BROOKHAVEN NATIONAL LABORATORY
Associated Universities, Inc.
Upton, New York

CRISP 74-16

ACCELERATOR DEPARTMENT
Informal Report

OBSERVABLES WITH LINEARLY COUPLED OSCILLATORS

A.W. Chao and M. Month

August 12, 1974

MASTER

ABSTRACT

The theory of linear coupled oscillators as it relates to a synchrotron or storage ring is reviewed and extended. A correspondence is made between certain derived variables and physical observables. Considered in particular are the beam envelope, the filtered response to the excitation of a coherent betatron oscillation, and the influence on the transverse beam shape (i.e. the two-dimensional transverse density distribution).

NOTICE

This report was prepared as an account of work sponsored by the United States Government. Neither the United States nor the United States Atomic Energy Commission, nor any of their employees, nor any of their contractors, subcontractors, or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness or usefulness of any information, apparatus, product or process disclosed, or represents that its use would not infringe privately owned rights.

NOTICE

This report was prepared as an account of work sponsored by the United States Government. Neither the United States nor the United States Atomic Energy Commission, nor any of their employees, nor any of their contractors, subcontractors, or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness or usefulness of any information, apparatus, product or process disclosed, or represents that its use would not infringe privately owned rights.

I. INTRODUCTION

The basic theory of linear coupled oscillators as it relates to a synchrotron or storage ring is reviewed and extended. A correspondence is made between certain derived variables and physical observables. In particular, we discuss the beam envelope, the filtered response to the excitation of a coherent betatron oscillation, and the influence on the transverse beam shape (i.e. the two-dimensional transverse density distribution). We will use betatron resonance theory to separate the slowly and rapidly oscillating components.¹ Only difference resonances will concern us, although the method is equally applicable to sum resonances. The point is that linear sum resonances are universally avoided by choice of the betatron tune, whereas difference resonances are not. When large tune spreads are required as in slow resonant extraction from synchrotrons in order to control the spill rate or in storage rings in order to suppress transverse beam instabilities, difference resonances cannot be avoided. In fact, in storage rings, stacking the beam along the main diagonal is desirable in providing the largest region in vertical-horizontal tune space between nonlinear sum resonances. (In electron storage rings the beam positioning near the coupling line is important because it is the linear coupling which creates vertical beam size which is necessary for transverse stability.) Although difference resonances cannot induce growing betatron oscillations, they can produce beam growth especially when the vertical and horizontal beam emittances differ substantially.

The difference resonances we will study are defined by the integer separation of the vertical and horizontal tunes. If the integer separation is zero, then only the 0th azimuthal harmonic of the skew gradient distribution excites the resonance. In this case, if we use the smoothing approximation for the β -function oscillations, the coupling equations can be solved exactly. However, the resonance technique gives the equivalent results in the case that both the tune split and the 0th azimuthal harmonic are small compared to the tune. This is always true. If the integer separation is nonzero, but some value k , then the excitation azimuthal harmonic is the k^{th} rather than the 0th.

We develop the theory for three situations. First, we consider a beam of particles and derive the change in the beam envelope when the coupling is

1. See, for example, A. Schoch, CERN Report, CERN 57-23 (1958).

introduced. We show that only three parameters define the physical situation: (1) the beat frequency, (2) the ratio of the coupling strength to the non-integer part of the tune split, and (3) the emittance ratio between horizontal and vertical planes. Secondly, we consider the problem of a beam excitation of free betatron oscillations when coupling is present. We include here the influence of tune spreads in the beam, both horizontal and vertical separately. Finally, we develop the formalism for an analysis of the effect of coupling on the beam density distribution, specifically discussing the influence on the horizontal-vertical beam shape.

II. THEORY OF LINEAR COUPLING

Linearized Equations of Motion

For x-z coupling provided by a skew quadrupole distribution, the equations of motion are

$$\begin{aligned} \frac{d^2 x}{ds^2} + K_x(s)x &= M(s)z \\ \frac{d^2 z}{ds^2} + K_z(s)z &= M(s)x \end{aligned} \quad (1)$$

Suppose we have solved the uncoupled case. Let v_x and v_z be the unperturbed tunes and

$$\theta_x = \frac{1}{v_x} \int_0^s \frac{ds'}{\beta_x(s')}, \quad \theta_z = \frac{1}{v_z} \int_0^s \frac{ds'}{\beta_z(s')}.$$

The Courant-Snyder transformation²

$$x = x_1 \sqrt{\beta_x}, \quad z = z_1 \sqrt{\beta_z}$$

then gives

$$\begin{aligned} \frac{d^2 x_1}{d\theta_x^2} + v_x^2 x_1 &= v_x^2 \beta_x^{3/2} \beta_z^{1/2} M z_1 \\ \frac{d^2 z_1}{d\theta_z^2} + v_z^2 z_1 &= v_z^2 \beta_z^{3/2} \beta_x^{1/2} M x_1 \end{aligned} \quad (2)$$

2. E.D. Courant and H.S. Snyder, Ann. of Phys. 3, 1 (1958).

Without coupling, (2) has the solution

$$\begin{aligned} x_1 &= a_x e^{i\nu_x \theta_x} + \text{c.c.} \\ z_1 &= a_z e^{i\nu_z \theta_z} + \text{c.c.} \end{aligned} \quad (3)$$

where a_x and a_z are complex constants determined by initial conditions. In the presence of coupling, (3) no longer holds. However, if the coupling is weak, (3) can still be an approximate solution of (2) with a_x and a_z now slowly varying functions of θ_x and θ_z respectively. Note that we have ignored synchrotron oscillations, which means, among other things, that the particle tunes do not change.

Consider a difference resonance, meaning

$$\begin{aligned} \nu_x - \nu_z &= k + \delta \quad (|\delta| \ll 1) \\ k &= \text{integer} \end{aligned} \quad (4)$$

Substituting (3) into (2), one obtains

$$\begin{aligned} \left(\frac{d^2 a_x}{d\theta_x^2} + 2i\nu_x \frac{da_x}{d\theta_x} \right) e^{i\nu_x \theta_x} + \text{c.c.} &= \nu_x^2 \beta_x^{3/2} \beta_z^{1/2} M a_z e^{i\nu_z \theta_z} + \text{c.c.} \\ \left(\frac{d^2 a_z}{d\theta_z^2} + 2i\nu_z \frac{da_z}{d\theta_z} \right) e^{i\nu_z \theta_z} + \text{c.c.} &= \nu_z^2 \beta_z^{3/2} \beta_x^{1/2} M a_x e^{i\nu_x \theta_x} + \text{c.c.} \end{aligned}$$

Take the first of these equations. Multiply both sides by $e^{-i\nu_x \theta_x}$. Discard the second order term $d^2 a_x / d\theta_x^2$. The remaining equation is

$$\begin{aligned} 2i \frac{da_x}{d\theta} - 2i \frac{da_x^*}{d\theta} e^{-2i\nu_x \theta_x} &= \\ = R \beta_x^{1/2} \beta_z^{1/2} M (a_z e^{i\nu_z \theta_z - i\nu_x \theta_x} + a_z^* e^{-i\nu_z \theta_z - i\nu_x \theta_x}) \end{aligned} \quad (5)$$

where we have made a change of variable from θ_x to

$$\theta = \int_0^s \frac{ds'}{R} \quad ,$$

with $C = 2\pi R$ = the circumference of the equilibrium orbit.

We now average (5) over many revolutions. Fast oscillating terms are thus removed. Keeping only slowly varying components, (5) becomes [note that a prime means taking the derivative with respect to θ]:

$$a'_x \approx iQ a_z e^{-i\theta\delta} \quad .$$

In exactly the same manner, we find

$$a'_z \approx iQ^* a_x e^{i\theta}$$

We have defined

$$Q = -\frac{R}{4\pi} \int_0^{2\pi} d\theta \, M \beta_x^{\frac{1}{2}} \beta_z^{\frac{1}{2}} e^{-ik\theta} e^{iv_z(\theta_z - \theta) - iv_x(\theta_x - \theta)} \quad (6)$$

as the coupling coefficient. Note that aside from the factor $e^{-ik\theta}$, the integrand of (6) is periodic in θ with period 2π and is Fourier analyzable.

Writing $Q = |Q| e^{iq}$, we have

$$a'_x = i|Q| a_z e^{-i\delta[q - (q/\delta)]}$$

$$a'_z = i|Q| a_x e^{i\delta[q - (q/\delta)]}$$

and making the variable change $\theta \rightarrow \theta + (q/\delta)$, the resulting equations are seen to be only dependent on the magnitude of Q . Therefore, without losing any generality, we shall assume Q is real and positive. The equations of motion are then

$$\begin{aligned} a'_x &= iQ a_z e^{-i\theta} \\ a'_z &= iQ a_x e^{i\theta} \end{aligned} \quad (7)$$

$$Q \geq 0$$

Constants of the Motion

Before solving (7), it is interesting to note that there exist two constants of the motion. Let

$$\begin{aligned} a_x &= \sqrt{I_x} e^{i[\phi_x - (\delta/2)\theta]} \\ a_z &= \sqrt{I_z} e^{i[\phi_z + (\delta/2)\theta]} \end{aligned} \quad (8)$$

or equivalently

$$\begin{aligned} x_1 &= 2\sqrt{I_x} \cos(v_x \theta_x + \phi_x - \frac{\delta}{2} \theta) \\ z_1 &= 2\sqrt{I_z} \cos(v_z \theta_z + \phi_z + \frac{\delta}{2} \theta) \end{aligned}$$

where I_x , I_z , ϕ_x , ϕ_z are slowly varying functions of θ . Note that with coupling, I_x and I_z are no longer equal to $\frac{1}{2}[x_1^2 + (dx_1/v_x d\theta)^2]$ and

$\frac{1}{2}[z_1^2 + (dz_1/v_z d\theta_z)^2]$. The association of I_x and I_z to "emittances" becomes ambiguous. The meaning of $\sqrt{I_x}$ and $\sqrt{I_z}$ is simply the oscillation envelopes of x_1 and z_1 for a given particle.

Substituting (8) into (7), and equating real and imaginary parts, one finds

$$I'_x = 2Q \sqrt{I_x I_z} \sin (\phi_x - \phi_z) = - I'_z$$

$$\phi'_x = \frac{\delta}{2} + Q \sqrt{\frac{I_z}{I_x}} \cos (\phi_x - \phi_z)$$

$$\phi'_z = -\frac{\delta}{2} + Q \sqrt{\frac{I_x}{I_z}} \cos (\phi_x - \phi_z) .$$

It can be shown by direct differentiation that

$$C_1 = I_x + I_z$$

$$C_2 = 2Q \sqrt{I_x I_z} \cos (\phi_x - \phi_z) + \delta I_x \quad (9)$$

are constants of the motion. It should be emphasized that C_1 and C_2 are constants only after averaging over many revolutions.

Solving the Linearized Equations

Now our job is to solve (7). Let

$$\tilde{a}_z = a_z e^{-i\delta\theta} \quad (10)$$

Equation (7) becomes

$$a'_x = iQ \tilde{a}_z$$

$$\tilde{a}'_z = iQ a_x - i\delta \tilde{a}_z .$$

Hence

$$a'' + i\delta a' + Q^2 a = 0 , \quad a = a_x, \tilde{a}_z .$$

The solution is then

$$a_x = A e^{i\lambda_+ \theta} + B e^{i\lambda_- \theta} \quad (11)$$

$$\tilde{a}_z = \frac{1}{Q} (A \lambda_+ e^{i\lambda_+ \theta} + B \lambda_- e^{i\lambda_- \theta}) ,$$

where A and B are complex constants to be determined by initial conditions and $\lambda_{\pm} = -\delta/2 \pm \sqrt{(\delta/2)^2 + Q^2}$.

The initial conditions can be expressed in two separate ways.

(i) Suppose $x_1(0)$, $x_1'(0)$, $z_1(0)$ and $z_1'(0)$ are known.

Taking explicitly into account the differences among θ_x , θ_z , and θ , it is straightforward to show that

$$\begin{aligned} \frac{x}{\sqrt{\beta_x}} &= \frac{1}{\lambda_+ - \lambda_-} \{ [Qz_1(0) - \lambda_- x_1(0)] \cos(\nu_x \theta_x + \lambda_+ \theta) + Qf_2 \sin(\nu_x \theta_x + \lambda_+ \theta) \\ &\quad - [Qz_1(0) - \lambda_+ x_1(0)] \cos(\nu_x \theta_x + \lambda_- \theta) - Qf_1 \sin(\nu_x \theta_x + \lambda_- \theta) \} \\ \frac{z}{\sqrt{\beta_z}} &= \frac{1}{\lambda_+ - \lambda_-} \{ [\lambda_+ z_1(0) + Qx_1(0)] \cos(\nu_z \theta_z - \lambda_- \theta) + \lambda_+ f_2 \sin(\nu_z \theta_z - \lambda_- \theta) \\ &\quad - [\lambda_- z_1(0) + Qx_1(0)] \cos(\nu_z \theta_z - \lambda_+ \theta) - \lambda_- f_1 \sin(\nu_z \theta_z - \lambda_+ \theta) \} \end{aligned} \quad (12)$$

where

$$\begin{aligned} f_1 &= \left[z_1'(0) \left(\frac{R}{\beta_x(0)} + \lambda_+ \right) + \frac{Q}{\lambda_-} x_1'(0) \left(\frac{R}{\beta_z(0)} - \lambda_- \right) \right] / \left(\frac{R^2}{\beta_x(0)\beta_z(0)} - Q^2 \right) \\ f_2 &= \left[z_1'(0) \left(\frac{R}{\beta_x(0)} + \lambda_- \right) + \frac{Q}{\lambda_+} x_1'(0) \left(\frac{R}{\beta_z(0)} - \lambda_+ \right) \right] / \left(\frac{R^2}{\beta_x(0)\beta_z(0)} - Q^2 \right) \end{aligned}$$

Another proof of (12) can be found in the Appendix.

(ii) Suppose I_{x0} , I_{z0} , $\tilde{\Phi}_{x0}$ and $\tilde{\Phi}_{z0}$ are known, where

$$\begin{aligned} a_x(0) &\equiv \sqrt{I_{x0}} e^{i\tilde{\Phi}_{x0}} = A + B \\ a_z(0) &\equiv \sqrt{I_{z0}} e^{i\tilde{\Phi}_{z0}} = \frac{1}{Q} (\lambda_+ A + \lambda_- B) \end{aligned} \quad (13)$$

We find then that

$$\begin{aligned} a_x &= \frac{1}{\lambda_+ - \lambda_-} \{ (Q\sqrt{I_{z0}} e^{i\tilde{\Phi}_{z0}} - \lambda_- \sqrt{I_{x0}} e^{i\tilde{\Phi}_{x0}}) e^{i\lambda_+ \theta} \\ &\quad - (Q\sqrt{I_{z0}} e^{i\tilde{\Phi}_{z0}} - \lambda_+ \sqrt{I_{x0}} e^{i\tilde{\Phi}_{x0}}) e^{i\lambda_- \theta} \} \\ \tilde{a}_z &= \frac{1}{\lambda_+ - \lambda_-} \{ (\lambda_+ \sqrt{I_{z0}} e^{i\tilde{\Phi}_{z0}} + Q\sqrt{I_{x0}} e^{i\tilde{\Phi}_{x0}}) e^{i\lambda_+ \theta} \\ &\quad - (\lambda_- \sqrt{I_{z0}} e^{i\tilde{\Phi}_{z0}} + Q\sqrt{I_{x0}} e^{i\tilde{\Phi}_{x0}}) e^{i\lambda_- \theta} \} \end{aligned} \quad (14)$$

and

$$\begin{aligned} \frac{x}{\sqrt{\beta_x}} &= \frac{2}{\lambda_+ - \lambda_-} \{ Q\sqrt{I_{z0}} [\cos (\nu_x \theta_x + \lambda_+ \theta + \phi_{z0}) - \cos (\nu_x \theta_x + \lambda_- \theta + \phi_{z0})] \\ &\quad + \sqrt{I_{x0}} [\lambda_+ \cos (\nu_x \theta_x + \lambda_- \theta + \phi_{x0}) - \lambda_- \cos (\nu_x \theta_x + \lambda_+ \theta + \phi_{x0})] \} , \\ (15) \\ \frac{z}{\sqrt{\beta_z}} &= \frac{2}{\lambda_+ - \lambda_-} \{ Q\sqrt{I_{x0}} [\cos (\nu_z \theta_z - \lambda_- \theta + \phi_{x0}) - \cos (\nu_z \theta_z - \lambda_+ \theta + \phi_{x0})] \\ &\quad + \sqrt{I_{z0}} [\lambda_+ \cos (\nu_z \theta_z - \lambda_- \theta + \phi_{z0}) - \lambda_- \cos (\nu_z \theta_z - \lambda_+ \theta + \phi_{z0})] \} . \end{aligned}$$

Envelope of a Single Particle

Equations (12) and (15) can be used to find the envelopes I_x and I_z . But a more direct method is to use Eq. (8):

$$\sqrt{I_x} = |a_x| , \quad \sqrt{I_z} = |a_z| = |\tilde{a}_z| .$$

Equation (14) then gives

$$\begin{aligned} I_x &= I_{x0} + \frac{4Q \sin \Theta}{\delta^2 + 4Q^2} \{ [Q(I_{z0} - I_{x0}) + \delta\sqrt{I_{x0}I_{z0}} \cos \phi] \sin \Theta \\ &\quad + \sqrt{I_{x0}I_{z0}} (\delta^2 + 4Q^2) \sin \phi \cos \Theta \} \end{aligned} \quad (16)$$

$$I_z = I_{x0} + I_{z0} - I_x ,$$

where $\Theta = \frac{1}{2}(\lambda_+ - \lambda_-)\theta$

$$\phi = \phi_{x0} - \phi_{z0} .$$

Equation (16) gives explicitly the beating of envelopes I_x and I_z for a given particle with initial conditions I_{x0} , I_{z0} and $\phi = \phi_{x0} - \phi_{z0}$. The beating frequency is $\lambda_+ - \lambda_-$. Since we have insisted that I_x and I_z be slowly varying, our approximation holds only when $\lambda_+ - \lambda_- = \sqrt{\delta^2 + 4Q^2}$ is small compared with unity.

Beam Envelope

Consider a set of particles, all having the same I_{x0} and I_{z0} but random ϕ_{x0} and ϕ_{z0} . By varying ϕ , we can find the maximum and minimum values of I_x at any given Θ . This will give us envelopes of I_x and I_z for the particle set. Note that because of (9), maxima of I_z occur when I_x assumes its minimum value. The envelopes will be functions of I_{x0} , I_{z0} and Θ . The extrema

of I_x occur when

$$\tan \bar{\phi} = \frac{\sqrt{\delta^2 + 4Q^2}}{\delta} \cot \Theta ,$$

with maxima and minima given by $\delta \cos \bar{\phi} > 0$ and < 0 respectively. Substituting into (16), we obtain the envelopes of the set:

$$\begin{aligned} \frac{(I_x)_{\text{env}}}{I_{x0}} &= 1 + \frac{1}{r^2} - \frac{1}{1+t} \left[\frac{1}{r} \sqrt{t + \cos^2 \Theta} - |\sin \Theta| \right]^2 \\ \frac{(I_z)_{\text{env}}}{I_{z0}} &= 1 + r^2 - \frac{1}{1+t} \left[r \sqrt{t + \cos^2 \Theta} - |\sin \Theta| \right]^2 , \end{aligned} \quad (17)$$

where $t = \delta^2/4Q^2$ and $r = \sqrt{I_{x0}/I_{z0}}$.

Note that the envelopes, when properly normalized, depend only on three variables r , t and Θ . Also note that because of (9) and the fact that I_x and I_z are positive, one immediately finds a "grand" maximum value for both I_x/I_{z0} and I_z/I_{z0} , namely $1 + r^2$. It is interesting to note that $(I_x)_{\text{env}}$ and $(I_z)_{\text{env}}$ can indeed reach this grand maximum. From (17), we find that

$$\frac{(I_x)_{\text{env}}}{I_{z0}} = 1 + r^2 \quad \Leftrightarrow \quad \sqrt{t + \cos^2 \Theta} = r |\sin \Theta| ,$$

which possesses a solution for Θ as long as

$$t \leq r^2 . \quad (18)$$

Similarly, $(I_z)_{\text{env}}/I_{z0}$ reaches $1 + r^2$ if and only if

$$t \leq \frac{1}{r^2} . \quad (19)$$

In Fig. 1(a)-(c), we have plotted the envelopes as a function of Θ for different values of r and t . If we interpret I_{x0} and I_{z0} to be the uncoupled beam emittances, then these curves correspond to the beam envelopes for the various conditions on t and r .

Beam Excitation of Free Betatron Oscillations

Consider a beam excited by a horizontal kick. The resulting coherent motion can be treated as a single particle free betatron oscillation with initial conditions given by

$$x_1 = 0 , \quad x'_1 = W , \quad z_1 = 0 \text{ and } z'_1 = 0 .$$

(a) $r = \sqrt{I_{x0}/I_{z0}} = 1$. The curves are for $(I_x)_{env}/I_{x0} = (I_z)_{env}/I_{z0}$. The dash-dot curve is for $t = r^2 = 1/r^2$. The grand maximum value is $1 + r^2 = 2$.

(b) $r = 2$. The curves are for $(I_z)_{env}/I_{z0}$. The dash-dot curve corresponds to $t = 1/r^2$.

(c) $r = 2$. The curves are for $(I_x)_{env}/I_{z0}$. The dash-dot curve corresponds to $t = r^2$.

Fig. 1. Beam envelopes with linear coupling.

The subsequent motion is described by (12), i.e.

$$\begin{aligned} \frac{x}{\beta_x} &= f_3 \left[\lambda_+ \left(\frac{R}{\beta_z(0)} - \lambda_- \right) \sin (\nu_x \theta_x + \lambda_- \theta) \right. \\ &\quad \left. - \lambda_- \left(\frac{R}{\beta_z(0)} - \lambda_+ \right) \sin (\nu_x \theta_x + \lambda_+ \theta) \right] \\ \frac{z}{\beta_z} &= f_3 Q \left[\left(\frac{R}{\beta_z(0)} - \lambda_+ \right) \sin (\nu_z \theta_z - \lambda_- \theta) \right. \\ &\quad \left. - \left(\frac{R}{\beta_z(0)} - \lambda_- \right) \sin (\nu_z \theta_z - \lambda_+ \theta) \right] , \end{aligned} \quad (20)$$

where

$$f_3 = \frac{W}{(\lambda_+ - \lambda_-) \left(\frac{R^2}{\beta_x(0)\beta_z(0)} - Q^2 \right)} .$$

We can also factorize (20) into a slow oscillating "envelope" component and a fast oscillating component. The result is

$$\begin{aligned} \frac{x}{\beta_x} &= \frac{2RQf_3}{\beta_z(0)} \left[1 + t - \sin^2 \Theta \left(1 + \frac{\delta \beta_z(0)}{R} - \frac{Q^2 \beta_z^2(0)}{R^2} \right) \right]^{\frac{1}{2}} \cos (\nu_x \theta_x - \frac{\delta}{2} \theta + f_4) \\ \frac{z}{\beta_z} &= 2f_3 Q^2 \left[1 + t + \sin^2 \Theta \left(\frac{R^2}{\beta_z^2(0)Q^2} + \frac{R\delta}{\beta_z(0)Q^2} - 1 \right) \right]^{\frac{1}{2}} \cos (\nu_z \theta_z + \frac{\delta}{2} \theta + f_5) , \end{aligned} \quad (21)$$

with f_4 and f_5 the appropriate phase angles. The beam will then oscillate with a frequency $\nu_x - \delta/2$ in the x-plane and a frequency $\nu_z + \delta/2$ in the z-plane. [Note that θ , θ_x and θ_z all run through 2π in one revolution.] The beating frequencies, however, are the same in both planes, namely $(\lambda_+ - \lambda_-) = 2Q/(1+t)$.

If we neglect δ and Q as compared with $R/\beta_x(0)$ and $R/\beta_z(0)$, Eq. (21) becomes

$$\begin{aligned} \frac{x}{\beta_x} &\approx \frac{2RQf_3}{\beta_z(0)} \sqrt{t + \cos^2 \Theta} \cos (\nu_x \theta_x - \frac{\delta}{2} \theta + f_4) \\ \frac{z}{\beta_z} &\approx \frac{2RQf_3}{\beta_z(0)} |\sin \Theta| \cos (\nu_z \theta_z + \frac{\delta}{2} \theta + f_5) . \end{aligned}$$

The envelope factor for z/β_z can always be approximately zero at some Θ , while the envelope factor for x/β_x can be approximately zero only when $t \ll 1$, i.e. $|\delta| \ll Q$.

Damping of an Excited Free Betatron Oscillation by Tune Spreads

If there are tune spreads in ν_x and ν_z , and we measure the betatron amplitude of the beam charge center, Eqs. (20) and (21) have to be averaged over the tune spread distributions. The measured $\langle x/\sqrt{\beta_x} \rangle$ of the beam will diminish in time because the factor $\cos [\nu_x \theta_x - (\delta/2)\theta + f_4]$, when averaged over a tune spread becomes small. The "damping time" will be roughly given by

$$\pi \sim \left[\Delta \left(\nu_x - \frac{\delta}{2} \right) \right] \theta_D \sim \frac{\Delta(\nu_x + \nu_z)}{2} \theta_D, \quad (22)$$

where $\Delta(\nu_x + \nu_z)$ is a spread in $\nu_x + \nu_z$. For a Gaussian distribution in both ν_x and ν_z , we have $\Delta(\nu_x + \nu_z) = \sqrt{(\Delta\nu_x)^2 + (\Delta\nu_z)^2}$. The same θ_D will be obtained if $z/\sqrt{\beta_z}$ is considered.

However, Eq. (22) is not valid when the beating envelope factors of (21) are taken into account. Indeed, in a time θ_D , Θ changes by $(\theta_D/2)\sqrt{\delta^2 + Q^2} > (\theta_D/2)|\delta|$. Since δ is invariably of the same order as $\Delta(\nu_x + \nu_z)$, one sees that the beating factors in (21) change significantly in a time θ_D and could interfere strongly with the measurements of the damping time. In order to properly include the tune spreads, we have to take the average of (20) over the tune spread distribution $N(\nu_x, \nu_z)$.

For simplicity, we neglect Q and δ compared to $R/\beta_x(0)$ or $R/\beta_z(0)$. The average displacements are found to be

$$\begin{aligned} \left\langle \frac{x}{\sqrt{\beta_x}} \right\rangle &\approx \frac{W\beta_x(0)}{R} \int d\nu_x \int d\nu_z N(\nu_x, \nu_z) \frac{1}{\sqrt{\delta^2 + 4Q^2}} \{ \lambda_+ \sin(\nu_x \theta_x + \lambda_- \theta) \\ &\quad - \lambda_- \sin(\nu_x \theta_x + \lambda_+ \theta) \} \\ \left\langle \frac{z}{\sqrt{\beta_z}} \right\rangle &\approx \frac{W\beta_z(0)Q}{R} \int d\nu_x \int d\nu_z N(\nu_x, \nu_z) \frac{1}{\sqrt{\delta^2 + 4Q^2}} \{ \sin(\nu_z \theta_z - \lambda_- \theta) \\ &\quad - \sin(\nu_z \theta_z - \lambda_+ \theta) \}, \end{aligned} \quad (23)$$

with $\delta = \nu_x - \nu_z - k$ and $\lambda_{\pm} = -\delta/2 \pm \sqrt{(\delta/2)^2 + Q^2}$.

As an example, we consider a factorized Gaussian form for $N(\nu_x, \nu_z)$:

$$N(\nu_x, \nu_z) d\nu_x d\nu_z = \frac{1}{2\pi\sigma_x\sigma_z} e^{-(\nu_x - \langle \nu_x \rangle)^2/2\sigma_x^2 - (\nu_z - \langle \nu_z \rangle)^2/2\sigma_z^2} d\nu_x d\nu_z \quad (24)$$

where σ_x and σ_z are the rms tune spreads and $\langle v_x \rangle - \langle v_z \rangle = k + \langle \delta \rangle$, $|\langle \delta \rangle| \ll 1$.

Since we are interested only in long-term behavior (e.g. beam damping in time $\theta \sim \theta_D$), we shall let $\theta_x = \theta_z = \theta$. Substituting (24) into (23), changing variables $v_x, v_z \rightarrow \delta, \epsilon$, where $\epsilon = v_x + v_z$, and then integrating out ϵ , we obtain

$$\begin{aligned} \langle \frac{x}{\beta_x} \rangle &= \frac{W\beta_x(0)}{\sqrt{2\pi} R\sigma} e^{-\sigma_x^2 \sigma_z^2 \theta^2 / 2\sigma^2} \int \frac{d\delta}{\sqrt{\delta^2 + 4Q^2}} e^{-(\delta - \langle \delta \rangle)^2 / 2\sigma^2} \\ &\quad \left\{ \lambda_+ \sin \left[f_6 - \sqrt{\left(\frac{\delta}{2}\right)^2 + Q^2} \right] \theta \right. \\ &\quad \left. - \lambda_- \sin \left[f_6 + \sqrt{\left(\frac{\delta}{2}\right)^2 + Q^2} \right] \theta \right\} \\ \langle \frac{z}{\beta_z} \rangle &= \frac{2W\beta_x(0)Q}{\sqrt{2\pi} R\sigma} e^{-\sigma_x^2 \sigma_z^2 \theta^2 / 2\sigma^2} \int \frac{d\delta}{\sqrt{\delta^2 + 4Q^2}} e^{-(\delta - \langle \delta \rangle)^2 / 2\sigma^2} \\ &\quad \sin \sqrt{\left(\frac{\delta}{2}\right)^2 + Q^2} \theta \cos f_6 \theta, \end{aligned} \quad (25)$$

where

$$f_6 = \frac{1}{2} \left[\left(\frac{\sigma_x^2 - \sigma_z^2}{\sigma^2} \right) (\delta - \langle \delta \rangle) + \langle v_x \rangle + \langle v_z \rangle + k \right] \text{ and } \sigma^2 = \sigma_x^2 + \sigma_z^2.$$

Our result (25) can be simplified in the following two cases:

(i) $Q = 0$, i.e. no coupling:

$$\begin{aligned} \langle \frac{x}{\beta_x} \rangle &= \frac{W\beta_x(0)}{R} e^{-\sigma_x^2 \theta^2 / 2} \sin \langle v_x \rangle \theta \\ \langle \frac{z}{\beta_z} \rangle &= 0. \end{aligned} \quad (26)$$

(ii) $Q = |\langle \delta \rangle| \gg \sigma$, i.e. strong coupling: We neglect δ^2 compared with Q^2 .

$$\begin{aligned} \langle \frac{x}{\beta_x} \rangle &\approx \frac{W\beta_x(0)}{R} e^{-\sigma^2 \theta^2 / 8} \cos Q\theta \sin (\langle v_x \rangle + \langle v_z \rangle + k) \frac{\theta}{2} \\ \langle \frac{z}{\beta_z} \rangle &\approx \frac{W\beta_x(0)}{R} e^{-\sigma^2 \theta^2 / 8} \sin Q\theta \cos (\langle v_x \rangle + \langle v_z \rangle + k) \frac{\theta}{2}. \end{aligned} \quad (27)$$

In general, numerical calculations are needed for (25). However, we can separate the fast oscillating sinusoidal factor from the slower time dependence. On the other hand, there is a complicated interference between the

damping due to tune spreads and the "energy exchange" due to the coupling, where explicit separation is not generally possible.

Equation (25) can be rewritten in a form explicitly showing its "slowly varying" and "fast oscillating" parts. Writing

$[u, v] = \sqrt{2\pi} R/W\beta_x(0) [\langle x/\beta_x \rangle, \langle z/\beta_z \rangle]$, we have

$$u, v = S_{u,v} \cos [\frac{1}{2}(\langle v_x \rangle + \langle v_z \rangle + k) \theta + T_{u,v}] , \quad (28)$$

where S and T are slowly varying amplitude and phase. An appropriate electronic filter deriving its input signal from a pick-up station can be designed to produce an output signal which is essentially the function (28), with θ interpreted as real time through

$$\theta = \omega_0 t , \quad \omega_0 = \text{revolution angular frequency.}$$

Of course, since there is only one pick-up station, the output frequency corresponds only to the noninteger part of the betatron frequency. Thus, we can show from (25) and (28) that the filtered signal is simply

$$u, v = S_{u,v} \cos [\bar{\mu} \bar{\theta} + T_{u,v}] , \quad (29)$$

where $\bar{\mu} = 1/Q \times \text{noninteger part of } \frac{1}{2}[\langle v_x \rangle + \langle v_z \rangle + k]$, while S and T are slowly varying functions of θ , and explicitly functions of

$$\begin{aligned} \bar{\theta} &= Q\theta \\ q &= Q/\sigma \\ \langle t \rangle &= \langle \delta \rangle / 2Q \\ s_z^2 &= \sigma_z^2 / \sigma^2 . \end{aligned} \quad (30)$$

and

In terms of these variables, from (23) and (24), we obtain explicitly for S and T ,

$$S_{u,v} = \exp \left[\frac{-s_z^2 \bar{\theta}^2}{2q^2} \right] (A_{u,v}^2 + B_{u,v}^2)^{\frac{1}{2}} , \quad (31)$$

$$T_{u,v} = \begin{cases} \tan^{-1} \left(\frac{B_{u,v}}{A_{u,v}} \right) , & \text{if } A_{u,v} > 0 \\ \pi + \tan^{-1} \left(\frac{B_{u,v}}{A_{u,v}} \right) , & \text{if } A_{u,v} < 0 ; \end{cases}$$

where,

$$\begin{aligned}
 A_u &= 2q \int_{-\infty}^{\infty} dt e^{-2q^2(t-\langle t \rangle)^2} \left\{ \sin(1 - 2s_z^2)(t - \langle t \rangle) \bar{\theta} \cos \sqrt{1+t^2} \bar{\theta} \right. \\
 &\quad \left. + \frac{t}{\sqrt{1+t^2}} \cos(1 - 2s_z^2)(t - \langle t \rangle) \bar{\theta} \sin \sqrt{1+t^2} \bar{\theta} \right\}, \\
 B_u &= 2q \int_{-\infty}^{\infty} dt e^{-2q^2(t-\langle t \rangle)^2} \left\{ \cos(1 - 2s_z^2)(t - \langle t \rangle) \bar{\theta} \cos \sqrt{1+t^2} \bar{\theta} \right. \\
 &\quad \left. - \frac{t}{\sqrt{1+t^2}} \sin(1 - 2s_z^2)(t - \langle t \rangle) \bar{\theta} \sin \sqrt{1+t^2} \bar{\theta} \right\}, \\
 A_v &= 2q \int_{-\infty}^{\infty} \frac{dt}{\sqrt{1+t^2}} e^{-2q^2(t-\langle t \rangle)^2} \cos(1 - 2s_z^2)(t - \langle t \rangle) \bar{\theta} \sin \sqrt{1+t^2} \bar{\theta}, \\
 B_v &= -2q \int_{-\infty}^{\infty} \frac{dt}{\sqrt{1+t^2}} e^{-2q^2(t-\langle t \rangle)^2} \sin(1 - 2s_z^2)(t - \langle t \rangle) \bar{\theta} \sin \sqrt{1+t^2} \bar{\theta}; \quad (32)
 \end{aligned}$$

and we have used the fact that

$$s_x^2 + s_z^2 = 1. \quad (33)$$

The coupling thus perturbs the betatron oscillations on a time scale which is $1/Q$ times slower than that for the natural betatron oscillations (i.e. taking only the nonintegral part of the tune). This means that the observed betatron frequencies are shifted by a small amount, proportional to the coupling strength Q . But, more importantly, the shift in frequency is time dependent. Thus, the presence of the phase shifting term $T(\theta)$ represents a fundamental limit to the measurement of betatron tunes inasmuch as T modulates with the coupling time scale. In addition to the phase shifting term, there is a slow (i.e. on the coupling time scale) amplitude modulation, represented by $S(\theta)$, which is superimposed on the fast betatron oscillation. As can be seen from the expression for $S_{u,v}(\theta)$, the time variation has both coupling and damping contributions, as we will illustrate.

In Figs. 2(a)-(d), we plot S_u and S_v as a function of $\bar{\theta}$ using parameter values which are roughly applicable to a synchrotron such as the AGS at BNL: $Q \approx 0.01$ (a large value), $\sigma \approx 0.01$ (i.e. $q \approx 1.0$), $s_z^2 = 0$, and $\langle t \rangle = \langle \delta \rangle / 2Q = 0, 1, 2$ and 5 . Note that these curves represent envelopes, i.e. the amplitude modulations of the fast betatron oscillations, observable at a pickup electrode, as we have already described. To show the variability in the shapes of these envelopes that can arise under different circumstances, we show in Figs. 3(a)-(d) a sequence similar to that in Figs. 2, but with $s_z^2 = \frac{1}{2}$

Fig. 2. Filtered coherent response to the excitation of a free horizontal betatron oscillation with coupling present. Horizontal and vertical responses are shown, with $S_v = 0$ at $\theta = 0$. The difference in central tunes is varied for (a) to (d). The vertical tune spread is zero and the coupling strength is moderate.

Fig. 3. Filtered coherent response to the excitation of a free horizontal betatron oscillation with coupling present. Horizontal and vertical responses are shown with $S_v = 0$ at $\theta = 0$. The difference in central tunes is varied for (a) to (d). The tune spreads are equal and coupling strength is moderate.

Fig. 4. Filtered coherent response to the excitation of a free horizontal betatron oscillation with coupling present. Horizontal and vertical responses are shown, with $S_v = 0$ at $\theta = 0$. The difference in central tunes is varied for (a) to (d). The vertical tune spread is zero and the coupling strength is large.

(i.e. $\sigma_x = \sigma_z$); while in Figs. 4(a)-(d), the plots are also similar to those in Figs. 2, but with a large coupling strength, $q = 5.0$.

Effect of Coupling on the Beam Distribution

Consider an unperturbed initial beam distribution $D(x_1, dx_1/d\theta_x, z_1, dz_1/d\theta_z)$ in betatron phase space, satisfying

$$D(x_1, \frac{dx_1}{d\theta_x}, z_1, \frac{dz_1}{d\theta_z}) = D[x_1^2 + \left(\frac{dx_1}{v_x d\theta_x}\right)^2, z_1^2 + \left(\frac{dz_1}{v_z d\theta_z}\right)^2] \quad (34)$$

If a skew coupling is switched on at time $\theta = 0$, the betatron distribution will change with time.

For simplicity, we shall assume $\theta_x = \theta_z = \theta$ and $k = 0$ in this section. From Eq. (12), it can be shown that

$$Y(0) = \frac{1}{(\lambda_+ - \lambda_-)} MY(0) \quad (35)$$

where

$$Y \equiv \begin{bmatrix} x_1 \\ x_1' \\ z_1 \\ z_1' \end{bmatrix}$$

$$M = \begin{bmatrix} \lambda_+ C_- - \lambda_- C_+ & \frac{\lambda_- S_+}{f_+} - \frac{\lambda_+ S_-}{f_-} & Q(C_+ - C_-) & Q\left(\frac{S_-}{f_-} - \frac{S_+}{f_+}\right) \\ \lambda_+ f_- S_- - \lambda_- f_+ S_+ & \lambda_+ C_- - \lambda_- C_+ & Q(f_+ S_+ - f_- S_-) & Q(C_+ - C_-) \\ Q(C_+ - C_-) & Q\left(\frac{S_-}{f_-} - \frac{S_+}{f_+}\right) & \lambda_+ C_+ - \lambda_- C_- & \frac{\lambda_- S_-}{f_-} - \frac{\lambda_+ S_+}{f_+} \\ Q(f_+ S_+ - f_- S_-) & Q(C_+ - C_-) & \lambda_+ f_+ S_+ - \lambda_- f_- S_- & \lambda_+ C_+ - \lambda_- C_- \end{bmatrix} \quad (36)$$

with $f_{\pm} = v_x + \lambda_{\pm}$

$$C_{\pm} = \cos f_{\pm} \theta$$

(37)

$$S_{\pm} = \sin f_{\pm} \theta$$

The matrix M, Eq. (36), can be obtained more elegantly by first performing a rotation in the x-z plane to find the principal axes. This is done in the Appendix.

Having obtained (35), we can find $[x_1(0)]^2 + [x'_1(0)]^2/v_x^2$ and $[z_1(0)]^2 + [z'_1(0)]^2/v_z^2$ in terms of x_1, x'_1, z_1 and z'_1 at time θ . Substituting into (34) gives the betatron distribution at any time θ .

As an example, consider an initial distribution

$$D dx_1 dx'_1 dz_1 dz'_1 = \frac{1}{4\pi^2 a^2 b^2 v_x v_z} \exp \left\{ -\frac{1}{2a^2} \left[x_1^2(0) + \frac{1}{v_x^2} x_1'^2(0) \right] - \frac{1}{2b^2} \left[z_1^2(0) + \frac{1}{v_z^2} z_1'^2(0) \right] \right\} dx_1 dx'_1 dz_1 dz'_1. \quad (38)$$

Using (35) in (38) yields the distribution density in the four-dimensional betatron phase space at time θ . If we are interested only in the two-dimensional x-z phase space, we must integrate over x'_1 and z'_1 .

The exponent of (38) can be written as

$$-\frac{1}{2} \tilde{Y}(0) R Y(0), \quad (39)$$

where

$$R = \begin{bmatrix} \frac{1}{a^2} & 0 & 0 & 0 \\ 0 & \frac{1}{a^2 v_x^2} & 0 & 0 \\ 0 & 0 & \frac{1}{b^2} & 0 \\ 0 & 0 & 0 & \frac{1}{b^2 v_z^2} \end{bmatrix} \quad (40)$$

Substituting (35) into (39), the exponent in (38) becomes

$$-\frac{1}{2} \sum_{i=1}^4 \sum_{j=1}^4 J_{ij} Y_i Y_j, \quad (41)$$

where J is the 4 x 4 symmetric matrix

$$J = \tilde{M} R M / (\lambda_+ - \lambda_-)^2. \quad (42)$$

We must now integrate (38) over $Y_2 = x_1'$ and $Y_4 = z_1'$. Substitute (41) into (38) and integrate over Y_2 and Y_4 . The result in terms of J_{ij} is

$$D dx_1 dz_1 = \frac{1}{2\pi a^2 b^2 v_x v_z \sqrt{J_{22}J_{44} - J_{24}^2}} \exp [-f_7 x_1^2 - f_8 z_1^2 - f_9 x_1 z_1] , \quad (43)$$

where

$$f_7 = \frac{1}{2} J_{11} - \frac{J_{14}^2}{2J_{44}} - \frac{(J_{12}J_{44} - J_{24}J_{14})^2}{2J_{44}(J_{22}J_{44} - J_{24}^2)}$$

$$f_8 = \frac{1}{2} J_{33} - \frac{J_{34}^2}{2J_{44}} - \frac{(J_{23}J_{44} - J_{24}J_{34})^2}{2J_{44}(J_{22}J_{44} - J_{24}^2)}$$

$$f_9 = J_{13} - \frac{J_{34}J_{14}}{J_{44}} - \frac{(J_{12}J_{44} - J_{24}J_{14})(J_{23}J_{44} - J_{24}J_{34})}{J_{44}(J_{22}J_{44} - J_{24}^2)} ,$$

with parameters J_{ij} obtained from (42).

Note that consistent with the spirit of this paper, we should keep only the slowly varying factors in the exponent of (43). But since the fast oscillating corrections are difficult to separate out and in any case are negligibly small, we will simply avoid the algebraic complications and perform all computations numerically. Illustrated in Fig. 5 (a) and (b) are sequences of snapshots of the beam density profiles in the x - z plane. Notice that except for fast oscillating terms, the tilt of the distribution vanishes when $\delta = 0$ or $a = b$. In fact, when $a = b$, the time evolution of the distribution is uninteresting in that it remains invariant except for the small fast oscillating terms. The graphs are drawn for $a \neq b$. As stated above, the tilt vanishes when $\delta = 0$ and we have a simple pulsation. In the cases where $\delta \neq 0$, a characteristic tilting results. The maximum tilting angle is equal to twice the angle of the principal axes, which is given by $\tan^{-1} (-Q/\lambda_-)$.

Fig. 5. Time evolution of beam distribution profiles. Parameters chosen are $a = 2$, $b = \frac{1}{2}$, $Q = 0.01$, $v_x = 8.7$ and (a) $\delta = 0$, (b) $\delta = 0.01$. Notice the pulsations in both cases and the tilting in case (b). The maximum tilting angle is $2\theta_0$, where θ_0 is the angle between the principal axes and the x-z axes and is given by $\theta_0 = \tan^{-1}(-Q/\lambda_-)$.

APPENDIX

In this appendix, we want to find the principal axes in the x-z plane and then give a more detailed proof of Eqs. (12) and (35).

From (3) and (7), it is easy to show, assuming $\theta_x = \theta_z = \theta$ and $k = 0$, that

$$\begin{aligned} x_1'' + (v_x^2 + Q^2)x_1 &= - (v_x + v_z)Qz_1 \\ z_1'' + (v_z^2 + Q^2)z_1 &= - (v_x + v_z)Qx_1 \end{aligned} \quad (A.1)$$

Let

$$\begin{bmatrix} y_+ \\ y_- \end{bmatrix} = \begin{bmatrix} \lambda_- & -Q \\ Q & \lambda_- \end{bmatrix} \begin{bmatrix} x_1 \\ z_1 \end{bmatrix} ,$$

then

$$y_{\pm}'' + f_{\pm}^2 y_{\pm} = 0 , \quad (A.2)$$

where $f_{\pm}$ are defined in (37). The variables y_+ and y_- define the principal axes in the x-z plane. The solution of Eq. (A.2) can be written in matrix form

$$\begin{bmatrix} y_+(\theta) \\ y_+'(\theta) \\ y_-(\theta) \\ y_-'(\theta) \end{bmatrix} = S(\theta) \begin{bmatrix} y_+(0) \\ y_+'(0) \\ y_-(0) \\ y_-'(0) \end{bmatrix} ,$$

where

$$S(\theta) = \begin{bmatrix} \cos f_+\theta & \frac{\sin f_+\theta}{f_+} & 0 & 0 \\ -f_+\sin f_+\theta & \cos f_+\theta & 0 & 0 \\ 0 & 0 & \cos f_-\theta & \frac{\sin f_-\theta}{f_-} \\ 0 & 0 & -f_-\sin f_-\theta & \cos f_-\theta \end{bmatrix} .$$

From

$$Y \equiv \begin{bmatrix} x_1' \\ x_1 \\ z_1' \\ z_1 \end{bmatrix} = T \begin{bmatrix} y_+ \\ y_+' \\ y_- \\ y_-' \end{bmatrix} ,$$

with

$$T = \frac{1}{\lambda_-^2 + Q^2} \begin{bmatrix} \lambda_- & 0 & Q & 0 \\ 0 & \lambda_- & 0 & Q \\ -Q & 0 & \lambda_- & 0 \\ 0 & -Q & 0 & \lambda_- \end{bmatrix} ,$$

we have

$$Y(\theta) = T S(\theta) T^{-1} Y(0)$$

or

$$Y(0) = T S(-\theta) T^{-1} Y(\theta) \quad .$$

Multiplying this equation out yields Eqs. (12) and (35).

Acknowledgment

We would like to thank K. Jellett for his assistance in the numerical work.

AWC:MM/ph