different portions of the seed through either the inner or outer blanket. The coupling between different portions of the seed must be through the seed itself and Eq (II-25) applies if the following conditions are met:

1. The thickness to mean diameter ratio of the seed is small.
(This implies the seed can be slit open and replaced by a slab region of length $2\pi r = L$ and width m , with the boundary condition that the flux and its spatial derivative at $y = 0$ and $y = L$ are identical. In other words, an infinite slab region with periodicity in y of L can represent the seed.)
2. The flux distribution across the thickness of the seed is constant.
3. The seed properties along the mean diameter of the seed are uniform. (This is approximately true if the control rods in the seed clusters are all at the same level.)

Expanding the disturbance $\delta \Sigma_a(y,s)$ in a series assuming separability of time and space we obtain

$$\delta \Sigma_a(y,s) = A(s) \sum_n a_n \sin \frac{2\pi ny}{L} \quad (II-26)$$

It is required that the total power of the region be constant and hence the integral of the perturbed flux over the length, L , must be zero. A solution for $\psi(y,s)$ of the following form satisfies the requirements for constant total power and the boundary conditions described above:

$$\psi(y,s) = \sum_n b_n(s) \sin \frac{2\pi ny}{L} \quad (\text{II-27})$$

Substituting Eqs (II-27) and (II-26) in Eq (II-25) we obtain

$$\sum_n \left\{ D \left(\frac{2\pi n}{L} \right)^2 - \frac{\partial \rho}{\partial T} \frac{[T_h(0) - T_c(0)]}{2} \nu \epsilon \rho \Sigma_f^{25} \right. \quad (\text{II-28})$$

$$\left. + \phi(0) \text{TF} [s, \phi(0)] \right\} b_n(s) \sin \frac{2\pi ny}{L} = \sum_n \phi(0) A(s) a_n \sin 2\pi n \frac{y}{L}$$

The individual terms $b_n(s)$ are then

$$b_n(s) = \frac{\phi(0) a_n}{- \frac{\partial \rho}{\partial T} \frac{[T_h(0) - T_c(0)]}{2} \nu \epsilon \rho \Sigma_f^{25} + D(2\pi n/L)^2 + \phi(0) \text{TF} [s, \phi(0)]} A(s) \quad (\text{II-29})$$

and the quantity

$$- \frac{\partial \rho}{\partial T} \frac{[T_h(0) - T_c(0)]}{2} \nu \epsilon \rho + \frac{D}{\Sigma_f^{25}} \left(\frac{2\pi n}{L} \right)^2 + \frac{\phi(0) \text{TF}}{\Sigma_f^{25}} [s, \phi(0)] = 0 \quad (\text{II-30})$$

represents the characteristic equation of the n th mode of flux perturbation. If the roots of the characteristic equation have negative real parts, the solution represented by the n th mode is stable (Ref 5). It might be mentioned at this point that for all of the linearized cases considered in this report the characteristic equation for the linearized case always reduces to the form of Eq (II-30). In the general case Eq (II-30) is

$$G + \frac{\phi(0) \text{TF} [s, \phi(0)]}{\Sigma_f^{25}} = 0 \quad (\text{II-31})$$

where the term G is a function of reactor design parameters and as is apparent when the value of $\text{TF} [s, \phi(0)]$ from Eq (II-23) is substituted above, the remainder of the equation depends only on the average flux $\phi(0)$ and the physics parameters, γ , γ_x , σ_x , λ_x , and λ_I which are more or less independent of the mechanical geometric configuration. When this substitution is made, Eq (II-31) becomes

$$G + \frac{\lambda_I \lambda_x \Sigma_x(0)/\Sigma_f^{25} + \phi(0) \sigma_x [\gamma_x - \Sigma_x(0)/\Sigma_f^{25}] s}{\sigma_x \lambda_I \phi(0) + \lambda_I \lambda_x + [\lambda_x + \lambda_I + \sigma_x \phi(0)] s + s^2} = 0 \quad (\text{II-32})$$

where

$$\frac{\Sigma_x(0)}{\Sigma_f^{25}} = \frac{\gamma \sigma_x \phi(0)}{\lambda_x + \sigma_x \phi(0)} \quad (\text{II-33})$$

A general stability (linear) criteria in terms of G can be obtained by putting Eq (II-32) in the form of a quadratic in s .

$$Gs^2 + \left\{ G \left[\lambda_x + \lambda_I + \sigma_x \phi(0) \right] + \phi(0) \sigma_x \left[\gamma_x - \Sigma_x(0)/\Sigma_f^{25} \right] \right\} s + \left\{ G \left[\sigma_x \lambda_I \phi(0) + \lambda_I \lambda_x \right] + \lambda_I \lambda_x \Sigma_x(0)/\Sigma_f^{25} \right\} = 0 \quad (\text{II-34})$$

A sufficient condition for instability is that the signs of all the coefficients of s in the above equation are not alike. The term G ,

$$G = - \frac{\partial \rho}{\partial T} \frac{[T_h(0) - T_c(0)]}{2} \nu_{ep} + \frac{D}{\Sigma_f^{25}} \left(\frac{2\pi m}{L} \right)^2 \quad (\text{II-35})$$

in the present case will always be positive except in the case of a very large positive temperature coefficient of reactivity (the reactor is likely to be inherently unstable and is of no interest in the present investigation). With the understanding that only positive values of G are admissible, instability is possible if the coefficient of s in Eq (II-34) is negative. That is, instability will occur if

$$\frac{\phi(0) \sigma_x \Sigma_x(0)/\Sigma_f^{25} - \phi(0) \sigma_x \gamma_x}{\left[\sigma_x \phi(0) + \lambda_I + \lambda_x \right]} > G \quad (\text{II-36})$$

It should be noted that the left hand term in Eq (II-36) depends only on the average flux and the xenon constants (yield, half-life, and absorption) which are independent of the size. Hence it is possible to represent the stability criteria as a relation between $\phi(0)$ and G (Fig. 2). It is apparent that for a given reactor size increasing the flux increases the likelihood of instability and that for a given flux increasing the reactor dimensions increases the likelihood of instability. It is also apparent that the higher modes are more difficult to excite than the lower modes and it is possible that several modes can be excited at once.

As would be expected, a large negative temperature coefficient increases the stability of the system.

The critical frequency, i.e., the frequency of oscillation under just critical conditions, can be found from Eq (II-34) substituting $s = i\omega_c$ to be

$$\omega_c^2 = \frac{G \left[\sigma_x \lambda_I \phi(0) + \lambda_I \lambda_x \right] + \lambda_I \lambda_x \Sigma_x(0) / \Sigma_f^{25}}{G} \quad (\text{II-37})$$

Since under critical conditions the term G is defined by an equality sign in Eq (II-36), it can be eliminated to give

$$\omega_c^2 = \sigma_x \lambda_I \phi(0) + \lambda_I \lambda_x + \frac{\lambda_I \lambda_x (\Sigma_x(0) / \Sigma_f^{25}) \left[\sigma_x \phi(0) + \lambda_I + \lambda_x \right]}{\phi(0) \sigma_x (\Sigma_x(0) / \Sigma_f^{25}) \phi(0) \sigma_x \gamma_x} \quad (\text{II-38})$$

The value of the critical frequency is plotted in Fig. 3. It should be noted that the critical frequency does not depend on reactor geometry. Since by Eq (II-33) $\Sigma_x(0) / \Sigma_f^{25}$ depends only on the equilibrium flux and the physical constants γ , σ_x and λ_x the critical frequency given by (II-38) and the critical value of G given by Eq (II-36) depend only on the flux level and these physical constants. The curves presented in Figs. 2 and 3 are based on the values given in Table I.

If $s = i\omega + \lambda$ is substituted in Eq (II-34), the real and imaginary equations resulting separated, it is possible to evaluate λ . The solution to the linear perturbation equations will be of the form

$$\phi = \phi(0) e^{\lambda t} \sin \omega t \quad (\text{II-39})$$

where λ is the attenuation factor. The value of λ can be expressed in terms of the actual value of G_{act} given by Eq (II-35) and the critical value of G_{crit} given by Eq (II-36) (when the inequality sign is replaced by an equality) for the actual value of the flux. This relation is

$$\lambda = \frac{\sigma_x \phi(0) + \lambda_I + \lambda_x}{2} \left[\frac{G_{crit}}{G_{act}} - 1 \right] \quad (\text{II-40})$$

In this section we have examined the case of xenon oscillations within a uniform region where the coupling is completely internal through the media itself. It has been shown that the disturbances will have a wave shape in the region and

that there are different wave shapes corresponding to different modes of possible oscillations. In the next section the two-point analysis of Henry and Germann (Ref 2) will be extended to several points (i.e., sides of an annular seed) which are coupled together through an exterior media. In this model, of course, there are no variations within the sides or seeds themselves. In a still later section the concepts of internal and external coupling will be combined.

III. FLUX OSCILLATIONS DUE TO EXTERIOR COUPLING BETWEEN REGIONS

A. Descriptive Equations

It is of interest to extend the two-point model of Henry and Germann (Ref 2) to a case where four regions are coupled to each other through an inner region as might be assumed to be the case of the rectangular seed core reactor if the corner control rods were fully inserted (Fig. 4). For convenience these four separated seeds will be designated north, south, east, and west with the corresponding subscripts N, S, E, and W. It is assumed that there is no coupling through the seeds themselves and that the coupling is through the inner blanket region (assumed to be a pure scatterer with diffusion constant D') as indicated by the arrows in Fig. 4. These arrows show the path of coupling between the north seed and the east, west, and south seeds. In extending the two-point model to the four-point model the equations of Ref 2 must be modified to include the leakage from each seed to the three other seeds. To indicate the fact that the leakage path between, say, the north and south seed is different from that between the north and west seed, pseudo distances ℓ_{NS} , ℓ_{NW} , etc., are used to indicate these respective distances when calculating the leakages. Again the quantity m represents the thickness of the seed.

Neglecting the temperature coefficient for the moment, the criticality equation for the west seed becomes

$$k_{\text{eff}} = 1 = \frac{L_W \nu_W \epsilon_W \rho_W \Sigma_{fW} \phi_W^m}{\left[\Sigma_{aW} + \Sigma_{xW} \right] \phi_W^m + D'(\phi_W - \phi_E)/\ell_{EW} + D'(\phi_W - \phi_N)/\ell_{WN} + D'(\phi_W - \phi_S)/\ell_{WS}} \quad (\text{III-1})$$

where $D'(\phi_i - \phi_j)/\ell_{ij}$ is the leakage from i to j . Similar equations for the east, south, and north seeds can be written. These equations can be extended to include the temperature coefficient to give

$$\frac{1}{k_{\text{eff}}} = \frac{\left[\Sigma_{aW} + \Sigma_{xW} \right] \phi_W^m + D_2(\phi_W - \phi_E)/\ell_{EW} + D_2(\phi_W - \phi_N)/\ell_{WN} + D_2(\phi_W - \phi_S)/\ell_{WS}}{L_W \nu_W \epsilon_W \rho_W \Sigma_{fW} \phi_W^m} \quad (\text{III-2})$$

$$- \frac{\partial \rho_W}{\partial T} \frac{1}{2} \left[T_n(0) - T_c(0) \right]_W \left[\frac{\phi_W}{\phi_W} - 1 \right]$$

where $\bar{\phi}_W$ is the initial unperturbed flux. Similar equations can then be written for the other seeds. The condition for constant total power is

$$\phi_W + \phi_S + \phi_N + \phi_E = 4\bar{\phi} \quad (\text{III-3})$$

where $\bar{\phi}$ is the initial flux assumed to be the same for all seeds.

As in the two-point model of Henry and German, it is assumed that ν , ϵ , p , Σ_f , ϕ , Σ_a , and $\partial \rho / \partial T$ are initially the same in the north, south, east, and west seed and that the axial leakage probabilities are zero. We now have three equations relating $1/k_{\text{eff}}$ (west to north), (west to east), and (west to south). The lengths of the coupling paths are designated as

$$l_3 = l_{EN} = l_{NE} = l_{ES} = l_{SE} = l_{SW} = l_{WS} = l_{NW} = l_{WN}$$

and

$$l_4 = l_{NS} = l_{SN} = l_{EW} = l_{WE}$$

(III-4)

$$\Sigma_{x_W} + \frac{\phi_W - \phi_E}{\phi_W} \frac{D'}{m l_4} + \frac{\phi_W - \phi_N}{\phi_W} \frac{D'}{m l_3} + \frac{\phi_W - \phi_S}{\phi_W} \frac{D'}{m l_3} - \frac{\partial \rho}{\partial T} \left(\frac{T_h - T_c}{2} \right) \nu \epsilon p \Sigma_f^{25} \left[\frac{\phi_W}{\bar{\phi}} - 1 \right]$$

$$= \Sigma_{a_N} - \Sigma_{a_W} + \Sigma_{x_N} + \frac{\phi_N - \phi_W}{\phi_N} \frac{D'}{m l_3} + \frac{\phi_N - \phi_S}{\phi_N} \frac{D'}{m l_4} - \frac{\partial \rho}{\partial T} \left(\frac{T_h - T_c}{2} \right) \nu \epsilon p \Sigma_f^{25} \left[\frac{\phi_W}{\bar{\phi}} - 1 \right]$$

(III-5)

$$\Sigma_{a_E} - \Sigma_{a_W} + \Sigma_{x_W} + \frac{\phi_W - \phi_E}{\phi_W} \frac{D'}{m l_4} + \frac{\phi_W - \phi_N}{\phi_W} \frac{D'}{m l_3} + \frac{\phi_W - \phi_S}{\phi_W} \frac{D'}{m l_3} - \frac{\partial \rho}{\partial T} \left(\frac{T_h - T_c}{2} \right) \nu \epsilon p \Sigma_f^{25} \left[\frac{\phi_W}{\bar{\phi}} - 1 \right]$$

$$= \Sigma_{x_E} + \frac{\phi_E - \phi_W}{\phi_E} \frac{D'}{m l_4} + \frac{\phi_E - \phi_S}{\phi_E} \frac{D'}{m l_3} + \frac{\phi_E - \phi_N}{\phi_E} \frac{D'}{m l_3} - \frac{\partial \rho}{\partial T} \left(\frac{T_h - T_c}{2} \right) \nu \epsilon p \Sigma_f^{25} \left[\frac{\phi_E}{\bar{\phi}} - 1 \right]$$

(III-6)

$$\Sigma_{a_S} - \Sigma_{a_W} + \Sigma_{x_W} + \frac{\phi_W - \phi_E}{\phi_W} \frac{D'}{m l_4} + \frac{\phi_W - \phi_N}{\phi_W} \frac{D'}{m l_3} + \frac{\phi_W - \phi_S}{\phi_W} \frac{D'}{m l_3} - \frac{\partial \rho}{\partial T} \left(\frac{T_h - T_c}{2} \right) \nu \epsilon p \Sigma_f^{25} \left[\frac{\phi_W}{\bar{\phi}} - 1 \right]$$

$$= \Sigma_{x_S} + \frac{\phi_S - \phi_W}{\phi_S} \frac{D'}{m l_3} + \frac{\phi_S - \phi_E}{\phi_S} \frac{D'}{m l_3} + \frac{\phi_S - \phi_N}{\phi_S} \frac{D'}{m l_4} - \frac{\partial \rho}{\partial T} \left(\frac{T_h - T_c}{2} \right) \nu \epsilon p \Sigma_f^{25} \left[\frac{\phi_S}{\bar{\phi}} - 1 \right]$$

The xenon cross sections in each seed are related to the flux in each seed by equations of the same form as Eqs (II-11) and (II-12). We now have four equations, Eq (III-4), (III-5), (III-6), and (III-3), and four unknowns, ϕ_N , ϕ_S , ϕ_E , and ϕ_W . Basically the four-point model is little different from the two-point model except for the greater complexity and the fact that several solutions are possible.

B. Linearized Results

A similar approach (differing only in algebraic manipulation) to that of Section II will be used to obtain the linear stability criteria for the four-point model in order to show how two modes of oscillations are possible. The following scheme of small perturbations will be used.

$$\left. \begin{aligned} \phi_i &= \bar{\phi} + \psi_i \\ \phi_j &= \bar{\phi} + \psi_j \\ \frac{\phi_i - \phi_j}{\phi_i} &= \frac{\psi_i - \psi_j}{\bar{\phi}} \\ \Sigma_{x_i} &= \Sigma_{x_i}(0) + \Sigma_{x_i} \end{aligned} \right\} \quad (\text{III-7})$$

Substituting these relations in Eqs (III-3), (III-4), (III-5), (III-6), neglecting second order quantities, taking the Laplace transform of the variables with respect to time as in Section II of this report, and eliminating the xenon perturbation again as in Section II we obtain

$$\begin{aligned} \psi_W(s) \left[\text{TF}(s) + \frac{D}{\phi_{m\ell_4}} + \frac{3D'}{\phi_{m\ell_3}} \right] - \psi_N(s) \left[\text{TF}(s) + \frac{3D'}{\phi_{m\ell_3}} + \frac{3D}{\phi_{m\ell_4}} \right] \\ - \psi_E(s) \left[\frac{D'}{m\ell_4 \bar{\phi}} - \frac{D'}{m\ell_3 \bar{\phi}} \right] - \psi_S(s) \left[\frac{D'}{m\ell_3 \bar{\phi}} - \frac{D'}{m\ell_4 \bar{\phi}} \right] - \frac{\partial \rho}{\partial T} \frac{T_h - T_c}{2} \end{aligned} \quad (\text{III-8})$$

$$\nu_{\text{ep}} \Sigma_f^{25} [\psi_W(s) - \psi_N(s)] = \Sigma_{a_N}(s) - \Sigma_{a_W}(s)$$

(III-9)

$$\begin{aligned} & \psi_W(s) \left[TF(s) + \frac{2D'}{\bar{\phi}_m \ell_4} + \frac{2D'}{\bar{\phi}_m \ell_3} \right] - \psi_E(s) \left[TF(s) + \frac{2D'}{\bar{\phi}_m \ell_4} + \frac{2D'}{\bar{\phi}_m \ell_3} \right] \\ & - \frac{\partial \rho}{\partial T} \left(\frac{T_h - T_c}{2} \right) \frac{\nu \epsilon p \Sigma_f^{25}}{\bar{\phi}} \left[\psi_W(s) - \psi_E(s) \right] = \Sigma_{a_E}(s) - \Sigma_{a_W}(s) \end{aligned}$$

(III-10)

$$\begin{aligned} & \psi_W(s) \left[TF(s) + \frac{D'}{\bar{\phi}_m \ell_4} + \frac{3D'}{\bar{\phi}_m \ell_3} \right] - \psi_N(s) \left[\frac{D'}{\bar{\phi}_m \ell_3} - \frac{D'}{\bar{\phi}_m \ell_4} \right] \\ & - \psi_E(s) \left[\frac{D'}{\bar{\phi}_m \ell_4} - \frac{D'}{\bar{\phi}_m \ell_3} \right] - \psi_S(s) \left[TF(s) + \frac{3D'}{\bar{\phi}_m \ell_3} + \frac{D'}{\bar{\phi}_m \ell_4} \right] \\ & - \frac{\partial \rho}{\partial T} \left(\frac{T_h - T_c}{2} \right) \frac{\nu \epsilon p \Sigma_f^{25}}{\bar{\phi}} \left[\psi_W(s) - \psi_S(s) \right] = \Sigma_{a_S}(s) - \Sigma_{a_W}(s) \end{aligned}$$

where $TF(s)$ is defined in Eq. (II-23) as

$$TF(s) = TF \left[s, \bar{\phi} \right] = \frac{\delta \Sigma_x(s)}{\psi(s)} \quad (III-11)$$

The constant power requirement is

$$\psi_N(s) + \psi_S(s) + \psi_E(s) + \psi_W(s) = 0 \quad (III-12)$$

The terms on the right of Eqs. (III-8), (III-9), and (III-10) may be considered as driving functions and their nature cannot affect the characteristic equation from which we will get our stability criteria. Therefore, the right hand sides of Eq. (III-8), (III-9), and (III-10) could be set equal to zero. The physical situation corresponds to finding the natural frequency of a system which is driven or disturbed by a vanishingly small disturbance. After setting the right hand terms of Eqs. (III-8), (III-9), and (III-10) to zero, it is apparent that from Eq. (III-9)

$$\psi_W(s) = \psi_E(s) \quad (\text{III-13})$$

and hence $\psi_E(s)$ can be eliminated from Eqs. (III-8), (III-10), and (III-12) to give

$$\psi_W(s) \left[\text{TF}(s) + \frac{4D'}{\bar{\phi}_m l_3} \right] - \psi_N(s) \left[\text{TF}(s) + \frac{3D'}{\bar{\phi}_m l_3} + \frac{D'}{\bar{\phi}_m l_4} \right] \quad (\text{III-14})$$

$$- \psi_S(s) \left[\frac{D'}{m l_3 \bar{\phi}} - \frac{D'}{m l_4 \bar{\phi}} \right] - \frac{\partial \rho}{\partial T} \left(\frac{T_h - T_c}{2} \right) \frac{\nu \epsilon p \Sigma_f^{25}}{\bar{\phi}} [\psi_W(s) - \psi_N(s)] = 0$$

$$\psi_W(s) \left[\text{TF}(s) + \frac{4D'}{\bar{\phi}_m l_3} \right] - \psi_N(s) \left[\frac{D'}{\bar{\phi}_m l_3} - \frac{D'}{\bar{\phi}_m l_4} \right] \quad (\text{III-15})$$

$$- \psi_S(s) \left[\text{TF}(s) + \frac{3D'}{\bar{\phi}_m l_3} + \frac{D'}{\bar{\phi}_m l_4} \right] - \frac{\partial \rho}{\partial T} \left(\frac{T_h - T_c}{2} \right) \frac{\nu \epsilon p \Sigma_f^{25}}{\bar{\phi}} [\psi_W(s) - \psi_S(s)] = 0$$

$$2\psi_W(s) + \psi_S(s) + \psi_N(s) = 0 \quad (\text{III-16})$$

Equating Eq. (III-14) to Eq. (III-15) it is apparent that

$$\psi_N(s) = \psi_S(s) \quad (\text{III-17})$$

and Eq. (III-15) becomes

$$\psi_W(s) \left[\text{TF}(s) + \frac{4D'}{\bar{\phi}_m l_3} \right] - \psi_N(s) \left[\text{TF}(s) + \frac{4D'}{\bar{\phi}_m l_3} \right] \quad (\text{III-18})$$

$$- \frac{\partial \rho}{\partial T} \left(\frac{T_h - T_c}{2} \right) \frac{\nu \epsilon p \Sigma_f^{25}}{\bar{\phi}} [\psi_W(s) - \psi_N(s)] = 0$$

and Eq (III-16) becomes

$$2\psi_W(s) + 2\psi_N(s) = 0 \quad (\text{III-19})$$

Combining Eqs (III-19) and (III-18) the characteristic equation in s is

$$\bar{\phi} \text{TF}(s) + \frac{4D'}{\bar{\phi} m \ell_3} - \frac{\partial \rho}{\partial T} \left(\frac{T_h - T_c}{2} \right) \frac{\nu \epsilon p \Sigma_f^{25}}{\bar{\phi}} = 0 \quad (\text{III-20})$$

When Eq (III-20) is multiplied by $\bar{\phi}/\Sigma_f^{25}$ the characteristic equation assumes the form

$$\frac{\bar{\phi} \text{TF}(s)}{\Sigma_f^{25}} + \frac{4D'}{m \ell_3 \Sigma_f^{25}} - \frac{\partial \rho}{\partial T} \left(\frac{T_h - T_c}{2} \right) \nu \epsilon p = 0 \quad (\text{III-21})$$

which is identical to the form of Eq (II-31). In this particular instance

$$G = \frac{4D'}{m \ell_3 \Sigma_f^{25}} - \frac{\partial \rho}{\partial T} \left(\frac{T_h - T_c}{2} \right) \nu \epsilon p \quad (\text{III-22})$$

and the statements made in Section II regarding criticality and critical frequency apply. The mode of instability represented above represents the case where the opposite seeds have the same value of flux. One pair having high fluxes while the other pair of opposites has low fluxes.

It should be noted that another mode of oscillation is possible. This mode is represented by a so-called "lost solution" obtained when $\psi_E(s) = \psi_W(s)$ of Eq (III-13) is set equal to zero. In this case

$$\psi_N(s) = -\psi_S(s) \quad (\text{III-23})$$

and the characteristic equation is

$$\frac{\bar{\phi} \text{TF}(s)}{\Sigma_f^{25}} + \frac{2D'}{m \Sigma_f^{25}} \left[\frac{1}{\ell_3} + \frac{1}{\ell_4} \right] - \frac{\partial \rho}{\partial T} \left(\frac{T_h - T_c}{2} \right) \nu \epsilon p = 0 \quad (\text{III-24})$$

and

$$G = \frac{2D'}{m\Sigma_f^{25}} \left[\frac{1}{\ell_3} + \frac{1}{\ell_4} \right] - \frac{\partial \rho}{\partial T} \left(\frac{T_h - T_c}{2} \right) \nu \epsilon p \quad (\text{III-25})$$

This mode is represented by one opposite pair of seeds not varying at all while the third seed has a high flux and the fourth has a low flux.

To estimate orders of magnitude we will arbitrarily say that the distance $\ell_3 = \ell_{EN}$ is the physical distance between the centers of two adjacent seeds and that the distance $\ell_4 = \ell_{NS}$ is the physical distance between two opposite seeds. For the case of the square type seed shown in Fig. 4, $\ell_3 = \sqrt{2}/2 \ell_4$ and the critical value of G for the second mode (high north and south, low east and west) in terms of ℓ_4 is

$$G = \frac{2D'}{m\ell_4 \Sigma_f^{25}} (2.828) - \frac{\partial \rho}{\partial T} \left(\frac{T_h - T_c}{2} \right) \nu \epsilon p \quad (\text{III-26})$$

The critical value of G for the first mode (the case high north, low south, and east and west undisturbed) in terms of ℓ_4 is

$$G = \frac{2D'}{m\ell_4 \Sigma_f^{25}} (2.414) - \frac{\partial \rho}{\partial T} \left(\frac{T_h - T_c}{2} \right) \nu \epsilon p \quad (\text{III-27})$$

The first mode will be excited first as the flux level or dimensions are increased. As the flux level is further increased the second mode will also be excited. When assessing the stability of a particular configuration the criteria for stability of the first mode is the most important.

As a matter of comparison let us recall that the value of G for the two-point model as inferred from Ref 2 is

$$G = \frac{2D'}{\Sigma_f^{25} m\ell_2} - \frac{\partial \rho}{\partial T} \left(\frac{T_h - T_c}{2} \right) \nu \epsilon p \quad (\text{III-28})$$

where ℓ_2 is the distance separating the two points. In comparing the two-point model to the four-point model and their respective linear stability criteria it is immediately apparent that the only difference is the manner in which G is related

to the dimensions of the reactor. It is further apparent that if the distance between the two slabs in the two-point model is $\ell_2 = \ell_4/2.414$ the linear stability criteria for the two-point model is identical to that of the four-point model. This similarity is of considerable practical importance. It is fairly simple to set up the nonlinear equations of the two-point model on a moderate size analog computer while setting up the four-point model would require considerably more equipment. Also it is likely that the nonlinear behavior of the four-point model is basically little different from that of the two-point model. Although the four-point model actually only approximates the practical situation, it might be possible to obtain very useful information just by "scaling" the distance between the two slabs on the analog computer to be $1/1.414$ times the actual distance across the four-sided seed type reactor.

In the next section a more satisfactory method of finding the coupling between different portions of the seed through the blanket will be investigated for the case where the corner control rods are not inserted and there is coupling from one seed to another internally through the seed. In addition, a continuously varying flux distribution along the seed is assumed instead of a single value so that the coupling of all portions of the seed with all other portions is accounted for. Thus, in the next section we combine the concepts derived in the second and the present section.

IV. COMBINED EXTERIOR AND INTERIOR SPATIAL COUPLING

In this section the effect of coupling within the region described in Section II is combined with the coupling effect through an exterior region described in Section III.

Consider the annular reactor surrounding a cylindrical scattering region with diffusion constant D' through which various portions of the annular reactor are also coupled to each other, as shown in Fig. 5. The leakage of neutrons from the volume of core $m dy$ to the volume of core $m dy'$ is taken as

$$\mathcal{L}(dy \rightarrow dy') = D' dy \left\{ \frac{\phi(y, t) - \phi(y', t)}{|\vec{\ell}|} \right\} \frac{dy'}{2\pi r} \quad (\text{IV-1})$$

where

$$|\vec{\ell}| = 2r \left| \sin \frac{\theta' - \theta}{2} \right| = 2r \sin \frac{\theta' - \theta}{2} \quad (\text{IV-2})$$

and D' is the diffusion constant of the central region.

The above expression for the leakage from one differential element of the seed to another is a simple expansion of the four-point model previously described to an infinite number of points. The remainder of the leakage, from one differential element to the next by leakage through the seed proper, is accounted for by a neutron balance. From the figure it is apparent that

$$r = \frac{L}{2\pi} \quad (\text{IV-3})$$

$$\theta' = \frac{y'}{r} = \frac{y' 2\pi}{L} \quad (\text{IV-4})$$

$$\theta = \frac{y}{r} = \frac{y 2\pi}{L} \quad (\text{IV-5})$$

and hence Eq (IV-1) can be written as

$$\mathcal{L}(dy \rightarrow dy') = D' dy \left\{ \frac{\phi(y, t) - \phi(y', t)}{L/\pi \sin (y' - y)/L \pi} \right\} \frac{dy'}{L} \quad (\text{IV-6})$$

The total leakage out of the volume $m dy$ is obtained by integrating over dy' to give

$$\mathcal{L}(y) = \frac{\pi D'}{L^2} \int_{y'} \frac{[\phi(y, t) - \phi(y', t)] dy'}{\sin (y' - y)/L \pi} \quad (\text{IV-7})$$

Again assuming that the annular core can be opened up and treated one-dimensionally with no flux variation across the seed, the equation of neutron conservation is

$$D \nabla^2 \phi(y, t) \, mdy - [\Sigma_a + \Sigma_x(y, t)] \phi(y, t) \, mdy \\ + \epsilon p \nu \Sigma_f^{25} \phi(y, t) \, mdy + \delta \Sigma_a(y, t) \phi(y, t) \, mdy \\ - \frac{\pi D'}{L^2} dy \int_{y'} \frac{[\phi(y, t) - \phi(y', t)]}{\sin \frac{(y' - y)}{L} \pi} dy' = 0$$

Dividing this equation by mdy we have an equation identical to Eq (II-1) with the exception of the integral term. Since the remaining development is identical to that in Section II, it is possible to write immediately an equation similar to Eq (II-25)

$$D \nabla^2 \psi(y, s) - \frac{\pi D'}{mL^2} \int_{y'} \frac{\psi(y, s) - \psi(y', s)}{\cos \frac{(y' - y)}{L} \pi} dy \quad (IV-9) \\ + \frac{\partial \rho}{\partial T} \frac{[T_h(0) - T_c(0)]}{2} \nu \epsilon p \Sigma_f^{25} \\ - \psi(y, s) \phi(0) TF[s, \phi(0)] = -\phi(0) \delta \Sigma_a(y, s)$$

It should be noted that the same assumptions have been made regarding the temperature coefficient of reactivity as in Section II.

Expanding the disturbance $\delta \Sigma_a(y, s)$ in a series assuming separability of space and time, we obtain as previously

$$\delta \Sigma_a(y, s) = A(s) \sum_n a_n \sin \frac{2n\pi y}{L} \quad (IV-10)$$

Again it is required that the total power of the reactor be constant and the solution for $\psi(y, s)$ used in Section II is valid.

$$\psi(y, s) = \sum_n b_n(s) \sin \frac{2n\pi y}{L} \quad (IV-11)$$

Substituting Eqs (IV-10) and (IV-11) in Eq (IV-9), we obtain

$$\begin{aligned}
 & \sum_n \left\{ D \left(\frac{2\pi n}{L} \right)^2 - \frac{\partial \rho}{\partial T} \frac{[T_h(0) - T_c(0)]}{2} \nu \epsilon_p \Sigma_f^{25} \right. \\
 & \quad \left. + \phi(0) \text{TF} [s, \phi(0)] \right\} b_n(s) \sin \frac{2\pi n y}{L} \\
 & \quad + \frac{\pi D'}{L^2 m} \sum_n b_n(s) \int_{y'} \frac{[\sin 2\pi n y/L - \sin 2\pi n y'/L] dy'}{\cos (y' - y)/L \pi} \\
 & = \sum_n \phi(0) A(s) a_n \sin \frac{2\pi n y}{L}
 \end{aligned} \tag{IV-12}$$

The integral in the above expression is shown in the appendix to be

$I_n \cdot \sin 2\pi n y/L$ where

$$I_n = \frac{1}{\sin 2\pi n y/L} \int_{y'} \frac{[\sin 2\pi n y/L - \sin 2\pi n y'/L] dy'}{\cos (y' - y)/L \pi} \tag{IV-13}$$

An expression for $b_n(s)$ can now be found from Eq (IV-12) to be

$$b_n(s) = \frac{\phi(0) a_n A(s)}{\pi D' I_n / L^2 m - \partial \rho / \partial T [T_h(0) - T_c(0)] / 2 \epsilon_p \nu \Sigma_f^{25} + D(2\pi n/L)^2 + \phi(0) \text{TF} [s, \phi(0)]} \tag{IV-14}$$

where the denominator represents the characteristic equation of the n th mode of flux perturbation. In the format of the previous sections the characteristic equation is

$$G_n + \frac{\phi(0) \text{TF} [s, \phi(0)]}{\Sigma_f^{25}} = 0 \tag{IV-15}$$

and

$$G_n = - \frac{\partial \rho}{\partial T} \frac{[T_h(0) - T_c(0)]}{2} \nu \epsilon_p + \frac{D}{\Sigma_f^{25}} \left(\frac{2\pi n}{L} \right)^2 + \frac{\pi D' I_n}{L^2 m \Sigma_f^{25}} \tag{IV-16}$$

The numerical values of I_n found in the appendix are

$$I_1 = \frac{L^4}{\pi} \tag{IV-17}$$

$$I_2 = \frac{L^4}{\pi} (1 + 1/3) \tag{IV-18}$$

$$I_3 = \frac{I_4}{\pi} (1 + 1/3 + 1/5) \quad (\text{IV-19})$$

$$I_n = \frac{I_4}{\pi} (1 + 1/3 + 1/5 + \dots + 1/2n-1) \quad (\text{IV-20})$$

From Section II the stability criteria can be written immediately as

$$\frac{\phi(0)\sigma_x \Sigma_x(0)/\Sigma_f^{25} - \phi(0)\sigma_x \gamma_x}{\sigma_x \phi(0) + \lambda_I + \lambda_x} > G_n \quad (\text{IV-21})$$

where G_n has been given in Eq (IV-16).

It is instructive to consider the relative contributions of the geometrical factors

$$D \left(\frac{2\pi m}{L} \right)^2 + \frac{D' I_4}{L m} (1 + 1/3 + 1/5 + \dots + 1/2n-1) \quad (\text{IV-22})$$

which appear in G_n . The higher the value of G_n the greater the stability. Thus, upon examination of the geometrical terms it is immediately apparent that the coupling through the seed itself (the first term in Eq (IV-22)) is extremely important in stabilizing the higher modes, compared to the coupling through the central region. For very large n the magnitude of the geometrical term representing coupling through the central region is bounded while the interior coupling term is unbounded.

It should be noted that if D is equal to zero there is no coupling through the seed proper, and the only coupling is through the inner blanket, G_n becomes

$$G_n = - \frac{\partial \rho}{\partial T} \frac{[T_h(0) - T_c(0)]}{2} \nu_{ep} + \frac{D' \pi I_n}{L^2 m \Sigma_f^{25}} \quad (\text{IV-23})$$

which should be compared to Eq (III-25)

$$G = - \frac{\partial \rho}{\partial T} \frac{[T_h(0) - T_c(0)]}{2} \nu_{ep} + \frac{2D'}{m \Sigma_f^{25}} \left[\frac{1}{\lambda_3} + \frac{1}{\lambda_4} \right] \quad (\text{IV-24})$$

if we were to arbitrarily assume that the annular core with $D = 0$ could be broken into four isolated quadrants as in Fig. 4. The equivalent four-point model will be assumed to surround the same volume of material, thus

$$\pi r^2 = \frac{L^2}{4\pi} = \lambda_4^2$$

and

$$\ell_4 = \frac{L}{2\sqrt{\pi}}$$

$$\ell_3 = \frac{L\sqrt{2}}{4\sqrt{\pi}}$$

From Eq (IV-24)

$$G = - \frac{\partial \rho}{T} \frac{[T_h(0) - T_c(0)]}{2} \nu \epsilon p + \frac{4D'(1 + \sqrt{2})\sqrt{\pi}}{L_m \Sigma_f^{25}}$$

for the four-point model which corresponds to $n = 1$. Comparison of these results to the continuous model indicates that the variation of flux along the sides is relatively important and the stability is over estimated considerably by the four-point model which assumes an average flux in each side or quadrant of the core.

The results of this section indicate that both the coupling through the central blanket and the seed itself affect the stability. In fact, increasing the dimensions affects the coupling through the seed in a much more adverse manner than the coupling through the central blanket.

V. ENERGY DEPENDENT COUPLING

In the previous sections a one-group diffusion theory has been used to describe the phenomena of xenon instability. The diffusion constant was defined as

$$D = D_{\text{thermal}} \left(1 = \frac{\tau}{L_S^2} \right)$$

in order to take into account approximately the effect of the fast flux on the coupling. While it is expected that the one-group formulation will describe the phenomena in general, it is instructive to note the effect on the stability criteria of using a multigroup scheme, and in particular a two-group scheme.

Let us consider the case described in Section II where the fluxes depend only on the y space coordinate. Instead of the single Eq (II-1) we now have two equations:

$$D_f \nabla^2 \phi_f - \Sigma_f \phi_f + \epsilon \nu \Sigma_f^{25} \phi_s = 0 \quad (V-1)$$

$$D_s \nabla^2 \phi_s - (\Sigma_a + \Sigma_x) \phi_s + \rho \Sigma_f \phi_f = 0 \quad (V-2)$$

where D_f and D_s represent the diffusion constants of the fast and slow fluxes, respectively. It is assumed the xenon is burned out only by the slow flux. These equations will be linearized and the characteristic equation will be found for the case of a vanishingly small disturbance.

Let

$$\phi_f = \phi_f(0) + \psi_f(y, t) \quad (V-3)$$

$$\phi_s = \phi_s(0) + \psi_s(y, t) \quad (V-4)$$

$$\Sigma_x = \Sigma_x(0) + \delta \Sigma_x(y, t) \quad (V-5)$$

since it is assumed that the fluxes initially are everywhere uniform. Substituting Eqs (V-3) and (V-4) in Eqs (V-1) and (V-2)

$$D_f \nabla^2 \psi_f(y, t) - \Sigma_f [\phi_f(0) + \psi_f(y, t)] + \epsilon \nu \Sigma_f^{25} [\phi_s(0) + \psi_s(y, t)] = 0 \quad (V-6)$$

$$\begin{aligned}
 & D_s \nabla^2 \psi_s(y,t) - \Sigma_a \phi_s(0) - \Sigma_a \psi_s(y,t) - \left[\Sigma_x(0) + \delta \Sigma_x(y,t) \right] \left[\phi_s(0) + \psi_s(y,t) \right] \\
 & + p \Sigma_f \phi_f(0) + p \Sigma_f \psi_f(y,t) = 0
 \end{aligned} \tag{V-7}$$

Under steady state conditions for uniform fluxes initially

$$- \Sigma_f \phi_f(0) + \epsilon \nu \Sigma_f^{25} \phi_s(0) = 0 \tag{V-8}$$

$$- (\Sigma_a + \Sigma_x) \phi_s(0) + p \Sigma_f \phi_f(0) = 0 \tag{V-9}$$

Substituting Eqs (V-8) and (V-9) in Eqs (V-6) and (V-7) to eliminate $\phi_f(0)$ and neglecting second order terms

$$D_f \nabla^2 \psi_f(y,t) - \Sigma_f \psi_f(y,t) + \epsilon \nu \Sigma_f^{25} \psi_s(y,t) = 0 \tag{V-10}$$

$$\begin{aligned}
 & D_s \nabla^2 \psi_s(y,t) - \delta \Sigma_x(y,t) \phi_s(0) - \left[\Sigma_a + \Sigma_x(0) \right] \psi_s(y,t) \\
 & + p \Sigma_f \psi_f(y,t) = 0
 \end{aligned} \tag{V-11}$$

Taking the Laplacian of Eq (V-11)

$$\begin{aligned}
 & D_s \nabla^4 \psi_s(y,t) - \phi_s(0) \nabla^2 \delta \Sigma_x(y,t) - \left[\Sigma_a + \Sigma_x(0) \right] \nabla^2 \psi_s(y,t) \\
 & + p \Sigma_f \nabla^2 \psi_f(y,t) = 0
 \end{aligned} \tag{V-12}$$

and eliminating $\psi_f(y,t)$ from Eqs. (V-10), (V-11), and (V-12) we obtain the following equation for $\psi_s(y,t)$

$$\begin{aligned}
 & - \frac{D_f D_s}{\Sigma_f} \nabla^4 \psi_s(y,t) + \left[\frac{D_f}{\Sigma_f} \left[\Sigma_a + \Sigma_x(0) \right] + D_s \right] \nabla^2 \psi_s(y,t) \\
 & - \phi_s(0) \delta \Sigma_x(y,t) + \frac{D_f}{\Sigma_f} \phi_s(0) \nabla^2 \delta \Sigma_x(y,t) = 0
 \end{aligned} \tag{V-13}$$

As in Section II the Laplace transform of Eq (V-13) is

$$\begin{aligned}
 & - \frac{D_f}{\Sigma_f} \nabla^4 \psi_s(y,s) + \left[\frac{D_f}{\Sigma_f} \left[\Sigma_a + \Sigma_x(0) \right] + D_s \right] \nabla^2 \psi_s(y,s) \\
 & - \phi_s(0) \delta \Sigma_x(y,s) + \frac{D_f}{\Sigma_f} \phi_s(0) \nabla^2 \delta \Sigma_x(y,s) = 0
 \end{aligned} \tag{V-14}$$

and

$$\delta \Sigma_x(y,s) = \text{TF} [s, \phi_s(0)] \psi_s(y,s) \quad (\text{V-15})$$

which gives the equation

$$\begin{aligned} & - \frac{D_f D_s}{\Sigma_f} \nabla^4 \psi_s(y,s) + \left[\frac{D_f}{\Sigma_f} [\Sigma_a + \Sigma_x(0)] + D_s \right] \nabla^2 \psi_s(y,s) \\ & - \phi_s(0) \text{TF} [s, \phi_s(0)] \psi_s(y,s) + \frac{D_f}{\Sigma_f} \phi_s(0) \text{TF} [s, \phi_s(0)] \nabla^2 \psi_s(y,s) = 0 \end{aligned} \quad (\text{V-16})$$

Let us again assume that the solution for $\psi_s(y,s)$ can be represented by a series

$$\psi_s(y,s) = \sum_n a_n(s) \xi_n(y) \quad (\text{V-17})$$

and choose $\xi_n(y)$ so that the total power remains constant. Let

$$\xi_n(y) = \sin \frac{2\pi n y}{L} \quad (\text{V-18})$$

(cf. Eq. (II-26)) and substituting in

$$\begin{aligned} & \sum_n \left[\frac{D_f D_s}{\Sigma_f} \left(\frac{2\pi n}{L} \right)^4 + \left(\frac{D_f}{\Sigma_f} [\Sigma_a + \Sigma_x(0)] + D_s \right) \left(\frac{2\pi n}{L} \right)^2 \right. \\ & \left. + \phi_s(0) \text{TF} [s, \phi_s(0)] \left(1 + \frac{D_f}{\Sigma_f} \left(\frac{2\pi n}{L} \right)^2 \right) \right] a_n \xi_n(y) = 0 \end{aligned} \quad (\text{V-19})$$

The term in the brackets is then the characteristic equation for the n th mode and should be compared to Eq. (II-30) after rearranging as follows

$$\frac{\frac{D_s}{\Sigma_f^{25}} \left(1 + \frac{\tau}{L_s^2} \right) \left(\frac{2\pi n}{L} \right)^2 + \frac{D_s}{\Sigma_f^{25}} \tau \left(\frac{2\pi n}{L} \right)^4}{\left[1 + \tau \left(\frac{2\pi n}{L} \right)^2 \right]} + \frac{\phi_s(0) \text{TF} [s, \phi_s(0)]}{\Sigma_f^{25}} = 0 \quad (\text{V-20})$$

or

$$G + \frac{\phi_s(0) \text{TF} [s, \phi_s(0)]}{\Sigma_f^{25}} = 0 \quad (\text{V-21})$$

which is of similar form to Eq. (II-31) and hence the stability criteria as a function of G developed in Section II can be used. The inclusion of the fourth power terms is due to the two-group formulation of the problem and more accurately represents the coupling of one region to another by the fast flux. In order to assess the importance of these terms G_n (two-group) will be expressed as a function of G_n (one-group) given by Eq. (II-35).

$$\frac{G_n(\text{two-group})}{G_n(\text{one-group})} = \frac{\left(1 + \frac{\tau}{L_S^2}\right) + \tau \left(\frac{2\pi n}{L}\right)^2}{\left(1 + \frac{\tau}{L_S^2}\right) \left[1 + \tau \left(\frac{2\pi n}{L}\right)^2\right]} = C.F._n \quad (V-22)$$

The correction factor $C.F._n$ is plotted in Figs. 6 and 7 as a function of τ/L^2 for constant values of τ/L_S^2 for $n = 1$ and $n = 2$. These results indicate that the larger the dimension L , i.e., the smaller the value of τ/L^2 , the smaller the difference between G calculated by the one-group approximation and the more correct value calculated with two-group theory. It is also true that the larger the value of L the greater chance for instability and hence it is possible but not necessary that in situations where instability is predicted by the one-group approximation the correction due to fast flux coupling will be small.

The above results indicate that the value of G for a particular geometry is actually smaller than one would expect from the one-group approximation. Recalling that G must be greater than

$$\frac{\phi(0)\sigma_{\Sigma_X}(0)/\Sigma_f^{25} - \phi(0)\sigma_{S\gamma_X}}{\sigma_X\phi(0) + \lambda_I + \lambda_{\bar{I}}}$$

for the system to be stable it is immediately apparent that the one-group approximation for G is not necessarily conservative.

VI. SUMMARY

It has been shown that the stability criteria for xenon oscillations related to coupling within the interior of a region or to coupling through exterior regions can be expressed in a form identical to each other:

$$G > \frac{\phi(0)\sigma_{x\Sigma_x}(0)/\Sigma_f^{25} - \phi(0)\sigma_{x\gamma_x}}{\sigma_x\phi(0) + \lambda_I + \lambda_x}$$

This expression, developed by a linearized approximation, must be satisfied in order to assure stability where G is a factor which is related to the geometry, type of coupling, and the local temperature coefficient of reactivity. In the case of the annular seed-blanket reactor with both interior and exterior coupling of various portions of the seed it has been shown that G is the sum of two terms; one due to interior and one due to exterior coupling. It is thus possible to assess the relative importance of these two coupling effects on the stability of the reactor to xenon oscillations.

Further studies along the following lines are indicated as a result of these studies. (1) Digital code calculations with spatial time dependent xenon concentration to assess the effect of nonlinearities on coupling within a uniform region. (2) Analysis of the effect of xenon formation due to fast fission and the depletion of xenon by fast flux on the stability, particularly in cases where the fast coupling might be expected to be important. (3) Development of overall reactor transfer functions when there is loose internal coupling within the reactor proper. (4) Development of stability concepts for cases where reactor is on a xenon transient such as maximum xenon override. (5) Development of stability concepts for cases where there are several external coolant flow loops which could act as disturbing elements.

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FIG. 1 TREATMENT OF SEED-BLANKET REACTOR

FIG. 2 STABILITY CRITERIA FOR XENON OSCILLATION

WAPD-TM-139

FIG. 3 PERIOD OF JUST CRITICAL XENON OSCILLATION

SEED-BLANKET REACTOR WITH
CORNER CONTROL RODS INSERTED

EQUIVALENT 4 POINT MODEL

FIG. 4 THE FOUR POINT MODEL

FIG. 5 DIFFERENTIAL TREATMENT OF ANNULAR SEED

FIG.6 CORRECTION FACTOR FOR FAST COUPLING: $n = 1$

FIG. 7 CORRECTION FACTOR FOR FAST COUPLING: $n = 2$

APPENDIX

EVALUATION OF THE LEAKAGE INTEGRAL

In Section IV it is necessary to evaluate the following integral

$$\int_{y'} \frac{\sin 2\pi ny/L - \sin 2\pi ny'/L}{\cos (y' - y)/L} dy' \quad (A-1)$$

This integral can be evaluated by a change of variable. Let

$$\xi = \frac{y' - y}{L} \pi \quad (A-2)$$

$$dy' = \frac{L d\xi}{\pi} \quad (A-3)$$

$$y' = y + \frac{L\xi}{\pi} \quad (A-4)$$

and

$$\sin \frac{2\pi ny'}{L} = \sin 2\xi n \cos \frac{2\pi ny}{L} + \cos 2\xi n \sin \frac{2\pi ny}{L} \quad (A-5)$$

The integral is then

$$\frac{L}{\pi} \int_0^\pi \frac{\sin 2\pi ny'/L - \sin 2\xi n \cos 2\pi ny/L - \cos 2\xi n \sin 2\pi ny/L}{\sin \xi} d\xi \quad (A-6)$$

If we are to obtain a solution for $b_n(s)$, the terms involving $\cos \frac{2\pi ny}{L}$ will have to drop out. Consequently, we will investigate them first. They are

$$\frac{L}{\pi} \int_0^\pi \frac{\sin 2\xi n d\xi}{\sin \xi} = I \quad (A-7)$$

This integral can be evaluated by letting $p = 2n$ and using the expansion

$$\begin{aligned}
\frac{\sin p x dx}{\sin x} &= 2 \int \cos (p-1) x dx + \int \frac{\sin(p-2)x dx}{\sin x} & (A-8) \\
&= 2 \int \cos (p-1) x dx + 2 \int \cos(p-3)x dx \\
&\quad + 2 \int \cos (p-5) x dx + 2 \int \cos(p-7)x dx \\
&\quad + \dots \text{etc.}
\end{aligned}$$

Let us consider the definite integral required

$$\begin{aligned}
I &= -\frac{L}{\pi} \int_0^{\pi(p-1)} \frac{\cos Z dZ}{(p-1)} & (A-9) \\
&\quad + \frac{L}{\pi} \int_0^{\pi(p-3)} \frac{\cos Z dZ}{(p-3)} \\
&\quad + \dots
\end{aligned}$$

This reduces to

$$\begin{aligned}
\frac{I\pi}{L} &= \sin\pi(p-1) - \sin 0 + \sin\pi(p-3) - \sin 0 & (A-10) \\
&\quad + \sin\pi(p-5) - \sin 0 + \dots
\end{aligned}$$

Now $p = 2n$ and hence will always be an integer and all the above terms are identically zero. Consequently, only the terms involving $\sin \frac{2\pi ny}{L}$ remain when the integral of Eq (IV-17) is evaluated. The nonzero part of the integral in Eq (A-1) will be designated as

$$\sin \frac{2\pi ny}{L} I_n \quad (A-11)$$

where

$$I_n = \frac{L}{\pi} \int_0^{\pi} \frac{1 - \cos 2n\xi}{\sin \xi} d\xi \quad (A-12)$$

It is now only necessary to evaluate the integral I_n . The values of I_n are

$$I_1 = \frac{L_1}{\pi} \quad (A-13)$$

$$I_2 = \frac{L_1}{\pi} (1 + 1/3) \quad (A-14)$$

$$I_3 = \frac{L_1}{\pi} (1 + 1/3 + 1/5) \quad (A-15)$$

$$I_n = \frac{L_1}{\pi} (1 + 1/3 + 1/5 + \dots + 1/2n-1) \quad (A-16)$$